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► **To cite this version:**

Eric Amar. The raising steps method. Applications to the L^r Hodge theory in a compact riemannian manifold.. 2017. hal-01158323v2

HAL Id: hal-01158323

<https://hal.science/hal-01158323v2>

Preprint submitted on 3 Oct 2017

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The raising steps method. Application to the L^r Hodge theory in a compact riemannian manifold.

Eric Amar

Abstract

Let X be a complete metric space and Ω a domain in X . The Raising Steps Method allows to get from local results on solutions u of a linear equation $Du = \omega$ global ones in Ω .

It was introduced in [1] to get good estimates on solutions of $\bar{\partial}$ equation in domains in a Stein manifold.

As a simple application we shall get a strong L^r Hodge decomposition theorem for p -forms in a compact riemannian manifold without boundary, and then we retrieve this known result by an entirely different and simpler method.

1 Introduction.

This work proposes a way for passing from local to global: the raising steps method, RSM for short. I introduce it precisely to get $L^r - L^s$ estimates for solutions of the $\bar{\partial}$ equation in Stein manifold in [1]. See also [2] to have result in case of intersection of domains in Stein manifolds.

The aim is to generalise it to the case where the $\bar{\partial}$ operator is replaced by an abstract linear operator D acting on a domain in a complete metric space.

We shall deal with the following situation: we have a complete metric space X admitting partitions of unity (see condition (ii) below) and a measure μ .

A *domain* of X will be a connected open set Ω of X , relatively compact.

We are interested on solutions u of a linear equation $Du = \omega$, in a domain Ω . Precisely fix a threshold $s > 1$. Suppose you have a *global* solution u on Ω of $Du = \omega$ with estimates $L^s(\Omega) \rightarrow L^s(\Omega)$. It may happen that we have a constrain:

$\exists K \subset L^{s'}(\Omega)$ with s' the conjugate exponent for s , such that $\forall u :: Du \in L^s(\Omega), \forall h \in K, \langle Du, h \rangle = 0$.

In this case, in order for a solution of $Du = \omega$ to exist, we need to have $\omega \perp K$. With no constrain, we take $K = \{0\}$.

Very often this threshold will be $s = 2$, since Hilbert spaces are usually more tractable.

Now suppose that we have, for $1 \leq r \leq s$, *local* solutions u on $U \cap \Omega$ of $Du = \omega$ with estimates $L^r(\Omega) \rightarrow L^t(U \cap \Omega)$ with a *strict increase* of the regularity, for instance $\frac{1}{t} = \frac{1}{r} - \tau, \tau > 0$ for any $r \leq s$, then the raising steps method gives a *global* solution v of $Dv = \omega$ which is essentially in $L^t(\Omega)$ if we start with a data ω in $L^r(\Omega)$.

In particular we prove:

Theorem 1.1. (*Raising steps theorem*) Under the assumptions above, there is a positive constant c such that for $1 \leq r \leq s$, if $\omega \in L^r(\Omega)$, $\omega \perp K$, there is a $u \in L^t(\Omega)$ with $\frac{1}{t} = \frac{1}{r} - \tau$, such that $Du = \omega + \tilde{\omega}$, with $\tilde{\omega} \in L^s(\Omega)$, $\tilde{\omega} \perp K$ and control of the norms.

From this theorem and the fact that there is a *global* solution $v \in L^s(\Omega)$ to $Dv = \tilde{\omega}$ we get that $u - v$ is the *global* solution we are searching for.

To illustrate the method, we shall apply it for the Poisson equation associated to the Hodge Laplacian in a compact riemannian boundary-less manifold (M, g) .

On (M, g) we can define Sobolev spaces $W^{k,r}(\Omega)$ (see [8]) and if M is compact these spaces are in fact independent of the metric. Moreover the Sobolev embeddings are true in this case and a chart diffeomorphism makes a correspondence between Sobolev spaces in \mathbb{R}^n and Sobolev spaces in M .

Let d be the exterior derivative on M and d^* its adjoint; we define the Hodge laplacian acting from p differential forms to p differential forms to be: $\Delta := dd^* + d^*d$. Because M is compact, we have that Δ is self adjoint. The Poisson equation on M is, for a given p -form ω on M , to find a p -form u on M such that $\Delta u = \omega$.

Let \mathcal{H}_p be the set of p -harmonic forms in M , i.e. $h \in \mathcal{H}_p \iff h \in C_p^\infty(M)$, $\Delta h = 0$. We have:

$$\forall h \in \mathcal{H}_p, \langle \Delta u, h \rangle = \langle u, \Delta h \rangle = 0$$

hence, in order to solve $\Delta u = \omega$, we need to have $\omega \perp \mathcal{H}_p$.

We derive, from a solution of the Poisson equation we get by use of the RSM, a L_p^r Hodge decomposition for p differential forms on M .

We shall use, for the local results, the classical ones. Let B be a ball in \mathbb{R}^n , then:

$$\forall \gamma \in L^r(B), \exists u \in W^{2,r}(B) :: \Delta_{\mathbb{R}^n} u = \gamma, \|u\|_{W^{2,r}(B)} \leq C \|\gamma\|_{L^r(B)}.$$

These non trivial estimates are coming from Gilbarg and Trudinger [6, Theorem 9.9, p. 230] and the constant $C = C(n, r)$ depends only on n and r .

Then, together with the R.S.M., we get a solution of the Poisson equation for the Hodge laplacian.

Theorem 1.2. Let M be a compact, C^∞ Riemannian manifold without boundary. For any r , $1 \leq r \leq n/2$, if g is a p -form in $L_p^r(M) \cap \mathcal{H}_p^\perp$ there is a p -form $v \in L^t(M)$ such that $\Delta v = g$ and $\|v\|_{L_p^t(M)} \leq c \|g\|_{L_p^r(M)}$ with $\frac{1}{t} = \frac{1}{r} - \frac{2}{n}$.

Moreover for any $r > 1$ and any p form u solution of $\Delta u = g$, we have u in $W_p^{2,r}(M)$.

From this result we deduce the L^r Hodge decomposition of p -forms.

