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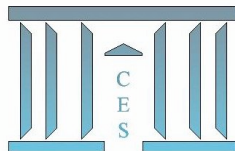
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Comonotonic Monte Carlo and its applications in option pricing and quantification of risk

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Abstract

Monte Carlo (MC) simulation is a technique that provides approximate solutions to a broad range of mathematical problems. A drawback of the method is its high computational cost, especially in a high-dimensional setting. Estimating the Tail Value-at-Risk for large portfolios or pricing basket options and Asian options for instance can be quite time-consuming. For these types of problems, one can construct an upper bound in the convex order by replacing the copula by the comonotonic copula. This comonotonic upper bound can be computed very quickly, but it gives only a rough approximation. In this paper we introduce the Comonotonic Monte Carlo (CoMC) simulation, which uses the best features of both approaches. By using the comonotonic approximation as a control variate we get more accurate estimates and hence the simulation is less time-consuming. The CoMC is of broad applicability and numerical results show a remarkable speed improvement. We illustrate the method for estimating Tail Value-at-Risk and pricing basket options and Asian options.

Keywords: Control Variate Monte Carlo, Comonotonicity, Option pricing

1. Introduction

The concept of comonotonicity has received a lot of interest in the recent actuarial and financial literature due to its interesting properties that can be used to facilitate various complicated problems, see Dhaene et al. (2014); Deelstra et al. (2010, 2011); Liu et al. (2013); Tsuzuki (2013). Using the properties of comonotonicity, which are described in the following sections,

it can be employed as a powerful tool for reducing the variance in Monte Carlo simulation.

The crude Monte Carlo method is based on the estimation of the expectation of a real-valued random variable X by generating many independent and identically distributed of X , denoted X_1, \dots, X_n . The natural unbiased estimator for $E(X)$ is then the sample mean $\bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i$.

A typical application of the Monte Carlo method in finance is the estimation of the no-arbitrage price of a specific derivative security (e.g. a call option), which can be expressed as the expected value of its discounted payoff under the risk neutral measure. For instance the price at time t of a European call option with strike price K and maturity date T on an underlying with price process S_t can be obtained as the expectation of its discounted payoff $e^{-r(T-t)}(S_T - K)_+$ under the risk-neutral probability Q ,

$$EC(K, T, t) = E^Q[e^{-r(T-t)}(S_T - K)_+].$$

For the computation of this price by Monte Carlo simulation, a large number of price paths S_T should be generated. Then the discounted payoffs and the sample mean are computed. The obtained result is an unbiased estimate of the option price.

Similarly, for the computation of the Value-at-Risk of a portfolio, a huge number of portfolio values should be generated for which the exceedance probabilities $Pr[X \geq x] = E[I(X \geq x)]$ are computed, where $I(\cdot)$ denotes the indicator function. The estimation of Tail Value-at-Risk can be done similarly, since the Tail Value-at-Risk of a portfolio at the probability level p is the arithmetic average of its quantiles from the threshold p to 1.

The main shortcoming of the Monte Carlo method is its high computational cost. Considering the central Limit Theorem, if X_1, \dots, X_n have finite variance σ^2 , then \bar{X}_n is approximately Gaussian and $\text{Var}(\bar{X}_n) = \frac{\sigma^2}{n}$. Consequently, the standard error of the crude Monte Carlo estimate is of order $O(\frac{1}{\sqrt{n}})$ and thus, to double the precision, one must run four times the number of simulations. Alternatively, strategies for reducing σ should be considered.

Several variance reduction techniques are utilized in companion with the Monte Carlo method. The most well-known are the antithetic variate method, the control variate approach and the importance sampling approach. A detailed survey of these techniques is given in Ripley (1987).

In this paper we focus on the application of the control variate method for variance reduction. The applications considered in this paper are simu-

lation problems based on multivariate random variables, e.g. basket options where the price depends on several underlying securities. In these problems the closed form expressions are often available for the univariate cases. For instance, the price of a European call option (which only depends on S_T) can be calculated with the Black-Scholes pricing formula.

As comonotonicity essentially reduces a multivariate problem to a univariate one and the marginal distributions remain intact, we propose to use the comonotonic approximation as a control variate in a so-called Comonotonic Monte Carlo (CoMC) framework. One further step which can be considered is utilizing the CoMC method in addition to the other existing control variates in the framework of a multi-variable control variate method.

Another application field of CoMC is estimating distorted risk measures. This is due to the fact that the comonotonicity concept is particularly useful for estimating the "distorted" expectation of a sum S of random variables. The distorted expectation of a random variable X is defined as:

$$\rho_g[X] = \int_{-\infty}^0 (g(\bar{F}_X(x)) - 1) dx + \int_0^{\infty} g(\bar{F}_X(x)) dx \quad (1)$$

where $\bar{F}_X(x) = 1 - F_X(x)$ denotes the tail function of $F_X(x)$ and the function $g(\cdot)$ is a so-called distortion function, i.e. a non-decreasing function $g : [0, 1] \rightarrow [0, 1]$ such that $g(0) = 0$ and $g(1) = 1$. According to Wang (1996), ρ_g is known as the distortion risk measure associated with distortion function g . Note that the distortion function g is assumed to be independent of the distribution function of X . For $g(x) = x$, we have the ordinary expectation $\rho_g[X] = E[X]$, while $g(x) = I(x > 1 - p)$ corresponds to $\rho_g[X] = F_X^{-1}(p)$.

