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MINIMALITY OF THE EHRENFEST WIND-TREE MODEL

ALBA MÁLAGA SABOGAL AND SERGE TROUBETZKOY

ABSTRACT. We consider aperiodic wind-tree models, and show that for a generic (in the sense of Baire) configuration the wind-tree dynamics is minimal in almost all directions, and has a dense set of periodic points.

1. INTRODUCTION

In 1912 Paul et Tatyana Ehrenfest proposed the wind-tree model in order to interpret the ergodic hypothesis of Boltzmann [EhEh]. In the Ehrenfest wind-tree model, a point particle (the “wind”) moves freely on the plane and collides with the usual law of geometric optics with irregularly placed identical square scatterers (the “trees”). Nowadays we would say “randomly placed”, but the notion of “randomness” was not made precise, in fact it would have been impossible to do so before Kolmogorov laid the foundations of probability theory in the 1930s.

From the mathematical rigorous point of view, there have been many recent results about the dynamical properties of a periodic version of wind-tree models, scatterers are identical square obstacles one obstacle centered at each lattice point. The periodic wind-tree model has been shown to be recurrent ([HaWe], [HuLeTr],[AvHu]), to have abnormal diffusion ([DeHuLe],[De]), and to have an absence of ergodicity in almost every direction ([FrUl]); furthermore the periodic wind-tree model can not have a minimal direction.¹ Periodic wind-tree models naturally yield infinite periodic translation surfaces, ergodicity in almost every direction for such surfaces have been obtained only in a few situations [HoHuWe], [HuWe], [RaTr].

For randomly placed obstacles the wind-tree model has been intensively studied by physicists, see for example [BiRo], [DeCoVB], [Ga], [HaCo], [VBHa], [WoLa] and the references therein. From the mathematically rigorous point of view, it has been shown that if at each point of the lattice \mathbb{Z}^2 we either center a square obstacle of fixed size or omit it in a random way, then the generic in the sense of Baire wind-tree model is recurrent and has a dense set of periodic points ([Tr1]).

In this article we continue the study of the Baire generic properties of wind-tree models. We study a random version of the wind-tree model: the plane is tiled by one by one cells with corners on the lattice \mathbb{Z}^2 , in each cell we place a square tree of a fixed size with the center chosen randomly. Our main result is that for the generic in the sense of Baire wind-tree model, for almost all directions the wind-tree model is minimal, in stark contrast to the situation for the periodic wind-tree model. This result can be viewed as a topological version of the Ehrenfests question.

¹K. Frączek explained to us that this follows from elementary arguments close to those in the article [Be].

Our proofs hold in a more general setting than the one described above, for example we can vary the size of the square, or use certain other polygonal trees. We discuss such extensions of our result in the last section of the article.

The method of proof is by approximation by finite wind-tree models where the dynamics is well understood. There is a long history of proving results about billiard dynamics by approximation which began with the article of Katok and Zemlyakov [KaZe]. This method was used in several of the results on wind-tree models mentioned above [HuLeTr],[AvHu],[Tr1], see [Tr] for a survey of some other usages in billiards. The idea of approximating infinite measure systems by compact systems was first studied in [MS].

2. STATEMENTS OF RESULTS

We consider the plane \mathbb{R}^2 tiled by one by one closed square *cells* with corners on the lattice \mathbb{Z}^2 . Fix $r \in [1/4, 1/2)$. We consider the set of $2r$ by $2r$ squares, with vertical and horizontal sides, centered at (a, b) contained in the unit cell $[0, 1]^2$, this set is naturally parametrized by

$$\mathcal{A} := \{t = (a, b) : r \leq a \leq 1 - r, r \leq b \leq 1 - r\}$$

with the usual topology inherited from \mathbb{R}^2 . Our parameter space is $\mathcal{A}^{\mathbb{Z}^2}$ with the product topology. It is a Baire space. Each parameter $g = (a_{i,j}, b_{i,j})_{(i,j) \in \mathbb{Z}^2} \in \mathcal{A}^{\mathbb{Z}^2}$ corresponds to a wind-tree table in the plane in the following manner: the tree inside the cell corresponding to the lattice point $(i, j) \in \mathbb{Z}^2$ is a $2r$ by $2r$ square with center at position $(a_{i,j}, b_{i,j}) + (i, j)$. The wind-tree table B^g is the plane \mathbb{R}^2 with the interiors of the union of these trees removed. Note that trees can intersect only at the boundary of cells.

Fix a direction $\theta \in \mathbb{S}^1$. The billiard flow in the direction θ is the free motion on the interior of B^g with elastic collision from the boundary of B^g (the boundary of the union of the trees). The *billiard map* T_θ^g in the direction θ on the table is the first return to the boundary. If the flow orbit arrives at a corner of the table, the collision is not well defined, and we choose not to define the billiard map, i.e. the orbit stops. Once launched in the direction θ , the billiard direction can only achieve four directions $\{\pm\theta, \pm(\theta - \pi)\}$; thus the phase space Ω_θ^g of the billiard map T_θ^g is a subset of the cartesian product of the boundary with these four directions. It contains precisely the pairs (s, ϕ) such that at s the direction ϕ points to the interior of the table.

