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The raising steps method. Applications to the L^r Hodge theory in a compact riemannian manifold.

Eric Amar

Abstract

Let X be a smooth manifold and Ω a domain in X . The Raising Steps Method allows to get from local results on solutions u of equation $Du = \omega$ global ones in Ω .

It was introduced in [1] to get good estimates on solutions of $\bar{\partial}$ equation in domains in a Stein manifold.

It is extended here to linear partial differential operator of any finite order.

As a simple application we shall get a L^r Hodge decomposition theorem for p forms in a compact riemannian manifold without boundary, and then we retrieve known results of C. Scott [7] by an entirely different method.

1 Introduction.

In all the sequel a *domain* of a smooth manifold X will be a connected open set Ω of X , relatively compact and with C^∞ smooth boundary.

The raising steps method allows to get from local results on solutions u of an equation $Du = \omega$, global ones in a domain Ω of a smooth manifold X . Precisely fix a threshold $s > 1$ such that you have a global solution u on Ω of $Du = \omega$ with estimates $L^s(\Omega) \rightarrow L^s(\Omega)$; very often this threshold will be $s = 2$, since Hilbert spaces are usually more tractable. Suppose that we have, for $1 \leq r < s$, local solutions u on $U \cap \Omega$ of $Du = \omega$ with estimates $L^r(\Omega) \rightarrow L^t(U \cap \Omega)$ with a strict increase of the regularity, i.e. $\frac{1}{t} = \frac{1}{r} - \delta$, $\delta > 0$ for any $r \leq s$, then you get a global solution v of $Dv = \omega$ which is essentially in $L^t(\Omega)$. I introduced this method in [1] to get solutions for the $\bar{\partial}$ equation with good estimates in domains in Stein manifold. I extend it here to linear partial differential operator D of any finite order m .

We shall apply this method for the Poisson equation associated to the Hodge Laplacian in a compact riemannian manifold (M, g) of dimension n . Then we derive from this result a L_p^r Hodge decomposition for p differential forms on M . This way we recover already known results of C. Scott [7] but by an entirely different method. The case $r = 2$ of this decomposition, which gives us $s = 2$ as a threshold, goes to Morrey and essentially all results in the L^2 case we use here are coming from the basic work of Morrey [6].

Precisely let $L_p^r(M)$ be the set of p forms in the Lebesgue space $L^r(M)$, $W_p^{k,r}(M)$ the Sobolev space of p forms which have all their (covariant) derivatives up to order k in $L^r(\Omega)$, and \mathcal{H}_p the set of harmonic p forms on M . We get the following Hodge decomposition :

Theorem 1.1 *Let (M, g) be a compact riemannian manifold without boundary ; we have the strong L^r Hodge decomposition :*

$$\forall r, 1 \leq r < \infty, L_p^r(M) = \mathcal{H}_p^r \oplus \text{Im}\Delta(W_p^{2,r}(M)) = \mathcal{H}_p^r \oplus \text{Im}d(W_p^{1,r}(M)) \oplus \text{Im}d^*(W_p^{1,r}(M)).$$

We shall denote $\Lambda^p(X)$ the set of \mathcal{C}^∞ p forms on X . Let d_p be the exterior differential on $\Lambda^p \xrightarrow{d_p} \Lambda^{p+1}$, then applications of this decomposition are well known and important. Fix a p in $0, 1, \dots, n$ and let $\Lambda^{p-1} \xrightarrow{d_{p-1}} \Lambda^p \xrightarrow{d_p} \Lambda^{p+1}$; now consider the cohomology class $H^p(M) := \ker d_p / \text{Im}d_{p-1}$ then

- any class in $H^p(M)$ contains exactly one harmonic p form ;
- the dimension $b_p(M)$ of $H^p(M)$ is finite and equals the dimension of the space of harmonic p forms ;

and many other applications in cohomology theory and in other areas, see for instance C. Scott [7].

Finally we use a nice construction due to Guneyasu and Pigola [4] to get good solutions for the Poisson equation associated to the Hodge Laplacian in a relatively compact domain in a complete, compact or non compact, riemannian manifold :

Theorem 1.2 *Let Ω be a domain in the smooth complete riemannian manifold M and $\omega \in L_p^r(\Omega)$, then there is a form $u \in W_p^{2,r}(\Omega)$, such that $\Delta u = \omega$ and $\|u\|_{W_p^{2,r}(\Omega)} \leq c(\Omega)\|\omega\|_{L_p^r(\Omega)}$.*

This work will be presented the following way :

- In the next section we state precisely the hypotheses we need for the method to work.
- In the third section we state and prove the raising steps theorem.
- In the fourth section we apply it in the case of a compact riemannian manifold.
- In section 5 we prove the L_p^r Hodge decomposition theorem.
- In the last section we get good solutions for the Poisson equation associated to the Hodge Laplacian in a relatively compact domain in a complete, compact or not, riemannian manifold.

2 Hypotheses.

We shall deal with the following situation : we have a \mathcal{C}^∞ smooth manifold X admitting partitions of unity and a measure μ equivalent to the Lebesgue measure in each coordinates patch, so we have the associated Lebesgue spaces $L^r(X)$.

We suppose we have a punctual norm on $\Lambda^p(X)$, $|\omega(x)|$ and if Ω is an open set in X , the closure of $\Lambda^p(X)$ for this norm in the Lebesgue space $L^r(\Omega)$ will be denoted $L_p^r(\Omega)$:

$$\omega \in L_p^r(\Omega) \iff \|\omega\|_{L_p^r(\Omega)}^r := \int_{\Omega} |\omega(x)|^r d\mu(x) < \infty.$$

Let $\mathcal{D}^p(\Omega)$ be the set of \mathcal{C}^∞ p forms of compact support in Ω . We are interested in solution of a linear equation $Du = \omega$, where $D = D_p$ is a linear operator acting on p forms, with eventually the constraint $C\omega = 0$, where C is also a linear operator such that $CD = 0$. One aim is to apply this to the Hodge Laplace equation with $D = \Delta$.

Now on the degree p of the forms will be fixed and all the hypotheses will be made for this precise degree ; so the explicit mention of the degree will be often omitted.