Theorem 1.3. Let (M, g) be a compact riemannian manifold without boundary. We have the strong L^r Hodge decomposition:

$$\forall r, 1 \leq r < \infty, L_p^r(M) = \mathcal{H}_p^r \oplus \text{Im} \Delta(W_p^{2,r}(M)) = \mathcal{H}_p^r \oplus \text{Im} d(W_p^{1,r}(M)) \oplus \text{Im} d^*(W_p^{1,r}(M)).$$

The case $r = 2$ of this decomposition which gives us $s = 2$ as a threshold, goes to Morrey in 1966 and essentially all results in the L^2 case we use here are coming from the basic work of Morrey [10].

This decomposition is an already known result of C. Scott [12] but proved here by an entirely different method. Critical to Scott's proof is a nice L^r Gaffney's inequality which he proved and used to get the L^r Hodge decomposition, the same way than Morrey [10] did with the L^2 Gaffney's inequality [5] to get the L^2 Hodge decomposition.

In the case of a compact manifold with boundary, G. Schwarz [11] proved also a L^r Gaffney's inequality to get the L^r Hodge decomposition in that case, then he deduced of it a global L^r solution for the equation $\Delta u = \omega$.

In the nice book by F.W. Warner [15], the author proved directly, without the use of Gaffney's inequality, a global L^2 solution for the equation $\Delta u = \omega$ in the case of a compact manifold without boundary. He deduced from it the L^2 Hodge decomposition.

Here we use the RSM plus the global L^2 solution for the equation $\Delta u = \omega$ given by Warner [15], to get a global L^r solution for the equation $\Delta u = \omega$ and then recover the L^r Hodge decomposition, without any Gaffney's inequalities. Hence we get a completely different proof of the known L^r Hodge decomposition.

Many important applications of the L^2 Hodge decomposition in cohomology theory and algebraic geometry are in the book by C. Voisin [14].

So it may be interesting to have a short proof of this important Hodge decomposition in the L^r case.

Finally, in the last section we prove, by use of the "double manifold" technique:

Theorem 1.4. *Let Ω be a domain in the smooth complete riemannian manifold M and $\omega \in L_p^r(\Omega)$, then there is a p -form $u \in W_p^{2,r}(\Omega)$, such that $\Delta u = \omega$ and $\|u\|_{W_p^{2,r}(\Omega)} \leq c(\Omega)\|\omega\|_{L_p^r(\Omega)}$.*

And we make a short incursion in the domain of manifold with boundary:

Corollary 1.5. *Let M be a smooth compact riemannian manifold with smooth boundary ∂M . Let $\omega \in L_p^r(M)$. There is a p -form $u \in W_p^{2,r}(M)$, such that $\Delta u = \omega$ and $\|u\|_{W_p^{2,r}(M)} \leq c\|\omega\|_{L_p^r(M)}$.*

Schwarz [11, Theorem 3.4.10, p. 137] proved a better theorem: you can prescribe the values of u on the boundary. But again the proof here is much lighter.

2 The Raising Steps Method.

We shall deal with the following situation: we have a complete metric space X admitting partitions of unity (see condition (ii) below) and a positive σ -finite measure μ .

2.1 Assumptions on the linear operator D .

We shall denote $E^p(X)$ the set of \mathbb{C}^p valued functions on X . This means that $\omega \in E^p(X) \iff \omega(x) = (\omega_1(x), \dots, \omega_p(x))$. We put a punctual norm on $\omega \in E^p(X)$, $|\omega(x)|^2 := \sum_{j=1}^p |\omega_j(x)|^2$ and if U is an open set in X , we consider the Lebesgue space $L_p^r(U)$:

$$\omega \in L_p^r(U) \iff \|\omega\|_{L_p^r(U)}^r := \int_U |\omega(x)|^r d\mu(x) < \infty.$$

The space $L_p^2(U)$ is a Hilbert space with the scalar product $\langle \omega, \omega' \rangle := \int_U \left(\sum_{j=1}^p \omega_j(x) \bar{\omega}'_j(x) \right) d\mu(x)$.

We are interested in solution of a linear equation $Du = \omega$, where $D = D_p$ is a linear operator acting on E^p .

In order to have $Du = \omega$, it may happen that we have a constrain: there is a subspace $K \subset L^{r'}(\Omega)$ such that

$$\forall h \in K, \forall u :: Du \in L^r(\Omega), \langle Du, h \rangle = 0 \iff Du \perp K.$$

The absence of constrain is done by setting $K = \{0\}$.

Now on the integer p will be fixed so the explicit mention of the integer p will be often omitted.

We shall make the following hypotheses.

Let Ω be a domain in X . There is a $\tau \geq \delta$ with $\frac{1}{t} = \frac{1}{r} - \tau$, and a positive constant c_l such that:

(i) Local Existence with Increasing Regularity (LEIR): for any $x \in \bar{\Omega}$, there is a ball $B := B(x, R_x)$ such that if $\omega \in L_p^r(B)$, we can solve $Du_x = \omega$ in $B' := B(x, R_x/2)$ with $L_p^r(B) - L_p^t(B')$ estimates, i.e. $\exists u_x \in L^t(B')$, $Du_x = \omega$ in B' and $\|u_x\|_{L^t(B')} \leq c_l \|\omega\|_{L^r(B)}$.

It may append, in the case X is a manifold, that we have a better regularity for the local existence:

(i') Sobolev regularity: if $\omega \in L_p^r(B)$, we can solve $Du_x = \omega$ in $B' := B(x, R_x/2)$ with $L_p^r(B) - W_p^{\alpha,r}(B')$ estimates, i.e. $\exists u_x \in W_p^{\alpha,r}(B')$, $Du_x = \omega$ in B' and $\|u_x\|_{W_p^{\alpha,r}(B')} \leq c_l \|\omega\|_{L^r(B)}$.

By compactness we can cover $\bar{\Omega}$ by a finite set of balls $\{B(x_j, R_j/2)\}_{j=1,\dots,N}$ of the previous form. Set $B_j := B(x_j, R_j)$, $B'_j := B(x_j, R_j/2)$. Set u_j the local solution of $Du_j = \omega$ with $\|u_j\|_{L^t(B'_j)} \leq c_l \|\omega\|_{L^r(B_j)}$.

(ii) Partition of unity: If $\{B'_j\}_{j=1,\dots,N}$ is a covering of $\bar{\Omega}$, then there is an associated set of functions $\{\chi_j\}_{j=1,\dots,N}$ such that χ_j has compact support in B'_j , $\forall j = 1, \dots, N$, $0 \leq \chi_j(x) \leq 1$ and $\sum_{j=1}^N \chi_j(x) = 1$ for $x \in \bar{\Omega}$.