Note that the billiard map $T := T_\theta^g$ can be decomposed as $T = R \circ D$, where $R = R_\theta^g$ acts only on the angle component (it is the reflexion map with respect to the normal) and $D = D_\theta^g$, called a displacement map, acts as follows: $D(s, \phi) := (s', \phi + \pi)$, where s' is the point that s hits when following direction ϕ . D is everywhere defined on Ω_θ^g , but R is not defined at the corners. Note that T^{-1} can also be decomposed through these same transformations: $T^{-1} = D \circ R$, since $R^2 = D^2 = Id$.

A saddle connection is loosely speaking a T_θ^g -orbit going from a corner of a tree to some corner (maybe the same one). However, we want to include the corners in this set. More precisely, we say that a saddle connection of length $k > 0$ from $x = (s, \phi)$ to $x' = (s', \phi')$, where s and s' are corners, exists if $y = D \circ T^{k-1}(x)$, then we define the *saddle connection* as the set

$$\{x, Tx, T^2(x), \dots, T^{k-1}(x), D \circ T^{k-1}(x)\}.$$

For any positive integer N , we define $R_{\overline{N}}$ to be the closed rhombus $\{(x, y) : |x| + |y| \leq N + \frac{1}{2}\}$ and we define then $\Omega_{\theta, \overline{N}}^g$ to be $\Omega_{\theta}^g \cap (R_{\overline{N}} \cap B^g) \times \{\pm\theta, \pm(\theta - \pi)\}$. In a similar way, we define R_N to be the open set $\{(x, y) : [|x|] + [|y|] < N - 1\}$ and then $\Omega_{\theta, N}^g$ to be $\Omega_{\theta}^g \cap (R_N \cap B^g) \times \{\pm\theta, \pm(\theta - \pi)\}$. Let E_N the set of pairs (i, j) so that the interior of the (i, j) -th cell is contained in R_{N+1} .

The set of periodic points is called *locally dense* if there exists a G_{δ} -subset of the boundary which is of full measure, such that for every s in this set, there is a dense set of inner-pointing directions $\theta \in \mathbb{S}^1$ for which (s, θ) is periodic. We call a forward (resp. backward) T_{θ}^g -orbit a *forward (resp. backward) escape orbit* if it visits any compact set only a finite number of times.

Theorem 1. There is a dense G_{δ} set of parameters \mathcal{G} such that for each $g \in \mathcal{G}$:

- for a dense- G_{δ} set of full measure of θ the billiard map T_{θ}^g is minimal and has forward and backward escape orbits,
- the map T^g has a dense set of periodic points,
- if r is rational, then the map T^g has a locally dense set of periodic points,
- no two trees intersect.

Since the map is minimal we conclude

Corollary 2. The backwards orbit of any forward escape orbit is dense (and vice versa) for each $g \in \mathcal{G}$.

Corollary 3. The billiard flow on the wind-tree table is also minimal for each $g \in \mathcal{G}$.

3. PROOF OF MINIMALITY

Proof. We begin by the proof of the minimality. For the map T_{θ}^g to be minimal is equivalent to the statement: for any interval $I \subset \Omega_{\theta}^g$ we have $\bigcup_{k \in \mathbb{Z}} (T_{\theta}^g)^k(I)$ covers the whole space Ω_{θ}^g . It is enough to show that this happens for a finite union of iterates of I :

$$\forall I, C \subset \Omega_{\theta}^g \text{ compact } \exists K, L \text{ s.t. } \bigcup_{k=K}^L (T_{\theta}^g)^k(I) \supset C.$$

Furthermore it suffices to show this for a countable basis of intervals and a countable exhaustion of Ω_{θ}^g by compact sets C_i . If we had been in a compact situation (which will occur in the proof) this sufficient condition becomes necessary with $C = \Omega_{\theta}^g$.

Suppose that N is an integer satisfying $N \geq 2$. We will call a parameter f *N-tactful* if for each cell inside the rhombus $R_{\overline{N}}$, the corresponding tree is contained in the interior of its cell. We will call an *N-tactful* parameter f *N-ringed*, if the boundary of $R_{\overline{N}}$ is completely covered by trees.

For *N-ringed* parameters there is a compact connected rational billiard table $R_{\overline{N}} \cap B^f$, called the *N-ringed table*, contained in the rhombus $R_{\overline{N}}$ (see Figure 1). The corresponding phase space is $\Omega_{\theta, \overline{N}}^f$. It contains $\Omega_{\theta, N}^f$ which is compact for any *N-tactful* parameter. A direction θ is called *(f, N)-exceptional* if there is a saddle connection inside $\Omega_{\theta, \overline{N}}^f$. There are at most countably many exceptional directions, and for all non-exceptional directions, $\Omega_{\theta, \overline{N}}^f$ is a minimal set for the billiard map T_{θ}^f [MaTa].

