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Weak and Strong Consistency of  
non-parametric estimate of potential function  
for stationary and isotropic pairwise  
interaction point process

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**Abstract**

A method is proposed for estimating the potential function of a non-parametric estimator for stationary and isotropic pairwise interaction point process. The relation between a pair potential and the corresponding Papangelou conditional intensity is considered. Consistency and strong consistency of non-parametric estimate are proved in case of finite-range interaction potential.

**Keywords:** Non parametric estimation, kernel-type estimator, pairwise interaction point process, Papangelou conditional intensity, consistency, rates of strong uniform consistency.

# 1 Introduction

Gibbs point processes are a natural class of models for point patterns exhibiting interactions between the points. By far the most widely applied form in practical analysis is that of pairwise interaction, where the scale and strength of interaction between two points are determined by a so-called pair potential function. For a stationary and isotropic process the pair potential is a function of the distance between the two points. Fields of applications for point processes are image processing, analysis of the structure of tissues in medical sciences, forestry (Matérn [18]), ecology (Diggle [7]), spatial epidemiology (Lawson [16]) and astrophysics (Neyman and Scott [21]).

Pairwise interaction point process densities are intractable as the normalizing constant is unknown and/or extremely complicated to approximate. However, we can resort to estimates of parameters using the conditional intensity. In this paper, we suggest a new non-parametric estimate of the pair potential function for stationary and isotropic pairwise interaction point process specified by a Papangelou conditional intensity on increasing regions single realization is observed. In this case a point process is defined as a random locally-finite counting measure on the  $d$ -dimensional Euclidean space  $\mathbb{R}^d$ . Consistency and strong consistency of the resulting estimator are established.

To our knowledge only one attempt to solve the problem of non-parametric estimation of the pair correlation function and its approximate relation to the pair potential through the Percus Yevick equation (Diggle et al. [8]). The approximation is a result of a cluster expansion method, and it is accurate only for sparse data. Many attempts have been tried to estimate the potential function from point pattern data in a parametric framework ; maximization of likelihood approximations (Ogata and Tanemura [23], Ogata and Tanemura [24], Penttinen [26]), pseudolikelihood maximization (Besag et al. [2], Jensen and Møller [13]) and also some ad hoc methods (Strauss [29], Ripley [27], Hanisch and Stoyan [11], Diggle and Gratton [9], Fiksel [10], Takacs [30], Billiot and Goulard [3]).

Our paper is organized as follows. Section 2 introduces basic notation and definitions. In Section 3, we briefly present some models satisfying the assumptions needed to prove our asymptotic results. In Section 4, we present our main results. Consistency of non-parametric estimator is proved in Section 5, it is based on the knowledge of Papangelou conditional intensity and the iterated Georgii-Nguyen-Zessin formula. Using Orlicz spaces we can obtain a strong consistency of non-parametric estimator in Section 6.

## 2 Basic notation and definitions

Throughout the paper we adopt the following notation. We denote the space of locally finite point configurations in  $\mathbb{R}^d$  by  $N_{lf}$ . The volume of a bounded Borel set  $W$  of  $\mathbb{R}^d$  is denoted by  $|W|$  and  $o$  denotes the origin. For all finite subset  $\Gamma$  of  $\mathbb{Z}^d$ , we denote  $|\Gamma|$  the number of elements in  $\Gamma$ .  $\|\cdot\|$  denotes Euclidean distance on  $\mathbb{R}^d$ .  $\sigma_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$  is the measure of the unit sphere in  $\mathbb{R}^d$ . Let  $\mathbb{S}^{d-1}$  be the unit sphere in  $\mathbb{R}^d$ .

Papangelou conditional intensity (Møller and Waagepetersen [19]) of pairwise interaction point process has the form

$$\lambda(u, \mathbf{x}) = \gamma_0(u) \exp\left(-\sum_{v \in \mathbf{x} \setminus u} \gamma_0(\{u, v\})\right).$$

If  $\gamma_0(u) = \beta$  is a constant and  $\gamma_0(\{u, v\}) = \gamma(\|u - v\|)$  is invariant under translations and rotations, then a pairwise interaction point process is said to be stationary and isotropic or homogeneous. The Papangelou conditional intensity can be interpreted as follows: for any  $u \in \mathbb{R}^d$  and  $\mathbf{x} \in N_{lf}$ ,  $\lambda(u, \mathbf{x}) du$  corresponds to the conditional probability of observing a point in a ball of volume  $du$  around  $u$  given the rest of the point process is  $\mathbf{x}$ . Fortunately does not contain a normalising factor.

For convenience, throughout in this paper, we consider stationary and isotropic pairwise interaction point process. Then its Papangelou conditional intensity at a location  $u$  is given by

$$\lambda(u, \mathbf{x}) = \beta^* \exp\left(-\sum_{v \in \mathbf{x} \setminus u} \gamma(\|v - u\|)\right), \quad \forall u \in \mathbb{R}^d, \mathbf{x} \in N_{lf} \quad (2.1)$$

where  $\beta^*$  is the true value of the Poisson intensity parameter,  $\gamma$  is called the pair potential, a name that originates in physics: it measures the potential energy caused by the interaction among pairs of points  $(u, v)$  as a function of their distance  $\|v - u\|$ . Usually a finite range of interaction,  $R$ , is assumed such that

$$\gamma(\|v - u\|) = 0 \quad \text{whenever } \|v - u\| > R. \quad (2.2)$$

We assume that  $\gamma(\|v - u\|) > 0$  for  $\|v - u\| \leq R$ , so that typical realizations will be more or less regular compared to a completely random arrangement. The pairwise interaction between points may also be described in terms of the pair potential function  $\gamma$  into the interaction function  $\Phi = \exp(-\gamma)$ . For  $\Phi > 1$ ,  $\lambda(u, \mathbf{x})$  is increasing in  $\mathbf{x}$ . For  $\Phi < 1$ ,  $\lambda(u, \mathbf{x})$  is decreasing in  $\mathbf{x}$  (the repulsive case). It can be computed for the case  $\Phi = 1$  which corresponds to the homogeneous Poisson point process with intensity  $\beta^*$ .

### 3 Examples of Papangelou conditional intensity

Examples of conditional intensities are presented in Baddeley et al [1], Møller and Waagepetersen ([19], [20]). The following presents some examples which have been applied in various contexts and satisfying the assumptions needed to prove our asymptotic results.

1. A special case of pairwise interaction is the Strauss process. It has Papangelou conditional intensity

$$\lambda(u, \mathbf{x}) = \beta \Phi^{n_{[0,R]}(u, \mathbf{x} \setminus u)}$$

where  $\beta > 0$ ,  $0 \leq \Phi \leq 1$  and  $n_{[0,R]}(u, \mathbf{x}) = \sum_{v \in \mathbf{x}} \mathbb{1}(\|v - u\| \leq R)$  is the number of pairs in  $\mathbf{x}$  with distance not greater than  $R$ .

2. Piecewise Strauss point process.

$$\lambda(u, \mathbf{x}) = \beta \prod_{j=1}^p \Phi_j^{n_{[R_{j-1}, R_j]}(u, \mathbf{x} \setminus u)}$$

where  $\beta > 0$ ,  $0 \leq \Phi_j \leq 1$ ,  $n_{[R_{j-1}, R_j]}(u, \mathbf{x}) = \sum_{v \in \mathbf{x}} \mathbb{1}(\|v - u\| \in [R_{j-1}, R_j])$  and  $R_0 = 0 < R_1 < \dots < R_p = R < \infty$ .

3. Triplets point process.

$$\lambda(u, \mathbf{x}) = \beta \Phi^{s_{[0,R]}(\mathbf{x} \cup u) - s_{[0,R]}(\mathbf{x} \setminus u)}$$

where  $\beta > 0$ ,  $0 \leq \Phi \leq 1$  and  $s_{[0,R]}(\mathbf{x})$  is the number of unordered triplets that are closer than  $R$ .