(iii) Commutator condition: We set $\Delta_j = \Delta(\chi_j, u_j) := \chi_j Du_j - D(\chi_j u_j)$. There is a constant $\delta > 0$ such that, with $\frac{1}{t} = \frac{1}{r} - \delta$, we have:

$$\|\Delta_j\|_{L^t(B'_j)} \leq c(\chi_j)(\|\omega\|_{L^r(B_j)} + \|u_j\|_{L^r(B_j)}).$$

(iv) Global resolvability: We can solve $Dw = \omega$ globally in Ω with $L^s - L^s$ estimates, i.e.

$$\exists c_g > 0, \exists w \text{ s.t. } Dw = \omega \text{ in } \Omega \text{ and } \|w\|_{L^s(\Omega)} \leq c_g \|\omega\|_{L^s(\Omega)}, \text{ provided that } \omega \perp K.$$

It may append, in the case X is a manifold, that we have a better regularity for the global existence:

(iv') Sobolev regularity: We can solve $Dw = \omega$ globally in Ω with $L_p^s - W_p^{\alpha,s}$ estimates, i.e.

$$\exists c_g > 0, \exists w \text{ s.t. } Dw = \omega \text{ in } \Omega \text{ and } \|w\|_{W_p^{\alpha,s}(\Omega)} \leq c_g \|\omega\|_{L^s(\Omega)}, \text{ provided that } \omega \perp K.$$

Theorem 2.1. (*Raising steps theorem*) Under the assumptions (i), (ii), (iii), (iv) above, there is a positive constant c_f such that for $1 \leq r \leq s$, if $\omega \in L^r(\Omega)$, $\omega \perp K$ there is a $u = u_s \in L^t(\Omega)$ with $\frac{1}{t} = \frac{1}{r} - \tau$, such that $Du = \omega + \tilde{\omega}$, with $\tilde{\omega} \in L^s(\Omega)$, $\tilde{\omega} \perp K$ and control of the norms.

If moreover we have (i') then $u \in W_p^{\alpha,r}(\Omega)$ with control of the norm.

Proof. Let $r \leq s$ and $\omega \in L^r(\Omega)$, $\omega \perp K$; we start with the covering $\{B'_j\}_{j=1,\dots,N}$ and the local solution $Du_j = \omega$ with $\frac{1}{t} = \frac{1}{r} - \tau$, $u_j \in L^t(B'_j)$ given by hypothesis (i).

If (i') is true, then we have $u_j \in W_p^{\alpha,r}(B'_j)$, $Du_j = \omega$.

Let $\{\chi_j\}_{j=1,\dots,N}$ be the partition of unity subordinate to $\{B'_j\}_{j=1,\dots,N}$ given by (ii). Because $0 \leq \chi_j(x) \leq 1$ we have $\chi_j u_j \in L^t(\Omega)$ hence

$$\|\chi_j u_j\|_{L^t(\Omega)} \leq \|u_j\|_{L^t(B'_j)} \leq c_l \|\omega\|_{L^r(\Omega)}. \quad (2.1)$$

Set $v_0 := \sum_{j=1}^N \chi_j u_j$. Then we have, setting now $\frac{1}{t_0} = \frac{1}{r} - \tau$:

• $v_0 \in L^{t_0}(\Omega)$ because $\chi_j u_j \in L^{t_0}(\Omega)$ for $j = 1, \dots, N$, and $\|v_0\|_{L^{t_0}(\Omega)} \leq c\|\omega\|_{L^r(\Omega)}$ with $c = Nc_l$ by (2.1). If (i') is true, i.e. X is a manifold, then we can choose $\chi_j \in \mathcal{D}(B'_j)$, i.e. in the space of \mathcal{C}^∞ functions of compact support in B_j , hence $\|\chi_j u_j\|_{W_p^{\alpha,r}(\Omega)} \leq c\|u_j\|_{W_p^{\alpha,r}(B'_j)} \leq c_l\|\omega\|_{L^r(\Omega)}$. So

• $v_0 \in W_p^{\alpha,r}(\Omega)$ with $\|v_0\|_{W_p^{\alpha,r}(\Omega)} \leq c\|\omega\|_{L^r(\Omega)}$.

We have

$$\bullet Dv_0 = \sum_{j=1}^N \chi_j Du_j + \sum_{j=1}^N \Delta(\chi_j, u_j).$$

Setting $\omega_1(x) := \sum_{j=1}^N \Delta(\chi_j, u_j)(x)$, we get

$$\forall x \in \Omega, Dv_0(x) = \sum_{j=1}^N \chi_j(x)\omega(x) + \sum_{j=1}^N \Delta(\chi_j, u_j)(x) = \omega(x) + \omega_1(x)$$

By hypothesis (iii), $\Delta(\chi_j, u_j) \in L^{s_0}(B'_j)$ with $\frac{1}{s_0} = \frac{1}{r} - \delta$, hence $\omega_1 \in L^{s_0}(\Omega)$, with

$$\|\omega_1\|_{L^{s_0}(\Omega)} \leq G\|\omega\|_{L^r(\Omega)} \quad (2.2)$$

and $G = c_{sl}N$.

The regularity of ω_1 is higher by one step $\delta > 0$ than that of ω . Moreover

$$\forall h \in K, \langle \omega_1, h \rangle = \langle \omega, h \rangle - \langle Dv_0, h \rangle = 0 \text{ because } \langle \omega, h \rangle = 0 \text{ and } \langle Dv_0, h \rangle = 0.$$

If $s_0 \geq s$ we notice that $\omega_1 \in L^s(\Omega)$ because $L^{s_0}(\Omega) \subset L^s(\Omega)$ for Ω is relatively compact. So we are done by setting $u_s = v_0$, $\tilde{\omega} = \omega_1$.

If $s_0 < s$ we proceed by induction: we set t_1 such that $\frac{1}{t_1} = \frac{1}{s_0} - \tau = \frac{1}{r} - \delta - \tau$, and we use the same covering $\{B'_j\}_{j=1,\dots,N}$ and the same partition of unity $\{\chi_j\}_{j=1,\dots,N}$ and with ω_1 in place of ω , we get $\exists v_1 \in L^{t_1}(\Omega)$, $Dv_1 = \omega_1 + \omega_2$ and, setting s_1 such that $\frac{1}{s_1} = \frac{1}{s_0} - \delta = \frac{1}{r} - 2\delta$, we have that the regularity of ω_2 raises of 2 times δ from that of ω .