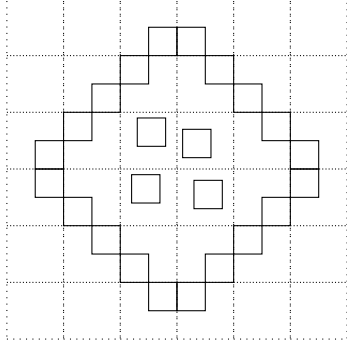


FIGURE 1. A 2-ringed configuration.

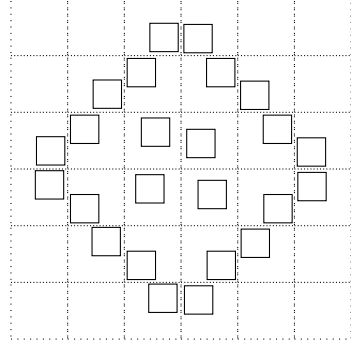


FIGURE 2. A small perturbation.

We need to describe $\Omega_{\theta,N}^g$ more concretely for any N -tactful g . We think of the contribution of each tree to $\Omega_{\theta,N}^g$ as the union of four closed intervals indexed by $\phi \in \{\pm\theta, \pm(\theta - \pi)\}$, each of these intervals corresponds to the cartesian product of the two intersecting sides of the tree with a fixed inner pointing direction ϕ (see Figure 3). We think of each of these intervals as $[0, 2\sqrt{2}r]$ since it corresponds to the diagonal (of length $2\sqrt{2}r$) of the tree centered at (a, b) . To make our map orientation preserving we choose the orientation of these interval in the following way; use the clockwise orientation inherited from the tree for $\phi \in \{\theta, \theta - \pi\}$ and the counterclockwise orientation of the other two values of ϕ . In particular this parametrization does not depend on the angle ϕ . (See figures 3,4).

For any N -tactful g , let \mathcal{J}_N^g be the collection of all the intervals as described above.

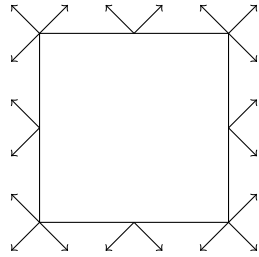


FIGURE 3. The phase space of one tree.

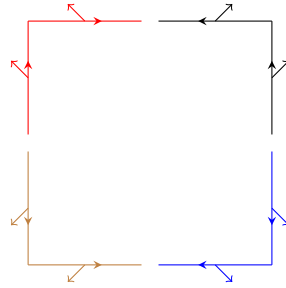


FIGURE 4. The phase space decomposes into four oriented "intervals".

Thus for any N -ringed f , any (f, N) -non-exceptional direction θ and any interval $I \subset \Omega_{\theta,N}^f$, there exists K, L such that

$$(1) \quad \bigcup_{k=K}^L (T_{\theta}^f)^k(I) \supset \Omega_{\theta,N}^f.$$

Now consider the following perturbation of f , the new configuration g is arbitrary outside R_{N+1} , and each tree $(a_{i,j}, b_{i,j})$ in R_{N+1} is replaced with a tree $(a'_{i,j}, b'_{i,j})$

which is sufficiently close to $(a_{i,j}, b_{i,j})$ in such a way that the new parameter is $(N + 1)$ -tactful (see Figure 2). We claim that if the perturbation is sufficiently small then Equation (1) still holds, to understand what this means we need to explain how we will choose I to depend on f .

Note that if the tree is contained in the interior of a cell then if s is a corner of this tree there are three directions pointing to the interior of the table, while for all other s there are only two such directions. Intersecting trees can have slightly different behavior, but we do not need to describe it.

We will consider the following covering of $[0, 2\sqrt{2}r]$:

$$\left\{ \left(\frac{\sqrt{2}r}{2^N}(i-1), \frac{\sqrt{2}r}{2^N}(i+1) \right) \cap [0, 2\sqrt{2}r] : i = 0, \dots, 2^{N+1} \right\}.$$

Note that this covering is an open covering even though there are two half-open intervals in the collection but they are open in the topology of $[0, 2\sqrt{2}r]$.

For any N -tactful g , $\mathcal{I}_{N,\theta}^g$ will be the union of all such families, where the interval $[0, 2\sqrt{2}r]$ corresponds to one of the intervals in \mathcal{J}_N^g . Note that $\mathcal{I}_{N,\theta}^g$ is a finite collection of open intervals in $\Omega_{\theta,N}^g$. Note also that $\bigcup_N \mathcal{I}_{N,\theta}^g$ is a topological basis of Ω_{θ}^g . We will call the endpoints of intervals in $\mathcal{I}_{N,\theta}^g$ *dyadic points*.

For our proof we need to enlarge the definition of a saddle connection. We will say that a **saddle connection* of length $k > 0$ from $x = (s, \phi)$ to $x' = (s', \phi')$, where each s and s' is a corner or a dyadic point, exists if $y = D \circ T^{k-1}(x)$, then we define the *saddle connection* as the set

$$\{x, Tx, T^2(x), \dots, T^{k-1}(x), D \circ T^{k-1}(x)\}.$$

Note that we could make the final reflection in the case when y is not a corner, in order to uniformize the notation in the proof, we chose not to do so.