Tail Value-at-Risk (TVaR) is also a distortion risk measure with $g(x) = \min\left(\frac{x}{1-p}, 1\right)$ $0 \leq x \leq 1$. The application domain of this method can be extended to the risk measures which can be written as a linear combination of distortion risk measures, such as the Expected Shortfall (ESF). As the ESF basically consists of a stop-loss transform, its mathematical concept is very similar to option pricing.

The structure of this paper is as follows. First the control variate method for reducing the variance is discussed in section two. Next, the application of the comonotonicity concept for defining the control variate is described. In the fourth section, the framework of using CoMC for computing Asian options, Basket options and TVaR is described.

Section five is devoted to the numerical examples. Illustrative numerical examples are given for Asian options, Basket options and TVaR. Also the efficiency improvement of the CoMC method is evaluated by comparing the estimated errors with the ones obtained for the crude Monte Carlo method. Finally, the obtained results and the relevant interpretations are concluded in the last section.

2. Control Variate Monte Carlo Method

A classical method for reducing the variance, and hence improving the efficiency, is the antithetic variate method. Basically for every draw X_i we also generate the opposite and equally likely realization X'_i and use both in our calculations. Due to the negative covariance between X_i and X'_i and the basic fact that $Var[X_i + X'_i] = Var[X_i] + Var[X'_i] + 2Cov[X_i, X'_i]$, the estimate $(\bar{X}_n + \bar{X}'_n)/2$ will have a smaller variance than if we would simply take twice as many samples. The antithetic variate method however is not very powerful. In the case of option pricing for instance, Boyle (1977) reports a very moderate efficiency improvement.

A different approach is the so-called control variate method, e.g. used by Kemna and Vorst (1990) for the pricing of arithmetic Asian options. When we generate the sample values to estimate $E[X]$, we could use the same values to estimate the expectation of a different but related random variable Y . Assuming that $E[Y]$ is known in advance, we can then determine the error of this last estimate and use it to correct the first estimate. For example, we could calculate the value of a geometric Asian call option using both the (analytical) Black-Scholes formula and Monte Carlo simulation.

If the simulation turns out to underestimate the real option price, one could argue that the corresponding estimate for the arithmetic Asian option will also be too low and adjust the Monte Carlo estimate accordingly.

In general, suppose that there exists a random variable Y , related to X and for which $E[Y]$ is known. With \bar{Y}_n being an unbiased estimator for $E[Y]$, the adjusted estimator,

$$\bar{X}_n(\lambda) = \bar{X}_n - \lambda(\bar{Y}_n - E[Y]), \lambda \in \mathbb{R}$$

will also be unbiased, i.e. $E[\bar{X}_n(\lambda)] = E[\bar{X}_n] = E[X]$. The control parameter λ is an arbitrary scalar, but in order to minimize the variance of $\bar{X}_n(\lambda)$ we

should set it to

$$\lambda^* = \frac{\text{Cov}(X, Y)}{\text{Var}[Y]} = \rho \sqrt{\frac{\text{Var}[X]}{\text{Var}[Y]}} \quad (2)$$

with ρ denoting the correlation between X and Y . This choice yields a minimum variance $(1 - \rho^2)\text{Var}[\bar{X}_n]$, which is obviously smaller than $\text{Var}[\bar{X}_n]$ as $-1 \leq \rho \leq 1$. Note that the optimal λ^* involves moments of X and Y that are generally unknown. Hence λ is often chosen to be -1 . This choice makes sense if the control variate Y is very similar to X , and thus ρ is close to 1 and $\text{Var}[X] \approx \text{Var}[Y]$. The optimal λ^* could also be estimated from the simulated data,

$$\lambda^* = \frac{\sum_{i=1}^n (X_i - \bar{X}_n)(Y_i - \bar{Y}_n)}{\sum_{i=1}^n (Y_i - \bar{Y}_n)^2}$$

but one should take into account that this introduces bias of order $O(1/n)$ to the estimator $\bar{X}_n(\lambda)$. A straightforward way to overcome this problem is to use different samples for the estimation of λ and $E[X]$.

3. Control Variate based on the Comonotonic Upper-Bound

3.1. Concept of Comonotonicity

In finance and actuarial science, one often encounters the problem of determining the distribution function of the sum of random variables $\mathbb{S} = \sum_{i=1}^n X_i$, like the aggregate claim of a portfolio, over a certain period. For evaluation convenience it is mostly assumed that the individual claims X_i of a portfolio are mutually independent, even though it has been recognized that the assumption of mutual independence is often violated and may result in an underestimate of the total claims.

