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**Comonotonic Monte Carlo and its applications in option  
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# Comonotonic Monte Carlo and its applications in option pricing and quantification of risk

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## Abstract

Monte Carlo (MC) simulation is a technique that provides approximate solutions to a broad range of mathematical problems. A drawback of the method is its high computational cost, especially in a high-dimensional setting, such as estimating the Tail Value-at-Risk for large portfolios or pricing basket options and Asian options. For these types of problems, one can construct an upper bound in the convex order by replacing the copula by the comonotonic copula. This comonotonic upper bound can be computed very quickly, but it gives only a rough approximation. In this paper we introduce the Comonotonic Monte Carlo (CoMC) simulation, by using the comonotonic approximation as a control variate. The CoMC is of broad applicability and numerical results show a remarkable speed improvement. We illustrate the method for estimating Tail Value-at-Risk and pricing basket options and Asian options when the logreturns follow a Black-Scholes model or a variance gamma model.

*Keywords:* Control Variate Monte Carlo, Comonotonicity, Option pricing

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## 1. Introduction

Monte Carlo (MC) simulation is a well known technique in different domains of mathematics such mathematical finance, see Glasserman (2003); Benninga (2014). The method is based on the estimation of the expectation of a real-valued random variable  $X$  by generating many independent and identically distributed samples of  $X$ , denoted  $X_1, \dots, X_n$ . The natural unbiased estimator for  $E(X)$  is then the sample mean  $\bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i$ .

A typical application of the Monte Carlo method in finance is the estimation of the no-arbitrage price of a specific derivative security (e.g. a call option), which can be expressed as the expected value of its discounted payoff under the risk neutral measure. For instance the price at time  $t$  of a European call option with strike price  $K$  and maturity date  $T$  on an underlying with price process  $S_t$  can be obtained as the expectation of its discounted payoff  $e^{-r(T-t)}(S_T - K)_+$  under the risk-neutral probability  $Q$ ,

$$EC(K, T, t) = E^Q[e^{-r(T-t)}(S_T - K)_+].$$

For the computation of this price by Monte Carlo simulation, we generate a large number of price paths  $S_T$  and compute the discounted payoffs and their sample mean. The obtained result is an unbiased estimate of the option price.

Another application of the Monte Carlo method in finance is estimating risk measures, such as Tail Value-at-Risk. The Tail Value-at-Risk of a portfolio at the probability level  $p$  is the arithmetic average of its quantiles from the threshold  $p$  to 1. The Monte Carlo method estimates these quantiles by generating a huge number of portfolio values for which the exceedance probabilities  $Pr[X \geq x] = E[I(X \geq x)]$  are computed, where  $I(\cdot)$  denotes the indicator function. A classical interpolation and inversion then gives an estimate for the quantile.

The main shortcoming of the Monte Carlo method is its high computational cost. By the Central Limit Theorem, if  $X_1, \dots, X_n$  have finite variance  $\sigma^2$ , then  $\bar{X}_n$  is approximately Gaussian and  $\text{Var}(\bar{X}_n) = \frac{\sigma^2}{n}$ . Consequently, the standard error of the crude Monte Carlo estimate is of order  $O(\frac{1}{\sqrt{n}})$  and thus, to double the precision, one must run four times the number of simulations. Alternatively, strategies for reducing  $\sigma$  should be considered.

Several variance reduction techniques can be used in companion with the Monte Carlo method, such as antithetic variables, control variates and importance sampling. A detailed survey of these techniques is given in Ripley (1987). In this paper we focus on the well-known control variate method for variance reduction.

The applications considered in this paper are simulation problems based on multivariate random variables, such as basket options where the price depends on several underlying securities. In these problems the closed form expressions are often available for the univariate cases. For instance, in a lognormal world the price of a European call option (which only depends on

$S_T$ ) can be calculated with the Black-Scholes pricing formula. As comonotonicity essentially reduces a multivariate problem to univariate ones, leaving the marginal distributions intact, we propose to use the comonotonic approximation as a control variate in a so-called Comonotonic Monte Carlo (CoMC) framework. One further step that can be considered is utilizing the CoMC method in addition to other existing control variates in the framework of a multi-variable control variate method.

The Comonotonic Monte Carlo method is particularly useful to estimate distortion risk measures for sums of random variables, such as Tail Value-at-Risk (TVaR). The application domain of this method can also be extended to the risk measures which can be written as a linear combination of distortion risk measures, such as the Expected Shortfall (ESF). As the ESF basically consists of a stop-loss transform, its mathematical concept is very similar to option pricing, so the technique is useful in this domain as well.

The structure of this paper is as follows. First we discuss the control variate method for reducing the variance. Next, we describe the application of the comonotonicity concept to construct the comonotonic control variate. In the fourth section, we illustrate the CoMC framework for Asian options, Basket options and TVaR. In the final section we conclude the results.

## 2. Control Variate Monte Carlo Method

The control variate method is a classical approach for reducing the variance, and hence improving the efficiency, in Monte Carlo simulation, see e.g. Kemna and Vorst (1990) for the pricing of arithmetic Asian options.