4. Lennard-Jones model

$$\lambda(u, \mathbf{x}) = \beta \prod_{v \in \mathbf{x} \setminus u} \Phi(\|v - u\|)$$

with  $\log \Phi(r) = (\theta^6 r^{-6} - \theta^{12} r^{-12}) \mathbb{1}_{[0,R]}(r)$ , for  $r = \|v - u\|$ , where  $\theta > 0$  and  $\beta > 0$  are parameters.

## 4 Main results

Suppose that a single realization  $\mathbf{x}$  of a point process  $\mathbf{X}$  is observed in a bounded window  $W_n$  where  $(W_n)_{n \geq 1}$  is a sequence of cubes growing up to  $\mathbb{R}^d$ . Throughout in this paper,  $\tilde{h}$  is a non-negative measurable function defined for all  $u \in \mathbb{R}^d$ ,  $\mathbf{x} \in N_{lf}$  by

$$\tilde{h}(u, \mathbf{x}) = \mathbb{1} \left( \inf_{v \in \mathbf{x}} \|v - u\| > R \right) = \mathbb{1} (d(u, \mathbf{x}) > R),$$

note that

$$\tilde{F}(o, rv) = \mathbb{E}[\tilde{h}(o, \mathbf{x})\tilde{h}(rv, \mathbf{x})]$$

and

$$J(r) = \int_{\mathbb{S}^{d-1}} \tilde{F}(o, rv) dv.$$

To estimate the function  $\beta^{*2}J(r)\Phi(r)$ , we introduce edge-corrected kernel-type estimator  $\hat{R}_n(r)$  defined by

$$\hat{R}_n(r) = \frac{1}{b_n |W_{n \ominus 2R}| \sigma_d} \sum_{\substack{\neq \\ u, v \in \mathbf{X} \\ \|v-u\| \leq R}} \frac{\mathbb{1}_{W_{n \ominus 2R}}(u)}{\|v-u\|^{d-1}} \tilde{h}(u, \mathbf{X} \setminus \{u, v\}) \tilde{h}(v, \mathbf{X} \setminus \{u, v\}) K_1 \left( \frac{\|v-u\| - r}{b_n} \right). \quad (4.3)$$

$\ominus$  will denote Minkowski subtraction, with the convention that

$W_{n \ominus 2R} = W_n \ominus B(u, 2R) = \{u \in W_n : \|u - v\| \leq 2R \text{ for all } v \in W_n\}$  denotes the  $2R$ -interior of the cubes  $W_n$ , with Lebesgue measure  $|W_{n \ominus 2R}| > 0$ .  $\sum^{\neq}$  signifies summation over distinct pairs.  $K_1 : \mathbb{R} \rightarrow \mathbb{R}$  is an univariate kernel function associated with a sequence  $(b_n)_{n \geq 1}$  of bandwidths satisfying the following:

**Condition**  $K(1, \alpha)$  : The sequence of bandwidths  $b_n > 0$  for  $n \geq 1$ , is chosen such that

$$\lim_{n \rightarrow \infty} b_n = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} b_n |W_{n \ominus 2R}| = \infty.$$

The kernel function  $K_1 : \mathbb{R} \rightarrow \mathbb{R}$  is non-negative and bounded with bounded support, such that:

$$\int_{\mathbb{R}} K_1(u) du = 1, \quad \int_{\mathbb{R}} u^j K_1(u) du = 0, \quad j = 0, 1, \dots, \alpha - 1, \quad \text{for } \alpha \geq 2.$$

To estimate the function  $\beta^* J(r)$  we introduce empiric estimator  $\widehat{J}_n(r)$  defined by

$$\widehat{J}_n(r) = \frac{1}{|W_{n \ominus 2R}|} \sum_{u \in \mathbf{X}} \mathbb{1}_{W_{n \ominus 2R}}(u) \widetilde{h}(u, \mathbf{X} \setminus \{u\}) h^*(u, \mathbf{X} \setminus \{u\}), \quad (4.4)$$

where  $h^*(u, \mathbf{x}) = \int_{\mathbb{S}^{d-1}} \widetilde{h}(rv - u, \mathbf{x}) dv$ . Using the spatial ergodic theorem of Nguyen and Zessin [22], estimator (4.4) turn out to be unbiased and strongly consistent. The natural estimator of Poisson intensity  $\beta^*$  is

$$\widehat{\beta}_n = \frac{\sum_{u \in \mathbf{X}} \mathbb{1}_{\Lambda_{n,R}}(u) \widetilde{h}(u, \mathbf{X} \setminus \{u\})}{\int_{\Lambda_{n,R}} \widetilde{h}(u, \mathbf{X}) du}. \quad (4.5)$$

This estimator turn out to be unbiased and strongly consistent and results on asymptotic normality were obtained by Morsli et al. [5].

Plugging in the above estimator (4.4) and (4.5), then the interaction function  $\Phi(r) = \exp(-\gamma(r))$  for  $r \in (0, R]$  can be estimated using edge-corrected non-parametric estimate by

$$\widehat{\Phi}_n(r) = \frac{\widehat{R}_n(r)}{\widehat{\beta}_n \widehat{J}_n(r)}. \quad (4.6)$$

The strong consistency of the estimators (4.4) and (4.5) implies the following:

**Proposition 1.** *Let  $\gamma$  be pairwise interaction potential defined in (2.1) satisfying condition (2.2). Let  $K_1$  kernel function satisfying Condition  $K(1, \alpha)$  and the function  $J(r) \exp(-\gamma(r))$  has bounded and continuous partial derivatives of order  $\alpha$  for all  $\alpha \geq 1$  in  $(r - \delta, r + \delta)$  for some  $\delta > 0$ . Then as  $n \rightarrow \infty$*

$$\widehat{\Phi}_n(r) \longrightarrow \exp(-\gamma(r)) \quad \text{in probability } \mathbb{P} \quad (\text{resp. } \mathbb{P}\text{-a.s.}) \quad \text{iff}$$

$$\widehat{R}_n(r) \longrightarrow \beta^{*2} J(r) \exp(-\gamma(r)) \quad \text{in probability } \mathbb{P} \quad (\text{resp. } \mathbb{P}\text{-a.s.}).$$

The convergence in probability (consistency) for a wide class of point process will be discussed in Section 5. Conditions ensuring uniform  $\mathbb{P}$ -a.s. convergence of kernel-type estimator of  $\widehat{R}_n(r)$  and the strong consistency  $\widehat{\Phi}_n(r)$  will be discussed in Section 6.

## 5 Consistency

### 5.1 Asymptotic behaviour mean squared error of the kernel-type estimator

In this section we will derive bounds for the mean squared error of the kernel estimator kernel-type estimator of  $\widehat{R}_n(r)$ . We consider the mean square error of  $\widehat{R}_n(r)$ ,  $\text{MSE}(\widehat{R}_n(r)) = \text{Var}(\widehat{R}_n(r)) + (\text{Bias}(\widehat{R}_n(r)))^2$ . So convergence in MSE implies that as  $n \rightarrow \infty$   $(\text{Bias}(\widehat{R}_n(r)))^2 = (\text{E}\widehat{R}_n(r) - \beta^{*2}J(r)\exp(-\gamma(r)))^2 \rightarrow 0$  and  $\text{Var}(\widehat{R}_n(r)) = \text{E}(\widehat{R}_n(r) - \text{E}\widehat{R}_n(r))^2 \rightarrow 0$ . Hence,  $\widehat{R}_n(r)$  is consistent in the quadratic mean and hence consistent. For doing this, we first determine the asymptotic behaviour of  $\text{E}\widehat{R}_n(r)$  and  $\text{Var}\widehat{R}_n(r)$ .