To avoid this underestimating we need to deal with a sum of dependent random variables, $\mathbb{S} = \sum_{i=1}^n X_i$, for which the marginal distributions of X_i are known but with an unspecified or less tractable joint distribution.

In such a case, it may be helpful to find the dependence structure of random vector (X_1, \dots, X_n) producing the least favorable aggregate claim S with the given marginal distribution. Therefore, given the marginal distributions of the random variables X_i 's in $\mathbb{S} = \sum_{i=1}^n X_i$, we will look for the joint distribution with the largest sum in the convex order sense, see Dhaene et al. (2002).

It was proved in Dhaene et al. (2002) that the convex-largest sum of the components of a random vector with given marginal distribution will be obtained in the case that the random vector $\underline{X} = (X_1, \dots, X_n)$ has a comonotonic distribution, which means that each two possible outcomes (x_1, \dots, x_n) and (y_1, \dots, y_n) of (X_1, \dots, X_n) are ordered component-wise. In other words, a set $A \subseteq \mathbb{R}^n$ is comonotonic if for any (x_1, \dots, x_n) and (y_1, \dots, y_n) in A , $x_i \leq y_i$ for some i implies that $x_j \leq y_j$ for all j . A random vector (X_1, \dots, X_n) is called comonotonic if it has a comonotonic support. The following theorem summarizes some characterizations of the comonotonicity concept, see Vyncke (2004):

Proposition 1 (*Equivalent conditions for comonotonicity*)

A random vector $\underline{X} = (X_1, \dots, X_n)$ is comonotonic if and only if one of the following equivalent conditions holds:

- (a) \underline{X} has a comonotonic support;
- (b) \underline{X} has a comonotonic copula, i.e. for all $\underline{x} = (x_1, \dots, x_n)$, we have

$$F_{\underline{X}}(\underline{x}) = \min \{F_{X_1}(x_1), F_{X_2}(x_2), \dots, F_{X_n}(x_n)\} \quad (3)$$

- (c) For $U \sim Uniform(0, 1)$, we have

$$\underline{X} \stackrel{d}{=} (F_{X_1}^{-1}(U), (F_{X_2}^{-1}(U), \dots, (F_{X_n}^{-1}(U)) \quad (4)$$

- (d) A random variable Z and non-decreasing functions $f_i (i = 1, \dots, n)$ exist such that

$$\underline{X} \stackrel{d}{=} (f_1(Z), f_2(Z), \dots, f_m(Z)) \quad (5)$$

For any random vector (X_1, \dots, X_n) , not necessarily comonotonic, we can construct its comonotonic counterpart $\underline{X}^c = (X_1^c, \dots, X_n^c)$ by using 4:

$$\underline{X}^c := (F_{X_1}^{-1}(U), (F_{X_2}^{-1}(U), \dots, (F_{X_n}^{-1}(U))$$

Clearly \underline{X} and \underline{X}^c have the same marginal distributions F_{X_i} , but they have a different copula. From 3 we see that the class of all n-dimensional random

vectors with given marginal distributions F_{X_i} is bounded from above by the comonotonic random vector \underline{X}^c .

Also the sum of its components gives an upper bound. In fact, replacing the copula by a comonotonic copula yields the largest sum in the convex order, see Dhaene et al. (2002).

Definition 1 (*Stop-loss order*)

Consider two random variables X and Y . Then X precedes in the stop-loss order sense, written $X \leq_{sl} Y$, if and only if X has lower stop-loss premium than Y (see Dhaene et al. (2002)):

$$E(X - d)_+ \leq E(Y - d)_+$$

for all real d . Notice that $E(X - d)_+$ is the stop-loss premium of X with retention d .

As we have already explained the goal is to replace the random variable X by less favorable one Y , for which the distribution function is easier to obtain. We can see that if $X \leq_{sl} Y$, then also $E[X] \leq E[Y]$ and it is clear that the best approximations arise in the borderline case where $E[X] = E[Y]$, see Dhaene et al. (2002). This leads to the so-called convex order.

Definition 2 (*Convex order*)

Consider two random variables X and Y . Then X precedes Y in the convex order sense, written $X \leq_{cx} Y$, if and only if

$$\begin{cases} E(X) = E(Y) \\ E(X - d)_+ \leq E(Y - d)_+ \end{cases}$$

for all real d .

It can be proven that $X \leq_{cx} Y$ if and only if $E(\nu(X)) \leq E(\nu(Y))$ for all convex functions ν , provided the expectations exist, see e.g. Kaas et al. (2001).

As it is shown in Dhaene et al. (2002), the sum of the components of a random vector precedes the sum of comonotonic counterpart in the convex order sense, written $\mathbb{S} \leq_{cx} \mathbb{S}^c$.

Calculating the distribution function of the sum of comonotonic random variables is straightforward. Indeed, the inverse distribution function of the sum turns out to be equal to the sum of the inverse marginal distribution functions.