In the control variate Monte Carlo method, when we generate the sample values to estimate  $E[X]$ , we use the same values to estimate the expectation of a different random variable  $Y$  which resembles  $X$  and for which the analytical calculation of its expectation,  $E[Y]$ , is straightforward. Assuming that  $E[Y]$  is known, we can then determine the error of estimating  $E[X]$  and use it to correct the estimate of  $E[X]$ . As an example, in the case of Asian option pricing, we can calculate the value of a geometric Asian call option using both the (analytical) Black-Scholes formula and Monte Carlo simulation. If the simulation turns out to underestimate the real option price, one could argue that the corresponding estimate for the arithmetic Asian option will also be too low and adjust the Monte Carlo estimate accordingly.

In general, the control variate method can be formulated as follows. Suppose that there exists a random variable  $Y$ , related to  $X$ , for which  $E[Y]$  is known.

Considering that the sample means  $\bar{X}_n$  and  $\bar{Y}_n$  are unbiased estimators for  $E(X)$  and  $E[Y]$  respectively, the adjusted estimator

$$\bar{X}_n(\lambda) = \bar{X}_n - \lambda(\bar{Y}_n - E[Y]), \quad \lambda \in \mathbb{R}$$

is also an unbiased estimator of  $E[X]$ , i.e.  $E[\bar{X}_n(\lambda)] = E[\bar{X}_n] = E[X]$ . The control parameter  $\lambda$  is an arbitrary scalar, but in order to minimize the variance of  $\bar{X}_n(\lambda)$  we should set it to

$$\lambda^* = \frac{Cov(X, Y)}{Var[Y]} = \rho \sqrt{\frac{Var[X]}{Var[Y]}}$$

with  $\rho$  denoting the correlation between  $X$  and  $Y$ . This choice yields a minimum variance  $(1 - \rho^2)Var[\bar{X}_n]$ , which is obviously smaller than  $Var[\bar{X}_n]$  as  $-1 \leq \rho \leq 1$ . Therefore the control variate unbiased estimator  $\bar{X}_n(\lambda)$  leads to a smaller variance compared to the obtained variance from the crude Monte Carlo unbiased estimator  $\bar{X}_n$ .

Note that the optimal  $\lambda^*$  involves moments of  $X$  and  $Y$  that are generally unknown. Hence  $\lambda$  is often chosen to be 1. This choice makes sense if the control variate  $Y$  is very similar to  $X$ , and thus if  $\rho$  is close to 1 and  $Var[X] \approx Var[Y]$ . The optimal  $\lambda^*$  could also be estimated from the simulated data, but one should take into account that this introduces bias of order  $O(1/n)$  to the estimator  $\bar{X}_n(\lambda)$ . A straightforward way to overcome this problem is to use different samples for the estimation of  $\lambda$  and  $E[X]$ .

### 3. Comonotonic Control Variate

The concept of comonotonicity has received a lot of interest in the recent actuarial and financial literature, mainly due to its interesting properties that can be used to facilitate various complicated problems, see Dhaene et al. (2014); Deelstra et al. (2011); Liu et al. (2013); Tsuzuki (2013). In the following sections we describe the properties of comonotonicity that can be used to construct a comonotonic control variate for a multivariate Monte Carlo simulation.

#### 3.1. Comonotonic Upper Bound

Consider a random vector  $\underline{X} = (X_1, \dots, X_n)$  for which the marginal distributions of  $X_i$ 's are known. In order to determine the distribution function of the sum of random variables,  $\mathbb{S} = \sum_{i=1}^n X_i$ , it is often assumed that the

individual random variables  $X_i$ 's are mutually independent. However, the assumption of mutual independence might be violated and may result in underestimating the sum  $\mathbb{S}$ . To avoid this underestimating, we need to consider the dependence structure of the random vector  $\underline{X}$ . If the joint distribution of  $\underline{X}$  is unspecified or less tractable, we can derive an upper bound for the sum  $\mathbb{S}$  in convex order<sup>1</sup>.

Dhaene et al. (2002) proved that the convex-largest sum of the components of a random vector  $\underline{X}$  with given marginal distributions will be obtained in the case that the random vector  $\underline{X}$  has a comonotonic distribution, which means that each two possible outcomes  $(x_1, \dots, x_n)$  and  $(y_1, \dots, y_n)$  of  $(X_1, \dots, X_n)$  are ordered component-wise.

*Definition 1*

A random vector  $\underline{X} = (X_1, \dots, X_n)$  is comonotonic if and only if it has a comonotonic copula i.e. for all  $\underline{x} = (x_1, \dots, x_n)$ , we have

$$F_{\underline{X}}(\underline{x}) = \min \{F_{X_1}(x_1), F_{X_2}(x_2), \dots, F_{X_n}(x_n)\}. \quad (1)$$

*Proposition 1*

If  $\underline{X}$  has a comonotonic copula then for  $U \sim \text{Uniform}(0, 1)$ , we have

$$\underline{X} \stackrel{d}{=} (F_{X_1}^{-1}(U), (F_{X_2}^{-1}(U), \dots, (F_{X_n}^{-1}(U))). \quad (2)$$