We still have

$$\forall h \in K, \langle \omega_2, h \rangle = \langle \omega_1, h \rangle - \langle Dv_1, h \rangle = 0$$

so by induction, after a finite number k of steps, we get a $s_k \geq s$. The linear combination $u := \sum_{j=0}^{k-1} (-1)^j v_j$ now gives $Du = \omega + (-1)^k \omega_k$ with $\omega_k \perp K$. And again we are done by setting $\tilde{\omega} = (-1)^k \omega_k$.

If (i') is true, then we have $v_0 \in W^{\alpha,r}(\Omega)$ and $v_1 \in W^{\alpha,s_0}(\Omega)$, $\|v_1\|_{W^{\alpha,s_0}(\Omega)} \leq c\|\omega_1\|_{L^{s_0}(\Omega)}$. Hence by (2.2) we get $\|v_1\|_{W^{\alpha,s_0}(\Omega)} \leq cG\|\omega\|_{L^r(\Omega)}$.

A fortiori, $v_1 \in W^{\alpha,r}(\Omega)$ because $r < s_0$ with the same control of the norm. Likewise we have $\forall j = 1, \dots, k$, $\|v_j\|_{W^{\alpha,r}(\Omega)} \leq c\|\omega\|_{L^r(\Omega)}$, hence the same control of the norm of u . \square

Corollary 2.2. *Under the assumptions of the raising steps theorem and with the global assumption (iv), there is a constant $c_f > 0$, such that for $r \leq s$, if $\omega \in L^r(\Omega)$, $\omega \perp K$ there is a $v \in L^t(\Omega)$ with $t := \min(s, t_0)$ and $\frac{1}{t_0} = \frac{1}{r} - \tau$, such that*

$$Dv = \omega \text{ and } v \in L^t(\Omega), \|v\|_{L^t(\Omega)} \leq c\|\omega\|_{L^r(\Omega)}.$$

If moreover (i') and (iv') are true then we have $v \in W^{\alpha,r}(\Omega) \cap L^t(\Omega)$ with $\|v\|_{W^{\alpha,r}(\Omega)} \leq c\|\omega\|_{L^r(\Omega)}$.

Proof. By the raising steps Theorem 2.1 we have a $u \in L^{t_0}(\Omega)$ such that $Du = \omega + \tilde{\omega}$ with $\tilde{\omega} \in L^s(\Omega)$ and $\tilde{\omega} \perp K$; by hypothesis (iv) we can solve $D\tilde{v} = \tilde{\omega}$ with $\tilde{v} \in L^s(\Omega)$ so it remains to set $v := u - \tilde{v}$ to get $v \in L^t(\Omega)$ and $Dv = \omega$.

If moreover (i') is true then we have $u \in W^{\alpha,r}(\Omega)$. If (iv') is true then we can solve $D\tilde{v} = \tilde{\omega}$ with $\tilde{v} \in W^{\alpha,s}(\Omega)$. Hence, because $r \leq s$, with $v := u - \tilde{v}$ we get $v \in W^{\alpha,r}(\Omega)$ and $Dv = \omega$. The proof is complete. \square

3 Application to Poisson equation on a compact riemannian manifold.

3.1 Local existence with increasing regularity.

In order to have the local result, we choose a chart $(V, \varphi := (x_1, \dots, x_n))$ so that $g_{ij}(y) = \delta_{ij}$ and $\varphi(V) = B$ where $B = B_e$ is an Euclidean ball centered at $\varphi(y) = 0$ and g_{ij} are the components of the metric tensor w.r.t. φ .

Because I was unable to find an easy proof of the following theorem in the literature, I reprove it for the reader's convenience.

Theorem 3.1. *For any $y \in M$, there are open sets $W \subset \bar{W} \subset V \subset M$, $y \in W$, such that we have:*

$$\forall \omega \in L^r_p(W), \exists u \in W^{2,r}_p(W) :: \Delta u = \omega, \|u\|_{W^{2,r}(W)} \leq C \|\omega\|_{L^r(W)}.$$

Proof. Of course the operator d on p -forms is local and so is d^* as a first order differential operator. We start with a chart (V, φ) of M such that $\varphi(y) = 0 \in \mathbb{R}^n$ and the metric tensor read in this chart at y is the identity.

Then the Hodge laplacian Δ_φ read by φ in a ball $B := B(0, R) \subset \mathbb{R}^n$ is not so different from that of \mathbb{R}^n in B when acting on p -forms in B . We set $\Delta_\varphi \omega_\varphi = \Delta_{\mathbb{R}} \omega_\varphi + A \omega_\varphi$, where ω_φ is the p -form ω read in the chart (V, φ) and A is a matrix valued second order operator with C^∞ smooth coefficients such that $A : W^{2,r}(B) \rightarrow L^r(B)$ with, for a R small enough $\|Av\|_{L^r(B)} \leq c\|v\|_{W^{2,r}(B)}$.

This is true because at the point $y \in V$ we are in the flat case and if R is small enough, the difference $A := \Delta_\varphi - \Delta_{\mathbb{R}}$ in operator norm $W^{2,r}(B) \rightarrow L^r(B)$ goes to 0 when R goes to 0, because $\varphi \in C^2$ and the metric tensor g is also C^2 .

We know that $\Delta_{\mathbb{R}}$ operates component-wise on the p -form γ , so we have

$$\forall \gamma \in L^r_p(B), \exists v_0 \in W^{2,r}_p(B) :: \Delta_{\mathbb{R}} v_0 = \gamma, \|v_0\|_{W^{2,r}(B)} \leq C \|\gamma\|_{L^r(B)},$$

simply setting the component of v_0 to be the Newtonian potential of the corresponding component of γ in U . This way v_0 is linear with respect to γ . These non trivial estimates are coming from Gilbarg and Trudinger [6, Theorem 9.9, p. 230] and the constant $C = C(n, r)$ depends only on n and r .