3.2. Additivity property of comonotonic variables

In the following propositions we discuss the additivity property of the quantile function for comonotonic variables and the notion of convex and stop-loss order, which will be used to derive the upper bound for the sum of dependent random variables with known marginal distributions. We will use this upper bound to obtain the exact expected value of our control variate in the numerical section.

Proposition 2 (*Additivity of quantile functions for comonotonic variables*)

The quantile function $F_{\mathbb{S}^c}^{-1}$ of a sum \mathbb{S}^c of comonotonic random variables with distribution functions F_{X_1}, \dots, F_{X_n} is given by

$$F_{\mathbb{S}^c}^{-1}(p) = \sum_{i=1}^n F_{X_i}^{-1}(p), \quad 0 < p < 1 \quad (6)$$

Proof. See Dhaene et al. (2002).

For the stop-loss transform of a sum of comonotonic variables we have the following result.

Proposition 3 (*Additivity of stop-loss transform for comonotonic variables*)

The stop-loss transform of the sum \mathbb{S}^c of the components of the comonotonic random vector with strictly increasing distribution functions F_{X_1}, \dots, F_{X_n} is given by $E[(\mathbb{S}^c - d)_+] = \sum_{i=1}^n [(X_i - F_{X_i}^{-1}(F_{\mathbb{S}^c}(d)))_+]$, for all $d \in \mathbb{R}$.

Proof. See Dhaene et al. (2002).

Next we discuss the additivity property of comonotonic variables for any distortion risk measure.

Proposition 4 (*Additivity of distortion risk measures for comonotonic variables*)

For any distortion function g and all random variables X_i , we have

$$\rho_g[\mathbb{S}^c] = \sum_{i=1}^n \rho_g[X_i] \quad (7)$$

Proof. This result is proved in Wang (1996) for non-negative random variables, but it can be easily extended to all real-valued variables. Substituting

$g(\bar{F}_X(x))$ by $\int_0^{\bar{F}_X(x)} dg(q)$ in (1) and changing the order of the integrations, we find that

$$\rho_g[X] = \int_0^1 \bar{F}_X^{-1}(q) dg(q) = \int_0^1 F_X^{-1}(1-q) dg(q) \quad (8)$$

for any distortion function g and any random variable X . Combining equations (5) and (8) yields

$$\rho_g[\mathbb{S}^c] = \int_0^1 F_{\mathbb{S}^c}^{-1}(1-q) dg(q) = \int_0^1 \sum_{i=1}^n F_{X_i}^{-1}(q) dg(q) = \sum_{i=1}^n \rho_g[X_i]$$

which proves the theorem. □

Considering the fact that the Tail Value-at-Risk (TVaR),

$$TVaR_X(p) = \frac{1}{1-p} \int_p^1 F_X^{-1}(q) dq \quad (9)$$

can be written as a distortion risk measure with distortion function $g(x) = \min\left(\frac{x}{1-p}, 1\right)$ $0 \leq x \leq 1$, we can conclude that the TVaR is additive for comonotonic risks.

Note that although the Expected Shortfall (ESF)

$$ESF_X(p) = E[(X - F_X^{-1}(p))_+]$$

is not a distortion risk measure, it is also additive for comonotonic risks. We can show this property as follows. Considering the following equality,

$$TVaR_X(p) = F_X^{-1}(p) + \frac{1}{1-p} ESF_X(p)$$

and using the additivity property of Tail Value-at-Risk and quantile function for comonotonic risks we have,

$$\begin{aligned} ESF_{\mathbb{S}^c}(p) &= (1-p)(TVaR_{\mathbb{S}^c}(p) - F_{\mathbb{S}^c}^{-1}(p)) \\ &= (1-p) \left(\sum_{i=1}^n TVaR_{X_i}(p) - \sum_{i=1}^n F_{X_i}^{-1}(p) \right) \\ &= \sum_{i=1}^n ESF_{X_i}(p), \quad 0 < p < 1 \end{aligned}$$

which proves the additivity property of ESF_X .

4. Comonotonic Monte Carlo Method for Basket Options, Asian Options and Tail Value-at-Risk

4.1. Basket Option

In this section we illustrate the method for pricing a basket option. A basket option is an option on a portfolio (or basket) of assets which allows to hedge the risk of a portfolio consisting of several assets. Consider a portfolio of n risky assets with price process $\{S_i(t), t \geq 0\}$, $i = 1, \dots, n$ and weights a_i . The weights a_i are assumed to be positive and to sum up to 1.

In a complete market, the no-arbitrage value of a basket call option with maturity date T and strike price K is given by

$$BC(n, K, T) = e^{-rT} E^Q [(\sum_{i=1}^n a_i S_i(T) - K)_+]$$

that is, the expected payoff of the call option under a martingale measure Q , discounted at the risk-free rate r .