**Proof.** See Dhaene et al. (2002). □

According to Proposition 1, for any random vector  $\underline{X} = (X_1, \dots, X_n)$ , not necessarily comonotonic, we can construct its comonotonic counterpart which will be denoted by  $\underline{X}^c = (X_1^c, \dots, X_n^c)$  as follows

$$\underline{X}^c := (F_{X_1}^{-1}(U), (F_{X_2}^{-1}(U), \dots, (F_{X_n}^{-1}(U))).$$

Clearly  $\underline{X}^c$  and  $\underline{X}$  have the same marginal distributions  $F_{X_i}$ , but they have a different copula. Also the sum of its components,  $\mathbb{S}^c = \sum_{i=1}^n X_i^c$ , gives an

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<sup>1</sup>A random variables  $X$  is said to precede a random variable  $Y$  in the convex order sense, written  $X \leq_{cx} Y$ , if and only if

$$\begin{cases} E(X) = E(Y) \\ E(X - d)_+ \leq E(Y - d)_+ \quad \text{for all real } d \end{cases}$$

upper bound for the sum  $\mathbb{S}$ . In fact, replacing the copula by a comonotonic copula yields the largest sum in the convex order, see Dhaene et al. (2002).

### 3.2. Additivity property

Here we discuss the additivity property of the quantile function and any distortion risk measure for a sum of comonotonic random variables. The additivity property will be used to compute the comonotonic upper bound.

#### *Proposition 2*

The quantile function  $F_{\mathbb{S}^c}^{-1}$  of a sum  $\mathbb{S}^c$  of comonotonic random variables with distribution functions  $F_{X_1}, \dots, F_{X_n}$  is additive

$$F_{\mathbb{S}^c}^{-1}(p) = \sum_{i=1}^n F_{X_i}^{-1}(p), \quad 0 < p < 1. \quad (3)$$

**Proof.** See Dhaene et al. (2002). □

By the additivity property exhibited in Proposition 2, calculating the distribution function of  $\mathbb{S}^c$  is straightforward. The distribution of  $\mathbb{S}^c$  simply follows from inverting its quantile function. This makes the comonotonic upper bound  $\mathbb{S}^c$  a natural control variate, namely comonotonic control variate, in a Monte Carlo simulation.

In the following propositions it will be shown that any distortion risk measure has the additivity property for comonotonic variables. This property facilitates deriving the comonotonic control variate for estimating the Tail Value-at-Risk (TVaR) and option pricing in a so-called comonotonic Monte Carlo (CoMC) framework.

#### *Definition 2*

The distorted expectation of a random variable  $X$  is defined by

$$\rho_g[X] = \int_{-\infty}^0 (g(\bar{F}_X(x)) - 1) dx + \int_0^{\infty} g(\bar{F}_X(x)) dx, \quad (4)$$

where  $\bar{F}_X(x) = 1 - F_X(x)$  denotes the tail function of  $F_X(x)$  and the function  $g(\cdot)$  is a so-called distortion function, i.e. a non-decreasing function  $g : [0, 1] \rightarrow [0, 1]$  such that  $g(0) = 0$  and  $g(1) = 1$ .

According to Wang (1996),  $\rho_g$  is known as the distortion risk measure associated with distortion function  $g$ . Note that the distortion function  $g$  is assumed to be independent of the distribution function of  $X$ .



*Proposition 3*

The distortion risk measure for a sum of comonotonic variables is additive i.e. for any distortion function  $g$  and all random variables  $X_i$  we have

$$\rho_g[\mathbb{S}^c] = \sum_{i=1}^n \rho_g[X_i]. \quad (5)$$

**Proof.** This result is proved in Wang (1996) for non-negative random variables, but it can be easily extended to all real-valued variables. Substituting  $g(\bar{F}_X(x))$  by  $\int_0^{\bar{F}_X(x)} dg(q)$  in (4) and changing the order of the integrations, we find that

$$\rho_g[X] = \int_0^1 \bar{F}_X^{-1}(q) dg(q) = \int_0^1 F_X^{-1}(1-q) dg(q) \quad (6)$$

for any distortion function  $g$  and any random variable  $X$ . Combining equations (3) and (6) yields

$$\rho_g[\mathbb{S}^c] = \int_0^1 F_{\mathbb{S}^c}^{-1}(1-q) dg(q) = \int_0^1 \sum_{i=1}^n F_{X_i}^{-1}(q) dg(q) = \sum_{i=1}^n \rho_g[X_i],$$

which completes the proof. □

*Corollary 1*

The Tail Value-at-Risk,  $TVaR_X(p)$ , at level  $p \in (0, 1)$  given by

$$TVaR_X(p) = \frac{1}{1-p} \int_p^1 F_X^{-1}(q) dq \quad (7)$$

is a distortion risk measure with distortion function

$$g(x) = \min\left(\frac{x}{1-p}, 1\right), 0 \leq x \leq 1,$$

hence it is additive for comonotonic random variables.

We remark that risk measures which can be written as a linear combination of distortion risk measures satisfy the additivity property as well. For instance the Expected Shortfall (ESF) defined as

$$ESF_X(p) = E[(X - F_X^{-1}(p))_+]$$

is not a distortion risk measure, but it is also additive for comonotonic random variables.