2.1 Local regularity.

Let Ω be a domain in X . We shall make the following assumptions on Ω and on D . First if $u \in \mathcal{D}^p(X)$, $Du \in \mathcal{D}^p(X)$. There is a $\tau \geq \delta$ with $\frac{1}{t} = \frac{1}{r} - \tau$, such that :

(i) there is a covering $\{U_j\}_{j=1,\dots,N}$ of $\bar{\Omega}$ such that if $\omega \in L^r(\Omega)$, $C\omega = 0$, we can solve $Du_j = \omega$ in $\Omega_j := U_j \cap \Omega$ with $L^r(\Omega) - L^t(\Omega_j)$ estimates, i.e. $\exists c_l > 0$ such that

$$\exists u_j \in L^t(\Omega_j), Du_j = \omega \text{ in } \Omega_j \text{ and } \|u_j\|_{L^t(\Omega_j)} \leq c_l \|\omega\|_{L^r(\Omega_j)}.$$

If we have Sobolev spaces $W_p^{k,r}(U)$ defined for p forms on $U \subset X$, then we may also suppose that :

(i') there is a $\beta \geq 1$ such that we can solve $Du_j = \omega$ in $\Omega_j := U_j \cap \Omega$ and

$$\exists c_{dl} > 0, \forall j = 1, \dots, N, \exists u_j \in W^{\beta,r}(\Omega_j), Du_j = \omega \text{ in } \Omega_j \text{ and } \|u_j\|_{W^{\beta,r}(\Omega_j)} \leq c_{dl} \|\omega\|_{L^r(\Omega)}.$$

Fix χ_j a C^∞ smooth partition of unity subordinate to $\{U_j\}_{j=1,\dots,N}$ and relative to $\bar{\Omega}$, i.e. :

$$\forall j = 1, \dots, N, \chi_j \in \mathcal{D}(U_j), 0 \leq \chi_j \leq 1, \forall x \in \bar{\Omega}, \sum_{j=1}^N \chi_j(x) = 1.$$

Now $\forall \chi \in \mathcal{D}^0(X)$, $\forall u \in L_p^t(\Omega)$, we set $B(\chi, u) := D(\chi u) - \chi D(u)$; this operator is linear in χ and u and we ask that :

(ii) we have $\forall j = 1, \dots, N$, $B(\chi_j, u_j) \in L_p^{t'}(\Omega_j)$, with $\frac{1}{t'} = \frac{1}{r} - \delta$; precisely, with the local solution u_j given by (i) :

$$\|B(\chi_j, u_j)\|_{L^{t'}(\Omega_j)} \leq c(\chi_j) \|\omega\|_{L^r(\Omega_j)}.$$

Again if we have Sobolev spaces $W_p^{k,r}(U)$ defined for p forms on $U \subset X$, then we may also suppose that there is a constant $c_{sl} > 0$:

(ii') $\forall j = 1, \dots, N$, $B(\chi_j, u_j) \in W_p^{1,r}(\Omega_j)$ and $\|B(\chi_j, u_j)\|_{W^{1,r}(\Omega_j)} \leq c_{sl} \|\omega\|_{L^r(\Omega_j)}$.

We notice here that the estimate $\|B(\chi_j, u_j)\|_{L^{t'}(\Omega_j)} \leq c(\chi_j) \|\omega\|_{L^r(\Omega_j)}$ cannot be true for any u : take $D = \Delta$ the laplacian and u a harmonic function, then $\omega = Du = 0$ but $B(\chi, u) \neq 0$ if $u \neq 0$. We ask this specifically for the given χ_j , u_j .

Remark 2.1 *Clearly the hypotheses done are valid for a strictly elliptic linear partial differential operator (or a system) of finite order m , and Ω relatively compact.*

2.2 Global solution.

(iii) We can solve $Dw = \omega$ globally in Ω with $L^s - L^s$ estimates, i.e.

$$\exists c_g > 0, \forall \omega \in L^s(\Omega) \exists w \in L^s(\Omega) \text{ s.t. } Dw = \omega \text{ in } \Omega \text{ and } \|w\|_{L^s(\Omega)} \leq c_g \|\omega\|_{L^s(\Omega)}$$

provided that $C\omega = 0$.

And again if we have Sobolev spaces defined on X ,

(iii') there is a $0 \leq \gamma$ such that we can solve $Dw = \omega$ globally in Ω with $L^s - W^{\gamma,s}$ estimates, i.e.

$$\exists c_{dg} > 0, \exists w \text{ s.t. } Dw = \omega \text{ in } \Omega \text{ and } \|w\|_{W^{\gamma,s}(\Omega)} \leq c_{dg} \|\omega\|_{L^s(\Omega)} \text{ provided that } C\omega = 0.$$

3 The "raising steps" method.

Theorem 3.1 *(Raising steps theorem) Under the assumptions above, there is a constant $c_f > 0$, such that for $r \geq 1$, $s \geq r$, if $\omega \in L^r(\Omega)$, $C\omega = 0$ there is a $u = u_s \in L^t(\Omega)$ with $\frac{1}{t} = \frac{1}{r} - \tau$, such that $Du = \omega + \tilde{\omega}$, with $\tilde{\omega} \in L^s(\Omega)$, $C\tilde{\omega} = 0$ and $\|u\|_{L^t(\Omega)} \leq c_f \|\omega\|_{L^r(\Omega)}$, $\|\tilde{\omega}\|_{L^s(\Omega)} \leq c_f \|\omega\|_{L^r(\Omega)}$.*

If (i'), (ii') are fulfilled we also have
 $u \in W^{\beta,r}(\Omega)$, $\|u\|_{W^{\beta,r}(\Omega)} \leq c_f \|\omega\|_{L^r(\Omega)}$.

Proof.

Let $r \leq s$ and $\omega \in L^r(\Omega)$, $C\omega = 0$; we start with the covering $\{U_j\}_{j=1,\dots,N}$ and the local solution $Du_j = \omega$ with $\frac{1}{t} = \frac{1}{r} - \tau$, $u_j \in L^t(\Omega_j)$ given by hypothesis (i).

Let χ_j be the C^∞ smooth partition of unity subordinate to $\{U_j\}_{j=1,\dots,N}$ and relative to $\bar{\Omega}$, fixed in the hypotheses; because $\Omega_j = \Omega \cap U_j$ and $\chi_j \in \mathcal{D}(U_j)$ we have $\chi_j u_j \in L^t(\Omega)$ with

$$\|\chi_j u_j\|_{L^t(\Omega)} \leq \|u_j\|_{L^s(\Omega_j)} \leq c_l \|\omega\|_{L^r(\Omega_j)} \leq c_l \|\omega\|_{L^r(\Omega)}. \quad (3.1)$$

Moreover if (i') is true, we have $u_j \in W^{\beta,r}(U_j) \Rightarrow \chi_j u_j \in W^{\beta,r}(\Omega)$ and :

$$\|\chi_j u_j\|_{W^{\beta,r}(\Omega)} \leq c(\chi_j) \|u_j\|_{W^{\beta,r}(\Omega_j)} \leq c(\chi_j) c_{dl} \|\omega\|_{L^r(\Omega_j)} \leq c(\chi_j) c_{dl} \|\omega\|_{L^r(\Omega)}.$$

Set $v_0 := \sum_{j=1}^N \chi_j u_j$. Then we have, setting now $\frac{1}{t_0} = \frac{1}{r} - \tau$,

• $v_0 \in L^{t_0}(\Omega)$ because $\chi_j u_j \in L^{t_0}(\Omega)$ for $j = 1, \dots, N$, and $\|v_0\|_{L^{t_0}(\Omega)} \leq c \|\omega\|_{L^r(\Omega)}$ with $c = N c_l$ by (3.1).