A direction θ is called $((f, N))$ -**exceptional* if there is a **saddle connection* inside $\Omega_{\theta,N}^f$. There are at most countably many **exceptional* directions, and since any non-**exceptional* direction is non-exceptional for all non-**exceptional* directions, $\Omega_{\theta,N}^f$ is a minimal set for the billiard map T_{θ}^f .

To prove that the billiard map T_{θ}^g in a given direction is minimal, it suffices to show that there exists infinitely many N such that

$$(2) \quad \exists K, L \forall I \in \mathcal{I}_{N,\theta}^g \bigcup_{k=K}^L (T_{\theta}^g)^k(I) \supset \Omega_{\theta,N}^g.$$

For any N -tactful g , let $\mathcal{C}_N^g(K, L)$ be the collection of all the connected components of $(T_{\theta}^g)^k(I) \cap \Omega_{\theta,N}^g$ where k varies from K to L . These intervals can be either open or half-open. Each interval I' in $\mathcal{C}_N^g(K, L)$ is a connected component of $(T_{\theta}^g)^k(I)$ for some k between K and L , and some $I \in \mathcal{I}_{N,\theta}^g$. Thus, for this k , $T^{-k}(I')$ is an interval in $\Omega_{\theta,N}^g$, we will call $\mathcal{D}_N^g(K, L)$ the collection of all such intervals.

For each tree $t \in \mathcal{A}$ let $U(t, \varepsilon)$ be the open set of trees (centers of trees) in the standard ε -neighborhood. For any parameter $g = (t_{i,j}) \in \mathcal{A}^{\mathbb{Z}^2}$, consider the open cylinder set $U_N(g, \varepsilon)$ of $(N + 1)$ -tactful parameters in $\prod_{(i,j) \in E_N} U(t_{i,j}, \varepsilon)$.

Define a homeomorphism $\psi = \psi^{g,f}$ between $\Omega_{\theta,N}^g$ and $\Omega_{\theta,N}^f$ for every N -ringed parameter f and every g in $U_N(f, \varepsilon)$ by mapping each interval $[0, 2\sqrt{2}r_{i,j}]$ in \mathcal{J}_N^g to the corresponding interval in \mathcal{J}_N^f by multiplying by the ratio of the corresponding

side-lengths $(r_{i,j}^f/r_{i,j}^g)$. Note that this homeomorphism induces a homeomorphism between $B^g \cap R_N$ and $B^f \cap R_N$ which we will note by the same letter ψ .

Let $\{f_i\}$ be a countable dense set of parameters such that each f_i is N_i -ringed for some N_i . Suppose ε_i are strictly positive. Let

$$\mathcal{G} := \bigcap_{N \geq 1} \bigcup_{\{i: N_i \geq N\}} U_{N_i}(f_i, \varepsilon_i).$$

Clearly \mathcal{G} is a dense G_δ . We claim that there is a choice for ε_i such that every parameter in \mathcal{G} gives rise to a wind tree which is minimal in almost all directions.

Fix f_i . We already proved that (2) holds for $g = f_i$, $N = N_i$ and θ any direction which is not (f_i, N_i) -exceptional (c.f. (1)). Let K_i, L_i be the two integers given by Equation (2). For sake of simplicity, we will denote $\mathcal{C}_i := \mathcal{C}_{N_i}^{f_i}(K_i, L_i)$ the collection of intervals in the covering in Equation (2) and we will denote $\mathcal{D}_i := \mathcal{D}_{N_i}^{f_i}(K_i, L_i)$. We denote by $\partial\mathcal{C}_i$ the set of endpoints of the intervals in \mathcal{C}_i .

Let us describe how the intervals in \mathcal{C}_i cover $\Omega_{\theta, N_i}^{f_i}$. A point in $\Omega_{\theta, N_i}^{f_i}$ can be covered by:

- an open interval,
- the interior of a half-open interval,
- or the boundary of a half-open interval.

Suppose $x \in \partial\mathcal{C}_i$ such that x is not the endpoint of an interval in \mathcal{J}_N^f . If x is not an interior point of any other interval in \mathcal{C}_i , then it is the endpoint of another interval in \mathcal{C}_i and there are two cases:

- (1) either x is the closed endpoint of precisely one half-closed interval,
- (2) or x is the closed endpoint of several half-closed intervals.

The second case implies that the direction θ is (f_i, N_i) -exceptional and thus cannot happen.

Let us analyze case (1). For concreteness we suppose that the half-closed interval is of the form $[a, b)$, the other case is similar. The point a must be the forward image of an endpoint of an interval in \mathcal{J}_N^f (note that the endpoints of intervals in \mathcal{J}_N^f do not have backwards images by definition). The case when a is equal to such an endpoint does not interest us, thus suppose that $a = T^k A$ for some endpoint A for some $k > 0$, thus $T^{-k}[a, b) = [A, B)$, thus $[A, B) \subset I$ where A is the left endpoint of the original interval I in (2).