In the classical Black-Scholes model, the stock prices are assumed to follow the risk-neutral stochastic differential equations

$$dS_i(t) = (r - q_i)S_i(t)dt + \sigma_i S_i(t)dB_i(t) \quad (10)$$

where the $B_i(t)$ are Brownian motions, q_i and σ_i denote respectively the dividend rate and the volatility of the underlying asset i . So, the random variable $S_i(T)/S_i(0)$ is log-normally distributed with parameters $(r - q_i - \sigma_i^2/2)T$ and $\sigma_i^2 T$. We assume that the Brownian motions B_i and B_j are correlated with a constant correlation ρ_{ij} .

Since the distribution of a sum of log-normally distributed random variables is not log-normal, the distribution of the weighted sum $\sum_{i=1}^n a_i S_i(T)$ is not known analytically and hence determining the price of the Basket option is not straightforward.

By using Proposition 3 we can easily construct a comonotonic control variate. Starting with $\mathbb{S}^c = \sum_{i=1}^n a_i S_i^c(T) = \sum_{i=1}^n a_i F_{S_i(T)}^{-1}(U)$ where $\sum_{i=1}^n a_i = 1$ and $a_i > 0$, results in

$$\begin{aligned}
BC_{com}(n, K, T) &= e^{-rT} E^Q \left[\left(\sum_{i=1}^n a_i S_i^c(T) - K \right)_+ \right] \\
&= e^{-rT} \sum_{i=1}^n a_i E^Q \left[\left(S_i(T) - F_{S_i(T)}^{-1}(F_{S^c}(K)) \right)_+ \right] \\
&= \sum_{i=1}^n a_i EC_i(K_i, T) \tag{11}
\end{aligned}$$

with $K_i = F_{S_i(T)}^{-1}(F_{S^c}(K))$. Note that $BC(n, K, T) \leq BC_{com}(n, K, T)$, so (11) gives a static super-hedging strategy consisting of European call options.

4.2. Asian Option

An Asian option is a path dependent option, for which the payoff depends on the average price of the asset in the considered time interval. We consider a discrete set of n time points along the time interval $[0, T]$ such that asset prices are observed at time points $0 = t_0 < t_1 < \dots < t_n = T$.

A European-style arithmetic Asian call option with maturity date T , n averaging dates and exercise price K has a pay-off

$$\left[\frac{1}{n} \sum_{i=1}^n S(t_i) - K \right]_+$$

In this type of option, if the average of the prices of the underlying risky asset at n dates in the time interval exceeds K , the pay-off equals the difference, otherwise the pay-off is zero.

In a complete market, the no-arbitrage price of the Asian is given by

$$AC(n, K, T) = e^{-rT} E^Q \left[\left(\frac{1}{n} \sum_{i=1}^n S(t_i) - K \right)_+ \right]. \tag{12}$$

For illustrating the formulation of the CoMC method we consider a Black-Scholes setting with constant drift r and constant volatility σ . This implies that under the equivalent martingale measure, the random variable $S(t)/S(0)$ is log-normally distributed with parameters $(r - \sigma^2)t$ and $\sigma^2 t$.

The corresponding comonotonic control variate can be obtained based on proposition 2 and using $\mathbb{S}^c = \sum_{i=1}^n S^c(t_i) = \sum_{i=1}^n F_{S(t_i)}^{-1}(U)$ we obtain,

$$\begin{aligned} AC_{com}(n, K, T) &= \frac{e^{-rT}}{n} E^Q \left[\left(\left(\sum_{i=1}^n S^c(t_i) \right) - nK \right)_+ \right] \\ &= \frac{e^{-rT}}{n} \sum_{i=1}^n E^Q \left[\left(S(t_i) - F_{S(t_i)}^{-1}(F_{\mathbb{S}^c}(nK)) \right)_+ \right] \\ &= \frac{1}{n} \sum_{i=1}^n e^{-r(T-t_i)} EC(K_i, t_i) \end{aligned} \quad (13)$$

with $K_i = F_{S(t_i)}^{-1}(F_{\mathbb{S}^c}(nK))$. Considering that $AC(n, K, T) \leq AC_{com}(n, K, T)$, it can be concluded that (13) gives a static super hedging strategy.

4.3. Tail Value-at-Risk

We start this section with recalling the well-known risk measure Value-at-Risk (VaR), which is the p -quantile risk measure and denoted by $Q_p[X]$. For any $p \in (0, 1)$, it is defined for the random variable X as follows,

$$Q_p[X] = \inf \{x \in \mathbb{R} | F_X(x) \geq p\} \quad (14)$$

where $F_X(x) = P(X \leq x)$ is the cumulative distribution function of X . Note that $Q_p[X]$ is equal to $F_X^{-1}(p)$ and $Q_p[-X]$ is equal to $-F_X^{-1}(1-p)$.

From a financial point of view, the Value-at-Risk measures the potential loss in value of a risky asset or portfolio for a given probability.

$TVaR_X(p)$ in equation (9) is actually the arithmetic average of the quantiles of X from the threshold p up to 1. It is obvious that $TVaR_X(p)$ is always larger than corresponding quantile.