Moreover if (i') is true we have $u_j \in W^{\beta,r}(\Omega_j)$, $\|u_j\|_{W^{\beta,r}(\Omega_j)} \leq c_{dl} \|\omega\|_{L^r(\Omega)}$ hence $\chi_j u_j \in W_p^{\beta,r}(\Omega)$,

with $\|\chi_j u_j\|_{W^{\beta,r}(\Omega)} \leq c(\chi_j) c_{dl} \|\omega\|_{L^r(\Omega)}$. But $v_0 := \sum_{j=1}^N \chi_j u_j$ so we get

$$\|v_0\|_{W^{\beta,r}(\Omega)} \leq \sum_{j=1}^N \|\chi_j u_j\|_{W^{\beta,r}(\Omega)} \leq c_{pu} c_{dl} \|\omega\|_{L^r(\Omega)},$$

with $c_{pu} := \sum_{j=1}^N c(\chi_j)$. Hence $v_0 \in W_p^{\beta,r}(\Omega)$ with $\|v_0\|_{W^{\beta,r}(\Omega)} \leq c_{pu} c_{dl} \|\omega\|_{L^r(\Omega)}$.

• $Dv_0 = \sum_{j=1}^N \chi_j Du_j + \sum_{j=1}^N B(\chi_j, u_j)$ hence

$$\forall x \in \Omega, Dv_0(x) = \sum_{j=1}^N \chi_j(x) \omega(x) + \sum_{j=1}^N B(\chi_j, u_j)(x) = \omega(x) + \omega_1(x)$$

with $\omega_1 := \sum_{j=1}^N B(\chi_j, u_j)(x)$.

By hypothesis (ii), $B(\chi_j, u_j) \in L^{s_0}(U_j)$ with $\frac{1}{s_0} = \frac{1}{r} - \delta$, hence $\omega_1 \in L^{s_0}(\Omega)$, with

$$\|\omega_1\|_{L^{s_0}(\Omega)} \leq G \|\omega\|_{L^r(\Omega)} \quad (3.2)$$

and $G = c_{sl} N$.

So the regularity of ω_1 is higher by one step $\delta > 0$ than that of ω . Moreover $C\omega_1 = CDv_0 - C\omega = 0$ because $CD = 0$ and $C\omega = 0$.

If $s_0 \geq s$ we notice that $\omega_1 \in L^s(\Omega)$ because then $L^{s_0}(\Omega) \subset L^s(\Omega)$ for Ω is relatively compact. So we are done by setting $u_s = v_0$, $\tilde{\omega} = \omega_1$.

If $s_0 < s$ we proceed by induction : setting t_1 such that $\frac{1}{t_1} = \frac{1}{s_0} - \tau = \frac{1}{r} - \delta - \tau$, using the same covering $\{U_j\}_{j=1,\dots,N}$ and the same partition of unity $\{\chi_j\}_{j=1,\dots,N}$ and with ω_1 in place of ω , we get $\exists v_1 \in L^{t_1}(\Omega)$, $Dv_1 = \omega_1 + \omega_2$ and, setting s_1 such that $\frac{1}{s_1} = \frac{1}{s_0} - \delta = \frac{1}{r} - 2\delta$, we have that the regularity of ω_2 raises of 2 times δ from that of ω .

We still have $C\omega_2 = CDv_1 - C\omega_1 = 0$, so by induction, after a finite number k of steps, we get a $s_k \geq s$. The linear combination $u := \sum_{j=0}^{k-1} (-1)^j v_j$ now gives $Du = \omega + (-1)^k \omega_k$ with $C\omega_k = 0$. So again we are done by setting $\tilde{\omega} = (-1)^k \omega_k \in L^s_p(\Omega)$.

If (i') is true, then we have $v_0 \in W^{\beta,r}(\Omega)$ and

$$v_1 \in W^{\beta,s_0}(\Omega), \quad \|v_1\|_{W^{\beta,s_0}(\Omega)} \leq c\|\omega_1\|_{L^{s_0}(\Omega)},$$

hence by (3.2) we get

$$\|v_1\|_{W^{\beta,s_0}(\Omega)} \leq cG\|\omega\|_{L^r(\Omega)}.$$

So, a fortiori, $v_1 \in W^{\beta,r}(\Omega)$ because $r < s_0$ with the same control of the norm. For the same reason we have $\forall j = 1, \dots, k$, $\|v_j\|_{W^{\beta,r}(\Omega)} \leq c\|\omega\|_{L^r(\Omega)}$ hence the same estimates for u . ■

Corollary 3.2 *Under the assumptions of the raising steps theorem and with the global assumption (iii), there is a constant $c_f > 0$, such that for $r \leq s$, if $\omega \in L^r(\Omega)$, $C\omega = 0$ there is a $v \in L^t(\Omega)$ with $\lambda := \min(\tau, \frac{1}{r} - \frac{1}{s})$, and $\frac{1}{t} = \frac{1}{r} - \lambda$, such that*

$$Dv = \omega \text{ and } v \in L^t(\Omega), \quad \|v\|_{L^t(\Omega)} \leq c\|\omega\|_{L^r(\Omega)}.$$

Moreover, if (i'), (ii') and (iii') are fulfilled we have, with $\alpha = \min(\beta, \gamma)$,

$$Dv = \omega \text{ and } v \in W^{\alpha,r}(\Omega), \quad \|v\|_{W^{\alpha,r}(\Omega)} \leq c_f\|\omega\|_{L^r(\Omega)}.$$

Proof.

By the raising steps theorem 3.1 we have a $u \in L^t(\Omega)$ such that $Du = \omega + \tilde{\omega}$ with $\tilde{\omega} \in L^s(\Omega)$ and $C\tilde{\omega} = 0$; by hypothesis (iii) we can solve $D\tilde{v} = \tilde{\omega}$ with $\tilde{v} \in L^s(\Omega) \subset L^t(\Omega)$ so it remains to set

$$v := u - \tilde{v} \text{ to get } v \in L^t(\Omega) \text{ and } Dv = \omega.$$

Now if (iii') is true then $\tilde{v} \in W^{\gamma,s}(\Omega)$ and because $s \geq r$, we have $\tilde{v} \in W^{\gamma,r}(\Omega)$, so finally, with $\alpha = \min(\beta, \gamma)$, we get $v \in W^{\alpha,r}(\Omega)$ with the control of the norm, and we are done. ■

4 Application to compact riemannian manifold.

Let (M, g) be a \mathcal{C}^∞ smooth compact riemannian manifold with metric g . We can define punctually, for $\omega, \varphi \in \Lambda^p(M)$, a scalar product $(\omega, \varphi)(x)$, see [6], hence a modulus $\forall x \in M$, $|\omega|(x) := \sqrt{(\omega, \omega)(x)}$. By use of the canonical volume dv_g on M we get a scalar product :

$$\langle \omega, \varphi \rangle := \int_M (\omega, \varphi)(x) dv_g(x),$$

for p forms in $L^2_p(M)$ i.e. such that

$$\|\omega\|_{L^2_p(M)}^2 := \int_M |\omega|^2(x) dv_g(x) < \infty.$$

The same way we define the spaces $L^r_p(M)$, and also $L^r_p(\Omega)$, for Ω any open set in M .