Let $\langle x, y \rangle$ denote any of the intervals contained in $[x, y]$ and containing (x, y) . In a similar way, we adopt the notations $\langle x, y \rangle$ and (x, y) .

By definition of \mathcal{C}_i , the interval $\langle c, a \rangle$ is a connected component of $T^j(I)$ for some j . If $j = 0$ then $a = A$ which we have already excluded, thus assume $j \neq 0$. We claim that $j > 0$. Otherwise $T^{-j}\langle c, a \rangle = \langle \bar{C}, \bar{A} \rangle \subset I$. If \bar{A} is not a the right endpoint of I , then for some $0 \leq j' < -j$, $D \circ T^{j'}(a)$ is on a corner of a tree. This means that θ is an exceptional direction, there exists a saddle connection, starting at A , passing through a , of length $k + j' + 1$. Similarly, if \bar{A} is the right endpoint of I , then there exists a *saddle connection, starting at A , passing through a , of length at most $k - j$. Since we assumed that θ is not *exceptional, we have proven that $j > 0$.

Let $\langle \bar{C}, \bar{A} \rangle \subset I$ such that $\langle c, a \rangle = T^j\langle \bar{C}, \bar{A} \rangle$. Let $\langle \hat{C}, \hat{A} \rangle := T^{j-k}\langle \bar{C}, \bar{A} \rangle$. \hat{A} belongs to some interval in \mathcal{C}_i . It can not be in the boundary of its covering interval, otherwise there would be a *saddle connection ending at A and passing

through \widehat{A} of length at most j . So there exists \widehat{D} and $\widehat{B} \neq \widehat{A}$ such that $\langle \widehat{D}, \widehat{B} \rangle$ belongs to \mathcal{C}_i and $\langle \widehat{C}, \widehat{A} \rangle \subset \langle \widehat{D}, \widehat{B} \rangle$.

We describe the set $\partial\mathcal{C}_i(\theta)$ exactly. Without loss of generality let suppose $K_i(\theta) < 0$ and $L_i(\theta) > 0$. First, let us consider the collection of points whose forward orbit arrive at a corner of a tree in time at most $-K_i(\theta)$, and similarly for backward orbits. Second, let us consider the following sub-collection of the forward orbits of corners of trees $\{T_\theta^k(x) : x \in \partial\mathcal{J}_{N_i}, 0 \leq k \leq L_i\}$, and similarly for backward orbits. Third, consider the iterates of the endpoints of I for times between $K_i(\theta)$ and $L_i(\theta)$. Then $\partial\mathcal{C}_i(\theta)$ is the restriction of the union of these three collections to $\Omega_{\theta, N_i}^{f_i}$. For each θ this is a finite collection of points. As we vary the parameters each point in $\partial\mathcal{C}_i(\theta)$ varies continuously with respect to g and θ .

Let Θ_i be the set of all directions θ who are not (f_i, N_i) -*exceptional. This set is of measure one since its complement is countable. For every $\theta \in \Theta$ the points in the collection $\partial\mathcal{C}_i(\theta)$ are all distinct. Recall that $\Omega_{\theta, N_i}^{f_i}$ is oriented, so it induces a strict partial order in $\partial\mathcal{C}_i(\theta)$ which is total on each interval in \mathcal{J}_{N_i} .

So, for every fixed $\theta \in \Theta$ it is possible to choose $\varepsilon_i(\theta) > 0$ such that the strict partial order on $\partial\mathcal{C}_i(\theta)$ is preserved for all $g \in U_{N_i}(f_i, \varepsilon_i)$; and thus, Equation (2) holds for every such g .

Let $\Theta_{K, L, i} := \{\theta \in \Theta_i : K_i(\theta) \geq K, L_i(\theta) \leq L\}$. Since $\Theta_i = \bigcup_{K \leq 0} \bigcup_{L \geq 0} \Theta_{K, L, i}$ is an increasing union of $\Theta_{K, L, i}$ and is of full measure, there exists \bar{K}_i, \bar{L}_i such that $\Theta_{\bar{K}_i, \bar{L}_i, i}$ is of measure larger than $\frac{1}{N_i}$. We can choose an open neighborhood $\widehat{\Theta}_i$ of $\Theta_{\bar{K}_i, \bar{L}_i, i}$ such that Equation (2) holds with the constants K_i, L_i for every $\theta \in \widehat{\Theta}_i$.

Thus

$$\bigcap_{N \geq 1} \bigcup_{\{i: N_i \geq N\}} \widehat{\Theta}_i$$

is a G_δ -dense set of full measure of minimal directions for all tables $g \in \mathcal{G}$. \square

4. PROOF OF THE EXISTENCE OF ESCAPE ORBITS.

Proof. Consider the set \mathcal{G} defined above. We additionally assume that $\varepsilon_i \rightarrow 0$. Fix $g \in \mathcal{G}$ and consider a minimal direction θ of $T = T_\theta^g$.