So we get $\Delta_{\mathbb{R}} v_0 + Av_0 = \gamma + \gamma_1$, with

$$\gamma_1 = Av_0 \Rightarrow \|\gamma_1\|_{L^r(B)} \leq c\|v_0\|_{W^{2,r}(B)} \leq cC\|\gamma\|_{L^r(B)}.$$

We solve again

$$\exists v_1 \in W^{2,r}_p(B) :: \Delta_{\mathbb{R}} v_1 = \gamma_1, \|v_1\|_{W^{2,r}(B)} \leq C\|\gamma_1\|_{L^r(B)} = C^2 c \|\gamma\|_{L^r(B)},$$

and we set

$$\gamma_2 := Av_1 \Rightarrow \|\gamma_2\|_{L^r(B)} \leq c\|v_1\|_{W^{2,r}(B)} \leq C\|\gamma_1\|_{L^r(B)} \leq C^2 c^2 \|\gamma\|_{L^r(B)}.$$

And by induction:

$$\forall k \in \mathbb{N}, \gamma_k := Av_{k-1} \Rightarrow \|\gamma_k\|_{L^r(B)} \leq c\|v_{k-1}\|_{W^{2,r}(B)} \leq C\|\gamma_{k-1}\|_{L^r(B)} \leq C^k c^k \|\gamma\|_{L^r(B)}$$

and

$$\exists v_k \in W^{2,r}_p(B) :: \Delta_{\mathbb{R}} v_k = \gamma_k, \|v_k\|_{W^{2,r}(B)} \leq C\|\gamma_k\|_{L^r(B)} \leq C^{k+1} c^k \|\gamma\|_{L^r(B)}.$$

Now we set $v := \sum_{j \in \mathbb{N}} (-1)^j v_j$. This series converges in norm $W^{2,r}(B)$, provided that we choose the radius of the ball B small enough to have $cC^2 < 1$, and we get:

$$\Delta_\varphi v = \Delta_{\mathbb{R}} v + Av = \sum_{j \in \mathbb{N}} (-1)^j (\Delta_{\mathbb{R}} v_j + Av_j) = \gamma,$$

the last series converging in $L^r(B)$.

In fact every step is linear and we get that v is linear in γ .

Going back to the manifold M with $\gamma := \omega_\varphi$ and setting $u_\varphi := v$, $W := \varphi^{-1}(B)$, we get the right estimates:

$$\exists u \in W^{2,r}(W) :: \Delta u = \omega \text{ in } W, \quad \|u\|_{W^{2,r}(W)} \leq C \|\omega\|_{L^r(W)},$$

because the Sobolev spaces for B go to the analogous Sobolev spaces for W in M . \square

Now the next corollary is precisely the result we are searching for.

Corollary 3.2. *For any $y \in M$, there are open sets $V, W \Subset V$, $y \in W$, such that we have for $r \leq 2$:*

$$\forall \omega \in L_p^r(V), \exists u \in W^{2,r}(W) \cap L_p^t(W) :: \Delta u = \omega, \quad \|u\|_{W^{2,r}(W)} \leq C \|\omega\|_{L^r(V)}, \quad \|u\|_{L_p^t(W)} \leq C \|\omega\|_{L_p^r(V)}$$

with $\frac{1}{t} = \frac{1}{r} - \frac{2}{n}$, and $\nabla u \in W_p^{1,r}(W)$.

Proof. The Theorem 3.1 gives $u \in W^{2,r}(W)$ such that $\Delta u = \omega$, $\|u\|_{W^{2,r}(W)} \leq C \|\omega\|_{L^r(V)}$, hence we get that $\nabla u \in W_p^{1,r}(W)$ with the same control: $\|\nabla u\|_{W^{1,r}(W)} \leq C \|\omega\|_{L^r(V)}$.

For the first statement it remains to apply the Sobolev embedding theorems which are true here. \square

So we are in a special case of the previous section with $D := \Delta$ and, because Δ is essentially self adjoint, we have here $K := \mathcal{H}_p(M)$ where $\mathcal{H}_p(M)$ is the space of \mathcal{C}^∞ harmonic p -forms in M .

We shall need the following lemma.

Lemma 3.3. *Let Δ_φ be a second order elliptic matrix operator with \mathcal{C}^∞ coefficients operating on p -forms v defined in $U \subset \mathbb{R}^n$. Let $B := B(0, R)$ a ball in \mathbb{R}^n , $B' := B(0, R/2)$ and suppose that $B \Subset U$. Then we have an interior estimate: there are constants c_1, c_2 depending only on $n = \dim_{\mathbb{R}} M$, r and the \mathcal{C}^1 norm of the coefficients of Δ_φ in \bar{B} such that*

$$\forall v \in W_p^{2,r}(B), \|v\|_{W^{2,r}(B')} \leq c_1 \|v\|_{L^r(B)} + c_2 \|\Delta_\varphi v\|_{L^r(B)}. \quad (3.3)$$

Proof. For a 0-form this lemma is exactly [6, Theorem 9.11].

For p -forms we cannot avoid the use of deep results on elliptic systems of equations.

Let v be a p -form in $B \subset \mathbb{R}^n$. We use the interior estimates in [10, §6.2, Thm 6.2.6]. In our context, second-order elliptic system, and with our notations, with $r > 1$, we get:

$$\exists C > 0, \forall v \in W_p^{2,r}(B), \|v\|_{W^{2,r}(B')} \leq c_1 R^{-2} \|v\|_{L^r(B)} + c_2 \|\Delta_\varphi v\|_{L^r(B)},$$

already including the dependence in R .

The constants c_1, c_2 depend only on r , $n := \dim M$ and the bounds and moduli of continuity of all the coefficients of the matrix Δ_φ . (In [10], p. 213: the constant depends only on E and on E' .)

In particular, if Δ_φ has its coefficients near those of $\Delta_{\mathbb{R}}$ in the \mathcal{C}^1 norm, then the constants c_1, c_2 are near the ones obtained for $\Delta_{\mathbb{R}}$. \square

Now we deduce from it local interior regularity for the laplacian on a smooth compact manifold without boundary.