The exterior differential $d : \Lambda^p \rightarrow \Lambda^{p+1}$ on p forms admits a formal adjoint d^* :

$$\forall \omega \in L_p^2(M), \forall \varphi \in \Lambda^p(M), \langle d^*\omega, \varphi \rangle = \langle \omega, d\varphi \rangle,$$

and we have

$$d^* : \Lambda^0 \rightarrow 0 ; 1 \leq p \leq n, \Lambda^p \rightarrow \Lambda^{p-1}.$$

Now we can define the Hodge laplacian :

$$\Delta = dd^* + d^*d.$$

This is a second order partial differential operator which is essentially self adjoint and positive and it sends p forms into p forms.

On (M, g) we can define Sobolev spaces (see [5]) and because M is compact these spaces are in fact independent of the metric. Moreover the Sobolev embeddings are true in this case and a chart diffeomorphism makes a correspondence between Sobolev spaces in \mathbb{R}^n and Sobolev spaces in M . Let Ω be any open set in M , we shall denote these Sobolev spaces $W_p^{s,r}(\Omega)$, meaning the set of p forms which have all their (covariant) derivatives up to order s in $L^r(\Omega)$.

4.1 Basic facts.

Let \mathcal{H}_p^2 be the set of harmonic p forms in $L^2(M)$, i.e. p form ω such that $\Delta\omega = 0$, which is equivalent here to $d\omega = d^*\omega = 0$.

The classical L^2 theory of Morrey [6] gives, on a compact manifold M without boundary :

$$\mathcal{H}_p := \mathcal{H}_p^2 \subset \mathcal{C}^\infty(M) \quad ([6], \text{(vi) p. 296})$$

$$\dim_{\mathbb{R}} \mathcal{H}_p < \infty \quad ([6], \text{theorem 7.3.1}).$$

This gives the existence of a linear projection from $L_p^r(M) \rightarrow \mathcal{H}_p$:

$$\forall v \in L_p^r(M), H(v) := \sum_{j=1}^{K_p} \langle v, e_j \rangle e_j$$

where $\{e_j\}_{j=1, \dots, K_p}$ is an orthonormal basis for \mathcal{H}_p . This is meaningful because $v \in L^r(M)$ can be integrated against $e_j \in \mathcal{H} \subset \mathcal{C}^\infty(M)$. Moreover we have $v - H(v) \in L_p^r(M) \cap \mathcal{H}^\perp$ in the sense that $\forall h \in \mathcal{H}_p, \langle v - H(v), h \rangle = 0$.

We shall prove, as a consequence of the Calderon Zygmund inequalities, that for $r < 2$, if $h \in L_p^r(M), \Delta h = 0$ then $h \in \mathcal{C}_p^\infty(M)$, hence in fact $h \in \mathcal{H}_p$.

4.2 Local estimates.

In order to have the local result, for $y \in M$, we choose a chart $(V \ni y, \varphi := (x_1, \dots, x_n))$ so that $g_{ij}(y) = \delta_{ij}$ and $\varphi(V) = B$ where B is a Euclidean ball centered at $\varphi(y) = 0$ and g_{ij} are the components of the metric tensor w.r.t. φ .

Lemma 4.1 *For any $y \in M$, there are open sets $V, W \Subset V, y \in W$, such that :*

$$\forall \omega \in L^r(V), \exists u \in W^{2,r}(W) :: \Delta u = \omega, \|u\|_{W^{2,r}(W)} \leq C \|\omega\|_{L^r(W)}.$$

And a Calderon Zygmund inequality : there are constants c_1, c_2 such that

$$\forall u \in W^{2,r}(V), \|u\|_{W^{2,r}(W)} \leq c_1 \|u\|_{L^r(V)} + c_2 \|\Delta u\|_{L^r(V)}$$

Proof.

Of course the operator d on p forms is local and so is d^* as a first order partial differential operator.

So the Hodge laplacian Δ_φ read by φ in B is not so different from that of \mathbb{R}^n in B when acting on forms in B ; hence we have

$$\Delta_\varphi \omega_\varphi = \Delta_{\mathbb{R}} \omega_\varphi + A \omega_\varphi,$$

where ω_φ is the p form ω read in the chart (V, φ) and A is a matrix valued second order operator with \mathcal{C}^∞ smooth coefficients such that $A : W^{2,r}(B) \rightarrow L^r(B)$ with, for a B with a small enough radius, $\|A\| \leq c \|\Delta_{\mathbb{R}}\|$ with $c < 1$.

This is true because at the point $y \in V$ we are in the flat case and if B is small enough, the difference $A := \Delta_\varphi - \Delta_{\mathbb{R}}$ in operator norm $W^{2,r}(B) \rightarrow L^r(B)$ goes to 0 when the radius of B goes to 0, because $\varphi \in \mathcal{C}^2$ and the metric tensor g is also \mathcal{C}^2 .

We know that $\Delta_{\mathbb{R}}$ operates component-wise on the p form γ , so we have

$$\forall \gamma \in L_p^r(B), \exists v_0 \in W_p^{2,r}(B) :: \Delta_{\mathbb{R}} v_0 = \gamma, \|v_0\|_{W^{2,r}(B)} \leq C \|\gamma\|_{L^r(B)},$$

simply setting the component of v_0 to be the Newtonian potential of the corresponding component of γ in U , these non trivial estimates coming from Gilbarg and Trudinger [3], Th 9.9, p. 230 and the constant $C = C(n, r)$ depends only on n and r .