In the definition of the set \mathcal{G} consider an approximating sequence f_{i_j} such the $g \in U_{N_{i_j}}(f_{i_j}, \varepsilon_{i_j})$. For simplicity of notation let $M_j := N_{i_j}$. Clearly θ is neither vertical nor horizontal, since these directions are exceptional. Then we can choose J so large that for any $j \geq J$, the ε_{i_j} is sufficiently small such that for any $x \in \Omega_{\theta, M_j}^g$ the T orbit of x must visit the set $\Omega_{\theta, M_{j+1}}^g \setminus \Omega_{\theta, M_j}^g$ before reaching $\Omega_{\theta, M_{j+2}}^g \setminus \Omega_{\theta, M_{j+1}}^g$ and the same for T^{-1} .

Consider the compact set Ω_{θ, M_J}^g . Since T is minimal at least one of the two open sets

$$A_{J,1}^+ := \{x \in \Omega_{\theta, M_J}^g : T^{\pm 1}x \notin \Omega_{\theta, M_J}^g\}$$

is non-empty. In fact, both $A_{J,1}^+$ and $A_{J,1}^-$ are non-empty because $A_{J,1}^+$ contains the singularities of T^{-1} (in Ω_{θ, M_J}^g) and $A_{J,1}^-$ contains the singularities of T (in Ω_{θ, M_J}^g). Now inductively define the open sets

$$A_{J,n+1}^+ := \left\{ x \in A_{J,n}^+ : \exists k > 0, \forall j = 1, 2, \dots, k-1, \text{ such that } T^j \in \Omega_{\theta, M_n}^g \setminus \Omega_{\theta, M_j}^g \text{ and } T^k x \notin \Omega_{\theta, M_n}^g \right\}$$

and in the analogous way $A_{J,n+1}^-$. For each $n \geq 1$ at least one of the two sets $A_{J,n}^\pm$ is non-empty by the minimality of T and the definition of J . In fact, by a similar reasoning as above, both $A_{J,n}^+ \neq \emptyset$ for all $n \geq 1$ and $A_{J,n}^- \neq \emptyset$ for all $n \geq 1$. Let us concentrate on the $+$ -sets and set $A_n := A_{J,n}^+$. Clearly $\bar{A}_{n+1} \subset \bar{A}_n$, thus $\bigcap_{n \geq 1} \bar{A}_n$ is non-empty. We claim that if x is in this intersection, then x is in the interior of all of the A_n , and thus a forward escape orbit.

Suppose not, then let $m := \min\{n \geq 1 : x \in \bar{A}_n \setminus A_n\}$. This implies that for some k , $T^j x \in \Omega_{\theta, M_m}^g \setminus \Omega_{\theta, M_j}^g$ for $j = 1, \dots, k-1$ and $T^k x$ is not defined. Chose a sequence $(x_\ell) \subset A_m$ such that $x_\ell \rightarrow x$ and let $y = \lim_{\ell \rightarrow \infty} T^k x_\ell$. Since our direction θ is non-exceptional, the forward orbit of y is infinite. Furthermore, we have $y \in A_{m,n}^+$ for all n , thus y is a forward escape orbit which is backwards singular. This contradicts the minimality of T , thus x was a forward escape orbit. The case $A_{J,n}^-$ is similar.

Finally we remark that if (s, θ) is a forward escape orbit, then $(s, \theta + \pi)$ is a backward escape orbit, and vice versa. \square

5. PROOF OF DENSITY AND LOCAL DENSITY OF PERIODIC POINTS

Proof. The idea behind the proof is similar to what has been done for minimality in Section 3. We first apply a known result to N -ringed parameters.

In this section, the direction θ varies in the proof, so we abandon the notation T_θ^g and we note the billiard transformation in the wind-tree by $T^g(s, \theta)$.

For each point $x = (s, \theta) \in \Omega_\theta^g$ and each p such that $(T^g)^p(x)$ exists, let us consider the following set :

$$\left\{ \theta' : (T^g)^i(s, \theta') \text{ and } (T^g)^i(s, \theta) \right. \\ \left. \text{lie on the same side of the same tree for } i = 1, \dots, p \right\}.$$

This set is an open interval, we will note by $\theta_-^g(x, p)$, and $\theta_+^g(x, p)$ the lower and the upper bound of this interval. We also consider the interval (t_-^g, t_+^g) where t_\pm is the spatial coordinate of $(T^g)^p(s, \theta_\pm^g)$.

Fix a N -ringed parameter f , and $x = (s, \theta) \in \Omega_{\theta, N}^f$ such that $(T^f)^p(x)$ exists. Let $\psi = \psi^{g,f}$ the map defined in section 3. Since ψ^{-1} is continuous with respect to g and since $(T^g)^p$ is locally continuous at $\psi^{-1}(x)$, both $\theta_-^g(\psi^{-1}(x), p)$ and $\theta_+^g(\psi^{-1}(x), p)$, and thus $t_-^g(\psi^{-1}(x), p)$ and $t_+^g(\psi^{-1}(x), p)$ vary continuously with respect to g in a sufficiently small neighborhood of f . Let $(s_*^g, \theta) := (T^g)^p(\psi^{-1}(s), \theta)$, then, in a sufficiently small neighborhood of f , s_*^g varies continuously with respect to g .