Lemma 3.4. *Let (M, g) be a riemannian manifold. For $x \in M$, $R > 0$, we take a geodesic ball $B(x, R)$ such that, read in a chart (V, φ) , $B(x, R) \Subset V$, the metric tensor at the center is the identity.*

We have a local Calderon Zygmund inequality on the manifold M . For any $r > 1$, there are constants c_1, c_2 depending only on $n = \dim_{\mathbb{R}} M$, r and R , such that:

$$\forall u \in W^{2,r}(B(x, R)), \quad \|u\|_{W^{2,r}(B(x, R/2))} \leq c_1 \|u\|_{L^r(B(x, R))} + c_2 \|\Delta u\|_{L^r(B(x, R))}.$$

Proof. We transcribe the problem in \mathbb{R}^n by use of a coordinates path (V, φ) exactly the same way we did to prove Theorem 3.1. The Hodge laplacian is the second order elliptic matrix operator Δ_{φ} with \mathcal{C}^{∞} coefficients operating in $\varphi(V) \subset \mathbb{R}^n$. By the choice of a R small enough we can apply Lemma 3.3, to the euclidean balls $B' := B_e(0, R'_e) \subset \varphi(B(x, R/2))$, $B := B_e(0, R_e) \subset \varphi(B(x, R))$ and we get, with u_{φ} the p -form u read in the chart (V, φ) ,

$$\|u_{\varphi}\|_{W^{2,r}(B')} \leq c_1 R^{-2} \|u_{\varphi}\|_{L^r(B)} + c_2 \|\Delta_{\varphi} u_{\varphi}\|_{L^r(B)}.$$

The Lebesgue measure on U and the canonical measure dv_g on $B(x, R)$ are equivalent; so the Lebesgue estimates and the Sobolev estimates up to order 2 on U are valid in $B(x, R)$ up to a constant.

So passing back to M , we get, with $A := \varphi^{-1}(B)$, $A' := \varphi^{-1}(B')$

$$\|u\|_{W^{2,r}(A')} \leq c_1 \|u\|_{L^r(A)} + c_2 \|\Delta u\|_{L^r(A)}.$$

So taking a smaller ball centered at x we end the proof of the lemma. \square

Theorem 3.5. *Let M be a compact \mathcal{C}^{∞} Riemannian manifold without boundary. We have:*

$$\forall \omega \in L_p^r(M) \cap \mathcal{H}_p(M)^{\perp}, \quad r \in (1, \frac{n}{2}); \quad \exists u \in L_p^s(M) :: \Delta u = \omega,$$

with $\frac{1}{s} = \frac{1}{r} - \frac{2}{n}$. Moreover $u \in W^{2,r}(M)$, $\|u\|_{W^{2,r}(M)} \leq c \|\omega\|_{L^r(M)}$.

Proof. First by duality we get the range $r > 2$. For this we shall proceed as we did in [1], using an avatar of the Serre duality [13]. We take t as in corollary 2.2.

Let $g \in L_p^{t'}(M) \cap \mathcal{H}_p(M)^{\perp}$, we want to solve $\Delta v = g$, with $t' > 2$ and t' conjugate to t .

We know by the previous part that, with $r \leq 2$,

$$\forall \omega \in L_p^r(M) \cap \mathcal{H}_p(M)^{\perp}, \quad \exists u \in L_p^t(M), \quad \Delta u = \omega, \quad \|u\|_{L^t(M)} \leq c \|\omega\|_{L^r(M)}. \quad (3.4)$$

Consider the linear form

$$\forall \omega \in L_p^r(M), \quad \mathcal{L}(\omega) := \langle u, g \rangle,$$

where u is a solution of (3.4); in order for $\mathcal{L}(\omega)$ to be well defined, we need that if u' is another solution of $\Delta u' = \omega$, then $\langle u - u', g \rangle = 0$; hence we need that g must be "orthogonal" to p -forms φ such that $\Delta \varphi = 0$ which is precisely our assumption.

Hence we have that $\mathcal{L}(f)$ is well defined and linear; moreover

$$|\mathcal{L}(f)| \leq \|u\|_{L^t(M)} \|g\|_{L^{t'}(M)} \leq c \|\omega\|_{L^r(M)} \|g\|_{L^{t'}(M)}.$$

So this linear form is continuous on $\omega \in L_p^r(M) \cap \mathcal{H}_p(M)^{\perp}$. By the Hahn Banach Theorem there is a form $v \in L_p^{r'}(M)$ such that:

$$\forall \omega \in L_p^r(M) \cap \mathcal{H}_p(M)^\perp, \mathcal{L}(\omega) = \langle \omega, v \rangle = \langle u, g \rangle.$$

But $\omega = \Delta u$, so we have, because Δ is essentially self adjoint and M is compact, $\langle \omega, v \rangle = \langle \Delta u, v \rangle = \langle u, \Delta v \rangle = \langle u, g \rangle$, for any $u \in L_p^t(M) :: \Delta u \in L_p^r(M)$. In particular for $u \in C_p^\infty(M)$. Now the hypothesis (iii) gives that $\Delta v = g$ in $L_p^{t'}(M)$, with $v \in L_p^{r'}(M)$. So we get:

$$\forall g \in L_p^{t'}(M) \cap \mathcal{H}_p(M)^\perp, \exists v \in L_p^{r'}(M), \Delta v = g.$$

It remains to prove the moreover. The condition (i') is true by the local existence Theorem 3.1. The condition (iv') is true in the case of the Hodge Laplacian on a compact boundary-less manifold by Morrey's results for $L^2(M)$, so we are done for $r \leq 2$.

For $r > 2$, Lemma 3.4 gives us by compactness that there is a smaller $R > 0$ and bigger constants c_1, c_2 such that:

$$\forall x \in M, \|u\|_{W^{2,r}(B(x,R/2))} \leq c_1 \|u\|_{L^r(B(x,R))} + c_2 \|\Delta u\|_{L^r(B(x,R))}. \quad (3.5)$$

We take for u our global solution in $L_p^r(M)$. We have that $\Delta u = \omega \in L_p^r(M)$ hence we can apply the estimate (3.5) to u :

$$\forall x \in M, \|u\|_{W^{2,r}(B(x,R/2))} \leq c_1 \|u\|_{L^r(B(x,R))} + c_2 \|\Delta u\|_{L^r(B(x,R))} \leq C \|u\|_{L^r(M)}.$$

Now it remains to cover M with a finite set of balls of the type $B(x, R/2)$ to end the proof. \square

3.2 The L^r Hodge decomposition.

In order to deduce the Hodge decomposition from the existence of a good solution to the Poisson equation, we shall need a little bit more material.

3.2.1 Basic facts.

Let \mathcal{H}_p^2 be the set of harmonic p -forms in $L^2(M)$, i.e. p -form ω such that $\Delta \omega = 0$, which is equivalent here to $d\omega = d^*\omega = 0$.