So we get $\Delta_{\mathbb{R}} v_0 + A v_0 = \gamma + \gamma_1$, with

$$\gamma_1 = A v_0 \Rightarrow \|\gamma_1\|_{L^r(B)} \leq c \|\Delta_{\mathbb{R}} v_0\|_{L^r(B)} = c \|\gamma\|_{L^r(B)}.$$

We solve again

$$\exists v_1 \in W_p^{2,r}(B) :: \Delta_{\mathbb{R}} v_1 = \gamma_1, \|v_1\|_{W^{2,r}(B)} \leq C \|\gamma_1\|_{L^r(B)} = C c \|\gamma\|_{L^r(B)},$$

and we set

$$\gamma_2 := A v_1 \Rightarrow \|\gamma_2\|_{L^r(B)} \leq c \|\Delta_{\mathbb{R}} v_1\|_{L^r(B)} = c \|\gamma_1\|_{L^r(B)} \leq c^2 \|\gamma\|_{L^r(B)}.$$

And by induction :

$$\forall k \in \mathbb{N}, \gamma_k := A v_{k-1} \Rightarrow \|\gamma_k\|_{L^r(B)} \leq c \|\Delta_{\mathbb{R}} v_{k-1}\|_{L^r(B)} = c \|\gamma_{k-1}\|_{L^r(B)} \leq c^k \|\gamma\|_{L^r(B)}$$

and

$$\exists v_k \in W_p^{2,r}(B) :: \Delta_{\mathbb{R}} v_k = \gamma_k, \|v_k\|_{W^{2,r}(B)} \leq C \|\gamma_k\|_{L^r(B)} = C c^k \|\gamma\|_{L^r(B)}.$$

Now we set

$$v := \sum_{j \in \mathbb{N}} (-1)^j v_j,$$

this series converges in norm $W^{2,r}(B)$ and we have

$$\Delta_\varphi v = \Delta_{\mathbb{R}} v + A v = \sum_{j \in \mathbb{N}} (-1)^j (\Delta_{\mathbb{R}} v_j + A v_j) = \gamma,$$

the last series converging in $L^r(B)$.

As an alternative proof, we could have used the estimates for elliptic system done by Douglis and Nirenberg [2] to work directly with Δ_φ but this is too heavy for a case so near to the usual laplacian in \mathbb{R}^n .

Going back to the manifold M with $\gamma := \omega_\varphi$ and setting $u_\varphi := v$, $W := \varphi^{-1}(B)$, we get the right estimates :

$$\exists u \in W^{2,r}(W) :: \Delta u = \omega \text{ in } W, \|u\|_{W^{2,r}(W)} \leq C \|\omega\|_{L^r(W)},$$

because the Sobolev spaces for B go to the analogous Sobolev spaces for W in M .

For the Calderon Zygmund (C-Z) inequality we proceed the same way ; by the classical C-Z inequality for the usual laplacian $\Delta_{\mathbb{R}}$ in \mathbb{R}^n , [3], Th. 9.11, p. 235, we have, with $\rho = 1/2$,

$$\forall u \in W_p^{2,r}(B), \|u\|_{W^{2,r}(\rho B)} \leq c_1 \|u\|_{L^r(B)} + c'_2 \|\Delta_{\mathbb{R}} u\|_{L^r(B)}$$

because the laplacian on forms in \mathbb{R}^n is diagonal. Because $\Delta_\varphi = \Delta_{\mathbb{R}} + A$ with A a matrix valued second order operator with \mathcal{C}^∞ smooth coefficients such that $A : W^{2,r}(B) \rightarrow L^r(B)$ with $\|A\| \leq c \|\Delta_{\mathbb{R}}\|$ with $c < 1$, we get

$$\forall u \in W_p^{2,r}(B), \|\Delta_{\mathbb{R}} u\|_{L^r(B)} \leq \|\Delta_\varphi u - A u\|_{L^r(B)} \leq \|\Delta_\varphi u\|_{L^r(B)} + \|A u\|_{L^r(B)} \leq \|\Delta_\varphi u\|_{L^r(B)} + c \|\Delta_{\mathbb{R}} u\|_{L^r(B)}.$$

So

$$\forall u \in W_p^{2,r}(B), (1 - c) \|\Delta_{\mathbb{R}} u\|_{L^r(B)} \leq \|\Delta_\varphi u\|_{L^r(B)}$$

and finally

$$\forall u \in W_p^{2,r}(B), \|u\|_{W^{2,r}(\rho B)} \leq c_1 \|u\|_{L^r(B)} + \frac{c'_2}{1-c} \|\Delta_\varphi u\|_{L^r(B)}.$$

It remains to set $c_2 := \frac{c'_2}{1-c}$ to get the C-Z inequality for Δ_φ , so passing back to M , we get the C-Z local inequality on M . ■

Remark 4.2 (i) *We notice that there is no compatibility condition for solving the system $\Delta u = \omega$ locally.*

(ii) *Of course we can regularize u by convolution in ρB so in fact we get :*

$\forall u \in L^r(B), \Delta_\varphi u \in L^r(B) \Rightarrow u \in W^{2,r}(\rho B)$ and $\|u\|_{W^{2,r}(\rho B)} \leq c_1 \|u\|_{L^r(B)} + c_2 \|\Delta_\varphi u\|_{L^r(B)}$
and the analogous in M :

$$\forall u \in L^r(V), \Delta u \in L^r(V) \Rightarrow u \in W^{2,r}(W) \text{ and } \|u\|_{W^{2,r}(W)} \leq c_1 \|u\|_{L^r(V)} + c_2 \|\Delta u\|_{L^r(V)}.$$

Proposition 4.3 *We have the hypo-ellipticity of the Hodge laplacian :*

$$\forall r > 1, \forall u \in L^r(M), \Delta u = 0 \Rightarrow u \in C^\infty(M).$$

Moreover if there is a solution $u \in W^{t,r}(M)$ of the equation $\Delta u = f$ in M then any solution $v \in L^r(M)$ of $\Delta v = f$ has the same regularity.

Proof.

By C-Z we have $\Delta u = 0 \Rightarrow \forall y \in M, \exists W \ni y :: u \in W^{2,r}(W)$; by Sobolev embeddings which are true here, we get $u \in L^t(W), \frac{1}{t} = \frac{1}{r} - \frac{2}{n}$, hence again, because $\Delta u = 0, u \in W^{2,t}(W)$; so by induction we get $u \in W^{2,2}(W)$ and by compactness $u \in W^{2,2}(M)$. Now the L^2 theory [6] implies that $u \in C^\infty(M)$.

Suppose that $\Delta v = f$ then $\Delta(v - u) = 0$ hence $v - u \in C^\infty(M)$ so v has the same regularity as u . ■

Corollary 4.4 *For any $y \in M$, there are open sets $V, W \Subset V, y \in W$, such that we have :*

$$\forall \omega \in L_p^r(V), \exists u \in L_p^t(W) :: \Delta u = \omega, \|u\|_{L_p^t(W)} \leq C \|\omega\|_{L_p^r(V)}$$

with $\frac{1}{t} = \frac{1}{r} - \frac{2}{n}$, and $\nabla u \in W_p^{1,r}(W)$.

Proof.

The lemma 4.1 gives $u \in W^{2,r}(W)$ such that $\Delta u = \omega, \|u\|_{W^{2,r}(W)} \leq C \|\omega\|_{L^r(V)}$, hence we get that $\nabla u \in W_p^{1,r}(W)$ with the same control : $\|\nabla u\|_{W_p^{1,r}(W)} \leq C \|\omega\|_{L^r(V)}$.

For the first statement it remains to apply the Sobolev embedding theorems which are true here. ■

4.3 Verification of the hypotheses.

Our operator is $D := \Delta$ on a compact manifold M without boundary and we set $\Omega = M$. The compatibility condition is given by the linear operator H , i.e. to be globally solvable a p form ω must be in \mathcal{H}^\perp , as we shall see in a moment, which is equivalent to $H(\omega) = 0$. So here we set $C = H$.