By definition of t_\pm^g , for all $s' \in (t_-^g, t_+^g)$, there exists an orbit starting at $(\psi^{-1}(s), \theta')$ and ending at (s', θ') for some $\theta' \in (\theta_-^g(x, p), \theta_+^g(x, p))$. Now, suppose that $x = (s, \theta)$ is T^f -periodic of period p . Note that $s_*^f = s \in (t_-^f, t_+^f)$. By continuity, $\psi^{-1}(s) \in (t_-^g, t_+^g)$ for all g in a neighborhood of f . So there exists $\theta_*^g(s)$ such that $(\psi^{-1}(s), \theta_*^g(s))$ is T^g -periodic and its period is a divisor of p .

Furthermore we can assume that this neighborhood $U_1(x)$ of f is so small that (s, θ_*^g) is $\frac{1}{N}$ -close to $(\psi(s), \theta)$ (with respect to a fixed usual norm).

Periodic points are dense in $\Omega_{\theta, N}^f$ by [BoGaKrTr]. Let $\{x_1, \dots, x_k\} \subset \Omega_{\theta, N}^f$ be a set of T^f -periodic points be such that $\{\psi^{-1}(x_1), \dots, \psi^{-1}(x_k)\}$ is $\frac{1}{N}$ -dense in $\Omega_{\theta, N}^g$ for each g in a neighborhood U_2 of f . Combining this with the previous paragraph,

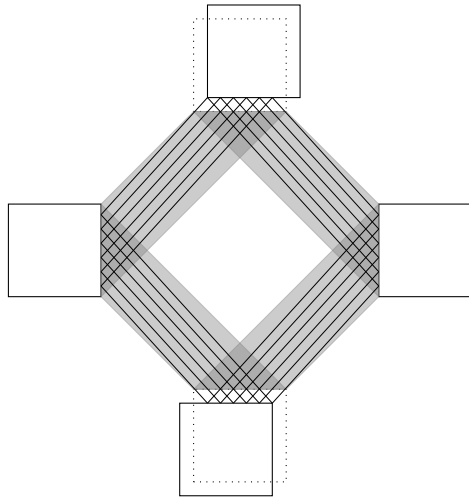


FIGURE 5. An example of periodic cylinder of length 4 (filled), and this cylinder after perturbation (striped).

we conclude that for every g in $U_N(f) = U_2 \cap \bigcap_i U_1(x_i)$, the set of T^g -periodic points is at least $\frac{2}{N}$ -dense in $\Omega_{\theta,N}^g$.

Let $\{f_i\} \subset \mathcal{A}^{\mathbb{Z}^2}$ be countable and dense, such that each f_i is N_i -ringed for some N_i . Let

$$\mathcal{G} := \bigcap_{N \geq 1} \bigcup_{\{i: N_i \geq N\}} U_{N_i}(f_i).$$

Clearly \mathcal{G} is a dense G_δ . We have shown that every parameter in \mathcal{G} gives rise to a wind tree with dense periodic points.

Now suppose r is rational. In this case, we can use a stronger property on periodic orbits. We will call a parameter f *rationally N -ringed* if f is N -ringed and all $a_{i,j}, b_{i,j}$ are rational for all $(i,j) \in E_N$. The key property here is that for any rationally N -ringed parameter the N -ringed table is Veech (moreover it is square-tiled) and thus there exists a countable dense set $\{\theta_j\} \subset \mathbb{S}^1$ such that every non-singular point of the form $(s, \theta_j) \in \Omega_{\theta_j,N}^f$ is periodic [Ve][MaTa]. We call such a direction a *periodic direction*.

We assume θ_j are enumerated so that the maximal combinatorial length of the periodic orbits inside $\Omega_{\theta_j,N}^f$ is increasing with j . Consider the smallest $\ell(f)$ such that $\theta_1, \dots, \theta_{\ell(f)}$ is $\frac{1}{N}$ -dense.

Let f be a rationally N -ringed parameter. For each periodic direction θ , we decompose all of $\Omega_{\theta,N}^f$ but points in saddle connections by the periodic orbit structure. More precisely this decomposition consists of a finite collection of intervals permuted by the dynamics such that the boundary of each interval from this decomposition is in a saddle connection. We call this collection of intervals $\mathcal{D}(f, \theta)$. For each $I \in \mathcal{D}(f, \theta)$, all points in I are periodic of the same period p , and we call $\bigcup_{i=0}^{p-1} (T^f)^i(I)$ a *periodic cylinder*.

In the general case, we presented a construction that associates to every T^f -periodic point $x = (s, \theta)$ and every g in a small enough neighborhood $U_1(s, \theta)$ of f , an angle $\theta_*^g(x)$ such that $(\psi^{-1}(s), \theta_*^g(x))$ is T^g -periodic.