**Theorem 1.** *Let  $\gamma$  be pairwise interaction potential defined in (2.1) satisfying condition (2.2). Let  $K_1$  kernel function satisfying Condition  $K(1, 1)$ . For all  $r \in (0, R]$ , we have*

$$\lim_{n \rightarrow \infty} \text{E}\widehat{R}_n(r) = \beta^{*2}J(r)\exp(-\gamma(r)).$$

*If Condition  $K(1, \alpha)$  is satisfied and the function  $\exp(-\gamma(r))J(r)$  has bounded and continuous partial derivatives of order  $\alpha$  in  $(r - \delta, r + \delta)$  for some  $\delta > 0$  and for all  $\alpha \geq 1$ . Then*

$$\text{E}\widehat{R}_n(r) = \beta^{*2}J(r)\exp(-\gamma(r)) + \mathcal{O}(b_n^\alpha) \quad \text{as } n \rightarrow \infty.$$

**Theorem 2.** *Let  $\gamma$  be pairwise interaction potential defined in (2.1) satisfying condition (2.2). Let  $K_1$  kernel function satisfying Condition  $K(1, \alpha)$  for all  $\alpha \geq 1$  such that  $\int_{\mathbb{R}} K_1^2(\rho)d\rho < \infty$ . For all  $r \in (0, R]$ , we have,*

$$\lim_{n \rightarrow \infty} b_n |W_{n \in 2R}| \text{Var}(\widehat{R}_n(r)) = \frac{2\beta^{*2}}{\sigma_d r^{d-1}} J(r)\exp(-\gamma(r)) \int_{\mathbb{R}} K_1^2(\rho)d\rho.$$

### 5.2 Proof of Theorem 1

*Proof.* We define

$$\widetilde{L}(u_1, \dots, u_s, \mathbf{X}) = \widetilde{h}(u_1, \mathbf{X}) \dots \widetilde{h}(u_s, \mathbf{X}), \quad \widetilde{F}(u_1, \dots, u_s) = \text{E}[\widetilde{h}(u_1, \mathbf{X}) \dots \widetilde{h}(u_s, \mathbf{X})]$$

$$\text{and } \widetilde{J}(\|u_1\|, \dots, \|u_s\|) = \mathbb{1}(\|u_1\| \leq R, \dots, \|u_s\| \leq R).$$



The calculation of expectation and variance of  $\widehat{R}_n(r)$  is based on the iterated Georgii-Nguyen-Zessin (GNZ) formula, see Papangelou [25]:

$$\mathbb{E} \sum_{\substack{\neq \\ u_1, \dots, u_s \in \mathbf{X}}} h(u_1, \dots, u_s, \mathbf{X} \setminus \{u_1, \dots, u_s\}) = \int \dots \int \mathbb{E} h(u_1, \dots, u_s, \mathbf{X}) \lambda(u_1, \dots, u_s, \mathbf{X}) du_1 \dots du_s \quad (5.7)$$

for non-negative functions  $h : (\mathbb{R}^d)^n \times N_{lf} \rightarrow \mathbb{R}$ , where  $\lambda(u_1, \dots, u_s, \mathbf{X})$  is Papangelou conditional intensity and is defined (not uniquely) by

$$\lambda(u_1, \dots, u_s, \mathbf{X}) = \lambda(u_1, \mathbf{X}) \lambda(u_2, \mathbf{X} \cup \{u_1\}) \dots \lambda(u_s, \mathbf{X} \cup \{u_1, \dots, u_{s-1}\}).$$

Applying the preceding formula (5.7) for  $s = 2$ , we derive

$$\begin{aligned} \mathbb{E} \widehat{R}_n(r) &= \frac{1}{b_n |W_{n \ominus 2R}| \sigma_d} \\ &\mathbb{E} \int_{\mathbb{R}^{2d}} \frac{\mathbb{1}_{W_{n \ominus 2R}}(u)}{\|v - u\|^{d-1}} \widetilde{J}(\|v - u\|) \widetilde{L}(u, v, \mathbf{X}) K_1\left(\frac{\|v - u\| - r}{b_n}\right) \lambda(u, v, \mathbf{X}) du dv. \end{aligned}$$

For an interaction radius  $R$ , the Papangelou conditional intensity satisfies

$$\lambda(u, \mathbf{x}) = \lambda(u, \emptyset) \quad \text{for all } \mathbf{x} \text{ with } d(u, \mathbf{x}) > R$$

since points further away from  $u$  than  $R$  do not contribute to the Papangelou conditional intensity at  $u$ . Using the finite range property (2.2), we get

$$\begin{aligned} \mathbb{E} \widehat{R}_n(r) &= \frac{\beta^{*2}}{b_n |W_{n \ominus 2R}| \sigma_d} \\ &\mathbb{E} \int_{\mathbb{R}^{2d}} \frac{\mathbb{1}_{W_{n \ominus 2R}}(u)}{\|v - u\|^{d-1}} \widetilde{J}(\|v - u\|) \widetilde{L}(u, v, \mathbf{X}) K_1\left(\frac{\|v - u\| - r}{b_n}\right) \Phi(\|v - u\|) du dv \\ &= \frac{\beta^{*2}}{b_n \sigma_d} \int_{\mathbb{R}^d} \frac{\widetilde{J}(\|s\|)}{\|s\|^{d-1}} \mathbb{E}[\widetilde{L}(o, s, \mathbf{X})] K_1\left(\frac{\|s\| - r}{b_n}\right) \Phi(\|s\|) ds. \end{aligned}$$

Recall a property of the integration theory (see Briane and Pagès [4] or Rudin [28]). Let  $\mathbb{S}^{d-1}$  be the unit sphere in  $\mathbb{R}^d$ , i.e.  $\mathbb{S}^{d-1} = \{u \in \mathbb{R}^d : \|u\| = 1\}$ , then for any Borel function  $f : \mathbb{R}^d \rightarrow \mathbb{R}_+$ ,

$$\int_{\mathbb{R}^d} f(u) du = \int_0^\infty \int_{\mathbb{S}^{d-1}} f(rz) r^{d-1} \sigma_d dr dz.$$

By combining the above result, we get so:

$$\mathbb{E} \widehat{R}_n(r) = \frac{\beta^{*2}}{\sigma_d} \int_{-r/b_n}^\infty \int_{\mathbb{S}^{d-1}} \widetilde{J}(b_n \varrho + r) \widetilde{F}(o, (b_n \varrho + r)v) K_1(\varrho) \Phi(b_n \varrho + r) \sigma_d d\varrho dv.$$

With bounded support on the kernel function and by dominated convergence theorem, we get as  $n \rightarrow \infty$ ,  $\mathbb{E} \widehat{R}_n(r) \rightarrow \beta^{*2} J(r) \exp(-\gamma(r))$ . Now, we are going to prove the second part of the Theorem 1. We have a product of two functions  $\widetilde{F}(o, (b_n \varrho + r)v) \Phi(b_n \varrho + r)$  and we approximate each one of them with a Taylor formula up to a certain  $\alpha$ . We use Taylor's formula to obtain for  $n \rightarrow \infty$ ,

$$\Phi(b_n \varrho + r) = \Phi(r) + \sum_{k=1}^{\alpha-1} \frac{(b_n \varrho)^k}{k!} \frac{d\Phi}{dr}(r) dr + \frac{b_n^\alpha}{\alpha!} \frac{d^\alpha \Phi}{dr^\alpha}(r + b_n \varrho \theta)$$

and

$$\widetilde{F}(o, (b_n \varrho + r)v) = \widetilde{F}(o, rv) + \sum_{k=1}^{\alpha-1} \frac{(b_n \varrho)^k}{k!} \frac{d\widetilde{F}(o, rv)}{dr} dr + \frac{b_n^\alpha}{\alpha!} \frac{d^\alpha \widetilde{F}}{dr^\alpha}(o, (r + b_n \varrho \theta)v).$$

So we denote this product by  $T_n(rv, r)$ , then we have as  $n \rightarrow \infty$

$$\widetilde{F}(o, (b_n \varrho + r)v) \Phi(b_n \varrho + r) = \widetilde{F}(o, rv) \Phi(r) + \sum_{k=1}^{\alpha-1} T_n(rv, r) (b_n \varrho)^k + \mathcal{O}(b_n^\alpha).$$

It follows that,

$$\begin{aligned} \mathbb{E} \widehat{R}_n(r) &= \beta^{*2} J(r) \Phi(r) \\ &+ \beta^{*2} \int_{\mathbb{S}^{d-1}} \sum_{k=1}^{\alpha-1} T_n(rv, r) b_n^k dv \int_{\mathbb{R}} \varrho^k K_1(\varrho) d\varrho \\ &+ \mathcal{O}(b_n^\alpha) \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Together with Condition  $K(1, \alpha)$  imply the second assertion of Theorem 1. ■

### 5.3 Proof of Theorem 2

*Proof.* The proof of Theorem 2 makes use of the following corollary.