*Corollary 2*

The ESF can be written as a linear combination of distortion risk measures given by

$$TVaR_X(p) = F_X^{-1}(p) + \frac{1}{1-p}ESF_X(p),$$

see Dhaene et al. (2006), thus it follows

$$\begin{aligned} ESF_{\mathbb{S}^c}(p) &= (1-p)(TVaR_{\mathbb{S}^c}(p) - F_{\mathbb{S}^c}^{-1}(p)) \\ &= (1-p) \left( \sum_{i=1}^n TVaR_{X_i}(p) - \sum_{i=1}^n F_{X_i}^{-1}(p) \right) \\ &= \sum_{i=1}^n ESF_{X_i}(p), \quad 0 < p < 1. \end{aligned}$$

It is worth noting that the Expected Shortfall basically consists of a stop-loss premium, so it is very closely related to the pricing of options. More generally, for the stop-loss transform of a sum of comonotonic variables we have the following result.

*Corollary 3*

By choosing  $p = F_{\mathbb{S}^c}(K)$  in Corollary 2, it follows that the stop-loss premium  $E[(\mathbb{S}^c - K)_+]$  of a sum  $\mathbb{S}^c$  of comonotonic random variables with strictly increasing distribution functions  $F_{X_1}, \dots, F_{X_n}$  can be written as

$$E[(\mathbb{S}^c - K)_+] = \sum_{i=1}^n [(X_i - F_{X_i}^{-1}(F_{\mathbb{S}^c}(K)))_+], \quad \forall K \in \mathbb{R}. \quad (8)$$

The additivity property of distortion risk measures for comonotonic variables reduces the multivariate problem to univariate ones.

Furthermore, replacing the copula by a comonotonic copula leaves the marginal distributions intact. Therefore the simulated samples in the univariate cases are readily available from the main simulation routine. Considering the mentioned properties, the comonotonic upper bound is an obvious control variate choice. In the next section we apply the CoMC method to Asian and Basket option pricing and to estimating the TVaR of a portfolio.

## 4. Comonotonic Control Variate for Asian Options, Basket Options and Tail Value-at-Risk

### 4.1. Asian Option

An Asian option is a path dependent option, for which the payoff depends on the average price of the underlying risky asset in the considered time interval. We consider a discrete set of  $n$  time points along the time interval  $[0, T]$  such that the asset price,  $S_t$ , is observed at time points  $0 = t_0 < t_1 < \dots < t_n = T$ . In a complete market, the no-arbitrage price of the Asian option at time 0 is its expected discounted pay-off under a martingale measure  $Q$  given by

$$AC(n, K, T) = e^{-rT} E^Q \left[ \left( \frac{1}{n} \sum_{i=1}^n S_{t_i} - K \right)_+ \right],$$

where  $r$  is the risk-free rate.

Since in general the distribution of the average  $\frac{1}{n} \sum_{i=1}^n S_{t_i}$  of dependent random variables is not available, it is not possible to derive a closed-form expression for the Asian option price. Therefore the comonotonic Monte Carlo simulation is a useful method for estimating the price of Asian option. In the following, we derive the comonotonic control variate for this estimation in the CoMC framework.

The comonotonic upper bound of  $AC(n, K, T)$ , which is obtained by replacing the price vector  $(S_{t_1}, \dots, S_{t_n})$  by its comonotonic counterpart  $(S_{t_1}^c, \dots, S_{t_n}^c)$ , reads

$$\begin{aligned} AC_{com}(n, K, T) &= e^{-rT} E^Q \left[ \left( \frac{1}{n} \sum_{i=1}^n S_{t_i}^c - K \right)_+ \right] \\ &= \frac{e^{-rT}}{n} E^Q \left[ (\mathbb{S}^c - nK)_+ \right], \end{aligned}$$

where  $\mathbb{S}^c = \sum_{i=1}^n S_{t_i}^c$ . Note that from Proposition 1 we have,  $\sum_{i=1}^n S_{t_i}^c = \sum_{i=1}^n F_{S_{t_i}}^{-1}(U)$ .

Using Corollary 3, we have

$$AC_{com}(n, K, T) = \frac{e^{-rT}}{n} \sum_{i=1}^n E^Q \left[ \left( S_{t_i} - F_{S_{t_i}}^{-1}(F_{\mathbb{S}^c}(nK)) \right)_+ \right].$$

Hence,  $AC_{com}(n, K, T)$  can be rewritten in terms of prices of European call options  $EC(k_i, t_i)$  at time 0 with exercising times  $t_i$  and strike prices  $k_i$

$$AC_{com}(n, K, T) = \frac{1}{n} \sum_{i=1}^n e^{-r(T-t_i)} EC(k_i, t_i), \quad (9)$$

where  $k_i = F_{S_{t_i}}^{-1}(F_{\mathbb{S}^c}(nK))$ .

For the practical determination of the strike prices  $k_i$ , the distribution function of the comonotonic sum  $F_{\mathbb{S}^c}$  has to be calculated and evaluated at  $nK$  by Proposition 2. Under the the risk-neutral probability, this can be done numerically in a straightforward way. The  $k_i$ 's are then obtained by evaluating the inverse distribution function of the marginals at  $F_{\mathbb{S}^c}(nK)$ .