Because the periodic points come in cylinders, as described above for f , the angles $\theta_*^g(s, \theta)$ and $\theta_*^g(s', \theta)$ will coincide for s' in an open interval around s (if $g \in U_1(s, \theta) \cap U_1(s', \theta)$).

For each interval I in $\mathcal{D}(f, \theta)$, we can thus find an interval $I' \subset \psi^{-1}(I)$ containing at least $1 - \frac{1}{\ell(f) \cdot N}$ proportion of points of $\psi^{-1}(I)$ such that the intersection $U_N(f) := \bigcap_{s' \in I'} U_1(s', \theta)$ is open. For all $g \in U_N(f)$ and all $s, s' \in I'$ we have $\theta_*^g(s, \theta) = \theta_*^g(s', \theta)$.

Furthermore we can assume that this neighborhood $U_N(f)$ of f is so small that θ_*^g is $\frac{1}{N}$ -close to θ (with respect to a fixed usual norm).

Let $\{f_i\} \subset \mathcal{A}^{\mathbb{Z}^2}$ be countable dense and such that each f_i is rationally N_i -ringed for some N_i . Let

$$\mathcal{G} := \bigcap_{N \geq 1} \bigcup_{\{i: N_i \geq N\}} U_{N_i}(f_i).$$

Clearly \mathcal{G} is a dense G_δ . We claim that every parameter in \mathcal{G} gives rise to a wind tree with locally dense periodic orbits. For each parameter $g \in \mathcal{G}$, there exists an infinite subsequence $(f_{i_k}) \subset (f_i)$ such that $g \in U_{N_{i_k}}(f_{i_k})$ for all k and N_{i_k} is increasing. For sake of simplicity we denote this subsequence by (f_k) .

Let m_k be the measure of $\bigcup_{I \in \mathcal{D}(f_k, \theta_j)} I$ (it does not depend on j). By definition of I' , $\bigcup_{I \in \mathcal{D}(f_k, \theta_j)} I'$ is of measure at least $(1 - \frac{1}{\ell(f_k) \cdot N_k}) m_k$. Thus $\bigcap_{j=1}^{\ell(f_k)} \bigcup_{I \in \mathcal{D}(f_k, \theta_j)} I'$ is of measure at least $(1 - \frac{1}{N_k}) m_k$ and thus the complement of the following infinite measure G_δ set:

$$\bigcap_K \bigcup_{k \geq K} \bigcap_{j=1}^{\ell(f_k)} \bigcup_{I \in \mathcal{D}(f_k, \theta_j)} I'$$

is of zero measure. □

6. GENERALIZATIONS

Our results hold in a much larger framework. In the proof of minimality we only used that N -ringed configurations are dense in the space of all configurations, and that they are rational polygonal billiard tables. For the local density of periodic orbits we also used that N -ringed configurations which are Veech polygonal billiard tables are dense. Now we give some examples where these properties hold.

1) We stay in the setup discussed in the article but additionally allow the empty tree denoted by \emptyset , thus the space of parameter is $\{\emptyset\} \cup \{(a, b) : r \leq a \leq 1 - r, r \leq b \leq 1 - r\}$. The Ehrenfests specifically required that the average distance A between neighboring squares is large compared to $2r$. For any probability distribution m on the continuous part of the space of parameters, if we add a δ function on the empty tree, then for $c < 1$ large enough, the distribution $c\delta + (1 - c)m$ verifies almost surely this requirement. However our result tells nothing about a full measure set of parameters for Lebesgue measure.

2) Instead of fixing $r \in [\frac{1}{4}, \frac{1}{2})$, we fix r between 0 and $\frac{1}{2}$. If $r \in [\frac{1}{2(n+1)}, \frac{1}{2n})$, place at most n^2 copies of trees in each cell. We can then form N -ringed configuration in a more general sense where we replace the rhombus by an appropriate

curve around the origin. One can do so using just $n + 1$ copies of the tree in each cell.

3) Instead of fixed size squares we use all vertical horizontal squares contained in the unit cell $[0, 1]^2$. This set is naturally parametrized by

$$\{t = (a, b, r) : 0 \leq a \leq 1, 0 \leq b \leq 1, 0 \leq r \leq \min(a, b, 1 - a, 1 - b)\}$$

where a $2r$ by $2r$ square tree is centered at the point (a, b) . More generally we call a polygon a VH-tree if the sides alternate between vertical and horizontal. For example a VH-tree with 4 sides is a rectangle, with 6 sides is a figure L. We can use various subsets of VH-trees, for example all VH-trees with at most $2M$ sides ($M \geq 4$ fixed) contained in the unit cell. Or we can use the VH-trees with 12 sides and fixed side length $r \in [1/4, 1/3]$ (called + signs). Many other interesting subclasses can be considered.

4) Fix a rational triangle P , and consider the set of all rescalings of P contained in the unit cell $[0, 1]^2$ oriented in such a way that they have either a vertical or horizontal side.

5) One can also change the cell structure to the hexagonal tiling and consider appropriate polygonal trees, for example one can use appropriate classes of equilateral triangular trees or hexagonal trees.

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