**Corollary 1.** *Consider any Gibbs point process  $\mathbf{X}$  in  $\mathbb{R}^d$  with Papangelou conditional intensity  $\lambda$ . For any non-negative, measurable and symmetric*

function  $f : \mathbb{R}^d \times \mathbb{R}^d \times N_{lf} \rightarrow \mathbb{R}$ , we have

$$\begin{aligned}
& \text{Var} \left( \sum_{u,v \in \mathbf{X}}^{\neq} f(u, v, \mathbf{X} \setminus \{u, v\}) \right) \\
&= 2 \mathbb{E} \int_{\mathbb{R}^{2d}} f^2(u, v, \mathbf{X}) \lambda(u, v, \mathbf{X}) \, du \, dv \\
&+ 4 \mathbb{E} \int_{\mathbb{R}^{3d}} f(u, v, \mathbf{X}) f(v, w, \mathbf{X}) \lambda(u, v, w, \mathbf{X}) \, du \, dv \, dw \\
&+ \mathbb{E} \int_{\mathbb{R}^{4d}} f(u, v, \mathbf{X}) f(w, y, \mathbf{X}) \lambda(u, v, w, y, \mathbf{X}) \, du \, dv \, dw \, dy \\
&- \int_{\mathbb{R}^{4d}} \mathbb{E}[f(u, v, \mathbf{X}) \lambda(u, v, \mathbf{X})] \mathbb{E}[f(w, y, \mathbf{X}) \lambda(w, y, \mathbf{X})] \, du \, dv \, dw \, dy.
\end{aligned}$$

*Proof.* Consider the decomposition (see Jolivet [14] and Heinrich [12])

$$\begin{aligned}
\left( \sum_{u,v \in \mathbf{X}}^{\neq} f(u, v, \mathbf{X} \setminus \{u, v\}) \right)^2 &= 2 \sum_{u,v \in \mathbf{X}}^{\neq} f^2(u, v, \mathbf{X} \setminus \{u, v\}) \\
&+ 4 \sum_{u,v,w \in \mathbf{X}}^{\neq} f(u, v, \mathbf{X} \setminus \{u, v, w\}) f(v, w, \mathbf{X} \setminus \{u, v, w\}) \\
&+ \sum_{u,v,w,y \in \mathbf{X}}^{\neq} f(u, v, \mathbf{X} \setminus \{u, v, w, y\}) f(w, y, \mathbf{X} \setminus \{u, v, w, y\}).
\end{aligned} \tag{5.8}$$

Applying the preceding (GNZ) formula (5.7) combining with (5.8), we obtain

$$\begin{aligned}
& \text{Var} \sum_{u,v \in \mathbf{X}}^{\neq} f(u, v, \mathbf{X} \setminus \{u, v\}) \\
&= \mathbb{E} \left( \sum_{u,v \in \mathbf{X}}^{\neq} f(u, v, \mathbf{X} \setminus \{u, v\}) \right)^2 - \left( \mathbb{E} \sum_{u,v \in \mathbf{X}}^{\neq} f(u, v, \mathbf{X} \setminus \{u, v\}) \right)^2 \\
&= 2 \mathbb{E} \int_{\mathbb{R}^{2d}} f^2(u, v, \mathbf{X}) \lambda(u, v, \mathbf{X}) \, du \, dv \\
&+ 4 \mathbb{E} \int_{\mathbb{R}^{3d}} f(u, v, \mathbf{X}) f(v, w, \mathbf{X}) \lambda(u, v, w, \mathbf{X}) \, du \, dv \, dw \\
&+ \mathbb{E} \int_{\mathbb{R}^{4d}} f(u, v, \mathbf{X}) f(w, y, \mathbf{X}) \lambda(u, v, w, y, \mathbf{X}) \, du \, dv \, dw \, dy \\
&- \int_{\mathbb{R}^{4d}} \mathbb{E}[f(u, v, \mathbf{X}) \lambda(u, v, \mathbf{X})] \mathbb{E}[f(w, y, \mathbf{X}) \lambda(w, y, \mathbf{X})] \, du \, dv \, dw \, dy.
\end{aligned}$$

We obtain the desired result. ■

Applying Corollary 1 to this function

$$f(u, v, \mathbf{X}) = \frac{\mathbb{1}_{W_{n\ominus 2R}}(u)}{\|v - u\|^{d-1}} \tilde{\mathcal{J}}(\|v - u\|) \tilde{L}(u, v, \mathbf{X}) K_1 \left( \frac{\|v - u\| - r}{b_n} \right),$$

it is easily seen that  $\text{Var } \hat{R}_n(r) = A_1 + A_2 + A_3 - A_4$ , where

$$\begin{aligned} A_1 &= \frac{2}{b_n^2 |W_{n\ominus 2R}|^2 \sigma_d^2} \\ &\quad \mathbb{E} \int_{\mathbb{R}^{2d}} \frac{\mathbb{1}_{W_{n\ominus 2R}}(u)}{\|v - u\|^{2(d-1)}} \tilde{\mathcal{J}}(\|v - u\|) \tilde{L}(u, v, \mathbf{X}) K_1^2 \left( \frac{\|v - u\| - r}{b_n} \right) \lambda(u, v, \mathbf{X}) du dv, \\ A_2 &= \frac{4}{b_n^2 |W_{n\ominus 2R}|^2 \sigma_d^2} \mathbb{E} \int_{\mathbb{R}^{3d}} \frac{\mathbb{1}_{W_{n\ominus 2R}}(u) \mathbb{1}_{W_{n\ominus 2R}}(v)}{\|v - u\|^{d-1} \|v - w\|^{d-1}} \tilde{\mathcal{J}}(\|v - u\|, \|v - w\|) \tilde{L}(u, v, w, \mathbf{X}) \\ &\quad K_1 \left( \frac{\|v - u\| - r}{b_n} \right) K_1 \left( \frac{\|v - w\| - r}{b_n} \right) \lambda(u, v, w, \mathbf{X}) du dv dw, \\ A_3 &= \frac{1}{b_n^2 |W_{n\ominus 2R}|^2 \sigma_d^2} \mathbb{E} \int_{\mathbb{R}^{4d}} \frac{\mathbb{1}_{W_{n\ominus 2R}}(u) \mathbb{1}_{W_{n\ominus 2R}}(w)}{\|v - u\|^{d-1} \|w - y\|^{d-1}} \tilde{\mathcal{J}}(\|v - u\|, \|w - y\|) \tilde{L}(u, v, w, y, \mathbf{X}) \\ &\quad \times K_1 \left( \frac{\|v - u\| - r}{b_n} \right) K_1 \left( \frac{\|w - y\| - r}{b_n} \right) \lambda(u, v, w, y, \mathbf{X}) du dv dw dy, \end{aligned}$$

and

$$\begin{aligned} A_4 &= \frac{1}{b_n^2 |W_{n\ominus 2R}|^2 \sigma_d^2} \\ &\quad \left( \mathbb{E} \int_{\mathbb{R}^d} \frac{\mathbb{1}_{W_{n\ominus 2R}}(u)}{\|v - u\|^{d-1}} \tilde{\mathcal{J}}(\|v - u\|) \tilde{L}(u, v, \mathbf{X}) K_1 \left( \frac{\|v - u\| - r}{b_n} \right) \lambda(u, v, \mathbf{X}) du dv \right)^2. \end{aligned}$$