Let's consider a portfolio of n risky assets with price process $\{S_i(t), t \geq 0\}$, $i = 1, \dots, n$ and the corresponding weights a_i satisfying $\sum_{i=1}^n a_i = 1$. The value of the portfolio at time T is $\sum_{i=1}^n a_i S_i(T)$ where each asset price follows the same risk-neutral stochastic differential equation in (10). Since the distribution of $\sum_{i=1}^n a_i S_i(T)$ is not known analytically, determining the Tail Value-at-Risk for the value of the portfolio or loss of portfolio at time T is not straightforward.

As we already mentioned in proposition 4, the Tail Value-at-Risk is additive for comonotonic risks. Using this property we can construct the comonotonic control variate.

By using the notation \mathbb{S}^c for comonotonic value of the portfolio and considering that $Q_p[-X] = -F_X^{-1}(1-p)$, the corresponding TVaR at level p of the portfolio loss is equal to

$$TVaR_{-\mathbb{S}^c}(p) = \frac{1}{1-p} \int_p^1 F_{-\mathbb{S}^c}^{-1}(q) dq = \sum_{i=1}^n a_i \left(-\frac{1}{1-p} \int_p^1 F_{S_i(T)}^{-1}(1-q) dq \right) \quad (15)$$

with (see Sandström (2010))

$$-\frac{1}{1-p} \int_p^1 F_{S_i(T)}^{-1}(1-q) dq = -\frac{E(S_i(T))}{1-p} \left(\Phi(\Phi^{-1}(1-p) - \sigma_i \sqrt{T}) \right) \quad (16)$$

where Φ denotes the standard normal distribution function.

5. Numerical Examples

In this section numerical examples are presented to evaluate the performance of the CoMC method for different applications. The results are compared against the crude Monte Carlo method based on their respective estimation error and computation time.

In the following numerical examples, to estimate the correlation between the CoMC and the original paths, we grouped the N_{Total} paths of prices into N_{Loop} sets of N_{Path} paths, where $N_{Total} = N_{Loop} \times N_{Path}$. Each group of N_{Path} paths gives a Monte Carlo estimate and the comonotonic price estimated by rearranging the original paths in increasing order, which will be transferred to the next computation stage where the control parameter is computed.

At the upper level of the Monte Carlo simulation loop, the control variate method is applied to the set of obtained results from the Monte Carlo estimate performed for each of the subgroups. At this final computation stage, the final result of the estimation, the accompanying error and the detailed computation time are computed which will be utilized for evaluating the performance of the method.

5.1. Asian Option

In this example we numerically illustrate CoMC for computing the price of an Asian option in a Black-Scholes setting, as described in the previous sections. The parameters that were used to generate the price paths, are similar to the parameters used in Jacques (1996).

The initial value of stock price is normalized to be 100. The considered risk free interest rate r equals 9% per year, three values (0.2, 0.3 and 0.4) are considered for the yearly volatility, and five values (80, 90, 100, 110 and 120) are assumed for the exercise price K . Denote that here we utilize the daily risk-free interest rate and volatility which are $\ln\left(\frac{1.09}{250}\right)$ and $\frac{\sigma}{\sqrt{250}}$, respectively.

In table 1 we compare the results of crude Monte Carlo (MC) and CoMC (based on 10,000,000 paths each, $N_{Loop} = 1000$ and $N_{Path} = 10,000$) in case $T = 120$ days and the number of averaging dates $n = 30$.

σ	K	MC	s.e.($\times 10^4$)	Time	CoMC	s.e.($\times 10^4$)	Time
0.2	80	21.2691	26	39.92	21.2717	0.3	61.1
	90	11.8658	25	38	11.8669	1	59.3
	100	4.3288	18	36.9	4.3281	1.9	60.4
	110	0.8612	8	36.6	0.8612	2	59.4
	120	0.0917	2.5	38.8	0.0919	0.9	60.3
0.3	80	21.3500	37	39	21.3523	1.1	59.1
	90	12.5416	35	40.1859	12.5437	2	61.6
	100	5.8832	27	38.5	5.8802	2.7	63.7
	110	2.1530	16	39.5	2.1525	3	58.4
	120	0.6257	8.9	38.8	0.6264	2.4	60.7
0.4	80	21.6600	49	39.6	21.6599	2	61.3
	90	13.5362	45	40.6	13.5396	3	59.5
	100	7.4459	35	38.1	7.4462	3.9	60.9
	110	3.6191	26	40.1	3.6211	4.1	60
	120	1.5809	18	40	1.5820	3.9	59

Table 1: Comparing the results of two methods for different volatilities and strike prices based on the computation time (in seconds) and error.

The estimated price based on the Monte Carlo and comonotonic control variate Monte Carlo method, their respective errors and computation time which are depicted in table 1, show the trend of the efficiency of the CoMC

by changing strike prices and volatilities.