Considering the Lévy market model for asset prices we derive the comonotonic upper bound (9). We assume that the price  $S_t$  of the risky asset follows a variance gamma process  $\{X_t^{(VG)}, t \geq 0\}$ , which is a popular class of Lévy process. The risk-neutral model for the asset price is then given by

$$S_t = S_0 \frac{\exp((r - q)t)}{E[\exp(X_t)]} \exp(X_t).$$

The factor  $\exp((r - q)t) / E[\exp(X_t)]$  guarantees that the risk-neutral setting holds by considering a mean correcting argument, see Albrecher et al. (2005).

The price  $EC(K, T)$  of a European call option with strike price  $K$  and maturity date  $T$  under the variance gamma model can be calculated by the Carr and Madan formula, see Madan et al. (1998); Albrecher et al. (2005), which formulates the price of European call option in terms of the characteristic function of the underlying Lévy process.

Let  $\alpha$  be a positive constant such that the  $\alpha$ th moment of the stock price exists and let  $\phi$  be the characteristic function of the variance gamma process. Then we have

$$EC(K, T) = \frac{\exp(-\alpha \log(K))}{\pi} \int_0^{+\infty} \exp(-iv \log(K)) \varrho(v) dv, \quad (10)$$

where

$$\varrho(v) = \frac{\exp(-rT) E[\exp(i(v - (\alpha + 1)i) \log(S_T))]}{\alpha^2 + \alpha - v^2 + i(2\alpha + 1)v}$$

$$= \frac{\exp(-rT)\phi(v - (\alpha + 1)i)}{\alpha^2 + \alpha - v^2 + i(2\alpha + 1)v}.$$

Hence the comonotonic upper bound can be obtained using the European option pricing formula (10) of Carr and Madan and (9). We consider this comonotonic upper bound as a control variate in the CoMC method for estimating the price of Asian options in a variance gamma model.

#### 4.1.1. Numerical example

We illustrate the performance of the CoMC method to estimate the price of an Asian option when the underlying asset follows a variance gamma process. We consider an arithmetic Asian option with maturity of 1 year and averaging every month (i.e. 12 averaging dates). The initial value of the stock price is normalized to be 100 and the yearly risk free interest rate is  $r = 0.02$ . The parameters of the variance gamma process that were used to generate the price paths are those from Albrecher et al. (2005). Five values (80, 90, 100, 110 and 120) are assumed for the strike price  $K$ .

In Table 1 we compare the performance of the crude Monte Carlo (MC) method and the CoMC method based on 10,000,000 simulated paths. The estimated price based on MC and CoMC is represented by  $AC_{MC}$  and  $AC_{CoMC}$  respectively. The performance of CoMC method is examined by comparing its computation time and obtained variance with the crude Monte Carlo method.

The ratio of computation times ( $T_{MC}/T_{CoMC}$ ) and Variances ( $V_{MC}/V_{CoMC}$ ) are depicted for each of strike prices in Table 1.

$K$	$AC_{MC}$	$AC_{CoMC}$	$V_{MC}/V_{CoMC}$	$T_{MC}/T_{CoMC}$
80	20.7295	20.7441	161.7179	0.5175
90	11.8211	11.7605	94.3999	0.5176
100	4.5661	4.5684	54.5796	0.5164
110	0.9405	0.9295	21.3384	0.5143
120	0.2039	0.2006	11.3167	0.5157

Table 1: Performance of the CoMC method in Asian option pricing

We observe that by increasing the strike price, the ratio of variance reduction  $V_{MC}/V_{CoMC}$  decreases while the ratio of computation time  $T_{MC}/T_{CoMC}$  is

almost constant. In other words, the CoMC method performs well when the option is in the money.

Since  $\mathbb{S}^c$  is larger than  $\mathbb{S}$  in convex order, they have the same expectation value,  $E(\mathbb{S}^c) = E(\mathbb{S})$ , but  $\mathbb{S}^c$  has heavier tails than  $\mathbb{S}$ , see Vyncke et al. (2001). Therefore, the difference of  $E[(\mathbb{S} - K)_+]$  and  $E[(\mathbb{S}^c - K)_+]$  is smaller for the in the money cases compared to the other cases where the strike price is comparatively larger. This results in a higher correlation between  $(\mathbb{S} - K)_+$  and  $(\mathbb{S}^c - K)_+$  when  $K$  is small. Consequently the comonotonic control variate method performs better for the in the money cases.

The efficiency of the method can be quantified by comparing the number of samples required for the crude Monte Carlo method to achieve the same degree of accuracy. For the different strike prices  $K = 80, 90, 100, 110, 120$  the number of samples required for the crude Monte Carlo to reach the same level of precision as the CoMC varies between 11 to 160 times the original number of samples.

Considering that the required computation time for the comonotonic control variate Monte Carlo method is almost twice the computation time of crude Monte Carlo method, it can be concluded that employing the CoMC method significantly increases the computation performance and efficiency.