The asymptotic behaviour of the leading term  $A_1$  is obtained by applying the second order Papangelou conditional intensity given by:

$$\lambda(u, v, \mathbf{x}) = \lambda(u, \mathbf{x}) \lambda(v, \mathbf{x} \cup \{u\}) \quad \text{for any } u, v \in \mathbb{R}^d \quad \text{and } \mathbf{x} \in N_{lf}.$$

Using the finite range property (2.2) for each function  $\lambda(u, \mathbf{x})$  and  $\lambda(v, \mathbf{x} \cup \{u\})$ , this implies that

$$\lambda(u, \emptyset) = \beta^* \quad \text{and} \quad \lambda(v, \emptyset \cup \{u\}) = \beta^* \Phi(\|v - u\|) \quad \text{for all } u, v \in \mathbb{R}^d.$$

And by stationarity of  $\mathbf{X}$ , it results

$$\begin{aligned}
A_1 &= \frac{2\beta^{*2}}{b_n^2 |W_{n\ominus 2R}|^2 \sigma_d^2} \\
&\int_{\mathbb{R}^{2d}} \frac{\mathbb{1}_{W_{n\ominus 2R}}(u)}{\|v-u\|^{2(d-1)}} \tilde{J}(\|v-u\|) \mathbb{E}[\tilde{H}(o, v-u, \mathbf{X})] K_1^2\left(\frac{\|v-u\|-r}{b_n}\right) \Phi(\|v-u\|) du dv \\
&= \frac{2\beta^{*2}}{b_n |W_{n\ominus 2R}| \sigma_d^2} \int_{-r/b_n}^{\infty} \int_{\mathbb{S}^{d-1}} \frac{\tilde{J}(b_n \varrho + r)}{(b_n \varrho + r)^{d-1}} \tilde{F}(o, (b_n \varrho + r)w) K_1^2(\varrho) \Phi(b_n \varrho + r) d\varrho d\sigma(w).
\end{aligned}$$

Dominated convergence theorem and assumption of  $K_1$  imply for all  $r \in (0, R]$

$$\lim_{n \rightarrow \infty} b_n |W_{n\ominus 2R}| A_1 = \frac{2\beta^{*2}}{\sigma_d r^{d-1}} J(r) \Phi(r) \int_{\mathbb{R}} K_1^2(\rho) d\rho.$$

We will now show that all other integrals to  $\text{Var} \hat{R}_n(r)$  converge to zero. For the asymptotic behaviour of the second term  $A_2$ , we remember the third order Papangelou conditional intensity by

$$\lambda(u, v, w, \mathbf{x}) = \lambda(u, \mathbf{x}) \lambda(v, \mathbf{x} \cup \{u\}) \lambda(w, \mathbf{x} \cup \{u, v\})$$

for any  $u, v, w \in \mathbb{R}^d$  and  $\mathbf{x} \in N_{lf}$ . Since  $\mathbf{X}$  is a point process to interact in pairs, the interaction terms due to triplets or higher order are equal to one, i.e. the potential  $\gamma(\mathbf{y}) = 0$  when  $n(\mathbf{y}) \geq 3$ , for  $\emptyset \neq \mathbf{y} \subseteq \mathbf{x}$ . Using the finite range property (2.2) for each function  $\lambda(u, \mathbf{x})$ ,  $\lambda(v, \mathbf{x} \cup \{u\})$  and  $\lambda(w, \mathbf{x} \cup \{u, v\})$  and after a elementary calculation, we have

$$\lambda(u, v, w, \emptyset) = \begin{cases} \beta^{*3} \Phi(\|v-u\|) \Phi(\|w-v\|) & \text{if } d(u, w) < R \\ \beta^{*3} \Phi(\|v-u\|) & \text{otherwise.} \end{cases}$$

Which ensures that  $\lambda(u, v, w, \emptyset)$  is a function that depends only variables  $\|v-u\|, \|w-v\|$ , denoted by  $\Phi_1(\|v-u\|, \|w-v\|)$ .

According to the stationarity of  $\mathbf{X}$ , it follows that

$$\begin{aligned}
A_2 &= \frac{4}{b_n^2 |W_{n\ominus 2R}|^2 \sigma_d^2} \mathbb{E} \int_{\mathbb{R}^{3d}} \frac{\mathbb{1}_{W_{n\ominus 2R}}(u) \mathbb{1}_{W_{n\ominus 2R}}(v)}{\|v-u\|^{d-1} \|v-w\|^{d-1}} \tilde{J}(\|v-u\|, \|v-w\|) \tilde{L}(u, v, w, \mathbf{X}) \\
&\times \Phi_1(\|v-u\|, \|w-v\|) K_1\left(\frac{\|v-u\|-r}{b_n}\right) K_1\left(\frac{\|v-w\|-r}{b_n}\right) du dv dw \\
&= \frac{4}{|W_{n\ominus 2R}| \sigma_d^2} \int_{-r/b_n}^{\infty} \int_{-r/b_n}^{\infty} \int_{\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} \frac{|W_{n\ominus 2R} \cap (W_{n\ominus 2R} - (b_n \varrho + r)z)|}{|W_{n\ominus 2R}|} \\
&\times \tilde{F}(o, (b_n \varrho + r)z, (b_n \varrho' + r)z') \Phi_1(b_n \varrho + r, b_n \varrho' + r) K_1(\varrho) K_1(\varrho') d\varrho d\varrho' d\sigma_d(z) d\sigma_d(z').
\end{aligned}$$

The asymptotic behaviour of the leading term  $A_2$  is obtained by applying the dominated convergence theorem. When multiplied by  $b_n|W_{n\ominus 2R}|$ , we get  $\lim_{n \rightarrow \infty} b_n|W_{n\ominus 2R}|A_2 = 0$ .

Next we introduce the finite range property (2.2) and reasoning analogous with the foregoing on  $\lambda(u, v, w, y, \emptyset)$ . which ensures that  $\lambda(u, v, w, y, \emptyset)$  is a function that depends only variables  $\|v - u\|, \|y - w\|, \|w - u\|, \|w - v\|$ , denoted by  $\Phi_2(\|v - u\|, \|y - w\|, \|w - u\|, \|w - v\|)$ . We find that

$$\begin{aligned} A_3 &= \frac{1}{b_n^2|W_{n\ominus 2R}|^2\sigma_d^2} \mathbb{E} \int_{\mathbb{R}^4} \frac{\mathbb{1}_{W_{n\ominus 2R}}(u)\mathbb{1}_{W_{n\ominus 2R}}(w)}{\|v - u\|^{d-1}\|w - y\|^{d-1}} \tilde{J}(\|v - u\|, \|w - y\|) \tilde{L}(u, v, w, y, \mathbf{X}) \\ &\quad \times K_1\left(\frac{\|v - u\| - r}{b_n}\right) K_1\left(\frac{\|w - y\| - r}{b_n}\right) \lambda(u, v, w, y, \mathbf{X}) \, dudvdwdy \\ &= \frac{1}{|W_{n\ominus 2R}|\sigma_d^2} \int_{\mathbb{R}^d} \int_{-r/b_n}^{\infty} \int_{-r/b_n}^{\infty} \int_{\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} \frac{|W_{n\ominus 2R} \cap (W_{n\ominus 2R} - w)|}{|W_{n\ominus 2R}|} K_1(\varrho) K_1(\varrho') \\ &\quad \times \Phi_2^*(b_n\varrho + r, b_n\varrho' + r, \|w\|, \|(b_n\varrho + r)z - w\|) \, dwd\varrho d\varrho' d\sigma_d(z) d\sigma_d(z'). \end{aligned}$$

Where

$$\Phi_2^*(b_n\varrho + r, b_n\varrho' + r, \|w\|, \|(b_n\varrho + r)z - w\|) = \tilde{J}(b_n\varrho + r, b_n\varrho' + r) \tilde{F}(o, (b_n\varrho + r)z, (b_n\varrho' + r)z') \Phi_2(b_n\varrho + r, b_n\varrho' + r, \|w\|, \|(b_n\varrho + r)z - w\|).$$

By dominated convergence theorem, we get  $\lim_{n \rightarrow \infty} b_n|W_{n\ominus 2R}|A_3 = 0$ .