It is observed that by increasing the strike prices the variance reduction decreases while the computation time ratio is almost constant. This can be attributed to the reduced correlation between the estimated price of the original path and the comonotonic price obtained for the comonotonic reordered path. In fact considering a greater strike price results in the larger number of incidences of zero pay-off in each subgroup price estimation. Consequently the correlation between the control variate and the original data is reduced.

Increasing the volatility obviously results in the higher error for both Monte Carlo and the CoMC method. This is directly the effect of increasing the variance of generated data due to higher volatility.

The efficiency of the method can be quantified by comparing the number of samples required for the crude Monte Carlo method to achieve the same degree of accuracy. For $\sigma = 0.2$ (with different strike prices $K = 80, 90, 100, 110, 120$), the number of samples required for the crude Monte Carlo to reach the same level of precision as the CoMC varies between 7 to 7500 times the original number of samples. For $\sigma = 0.3$, the number of samples should increase between 14 to 1100 times the original number of samples, while this ratio for $\sigma = 0.4$ varies between 20 to 390. Considering that the required computation time for the comonotonic control variate Monte Carlo method is almost 1.5 times the crude Monte Carlo method, it can be concluded that employing the CoMC method significantly increases the computation performance and efficiency.

5.2. Basket Option

In this section, the performance of the CoMC method is evaluated for pricing basket options. First we consider a benchmark case of a basket option with two assets. We investigate the effect of the correlation structure on the efficiency of the CoMC method by considering the first two assets of the basket option presented in Milevsky and Posner (1998a,b) and table 4. The properties of these two assets are given in table 2.

Asset No.	S_0	σ	Weight	risk free interest rate	dividend yield
1	100	0.1155	$\frac{1}{2}$	0.063	0.0169
2	100	0.1453	$\frac{1}{2}$	0.063	0.0136

Table 2: Details of the Basket assets for the benchmark example.

ρ	K	MC	s.e.($\times 10^4$)	CoMC	s.e.($\times 10^4$)
-0.5	80	23.3736	21	23.3715	1.5
	90	14.0010	21	14.0032	4.9
	100	5.4836	18	5.4840	8
-0.25	80	23.3723	25	23.3720	1.5
	90	14.0601	25	14.0635	4.7
	100	5.9218	20	5.9253	7.5
0.25	80	23.3792	33	23.3834	1
	90	14.2700	31	14.2698	3
	100	6.6811	25	6.6829	5
0.5	80	23.4013	35	23.3979	1
	90	14.3980	32	14.3991	2.5
	100	7.0139	26	7.0166	4
0.75	80	23.4188	38	23.4192	0.6
	90	14.5317	35	14.5374	1.6
	100	7.3295	28	7.3311	2

Table 3: Influence of the correlation on the efficiency of CoMC .

The performance of the method is compared with the crude Monte Carlo method for different correlation coefficients and strike prices in table 3. As it is expected that the performance of the comonotonic control variate method is improved by increasing the correlation, it can be seen in table 3 that the error of the method is decreasing by increasing the correlation while the error of the crude Monte Carlo remains almost unchanged.

Next we consider a multi-asset option written on a basket of seven assets. The data used for this purpose is based on the basket of seven stock indices underlying the G-7 index-linked guaranteed investment certificates offered by Canada Trust Co. More details about this set of data can be found in Milevsky and Posner (1998a,b).

The initial value of each asset in the basket is normalized to be 100. The considered risk free interest rate r equals 6.3%, $T = 1$ year and the remaining parameters considered in the simulations are given in table 4 and table 5.

country	index	weight (in%)	volatility (in%)	dividend yield (in%)
Canada	TSE 100	10	11.55	1.69
Germany	DAX	15	14.53	1.36
France	CAC 40	15	10.68	2.39
U.K.	FSTE 100	10	14.62	3.62
Italy	MIB 30	5	17.99	1.92
Japan	Nikkei 225	20	15.59	0.81
U.S.	S&P 500	25	15.68	1.66

Table 4: Weightings for G-7 index linked guaranteed investment certificate.

	Canada	Germany	France	U.K.	Italy	Japan	U.S.
Canada	1.00	0.35	0.10	0.27	0.04	0.17	0.71
Germany	0.35	1.00	0.39	0.27	0.50	-0.08	0.15
France	0.10	0.39	1.00	0.53	0.70	-0.23	0.09
U.K.	0.27	0.27	0.53	1.00	0.45	-0.22	0.32
Italy	0.04	0.50	0.70	0.45	1.00	-0.29	0.13
Japan	0.17	-0.08	-0.23	-0.22	-0.29	1.00	-0.03
U.S.	0.71	0.15	0.09	0.32	0.13	-0.03	1.00

Table 5: Correlation structure of G-7 index linked guaranteed investment certificate.

The estimation of the basket option price, the accompanying errors and computation times, for both crude Monte Carlo and comonotonic control variate Monte Carlo method, are reported in table 6.