#### 4.2. Basket Option

A Basket option is an option on a portfolio (or basket) of several underlying assets whose payoff is dependent on the value of a weighted sum of the underlying assets. Consider a portfolio of  $n$  risky assets with price process  $\{S_i(t), t \geq 0\}$ ,  $i = 1, \dots, n$  and positive weights  $a_i$ ,  $\sum_{i=1}^n a_i = 1$ .

In a complete market, the no-arbitrage price of a Basket call option with maturity date  $T$  and strike price  $K$  at time 0 is given by

$$BC(n, K, T) = e^{-rT} E^Q \left[ \left( \sum_{i=1}^n a_i S_i(T) - K \right)_+ \right], \quad (11)$$

which is the expected payoff of the call option under a martingale measure  $Q$ , discounted at the risk-free rate  $r$ .

In the classical Black-Scholes model, the price process of assets are assumed to follow the risk-neutral stochastic differential equations

$$dS_i(t) = (r - q_i)S_i(t)dt + \sigma_i S_i(t)dB_i(t), \quad (12)$$

where the  $B_i(t)$  are Brownian motions,  $q_i$  and  $\sigma_i$  denote the dividend rate and the volatility of the underlying asset  $i$  respectively. Given the above dynamics, the price of the  $i$ th asset at time  $T$  equals

$$S_i(T) = S_i(0)e^{(r-q_i-\sigma_i^2/2)T+\sigma_i B_i(T)}.$$

Thus, the random variable  $S_i(T)/S_i(0)$  is log normally distributed with parameters  $(r - q_i - \sigma_i^2/2)T$  and  $\sigma_i^2 T$ . We assume that the Brownian motions  $B_i$  and  $B_j$  are correlated with a constant correlation  $\rho_{ij}$ .

Since the distribution of a sum of log-normally distributed random variables is not log normal, the distribution of the weighted sum  $\sum_{i=1}^n a_i S_i(T)$  is not known analytically and hence determining the price of the Basket option is not straightforward.

In order to estimate the price of a Basket option in the comonotonic Monte Carlo framework, the corresponding comonotonic control variate can be constructed as follows.

By replacing the weighted average  $\sum_{i=1}^n a_i S_i(T)$  with the comonotonic weighted average  $\sum_{i=1}^n a_i S_i^c(T)$  in (11), the comonotonic upper bound of  $BC(n, K, T)$  is then given by

$$BC_{com}(n, K, T) = e^{-rT} E^Q \left[ (\mathbb{S}^c - K)_+ \right], \quad (13)$$

where  $\mathbb{S}^c = \sum_{i=1}^n a_i S_i^c(T) = \sum_{i=1}^n a_i F_{S_i(T)}^{-1}(U)$ , see Proposition 1.

Note that by using Corollary 3, the comonotonic upper bound (13) can be written in terms of a weighted sum of European call options,

$$\begin{aligned} BC_{com}(n, K, T) &= e^{-rT} \sum_{i=1}^n a_i E^Q \left[ \left( S_i(T) - F_{S_i(T)}^{-1}(F_{\mathbb{S}^c}(K)) \right)_+ \right] \\ &= \sum_{i=1}^n a_i EC_i(k_i, T), \end{aligned} \quad (14)$$

where  $k_i = F_{S_i(T)}^{-1}(F_{\mathbb{S}^c}(K))$ .

We know from Proposition 2 that the quantile function of a sum of comonotonic random variables is simply the sum of the quantile functions of the marginal distributions. Moreover, in case of strictly increasing and continuous marginals, the cumulative distribution function  $F_{\mathbb{S}^c}(x)$  is uniquely determined by

$$F_{\mathbb{S}^c}^{-1}(F_{\mathbb{S}^c}(x)) = \sum_{i=1}^n a_i F_{S_i(T)}^{-1}(F_{\mathbb{S}^c}(x)) = x \quad F_{\mathbb{S}^c}^{-1}(0) < x < F_{\mathbb{S}^c}^{-1}(1), \quad (15)$$

see Kaas et al. (2000). Hence using the inverse distribution function of  $S_i(T)$  given by

$$F_{S_i(T)}^{-1}(p) = S_i(0)e^{(r-q_i-\sigma_i^2/2)T+\sigma_i\sqrt{T}\Phi^{-1}(p)}, \quad \forall p \in (0, 1), \quad (16)$$

where  $\Phi$  is the cdf of the standard normal distribution, (15) results in

$$\sum_{i=1}^n a_i S_i(0) e^{(r-q_i-\sigma_i^2/2)T+\sigma_i\sqrt{T}\Phi^{-1}(F_{\text{Sc}}(K))} = K, \quad (17)$$

from which  $F_{\text{Sc}}(K)$  can be obtained numerically. Therefore the strike prices  $k_i$  for asset  $i$  can be determined by evaluating (16) at  $F_{\text{Sc}}(K)$ .

Having obtained the  $k_i$ 's, the price of European call options with strike price  $k_i$  and maturity date  $T$  at time 0 reads

$$EC_i(k_i, T) = S_i(0)\Phi(d_{i,1}) - k_i e^{-rT}\Phi(d_{i,2}),$$

where

$$d_{i,1} = \frac{\ln(S_i(0)/k_i) + (r_i + \sigma_i^2/2)T}{\sigma_i\sqrt{T}}, \quad d_{i,2} = d_{i,1} - \sigma_i\sqrt{T}.$$

Thus, the comonotonic control variate for a Basket option pricing in Black-Scholes setting can be determined by the weighted summation of  $EC_i(k_i, T)$  in (14).