For asymptotic behaviour of the leading term  $A_4$ , it then suffices to repeat the arguments developed previously to conclude the following result.

$$\begin{aligned} A_4 &= \frac{\beta^{*4}}{b_n^2|W_{n\ominus 2R}|^2\sigma_d^2} \\ &\quad \int_{\mathbb{R}^{4d}} \mathbb{1}_{W_{n\ominus 2R}}(u)\mathbb{1}_{W_{n\ominus 2R}}(w) \frac{\tilde{J}(\|v - u\|, \|w - y\|)}{\|v - u\|^{d-1}\|w - y\|^{d-1}} \mathbb{E}[\tilde{L}(u, v, \mathbf{X})] \mathbb{E}[\tilde{L}(w, y, \mathbf{X})] \\ &\quad \times \Phi(\|v - u\|)\Phi(\|y - w\|) K_1\left(\frac{\|v - u\| - r}{b_n}\right) K_1\left(\frac{\|w - y\| - r}{b_n}\right) \, dudvdwdy \\ &= \frac{\beta^4}{b_n^2|W_{n\ominus 2R}|\sigma_d^2} \\ &\quad \int_{\mathbb{R}^{3d}} \frac{|W_{n\ominus 2R} \cap (W_{n\ominus 2R} - w)|}{|W_{n\ominus 2R}|\|v - u\|^{d-1}\|w - y\|^{d-1}} \tilde{J}(\|v\|, \|w - y\|) \mathbb{E}[\tilde{H}(o, v, \mathbf{X})] \mathbb{E}[\tilde{L}(w, y, \mathbf{X})] \\ &\quad \times \Phi(\|v\|)\Phi(\|y - w\|) K_1\left(\frac{\|v\| - r}{b_n}\right) K_1\left(\frac{\|w - y\| - r}{b_n}\right) \, dvdwdy \\ &= \frac{\beta^4}{|W_{n\ominus 2R}|\sigma_d^2} \\ &\quad \times \int_{\mathbb{R}^d} \int_{-r/b_n}^{\infty} \int_{-r/b_n}^{\infty} \int_{\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} \frac{|W_{n\ominus 2R} \cap (W_{n\ominus 2R} - w)|}{|W_{n\ominus 2R}|} \tilde{J}(b_n\varrho + r, b_n\varrho' + r) \tilde{F}(o, (b_n\varrho + r)z) \\ &\quad \times \tilde{F}(o, (b_n\varrho' + r)z') \Phi(b_n\varrho + r) \Phi(b_n\varrho' + r) K_1(\varrho) K_1(\varrho') \, dwd\varrho d\varrho' d\sigma_d(z) d\sigma_d(z'). \end{aligned}$$

Then by dominated convergence theorem, we get  $\lim_{n \rightarrow \infty} b_n |W_{n \in 2R}| A_4 = 0$ .  $\blacksquare$

## 6 Strong consistency

### 6.1 Rates uniform strong convergence of the kernel-type estimator

Before realizing the strong consistency  $\widehat{\Phi}_n(r)$  we introduce some necessary definitions and notation. A Young function  $\psi$  is a real convex nondecreasing function defined on  $\mathbb{R}^+$  which satisfies  $\lim_{t \rightarrow \infty} \psi(t) = +\infty$  and  $\psi(0) = 0$ . We define the Orlicz space  $L_\psi$  as the space of real random variables  $Z$  defined on the probability space  $(N_{lf}, \mathcal{F}, \mathbb{P})$  such that  $E[\psi(|Z|/c)] < +\infty$  for some  $c > 0$ . The Orlicz space  $L_\psi$  equipped with the so-called Luxemburg norm  $\|\cdot\|_\psi$  defined for any real random variable  $Z$  by

$$\|Z\|_\psi = \inf\{c > 0; E[\psi(|Z|/c)] \leq 1\}$$

is a Banach space. For more about Young functions and Orlicz spaces one can refer to Krasnosel'skii and Rutickii [15]. Let  $\theta > 0$ . We denote by  $\psi_\theta$  the Young function defined for any  $x \in \mathbb{R}^+$  by

$$\psi_\theta(x) = \exp((x + \xi_\theta)^\theta) - \exp(\xi_\theta^\theta) \quad \text{where} \quad \xi_\theta = ((1 - \theta)/\theta)^{1/\theta} \mathbb{1}\{0 < \theta < 1\}.$$

On the lattice  $\mathbb{Z}^d$  we define the lexicographic order as follows: if  $i = (i_1, \dots, i_d)$  and  $j = (j_1, \dots, j_d)$  are distinct elements of  $\mathbb{Z}^d$ , the notation  $i <_{lex} j$  means that either  $i_1 < j_1$  or for some  $p$  in  $\{2, 3, \dots, d\}$ ,  $i_p < j_p$  and  $i_q = j_q$  for  $1 \leq q < p$ . Let the sets  $\{V_i^k; i \in \mathbb{Z}^d, k \in \mathbb{N}^*\}$  be defined as follows:

$$V_i^1 = \{j \in \mathbb{Z}^d; j <_{lex} i\},$$

and for  $k \geq 2$

$$V_i^k = V_i^1 \cap \{j \in \mathbb{Z}^d; |i - j| \geq k\} \quad \text{where} \quad |i - j| = \max_{1 \leq l \leq d} |i_l - j_l|.$$

For any subset  $\Gamma$  of  $\mathbb{Z}^d$  define  $\mathcal{F}_\Gamma = \sigma(\varepsilon_i; i \in \Gamma)$  and set

$$E_{|k|}(\varepsilon_i) = E(\varepsilon_i | \mathcal{F}_{V_i^{|k|}}), \quad k \in V_i^1.$$

Denote  $\theta(q) = 2q/(2 - q)$  for  $0 < q < 2$  and by convention  $1/\theta(2) = 0$ .

Next we list a set of conditions which are needed to obtain (rates of) uniform strong consistency over some compact set  $[r_1, r_2]$  in  $(0, R]$  of the estimator  $\widehat{R}_n(r)$  to the function  $\beta^{\star 2} J(r) \Phi(r)$ . The following assumption is imposed:

**Condition  $\mathcal{L}p$**  : The kernel function  $K$  is a Lipschitz condition, i.e. there exists a constant  $\eta > 0$  such that

$$|K_1(\rho) - K_1(\rho')| \leq \eta |\rho - \rho'| \quad \text{for all } \rho, \rho' \in [r_1, r_2].$$

Strong uniform consistency for the resulting estimator are obtained via assumptions of belonging to Orlicz spaces induced by exponential Young functions for stationary real random fields which allows us to derive the Kahane-Khintchine inequalities by El Machkouri [17]. Our results also carry through the most important particular case of Orlicz spaces random fields, we use the inequality follows from a Marcinkiewicz-Zygmund type inequality by Dedecker [6].

Now, we split up the sampling window  $W_{n \in 2R}$  into cubes such as  $W_{n \in 2R} = \cup_{i \in \Gamma_n} \Lambda_i$ , where  $\Lambda_i$  are centered at  $i$  and assume that  $\Gamma_n = \{-n, \dots, 0, \dots, n\}^d$  increases towards  $\mathbb{Z}^d$ . We split up  $\widehat{R}_n(r)$  as follows:

$$\widehat{R}_n(r) = \frac{1}{b_n |W_{n \in 2R}| \sigma_d} \sum_{i \in \Gamma_n} R_k(r)$$

$$R_k(r) = \sum_{\substack{\neq \\ u, v \in \mathbf{X} \\ \|v-u\| \leq R}} \frac{\mathbb{1}_{\Lambda_k}(u)}{\|v-u\|^{d-1}} \tilde{h}(u, \mathbf{X} \setminus \{u, v\}) \tilde{h}(v, \mathbf{X} \setminus \{u, v\}) K_1 \left( \frac{\|v-u\| - r}{b_n} \right).$$

Note for all  $k \in \Gamma_n$ ,  $\bar{R}_k = R_k(r) - \mathbb{E} R_k(r)$  and  $S_n = \sum_{k \in \Gamma_n} \bar{R}_k(r)$ .