K	MC	s.e.($\times 10^4$)	Time	CoMC	s.e.($\times 10^4$)	Time
60	41.9138	27	96.7	41.9155	0.06	100
70	32.5235	26	93.7	32.5261	0.5	97.6
80	23.1340	26	103.6	23.1382	2	104.3
90	13.8623	25	101.2	13.8649	4.4	103.8
100	5.9051	21	103.1	5.9034	6.5	99.7
110	1.5036	12	99.6	1.5047	5.9	101.8

Table 6: Comparing the results of two methods for different volatilities and strike prices based on the computation time (in seconds) and error.

According to table 6, the ratio of Monte Carlo error to the comonotonic control variate method decreases by increasing the strike price while the ratio of the computation times remains almost constant. This is due to the decrease of correlation between estimate of pay-off and its counterpart, which results in reduction of the efficiency of the control variate method.

In this example it is observed that, based on the estimation error, the number of samples required for the crude MC to reach the same level of accuracy as the precision of the CoMC, varies between 16 to 200000 times the original number of samples, while the computation time of CoMC is almost the same as the crude MC.

5.3. Tail Value-at-Risk

The performance of the CoMC method is evaluated here for the calculation of the TVaR risk measure. We consider a Black-Scholes setting for generating the price paths.

First the influence of the correlation structure on the efficiency of the CoMC method is evaluated. To this end, a portfolio with two assets is considered with the parameters defined in table 2. The TVaR for the value of the portfolio is computed for different correlation coefficients (ρ) and probabilities (p). The results of this study are shown in table 7.

The estimated errors for CoMC are reduced by decreasing p . Moreover, increasing the correlation coefficient improves the performance of the method. This performance improvement is considerable for TVaR. It is due to the fact that $VaR_X(p)$ is computed at one specific probability level p , while $TVaR_X(p)$ is estimated based on the arithmetic average of the quantiles

of X from the threshold p up to 1. This averaging scheme, embedded in the calculation of TVaR, increases the correlation between the estimations obtained for the original paths and the comonotonic estimations, computed for the comonotonic reordered paths.

ρ	p	MC	s.e. ($\times 10^4$)	CoMC	s.e. ($\times 10^4$)	λ
-0.50	0.90	93.1595	36	93.1583	34	-0.30
	0.95	91.2874	45	91.2867	44	-0.24
	0.99	87.7423	81	87.7415	81	-0.10
-0.25	0.90	90.7984	45	90.7975	40	-0.53
	0.95	88.58124	54	88.5794	50	-0.41
	0.99	84.4167	97	84.4120	95	-0.19
0.25	0.90	87.1016	57	87.0986	33	-0.88
	0.95	84.3712	68	84.3678	45	-0.80
	0.99	79.2966	111	79.2780	92	-0.71
0.50	0.90	85.5694	58	85.5608	25	-1.00
	0.95	82.6334	72	82.6260	36	-1.00
	0.99	77.1786	130	77.1743	80	-0.98
0.75	0.90	84.1635	61	84.1587	15	-1.05
	0.95	81.0489	78	81.0468	22	-1.07
	0.99	75.3137	130	75.2999	50	-1.06
0.95	0.90	83.1302	62	83.1163	4.8	-1.02
	0.95	79.8813	79	79.8772	6.9	-1.03
	0.99	73.9324	137	73.9126	16	-1.04

Table 7: Influence of the correlation structure on the efficiency of the CoMC method for computing TVaR.

Next the multi asset portfolio is considered for evaluating the performance of the method in more realistic conditions. The parameters used for these numerical illustrations are exactly the same as the ones used for the numerical illustration of basket option pricing in tables 4 and 5.

risk measure	MC	s.e.($\times 10^4$)	Time	CoMC	s.e.($\times 10^4$)	Time
TVaR(0.90)	89.8474	44	17.6	89.8495	36	20
TVaR(0.95)	87.5451	55	16	87.5393	47	19.7
TVaR(0.99)	83.2276	98	18.5	83.2155	92	19.9

Table 8: Performance of the crude Monte Carlo and the CoMC Monte Carlo for TVaR.

The results of the CoMC method are compared with the crude Monte Carlo in table 8 for $p = 0.90$, $p = 0.95$ and $p = 0.99$. For this specific correlation structure, the estimated error obtained by the CoMC method is not reduced considerably.

6. Conclusion

In this paper a novel control variate Monte Carlo method based on the comonotonic upper bound is presented and the relevant theories are formulated. The framework of applying the CoMC method is explained for basket options, Asian options and TVaR.

Numerical examples are given for evaluating the performance of the method in realistic cases. The parametric study revealed the strong dependence of the method performance on the correlation between assets for Basket option and TVaR. Moreover, it is shown that increasing the strike price reduces the efficiency of the method in Asian option and Basket option.

The realistic benchmark examples show that the precision of estimating the price of Asian option and Basket option is drastically increased by employing the CoMC method while the computation time is not increased considerably compared to the crude Monte Carlo method.

On the other hand, utilizing the realistic correlation structure for computing TVaR, does not result in a considerable improvement of precision, while decreasing the probability level or increasing the correlation, increases the performance of the method.

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