It should be noted that an alternative control variate can be obtained by replacing the weighted arithmetic average with the geometric average. Since the geometric average of the log-normally distributed variables is also log-normally distributed, obtaining the closed-form formulation for the geometric control variate is trivial, see Kemna and Vorst (1990).

#### 4.2.1. Numerical example

In this section, the performance of the CoMC method is evaluated for pricing basket options. We consider a Basket option consisting of seven assets. The data used for this purpose is based on the basket of seven stock indices underlying the G-7 index-linked guaranteed investment certificates offered by Canada Trust Co, see Milevsky and Posner (1998a,b).

The risk free interest rate is  $r = 0.063$  and the maturity date is set to 1 year. The initial value of each asset in the basket is normalized to be 100. The other considered parameters are given in Table 2 and 3.



country	index	weight (in%)	volatility (in%)	dividend yield (in%)
Canada	TSE 100	10	11.55	1.69
Germany	DAX	15	14.53	1.36
France	CAC 40	15	10.68	2.39
U.K.	FSTE 100	10	14.62	3.62
Italy	MIB 30	5	17.99	1.92
Japan	Nikkei 225	20	15.59	0.81
U.S.	S&P 500	25	15.68	1.66

Table 2: G-7 index linked guaranteed investment certificate weightings

	Canada	Germany	France	U.K.	Italy	Japan	U.S.
Canada	1.00	0.35	0.10	0.27	0.04	0.17	0.71
Germany	0.35	1.00	0.39	0.27	0.50	-0.08	0.15
France	0.10	0.39	1.00	0.53	0.70	-0.23	0.09
U.K.	0.27	0.27	0.53	1.00	0.45	-0.22	0.32
Italy	0.04	0.50	0.70	0.45	1.00	-0.29	0.13
Japan	0.17	-0.08	-0.23	-0.22	-0.29	1.00	-0.03
U.S.	0.71	0.15	0.09	0.32	0.13	-0.03	1.00

Table 3: Correlation structure of the G-7 index

The performance of the CoMC method is examined by comparing its computation time and obtained variance with the crude Monte Carlo method. The ratio of variances  $V_{MC}/V_{CoMC}$  is depicted for different strike prices in Table 4. The estimated prices for the considered Basket option based on MC and CoMC methods are represented by  $BC_{MC}$  and  $BC_{CoMC}$  respectively. The obtained results for both methods are based on 10,000,000 simulated paths.

$K$	$BC_{MC}$	$BC_{CoMC}$	$V_{MC}/V_{CoMC}$	$T_{MC}/T_{CoMC}$
80	23.1366	23.1387	273.6565	0.5191
90	13.8112	13.8166	35.2039	0.5778
100	5.6312	5.6440	10.7839	0.5577
110	1.2320	1.2387	3.2520	0.5260
120	0.1334	0.1336	1.4792	0.5339

Table 4: Performance of the CoMC method in Basket option pricing

According to Table 4, the variance reduction capability of CoMC decreases by increasing the strike price, while the required computation resource for the CoMC method is almost twice the crude Monte Carlo method. Therefore the method is best suited for the in the money cases with the same reasoning given in section 4.1.1.

In this example it is observed that, based on the estimation error, the number of samples required for the crude MC to reach the same level of accuracy as the precision of the CoMC, varies between 3 to 273 times the original number of samples.

Considering that the comonotonicity assumption induces the strongest positive dependency, it is expected that the correlation structure has a strong influence on the performance of the CoMC method. Therefore it is worth to examine these effects quantitatively in Basket option pricing. To this end, we consider a Basket option consists of the first two assets of Table 2 with the equal weights. The performance of the CoMC method is evaluated for different strike prices and correlations  $\rho$ .

$\rho$	$K$	$BC_{MC}$	$BC_{CoMC}$	$V_{MC}/V_{CoMC}$
0.75	80	23.4258	23.4196	2910.4
	90	14.5317	14.5374	534.91
	100	7.3315	7.3320	147.17
0.25	80	23.3801	23.3843	706.49
	90	14.2747	14.2701	87.210
	100	6.6607	6.6828	19.961
-0.25	80	23.3826	23.3734	313.20
	90	14.0618	14.0633	30.654
	100	5.9161	5.9256	6.6869
-0.75	80	23.3657	23.3714	100.95
	90	13.9747	13.9855	9.4026
	100	4.9893	4.9947	2.7801

Table 5: Influence of the correlation on the efficiency of CoMC

For the constant strike price  $K$ , the variance reduction ratio  $V_{MC}/V_{CoMC}$  increases for higher level of positive assets correlation  $\rho$ , see Table 5. On the other hand, in case of a negative correlation, which is in contradiction

with the comonotonicity assumption, the performance of the CoMC method is even worse than crude Monte Carlo method.