The classical L^2 theory of Morrey [10] gives, on a compact manifold M without boundary:

$$\mathcal{H}_p := \mathcal{H}_p^2 \subset C^\infty(M) \text{ [[10], (vi) p. 296]}$$

$$\dim_{\mathbb{R}} \mathcal{H}_p < \infty \text{ [[10], Theorem 7.3.1].}$$

This gives the existence of a linear projection from $L_p^r(M) \rightarrow \mathcal{H}_p$:

$$\forall v \in L_p^r(M), H(v) := \sum_{j=1}^N \langle v, e_j \rangle e_j$$

where $\{e_j\}_{j=1, \dots, N}$ is an orthonormal basis for \mathcal{H}_p . This is meaningful because $v \in L_p^r(M)$ can be integrated against $e_j \in \mathcal{H}_p \subset C^\infty(M)$. Moreover we have $v - H(v) \in L_p^r(M) \cap \mathcal{H}_p^\perp$ in the sense that $\forall h \in \mathcal{H}_p, \langle v - H(v), h \rangle = 0$; it suffices to test on $h := e_k$. We get

$$\langle v - H(v), e_k \rangle = \langle v, e_k \rangle - \left\langle \sum_{j=1}^N \langle v, e_j \rangle e_j, e_k \right\rangle = \langle v, e_k \rangle - \langle v, e_k \rangle = 0.$$

Let $v \in L_p^r(M)$. Set $h := H(v) \in \mathcal{H}_p$, and $\omega := v - h$. We have that $\forall k \in \mathcal{H}_p, \langle \omega, k \rangle = \langle v - H(v), k \rangle = 0$. Hence we can solve $\Delta u = \omega$ with $u \in W_p^{2,r}(M) \cap L_p^s(M)$. So we get $v = h + \Delta u$ which means:

$$L_p^r(M) = \mathcal{H}_p^r \oplus \text{Im} \Delta(W_p^{2,r}(M)).$$

We have a direct decomposition because if $\omega \in \mathcal{H}_p \cap \text{Im} \Delta(W_p^{2,r}(M))$, then $\omega \in C^\infty(M)$ and

$$\omega = \Delta u \Rightarrow \forall k \in \mathcal{H}_p, \langle \omega, k \rangle = 0$$

so choosing $k = \omega \in \mathcal{H}_p$ we get $\langle \omega, \omega \rangle = 0$ hence $\omega = 0$.

Now we are in position to prove:

Theorem 3.6. *If (M, g) is a compact riemannian manifold without boundary; we have the strong L^r Hodge decomposition:*

$$\forall r, 1 \leq r < \infty, L_p^r(M) = \mathcal{H}_p^r \oplus \text{Im}d(W_p^{1,r}(M)) \oplus \text{Im}d^*(W_p^{1,r}(M)).$$

Proof. We already have

$$L_p^r(M) = \mathcal{H}_p^r \oplus \text{Im}\Delta(W_p^{2,r}(M))$$

where \oplus means uniqueness of the decomposition.

So: $\forall \omega \in L_p^r(M) \cap \mathcal{H}_p^{r\perp}, \exists u \in W_p^{2,r}(M) :: \Delta u = \omega$.

From $\Delta = dd^* + d^*d$ we get $\omega = d(d^*u) + d^*(du)$ with $du \in W_p^{1,r}(M)$ and $d^*u \in W_p^{1,r}(M)$, so

$$\forall \omega \in L_p^r(M) \cap \mathcal{H}_p^{r\perp}, \exists \alpha \in W_{p-1}^{1,r}(M), \exists \beta \in W_{p+1}^{1,r}(M) :: \omega = \alpha + \beta,$$

simply setting $\alpha = d(d^*u)$, $\beta = d^*(du)$, which gives $\alpha \in d(W_{p-1}^{1,r}(M))$ and $\beta \in d^*(W_{p+1}^{1,r}(M))$ and this proves the existence.

The uniqueness is given by Lemma 6.3 in [12] and I copy this simple (but nice) proof here for the reader's convenience.

Suppose that $\alpha \in W_{p-1}^{1,r}(M)$, $\beta \in W_{p+1}^{1,r}(M)$, $h \in \mathcal{H}_p$ satisfy $d\alpha + d^*\beta + h = 0$.

Let $\varphi \in \mathcal{C}_p^\infty(M)$, because of the classical \mathcal{C}^∞ -Hodge decomposition, there are $\eta \in \mathcal{C}_{p-1}^\infty(M)$, $\omega \in \mathcal{C}_{p+1}^\infty(M)$ and $\tau \in \mathcal{H}_p$ satisfying $\varphi = d\eta + d^*\omega + \tau$.

Notice that $\langle d^*\beta, d\eta \rangle = \langle \beta, d^2\eta \rangle = \langle \beta, 0 \rangle = 0$ and $\langle h, d\eta \rangle = \langle d^*h, \eta \rangle = 0$, by the duality between d and d^* . Linearity then gives

$$\langle d\alpha, d\eta \rangle = \langle d\alpha + d^*\beta + h, d\eta \rangle = \langle 0, d\eta \rangle = 0. \quad (3.6)$$

Finally we have

$$\begin{aligned} \langle d\alpha, \varphi \rangle &= \langle d\alpha, d\eta \rangle + \langle d\alpha, d^*\omega \rangle + \langle d\alpha, \tau \rangle \\ &= 0 + \langle \alpha, d^{*2}\omega \rangle + \langle \alpha, d\tau \rangle \text{ by (3.6)} \\ &= \langle \alpha, 0 \rangle + \langle \alpha, 0 \rangle \text{ because } d^{*2} = 0 \text{ and } \tau \in \mathcal{H}_p \\ &= 0. \end{aligned}$$

Since $\mathcal{C}_p^\infty(M)$ is dense in $L_p^{r'}(M)$, r' being the conjugate exponent of r , and φ is arbitrary, we see that $d\alpha = 0$. Analogously, we see that $d^*\beta = 0$ and it follows that $h = 0$. \square

4 Case of Ω a domain in M .

Let Ω be a domain in a \mathcal{C}^∞ smooth complete riemannian manifold M , compact or non compact; we want to show how the results in case of a compact boundary-less manifold apply to this case.

A classical way to get rid of a "annoying boundary" of a manifold is to use its "double". For instance: Duff [4], Hörmander [9, p. 257]. Here we copy the following construction from [7, Appendix B].

Let N be a relatively compact domain of M such that ∂N is a smooth hypersurface and $\bar{\Omega} \subset N$. The "Riemannian double" $D := D(N)$ of N , obtained by gluing two copies of N along ∂N , is a compact Riemannian manifold without boundary. Moreover, by its very construction, it is always possible to assume that D contains an isometric copy of the original domain N , hence of the original Ω . We shall also write Ω for its isometric copy to ease notations.