**Theorem 3.** *Under Conditions  $K(1, \alpha)$  and  $\mathcal{L}p$ . Further, assume that  $J(r) \exp(-\gamma(r))$  has bounded and continuous partial derivatives of order  $\alpha$  in  $[r_1 - \delta, r_2 + \delta]$  for some  $\delta > 0$ .*

1) *If there exists  $0 < q < 2$  such that  $\bar{R}_0 \in \mathbb{L}_{\psi_{\theta(q)}}$  and*

$$\sum_{k \in V_0^1} \left\| \left\| \sqrt{|\bar{R}_k E_{|k}|(\bar{R}_0)} \right\| \right\|_{\psi_{\theta(q)}}^2 < \infty. \quad (6.9)$$

*Then*

$$\sup_{r_1 \leq r \leq r_2} \left| \widehat{R}_n(r) - \beta^{\star 2} J(r) \exp(-\gamma(r)) \right| = \mathcal{O}_{a.s.} \left( \frac{(\log n)^{1/q}}{b_n n^{d/2}} \right) + \mathcal{O}(b_n^\alpha) \quad \text{as } n \rightarrow \infty.$$



2) If  $\bar{R}_0 \in \mathbb{L}^\infty$  and

$$\sum_{k \in V_0^1} \|\bar{R}_k E_{|k|}(\bar{R}_0)\|_\infty < \infty. \quad (6.10)$$

Then

$$\sup_{r_1 \leq r \leq r_2} |\widehat{R}_n(r) - \beta^{*2} J(r) \exp(-\gamma(r))| = \mathcal{O}_{a.s.} \left( \frac{(\log n)^{1/2}}{b_n n^{d/2}} \right) + \mathcal{O}(b_n^\alpha) \quad \text{as } n \rightarrow \infty.$$

3) If there exists  $p > 2$  such that  $\bar{R}_0 \in \mathbb{L}^p$  and

$$\sum_{k \in V_0^1} \|\bar{R}_k E_{|k|}(\bar{R}_0)\|_{\frac{p}{2}} < \infty. \quad (6.11)$$

Assume that  $b_n = n^{-q_2} (\log n)^{q_1}$  for some  $q_1, q_2 > 0$ . Let  $a, b \geq 0$  be fixed and if  $a(p+1) - d^2/2 - q_2 > 1$  and  $b(p+1) + q_1 > 1$ . Then

$$\sup_{r_1 \leq r \leq r_2} |\widehat{R}_n(r) - \beta^{*2} J(r) \exp(-\gamma(r))| = \mathcal{O}_{a.s.} \left( \frac{n^a (\log n)^b}{b_n n^{d/2}} \right) + \mathcal{O}(b_n^\alpha) \quad \text{as } n \rightarrow \infty.$$

**Remark 1.** From the Markov property of  $\mathbf{X}$  entails that for  $i \neq 0$  are not neighborhoods, then  $\bar{R}_i$  et  $\bar{R}_o$  are conditionally independent, i.e  $\mathbb{E}[\bar{R}_0 | (X_{\Lambda_i}; i \neq 0)] = 0$ . Since  $\sigma(R_i, i \in V_0^k)$  is contained in  $\sigma(X_{\Lambda_i}, i \neq 0)$  for  $k > l$ , for some integer  $l$ , it follows immediately that conditions (6.9), (6.10), (6.11) are satisfied.

## 6.2 Proof of Theorem 3

*Proof.* To establish rates of the uniform  $\mathbb{P}$ - a.s. convergence for the estimator  $\widehat{R}_n(r)$ , we apply a triangle inequality decomposition allows for

$$\begin{aligned} \sup_{s_{i-1} \leq r \leq s_i} \left| \widehat{R}_n(r) - \mathbb{E} \widehat{R}_n(r) \right| &\leq \sup_{s_{i-1} \leq r, \rho \leq s_i} \left| \widehat{R}_n(r) - \widehat{R}_n(\rho) \right| \\ &+ \sup_{s_{i-1} \leq r, \rho \leq s_i} \left| \mathbb{E} \widehat{R}_n(r) - \mathbb{E} \widehat{R}_n(\rho) \right| \\ &+ \sup_{s_{i-1} \leq \rho \leq s_i} \left| \widehat{R}_n(\rho) - \mathbb{E} \widehat{R}_n(\rho) \right|. \end{aligned}$$

The compact set  $[r_1, r_2]$  is covered by the intervals  $C_i = [s_{i-1}, s_i]$ , where  $s_i = r_1 + i(r_2 - r_1)/N, i = 1, \dots, N$ . Choosing  $N$  as the largest integer satisfying  $N \leq c/l_n$  and  $l_n = r_n b_n^2$ . Under the condition  $\mathcal{L}p$ , we deduce that there exists a constant  $\eta > 0$  such that for any  $n$  sufficiently large

$$\begin{aligned} \sup_{s_{i-1} \leq r, \rho \leq s_i} \left| \widehat{R}_n(r) - \widehat{R}_n(\rho) \right| &\leq \frac{1}{b_n^2} \eta |r - \rho| \widetilde{R}_n \\ &\leq r_n \widetilde{R}_n \end{aligned}$$

where

$$\widetilde{R}_n = \frac{1}{\sigma_d |W_{n \ominus 2R}|} \sum_{\substack{u, v \in \mathbf{X} \\ \|v-u\| \leq R}}^{\neq} \frac{\mathbb{1}_{W_{n \ominus 2R}}(u)}{\|v-u\|^{d-1}} \widetilde{h}(u, \mathbf{X} \setminus \{u, v\}) \widetilde{h}(v, \mathbf{X} \setminus \{u, v\}).$$

Follows from the last inequalities and the Nguyen and Zessin ergodic theorem [22]:

$$\sup_{s_{i-1} \leq r, \rho \leq s_i} \left| \widehat{R}_n(r) - \widehat{R}_n(\rho) \right| = \mathcal{O}_{p.s.}(r_n) \quad \text{as } n \rightarrow \infty.$$

As well

$$\sup_{s_{i-1} \leq r, \rho \leq s_i} \left| \mathbb{E} \widehat{R}_n(r) - \mathbb{E} \widehat{R}_n(\rho) \right| = \mathcal{O}_{p.s.}(r_n) \quad \text{as } n \rightarrow \infty.$$

**Lemma 1.** *Assume that either (6.9) holds for some  $0 < q < 2$  such that  $\bar{R}_0 \in \mathbb{L}_{\psi_{\theta(q)}}$  and  $r_n = (\log n)^{1/q} / b_n (\sqrt{n})^d$  or (6.10) holds such that  $\bar{R}_0 \in \mathbb{L}_\infty$  and  $r_n = (\log n)^{1/2} / b_n (\sqrt{n})^d$ . Then*

$$\sup_{s_{i-1} \leq \rho \leq s_i} \left| \widehat{R}_n(\rho) - \mathbb{E} \widehat{R}_n(\rho) \right| = \mathcal{O}_{p.s.}(r_n) \quad \text{as } n \rightarrow \infty.$$

*Proof.* For  $\varepsilon > 0$ , using Markov's inequality, we get

$$\begin{aligned} \mathbb{P} \left( |\widehat{R}_n(r) - \mathbb{E} \widehat{R}_n(r)| > \varepsilon r_n \right) &= \mathbb{P} \left( |S_n| > \varepsilon r_n b_n n^d \right) \\ &\leq \exp \left[ - \left( \frac{\varepsilon r_n b_n n^d}{\|S_n\|_{\psi_{\theta(q)}}} + \xi_q \right)^q \right] \mathbb{E} \exp \left[ \left( \frac{|S_n|}{\|S_n\|_{\psi_{\theta(q)}}} + \xi_q \right)^q \right]. \end{aligned}$$