4.3.1 Verification of the local hypotheses.

We have, in the sense of distributions, for any open set U :

$$\forall \chi \in \mathcal{D}(U), \forall u \in W_p^{2,r}(U), B(\chi, u) := \Delta(\chi u) - \chi \Delta u,$$

and because Δ is a second order differential operator, we have that $B(\chi, u)$ is a first order differential operator in u with \mathcal{C}^∞ coefficients.

By corollary 4.4 and the fact that M is compact, we can cover M by a finite set $\{W_j, V_j\}_{j=1,\dots,N}$ such that, with $\frac{1}{t} = \frac{1}{r} - \frac{2}{n}$,

$$\forall \omega \in L_p^r(M), \exists u_j \in L_p^t(W_j) :: \Delta u_j = \omega, \|u_j\|_{L_p^t(W_j)} \leq C \|\omega\|_{L^r(M)}.$$

So we set $\forall j = 1, \dots, N$, $U_j = W_j$ and we have here $\tau = \frac{2}{n}$ hence the (i).

The lemma 4.1 gives $u_j \in W^{2,r}(W_j)$ such that

$$\Delta u_j = \omega, \|u_j\|_{W^{2,r}(W_j)} \leq C \|\omega\|_{L^r(M)}, \text{ hence we have (i')} \text{ with } \beta = 2.$$

The corollary 4.4 gives $\|\nabla u_j\|_{W^{1,r}(W_j)} \leq c \|\omega\|_{L^r(M)}$ but $B(\chi, u)$ is a first order differential operator in u with \mathcal{C}^∞ coefficients so we have (ii') :

$$\|B(\chi_j, u_j)\|_{W^{1,r}(W_j)} \leq c \|\omega\|_{L^r(M)}.$$

Again we use the Sobolev embedding theorem to get

$$\|B(\chi_j, u_j)\|_{L^t(W_j)} \leq c \|B(\chi_j, u_j)\|_{W^{1,r}(W_j)} \leq c \|\omega\|_{L^r(M)},$$

with $\frac{1}{t} = \frac{1}{r} - \frac{1}{n}$. Hence we have here (ii) with $\delta = \frac{1}{n}$.

It remains to check that $H\Delta = 0$ which is easy :

$\forall u \in W^{2,r}(M), \forall h \in \mathcal{H}_p, \langle \Delta u, h \rangle = \langle u, \Delta h \rangle = 0$ because Δ is essentially self adjoint and $h \in \mathcal{C}_c^\infty(M)$, M being compact. So $H\Delta = 0$.

4.3.2 Verification of the global hypotheses.

The threshold here is $L_p^2(M)$ where the Hodge decomposition is well known [6], we have

Theorem 4.5 (*The L^2 Hodge Decomposition*). *Let M be a compact, \mathcal{C}^∞ , riemannian manifold without boundary ; we have :*

$$L^2(\wedge^l M) = \text{Im} \Delta \oplus \mathcal{H}.$$

Moreover Δ^{-1} is bounded from $L_p^2(M) \cap \mathcal{H}^\perp$ into $W_p^{2,2}(M) \cap \mathcal{H}^\perp$.

This implies (iii) and (iii') with $\gamma = 2$:

$$\forall \omega \in L_p^2(M), H(\omega) = 0, \exists u \in W^{2,2}(M) \cap \mathcal{H}^\perp :: \Delta u = \omega.$$

4.4 Application of the raising steps method.

Now we are in position to apply the raising steps method, namely corollary 3.2, and it gives :

Theorem 4.6 *Let M be a compact \mathcal{C}^∞ , Riemannian manifold without boundary ; we have :*

$$\forall \omega \in L_p^r(M), \omega \in \mathcal{H}_p(M)^\perp, r < 2; \exists u \in L_p^t(M) :: \Delta u = \omega,$$

with $\lambda := \min(\frac{2}{n}, \frac{1}{r} - \frac{1}{2})$, $\frac{1}{t} = \frac{1}{r} - \lambda$.

Moreover $u \in W^{2,r}(\Omega)$, $\|u\|_{W^{2,r}(\Omega)} \leq c \|\omega\|_{L^r(\Omega)}$.

In order to get a L^p Hodge decomposition theorem, we shall improve a little bit this result :

Corollary 4.7 *In the previous theorem we can choose $v \perp \mathcal{H}$, $\Delta v = \omega$, and keeping the same bounds.*

Proof.

Take the previous solution u and set $v := u - H(u)$. Then clearly v belongs to the same spaces as u because $H(u) \in \mathcal{H}_p \subset \mathcal{C}^\infty(M)$ and the norms are just slightly modified and we have

$$\Delta v = \Delta u - \Delta(H(u)) = \Delta u = \omega. \quad \blacksquare$$

4.4.1 Case $r > 2$.

For this case we shall proceed by duality as we did in [1].

Let $g \in L_p^{s'}(\Omega) \cap \mathcal{H}_p^\perp$, we want to solve $\Delta v = g$, with $s' \geq 2$ and s' conjugate to s .

We know by the previous part that :

$$\forall f \in L_p^r(\Omega) \cap \mathcal{H}_p^\perp, \exists u \in L_p^s(\Omega), \Delta u = f. \quad (4.3)$$

Consider the linear form

$$\forall f \in L_p^r(\Omega), \mathcal{L}(f) := \langle u, g \rangle,$$

where u is a solution of (4.3) ; in order for $\mathcal{L}(f)$ to be well defined, we need that if u' is another solution of $\Delta u' = f$, then $\langle u - u', g \rangle = 0$; hence we need that g must be "orthogonal" to harmonic p forms which is precisely our assumption.

Hence we have that $\mathcal{L}(f)$ is well defined and linear ; moreover

$$|\mathcal{L}(f)| \leq \|u\|_{L^s(\Omega)} \|g\|_{L^{s'}(\Omega)} \leq c \|f\|_{L^r(\Omega)} \|g\|_{L^{s'}(\Omega)}$$

so this linear form is continuous on $f \in L_p^r(\Omega) \cap \mathcal{H}_p^\perp$. By Hahn Banach it can be extended to the whole of $L_p^r(\Omega)$, hence there is a p form $v \in L_p^{r'}(\Omega)$ such that :

$$\forall f \in L_p^r(\Omega) \cap \mathcal{H}_p^\perp, \mathcal{L}(f) = \langle f, v \rangle = \langle u, g \rangle.$$

But $f = \Delta u$, and Δ is essentially self adjoint so we have

$$\langle f, v \rangle = \langle \Delta u, v \rangle = \langle u, \Delta v \rangle = \langle u, g \rangle,$$

for any $u \in \mathcal{C}_p^\infty(M)$, hence we solved $\Delta v = g$ in the sense of distributions with $v \in L_p^{r'}(\Omega)$. So the theorem

Theorem 4.8 *For any r , $2 \leq r \leq n/2$, if $g \in L_p^r(M) \cap \mathcal{H}_p^\perp$ there is a $v \in L^t(M)$ such that*

$$\Delta v = g, \|v\|_{L_p^t(M)} \leq c \|g\|_{L_p^r(M)} \text{ with } \frac{1}{t} = \frac{1}{r} - \frac{2}{n}.$$

If $r > n/2$, there is a $v \in W^{2,r}(M)$ such that $\Delta v = g$, $\|v\|_{W^{2,r}(M)} \leq c \|g\|_{L_p^r(M)}$.