As mentioned in the theoretical background, the geometric control variate is an alternative control variate in Basket option pricing. The second numerical example is aimed at comparing the performance of the comonotonic control variate with its competent alternative, the geometric control variate. For this purpose, similar to the previous example, we consider a two asset basket and compute the efficiency of the methods for different weights  $a_i$  and initial prices  $S_0$  while the strike price is the initial value of the portfolio and correlation coefficient is considered to be constant,  $\rho = 0.35$ .

$a_1/a_2$	$(S_{01}, S_{02})$	$V_{MC}/V_{CoMC}$	$V_{MC}/V_G$
1	(100, 100)	29.7870	539.7020
	(100, 50)	34.1525	7.2490
	(50, 100)	50.0775	8.1827
0.25	(100, 100)	105.2138	2251.6
	(100, 50)	48.7746	17.9618
	(50, 100)	377.3903	41.6317

Table 6: Comparison of CoMC and geometric control variate

Table 6 compares the variance reduction  $V_{MC}/V_G$  obtained by the geometric control variate to the variance reduction  $V_{MC}/V_{CoMC}$  of the CoMC method. The results show that for the cases where the initial prices are equal the geometric control variate performs much better than the CoMC method. In the other cases, the performance of the comonotonic control variate method surpasses the variance reduction obtained by the geometric control variate method. We conclude that for non-equal initial prices, the CoMC method has real added value.

#### 4.3. Tail Value-at-Risk

The Tail Value-at-Risk (TVaR) of a portfolio at a given level of probability  $p \in (0, 1)$ , is defined as the arithmetic average of its quantiles from the threshold  $p$  up to 1, see Corollary 1.

Consider a portfolio consisting of  $n$  risky assets where each asset price  $S_i(t)$  follows the risk-neutral stochastic differential equation in (10). The value of the portfolio at time  $T$  equals  $\mathbb{S} = \sum_{i=1}^n a_i S_i(T)$ . Since the distribution

function of  $\mathbb{S}$  is unknown, determining the Tail Value-at-Risk of the loss of the portfolio,  $TVaR_{-\mathbb{S}}(p)$ , is not straightforward. Therefore, the comonotonic Monte Carlo method can be employed for estimating TVaR.

As already mentioned in Corollary 1, the Tail Value-at-Risk is additive for a sum of comonotonic random variables. Hence, the comonotonic control variate for estimating the TVaR for the loss of portfolio in the CoMC framework is given by

$$\begin{aligned} TVaR_{com} &= TVaR_{-\mathbb{S}^c}(p) = \frac{1}{1-p} \int_p^1 F_{-\mathbb{S}^c}^{-1}(q) dq \\ &= \sum_{i=1}^n a_i \left( -\frac{1}{1-p} \int_p^1 F_{S_i(T)}^{-1}(1-q) dq \right), \end{aligned} \quad (18)$$

where  $\mathbb{S}^c = \sum_{i=1}^n a_i S_i^c(T) = \sum_{i=1}^n a_i F_{S_i(T)}^{-1}(U)$ , see Proposition 1.

Considering that the price  $S_i(T)$  of each asset at time  $T$  is log normally distributed, we have for (18)

$$-\frac{1}{1-p} \int_p^1 F_{S_i(T)}^{-1}(1-q) dq = -\frac{E(S_i(T))}{1-p} \left( \Phi(\Phi^{-1}(1-p) - \sigma_i \sqrt{T}) \right), \quad (19)$$

where  $\Phi$  denotes the standard normal distribution function and  $\sigma_i$  is the volatility of asset  $i$ , see Sandström (2010).

#### 4.3.1. Numerical example

The performance of the CoMC method is evaluated here for the calculation of the TVaR risk measure. We consider a portfolio consisting of the first two assets, Canada and Germany, of Table 2. We generate the price paths in a Black-Scholes setting using the parameters given in Tables 2 and 3.

risk measure	$TVaR_{MC}$	$TVaR_{CoMC}$	$V_{MC}/V_{CoMC}$	$T_{MC}/T_{CoMC}$
TVaR(0.90)	86.4584	86.4631	3.5193	0.5321
TVaR(0.95)	83.6405	83.6446	2.6940	0.5227
TVaR(0.99)	78.4445	78.4150	1.8013	0.5600

Table 7: The performance of CoMC method for TVaR

The results of the CoMC method are compared with the ones obtained from the crude Monte Carlo method for the different levels of probability  $p$ , see Table 7. For this specific correlation structure, the variance reduction ratio,  $V_{MC}/V_{CoMC}$ , obtained by the CoMC method is rather limited.

## 5. Conclusion

In this paper, we presented a novel control variate Monte Carlo method based on the concept of comonotonicity. The CoMC method is explained for basket options, Asian options and TVaR.

We evaluated the performance of the method in realistic cases by the illustrative numerical examples. The parametric study revealed the strong dependence of the method performance on the correlation between assets for Basket option pricing. Moreover, we showed that increasing the strike price reduces the efficiency of the method in Asian option and Basket option pricing. Thus the CoMC method is best suited for the in the money options. The realistic benchmark examples show that the precision of estimating the price of Asian option and Basket option is drastically increased by employing the CoMC method while the computation time is not increased considerably compared to the crude Monte Carlo method.

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