We shall need the following difficult result by N. Aronszajn, A. Krzywicki and J. Szarski [3], a strong continuation property, which says that if, for any compact set $K \subset M$, the p -form ω satisfies the following inequality

$$\langle d\omega, d\omega \rangle + \langle d^*\omega, d^*\omega \rangle \leq C(K)\langle \omega, \omega \rangle \quad (4.7)$$

uniformly on K , then if ω is zero to infinite order at a point $x_0 \in M$, we have that $\omega \equiv 0$. The regularity conditions on ω are to be $L^2(M)$ with strong L^2 derivatives. The p -form ω must vanish at x_0 with all derivatives in "L¹ mean", which is also much weaker than the usual notion.

We shall apply it to the compact manifold D and with a C^∞ harmonic p -form h , hence which satisfies inequality (4.7), and which is zero on an open non void set, which also implies a zero of infinite order.

The main lemma of this section is:

Lemma 4.1. *Let $\omega \in L_p^r(\Omega)$, then we can extend it to $\omega' \in L_p^r(D)$ such that: $\forall h \in \mathcal{H}_p(D)$, $\langle \omega', h \rangle_D = 0$.*

Proof. Recall that $\mathcal{H}_p(D)$ is the vector space of harmonic p -form in D , it is of finite dimension K_p and $\mathcal{H}_p(D) \subset C^\infty(D)$.

Make an orthonormal basis $\{e_1, \dots, e_{K_p}\}$ of $\mathcal{H}_p(D)$ with respect to $L_p^2(D)$, by the Gram-Schmidt procedure. We get $\langle e_j, e_k \rangle_D := \int_D e_j e_k dV = \delta_{jk}$.

Set $\lambda_j := \langle \omega \mathbb{1}_\Omega, e_j \rangle = \langle \omega, e_j \mathbb{1}_\Omega \rangle$, $j = 1, \dots, K_p$, which makes sense since $e_j \in C^\infty(D) \Rightarrow e_j \in L^\infty(D)$, because D is compact.

We shall see that the system $\{e_k \mathbb{1}_{D \setminus \Omega}\}_{k=1, \dots, K_p}$ is a free one. Suppose this is not the case, then it will exist $\gamma_1, \dots, \gamma_{K_p}$, not all zero, such that $\sum_{k=1}^{K_p} \gamma_k e_k \mathbb{1}_{D \setminus \Omega} = 0$ in $D \setminus \Omega$. But the function $h := \sum_{k=1}^{K_p} \gamma_k e_k$ is in $\mathcal{H}_p(D)$ so if h is zero in $D \setminus \Omega$ which is non void, then $h \equiv 0$ in D by the N. Aronszajn, A. Krzywicki and J. Szarski [3] result. This is not possible because the e_k make a basis for $\mathcal{H}_p(D)$. So the system $\{e_k \mathbb{1}_{D \setminus \Omega}\}_{k=1, \dots, K_p}$ is a free one.

We set $\gamma_{jk} := \langle e_k \mathbb{1}_{D \setminus \Omega}, e_j \mathbb{1}_{D \setminus \Omega} \rangle$ hence we have that $\det\{\gamma_{jk}\} \neq 0$. So we can solve the linear system to get $\{\mu_k\}$ such that

$$\forall j = 1, \dots, K_p, \sum_{k=1}^{K_p} \mu_k \langle e_k \mathbb{1}_{D \setminus \Omega}, e_j \rangle = \lambda_j. \quad (4.8)$$

We put $\omega'' := \sum_{j=1}^{K_p} \mu_j e_j \mathbb{1}_{D \setminus \Omega}$ and $\omega' := \omega \mathbb{1}_\Omega - \omega'' \mathbb{1}_{D \setminus \Omega} = \omega - \omega''$. From (4.8) we get

$$\forall j = 1, \dots, K_p, \langle \omega', e_j \rangle_D = \langle \omega, e_j \rangle - \langle \omega'', e_j \rangle = \lambda_j - \sum_{k=1}^{K_p} \mu_k \langle e_k \mathbb{1}_{D \setminus \Omega}, e_j \rangle = 0.$$

So the p -form ω' is orthogonal to \mathcal{H}_p . Moreover $\omega'|_\Omega = \omega$ and clearly $\omega'' \in L_p^r(D)$ being a finite combination of $e_j \mathbb{1}_{D \setminus \Omega}$, so $\omega' \in L_p^r(D)$ because ω itself is in $L_p^r(D)$. The proof is complete. \square

Now let $\omega \in L^r(\Omega)$ and see Ω as a subset of D ; then extend ω as ω' to D by Lemma 4.1.

By the results on the compact manifold D , because $\omega' \perp \mathcal{H}_p(D)$, we get that there exists $u' \in W_p^{2,r}(D)$, $u' \perp \mathcal{H}_p(D)$, such that $\Delta u' = \omega'$; hence if u is the restriction of u' to Ω we get $u \in W_p^{2,r}(\Omega)$, $\Delta u = \omega$ in Ω .

Hence we proved

Theorem 4.2. *Let Ω be a domain in the smooth complete riemannian manifold M and $\omega \in L_p^r(\Omega)$, then there is a p -form $u \in W_p^{2,r}(\Omega)$, such that $\Delta u = \omega$ and $\|u\|_{W_p^{2,r}(\Omega)} \leq c(\Omega)\|\omega\|_{L_p^r(\Omega)}$.*

As for domains in \mathbb{R}^n , there is no constrain for solving the Poisson equation in this case.

Corollary 4.3. *(Of the proof) Let M be a smooth compact riemannian manifold with smooth boundary ∂M . Let $\omega \in L_p^r(M)$. There is a form $u \in W_p^{2,r}(M)$, such that $\Delta u = \omega$ and $\|u\|_{W_p^{2,r}(M)} \leq c\|\omega\|_{L_p^r(M)}$.*

Proof. We can build the "double manifold" $D := D(M)$ which is compact without boundary. Copying the proof of Theorem 4.2 we extend the form ω defined on M , viewed as a subset in D , to a form ω' in $L_p^r(D)$ orthogonal to $\mathcal{H}_p(D)$ so there is a $u' :: \Delta u' = \omega'$ in D . We just take $u := u'|_M$ to finish the proof. \square

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