For any $r \geq 2$ any solution v of $\Delta v = g$ is in $W^{2,r}(M)$.

It remains to prove the last assertion and for this if w is another solution $\Delta w = \omega$, then $\Delta(v-w) = 0$, hence

$$h := v - w \in \mathcal{H}_p \Rightarrow h \in \mathcal{C}_p^\infty(M) \Rightarrow w = v - h \in W_p^{2,r}(M). \quad \blacksquare$$

Now, suppose that $g \in L_p^t(M) \cap \mathcal{H}_p^\perp$ we shall apply the raising steps theorem with, as threshold global solution, the theorem 4.8 with $s > t$. So we get the theorem :

Theorem 4.9 For any r , $1 \leq r \leq n/2$, if $g \in L_p^r(M) \cap \mathcal{H}_p^\perp$ there is a $v \in L^t(M)$ such that $\Delta v = g$, $\|v\|_{L_p^t(M)} \leq c\|g\|_{L_p^r(M)}$ with $\frac{1}{t} = \frac{1}{r} - \frac{2}{n}$.
If $r > n/2$, there is a $v \in W^{2,r}(M)$ such that $\Delta v = g$, $\|v\|_{W^{2,r}(M)} \leq c\|g\|_{L_p^r(M)}$.
For any $r \geq 1$ any solution v of $\Delta v = g$ is in $W^{2,r}(M)$.

5 The Hodge decomposition.

We proved that

$$\omega \in L_p^r(M), \omega \in \mathcal{H}_p(M)^\perp, \exists u \in L_p^s(M) \cap \mathcal{H}_p(M)^\perp :: \Delta u = \omega ;$$

we want to see that Δ^{-1} is bounded from $L_p^r(M)$ to $W^{2,r}(M)$.

First Δ is injective from $\mathcal{H}_p(M)^\perp$ to $\mathcal{H}_p(M)^\perp$: let $u \in \mathcal{H}_p(M)^\perp$ be such that $\Delta u = 0$, the ellipticity of Δ implies that $u \in C^\infty(M)$ by proposition 4.3 and $u \in \mathcal{H}_p(M)$ but $u \in \mathcal{H}_p(M)^\perp$ hence $\langle u, u \rangle = 0 \Rightarrow u = 0$.

Now we have the operator Δ which is bounded from $W^{2,r}(M) \cap \mathcal{H}^\perp$ to $L_p^r(M) \cap \mathcal{H}^\perp$ and which is injective and onto by the corollary 4.7, so its inverse is also bounded.

This proves the L^r Hodge decomposition :

$$L_p^r(M) = L_p^r(M) \cap \mathcal{H}_p(M)^\perp \oplus L_p^r(M) \cap \mathcal{H}_p(M),$$

with Δ^{-1} continuous from $L_p^r(M) \cap \mathcal{H}_p(M)^\perp \rightarrow W^{2,r}(M) \cap \mathcal{H}_p(M)^\perp$.

Theorem 5.1 If (M, g) is a compact riemannian manifold without boundary ; we have the strong L^r Hodge decomposition :

$$\forall r, 1 \leq r < \infty, L_p^r(M) = \mathcal{H}_p^r \oplus \text{Im}\Delta(W_p^{2,r}(M)) = \mathcal{H}_p^r \oplus \text{Im}d(W_p^{1,r}(M)) \oplus \text{Im}d^*(W_p^{1,r}(M)).$$

Proof.

We already have

$$L_p^r(M) = \mathcal{H}_p^r \oplus \text{Im}\Delta(W_p^{2,r}(M))$$

where \oplus means uniqueness of the decomposition, and is given here by the fact that the operator H is bounded on $L_p^r(M)$.

So :

$$\forall \omega \in L_p^r(M) \cap \mathcal{H}_p^{r\perp}, \exists u \in W_p^{2,r}(M) :: \Delta u = \omega.$$

From $\Delta = dd^* + d^*d$ we get $\omega = d(d^*u) + d^*(du)$ with $du \in W_p^{1,r}(M)$ and $d^*u \in W_p^{1,r}(M)$, so

$$\forall \omega \in L_p^r(M) \cap \mathcal{H}_p^{r\perp}, \exists \alpha \in W_{p-1}^{1,r}(M), \exists \beta \in W_{p+1}^{1,r}(M) :: \omega = \alpha + \beta,$$

simply setting $\alpha = d(d^*u)$, $\beta = d^*(du)$, which gives $\alpha \in d(W_{p-1}^{1,r}(M))$ and $\beta \in d^*(W_{p+1}^{1,r}(M))$ and this proves the existence.

The uniqueness is given by lemma 6.3 in [7] and I copy this simple (but nice) proof here for the reader's convenience.

Suppose that $\alpha \in W_{p-1}^{1,r}(M)$, $\beta \in W_{p+1}^{1,r}(M)$, $h \in \mathcal{H}$ satisfy

$$d\alpha + d^*\beta + h = 0.$$

Let $\varphi \in C_p^\infty(M)$, because of the classical C^∞ -Hodge decomposition, there are $\eta \in C_{p-1}^\infty(M)$, $\omega \in C_{p+1}^\infty(M)$ and $\tau \in \mathcal{H}$ satisfying

$$\varphi = d\eta + d^*\omega + \tau.$$

Notice that $\langle d^*\beta, d\eta \rangle = \langle \beta, d^2\eta \rangle = \langle \beta, 0 \rangle = 0$ and $\langle h, d\eta \rangle = \langle d^*h, \eta \rangle = 0$, by the duality between d and d^* . Linearity then gives

$$\langle d\alpha, d\eta \rangle = \langle d\alpha + d^*\beta + h, d\eta \rangle = \langle 0, d\eta \rangle = 0. \quad (5.4)$$

Finally we have

$$\begin{aligned} \langle d\alpha, \varphi \rangle &= \langle d\alpha, d\eta \rangle + \langle d\alpha, d^*\omega \rangle + \langle d\alpha, \tau \rangle \\ &= 0 + \langle \alpha, d^{*2}\omega \rangle + \langle \alpha, d\tau \rangle \text{ by (5.4)} \\ &= \langle \alpha, 0 \rangle + \langle \alpha, 0 \rangle \text{ because } d^{*2} = 0 \text{ and } \tau \in \mathcal{H} \\ &= 0. \end{aligned}$$

Since $\mathcal{C}_p^\infty(M)$ is dense in $L_p^{r'}(M)$, r' being the conjugate exponent of r , and φ is arbitrary, we see that $d\alpha = 0$. Analogously, we see that $d^*\beta = 0$ and it follows that $h = 0$. \blacksquare

6 Case of Ω a domain in M .

Let Ω be a domain in a \mathcal{C}^∞ smooth complete riemannian manifold M , compact or non compact; we want to show how the results in case of a compact boundary-less manifold apply to this case. We copy the following construction from [4].