Therefore, we assume that there exists a real  $0 < q < 2$ , such that  $\bar{R}_0 \in \mathbb{L}_{\psi_{\theta(q)}}$  and using Kahane-Khintchine inequalities (cf. El Machkouri [17], Theorem 1), we have

$$\begin{aligned} \mathbb{P} \left( |\widehat{R}_n(r) - \mathbb{E} \widehat{R}_n(r)| > \varepsilon r_n \right) &\leq \mathbb{P} \left( |S_n| > \varepsilon r_n b_n n^d \right) \\ &\leq (1 + e^{\xi_q^q}) \exp \left[ - \left( \frac{\varepsilon r_n b_n n^d}{M(\sum_{i \in \Gamma_n} b_{i,q}(\bar{R}))^{1/2}} + \xi_q \right)^q \right] \end{aligned}$$

denote

$$b_{i,q}(\bar{R}) = \|\bar{R}_0\|_{\psi_{\theta(q)}}^2 + \sum_{k \in V_0^1} \left\| \sqrt{|\bar{R}_k E_{|k|}(\bar{R}_0)|} \right\|_{\psi_{\theta(q)}}^2.$$

We derive that if condition (6.9) holds, then there exist constant  $C > 0$  and so if  $r_n = (\log n)^{1/q}/b_n(\sqrt{n})^d$

$$\sup_{r_1 \leq r \leq r_2} \mathbb{P}(|\widehat{R}_n(r) - \mathbb{E} \widehat{R}_n(r)| > \varepsilon r_n) \leq (1 + e^{\varepsilon^q}) \exp \left[ -\frac{\varepsilon^q \log n}{C^q} \right].$$

Now, we will accomplish the second step the proof of Proposition 3. Using Kahane-Khintchine inequalities (cf. El Machkouri [17], Theorem 1) with  $q = 2$ , such that  $\bar{R}_0 \in \mathbb{L}_\infty$ , we have

$$\mathbb{P} \left( |\widehat{R}_n(r) - \mathbb{E} \widehat{R}_n(r)| > \varepsilon r_n \right) \leq 2 \exp \left[ - \left( \frac{\varepsilon r_n b_n n^d}{M(\sum_{i \in \Gamma_n} b_{i,2}(\bar{R}))^{1/2}} \right)^2 \right]$$

denote

$$b_{i,2}(\bar{R}) = \|\bar{R}_0\|_\infty^2 + \sum_{k \in V_0^1} \|\bar{R}_k E_{|k|}(\bar{R}_0)\|_\infty.$$

We derive that if condition (6.10) holds and so if  $r_n = (\log n)^{1/2}/b_n(\sqrt{n})^d$ , there exists  $C > 0$  such that

$$\sup_{r_1 \leq r \leq r_2} \mathbb{P}(|\widehat{R}_n(r) - \mathbb{E} \widehat{R}_n(r)| > \varepsilon r_n) \leq 2 \exp \left[ -\frac{\varepsilon^2 \log n}{C^2} \right].$$

choosing  $\varepsilon$  sufficiently large, therefore, it follows with Borel-Cantelli's lemma

$$\mathbb{P}(\limsup_{n \rightarrow \infty} \sup_{s_{i-1} \leq \rho \leq s_i} |\widehat{R}_n(\rho) - \mathbb{E} \widehat{R}_n(\rho)| > \varepsilon r_n) = 0.$$

■

Now, we will accomplish the last step the proof of Theorem 3.

**Lemma 2.** *Assume (6.11) holds for some  $p > 2$  such that  $\bar{R}_0 \in \mathbb{L}^p$  and  $b_n = n^{-q_2}(\log n)^{q_1}$  for some constants  $q_1, q_2 > 0$ . Let  $a, b \geq 0$  be fixed and denote  $r_n = n^a(\log n)^b/b_n(\sqrt{n})^d$ . If*

$$a(p+1) - d/2 - q_2 > 1 \quad \text{et} \quad b(p+1) + q_1 > 1,$$

then

$$\sup_{s_{i-1} \leq \rho \leq s_i} |\widehat{R}_n(\rho) - \mathbb{E} \widehat{R}_n(\rho)| = \mathcal{O}_{p,s}(r_n) \quad \text{as} \quad n \rightarrow \infty.$$

*Proof.* Let  $p > 2$  be fixed, such that  $\bar{R}_0 \in \mathbb{L}^p$  and for any  $\varepsilon > 0$ ,

$$\begin{aligned} \mathbb{P}(|\widehat{R}_n(r) - \mathbb{E} \widehat{R}_n(r)| > \varepsilon r_n) &= \mathbb{P}(|S_n| > \varepsilon r_n b_n n^d) \\ &\leq \frac{\varepsilon^{-p} \mathbb{E} |S_n|^p}{r_n^p b_n^p n^{pd}} \\ &\leq \frac{\varepsilon^{-p}}{r_n^p b_n^p n^{pd}} \left( 2p \sum_{i \in \Gamma_n} c_i(\bar{R}) \right)^{p/2}. \end{aligned}$$

The last inequality follows from a Marcinkiewicz-Zygmund type inequality by Dedecker [6], where

$$c_i(\bar{R}) = \|\bar{R}_i\|_p^2 + \sum_{k \in V_i^1} \|\bar{R}_k E_{|k-i|}(\bar{R}_i)\|_2^2.$$

Under assumption (6.11) and with the stationarity of  $\mathbf{X}$ , we derive that there exists  $C > 0$  such that

$$\begin{aligned} \mathbb{P} \left( \sup_{s_{i-1} \leq \rho \leq s_i} \left| \widehat{R}_n(\rho) - \mathbb{E} \widehat{R}_n(\rho) \right| > \varepsilon r_n \right) &\leq N \sup_{r_1 \leq r \leq r_2} \mathbb{P}(|\widehat{R}_n(r) - \mathbb{E} \widehat{R}_n(r)| > \varepsilon r_n) \\ &\leq N \frac{\kappa \varepsilon^{-p}}{r_n^p b_n^p (\sqrt{n})^{pd}}. \end{aligned}$$

As  $N \leq c/l_n$  and  $l_n = r_n b_n^2$ , then for  $r_n = n^a (\log n)^b / b_n (\sqrt{n})^d$ , it results for  $n$   $\varepsilon$  sufficiently large,

$$\begin{aligned} \mathbb{P} \left( \sup_{s_{i-1} \leq \rho \leq s_i} \left| \widehat{R}_n(\rho) - \mathbb{E} \widehat{R}_n(\rho) \right| > \varepsilon r_n \right) &\leq \frac{\kappa \varepsilon^{-p}}{n^{a(p+1)-d/2} (\log n)^{b(p+1)} b_n} \\ &\leq \frac{\kappa \varepsilon^{-p}}{n^{a(p+1)-d/2-q_2} (\log n)^{b(p+1)+q_1}}. \end{aligned}$$

For  $a(p+1) - d/2 - q_2 > 1$  et  $b(p+1) + q_1 > 1$ , we get for any  $\varepsilon > 0$   $\varepsilon > 0$

$$\sum_{n \geq 1} \mathbb{P} \left( \sup_{s_{i-1} \leq \rho \leq s_i} \left| \widehat{R}_n(\rho) - \mathbb{E} \widehat{R}_n(\rho) \right| > \varepsilon r_n \right) < \infty.$$

■

Considering these arguments the proofs of Theorem 3 are completed, it results from a direct application of the theorem of Borel-Cantelli and by Theorem 1 we have

$$\sup_{r_1 \leq r \leq r_2} |\mathbb{E} \widehat{R}_n(r) - R(r)| = \mathcal{O}(b_n^\alpha) \quad \text{as } n \rightarrow \infty.$$

■

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