Let N be a relatively compact domain of M such that ∂N is a smooth hypersurface and $\bar{\Omega} \subset N$. The "Riemannian double" $D := D(N)$ of N , obtained by gluing two copies of N along ∂N , is a compact Riemannian manifold without boundary. Moreover, by its very construction, it is always possible to assume that D contains an isometric copy Ω_N of the original domain Ω . We shall write Ω for Ω_N to ease notations.

Lemma 6.1 *Let $\omega \in L_p^r(\Omega)$, then we can extend it to $\omega' \in L_p^r(D)$ such that*

$$\forall h \in \mathcal{H}_p(D), \langle \omega', h \rangle_D = 0.$$

Proof.

Recall that $\mathcal{H}_p(D)$ is the vector space of harmonic p forms in D , it is of finite dimension and $\mathcal{H}_p(D) \subset \mathcal{C}^\infty(D)$.

Make an orthonormal basis $\{e_1, \dots, e_{K_p}\}$ of $\mathcal{H}_p(D)$ with respect to $L_p^2(D)$, by the Gram-Schmidt procedure, so we get :

$$\langle e_j, e_k \rangle_D := \int_D e_j e_k dV = \delta_{jk}.$$

Set $\lambda_j := \langle \omega \mathbb{1}_\Omega, e_j \rangle = \langle \omega, e_j \mathbb{1}_\Omega \rangle$, $j = 1, \dots, K_p$, which makes sense since $e_j \in \mathcal{C}^\infty(D) \Rightarrow e_j \in L^\infty(D)$,

because D is compact and put $\omega'' := \sum_{j=1}^{K_p} \mu_j e_j \mathbb{1}_{D \setminus \Omega}$. Then we set

$$\omega' := \omega \mathbb{1}_\Omega - \omega'' \mathbb{1}_{D \setminus \Omega} = \omega - \omega''.$$

We have

$$\forall j = 1, \dots, K_p, \langle \omega', e_j \rangle_D = \langle \omega, e_j \rangle - \langle \omega'', e_j \rangle = \lambda_j \|e_j\|_{L^2(\Omega)}^2 - \mu_j \|e_j\|_{L^2(D \setminus \Omega)}^2 = 0,$$

by the choice of

$$\forall j = 1, \dots, K_p, \mu_j := \lambda_j \frac{\|e_j\|_{L^2(\Omega)}^2}{\|e_j\|_{L^2(D \setminus \Omega)}^2}.$$

This choice is relevant because if $\|e_j\|_{L^2(D\setminus\Omega)} = 0$, then $e_j \equiv 0$ for e_j is harmonic and $D\setminus\bar{\Omega}$ is open non void. In fact set $\delta := \inf_j \|e_j\|_{L^2(D\setminus\Omega)} \Rightarrow \delta > 0$ and

$$H\omega = \sum_{j=1}^{K_p} \lambda_j e_j \Rightarrow \|H\omega\|_2^2 \leq \|\omega\|_2^2 \Rightarrow \sum_{j=1}^{K_p} |\lambda_j|^2 \leq \|\omega\|_2^2 \Rightarrow \|\omega''\|_2^2 \leq \frac{1}{\delta^2} \|\omega\|_2^2.$$

Of course $\omega'' \in L_p^r(D)$ because

$$|\lambda_j| \leq \|\omega\|_{L^r(\Omega)} \|e_j\|_{L^{r'}(\Omega)} \Rightarrow |\mu_j| \leq \frac{1}{\delta^2} \|\omega\|_{L^r(\Omega)} \|e_j\|_{L^{r'}(\Omega)},$$

and $\|e_j\|_{L^{r'}(\Omega)} \leq C_p(\Omega)$. Because $\|e_j\|_{L^2(D)} = 1$, a priori $\|e_j\|_{L^{r'}(D)}$ is bigger than 1 for $r < 2$, hence $C_p(\Omega)$ depends actually on Ω unless $r \geq 2$.

So we have $\|\omega''\|_{L^r(D)} \leq K_p C_p(\Omega) \|\omega\|_{L_p^r(\Omega)}$. ■

Now let $\omega \in L^r(\Omega)$ and see it on $L^r(\Omega_N)$; then extend it as ω' to D by lemma 6.1.

By the results on the compact manifold D , because $\omega' \perp \mathcal{H}_p(D)$, we get that there exists $u' \in W_p^{2,r}(D)$, $u' \perp \mathcal{H}_p(D)$, such that $\Delta u' = \omega'$; hence if u is the restriction of u' to Ω we get

$$u \in W_p^{2,r}(\Omega), \Delta u = \omega \text{ in } \Omega.$$

Hence we proved

Theorem 6.2 *Let Ω be a domain in the smooth complete riemannian manifold M and $\omega \in L_p^r(\Omega)$, then there is a form $u \in W_p^{2,r}(\Omega)$, such that $\Delta u = \omega$ and $\|u\|_{W_p^{2,r}(\Omega)} \leq c(\Omega) \|\omega\|_{L_p^r(\Omega)}$.*

As in domains in \mathbb{R}^n , there is no constraint for solving the Poisson equation in this case.

References

- [1] E. Amar. The raising steps method. Application to the $\bar{\partial}$ equation in Stein manifolds. *J. Geometric Analysis*, 2015. To appear.
- [2] A. Douglis and L. Nirenberg. Interior estimates for elliptic systems of partial differential equations. *Comm. Pure Appl. Math.*, 8:503–538, 1955.
- [3] D. Gilbarg and N. Trudinger. *Elliptic Partial Differential equations*, volume 224 of *Grundlehren der mathematischen Wissenschaften*. Springer, 1998.
- [4] B. Guneyusu and S. Pigola. Calderon-Zygmund inequality on riemannian manifolds. *Preprint*, 2014.
- [5] E. Hebey. *Sobolev spaces on Riemannian manifolds.*, volume 1635 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1996.
- [6] C. B. Morrey. *Multiple Integrals in the Calculus of Variations*, volume 130 of *Die Grundlehren der mathematischen Wissenschaften*. Springer-Verlag Berlin Heidelberg New York, 1966.
- [7] C. Scott. L^p theory of differential forms on manifolds. *Transactions of the American Mathematical Society*, 347(6):2075–2096, 1995.