Modelling sovereign risks: from a hybrid model to the generalized density approach

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Abstract

Motivated by the European sovereign debt crisis, we study the sovereign risk by analyzing the solvency and the sovereign bond yield and propose a hybrid model which takes into account the movement of the sovereign solvency and the impact of critical political events. This model combines the structural and the reduced-form approaches in the credit risk modelling and the sovereign default time can be decomposed into an accessible part with predictable components and a totally inaccessible part. As a consequence, the probability of default at a critical political event date is nonzero and the probability law admits atoms. We study this model in a generalized density framework to deduce the compensator process of default and show that the intensity process does not necessarily exist. We also apply the model to the valuation of sovereign bond and explain the significant jumps in the long-term government bond yield during the sovereign crisis.

Keywords : sovereign risk, sovereign solvency, decomposition of stopping times, generalized density of default, long-term government bond.

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1 Introduction

The European sovereign debt crisis started at the end of 2009, when the long-term interest rates of euro area countries began to diverge significantly. This has made it difficult for several member states in the euro area (e.g., Greece, Ireland, Portugal, Cyprus) to refinance their public debts without aid of third parties. The crisis has also led to a growing amount of attention to sovereign risks from governments and financial markets.

Sovereign risk is the possibility that the government of a country may default on its debt or other obligations. It belongs to the family of credit risks, and is a fundamental component of risks in government bond yield curves. The modelling of sovereign risks is a challenging subject and can differ from the corporate credit risk. Firstly, sovereign default is usually influenced by macroeconomic factors such as GDP, public debt, government revenue and expenditure, etc. Secondly, political events and decisions have important impacts on the sovereign default, especially for a European Union country.

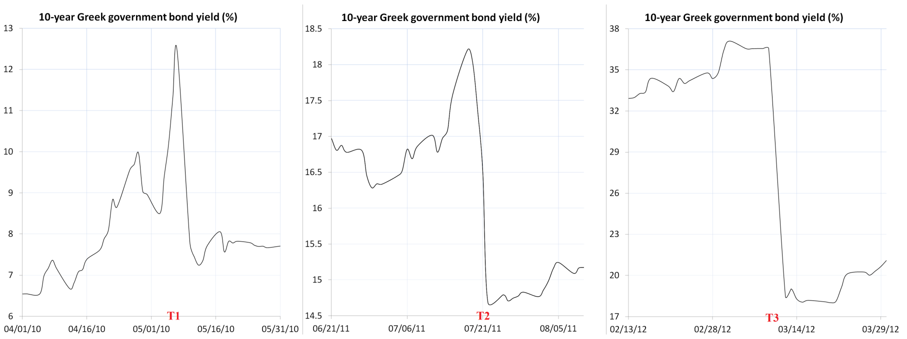
In literature (see e.g., Alogoskoufis [2]), the determinant macroeconomic variables can be summarized by a single one known as the sovereign solvency, which can be measured and monitored easily. Concerning the political decisions, we are notably interested in the impact when a critical political event happens. In practice, we observe that on a critical date when important political events hold, the sovereign default probability can become significant. This point may be justified by the following intuitive argument: when a sovereign is unable to repay its public debt, it solicits an international financial aid as a last resort; if the sovereign is not able to receive the financial support, it can end up in default. We have chosen as example three critical dates, noted as T_1, T_2 and T_3, all of which concern the financial aid packages for Greece:

- on 2 May 2010 (T_1), the euro area member states and International Monetary Fund (IMF) agree on a 110-billion-euro financial aid package for Greece;
- on 21 July 2011 (T_2), the government heads of the euro area agree to support a new financial aid program of 109-billion-euro for Greece;
- on 8 March 2012 (T_3), the European Central Bank (ECB) governing council acknowledges the activation of the buy-back scheme for Greece and decides that debt instruments issued or fully guaranteed by Greece will be again accepted as collateral in European credit operations.

We notice that T_1, T_2 and T_3 are predetermined dates publicly known to investors since political events are in general arranged in advance and these dates can be found on the official website of ECB. The impact of these political events can be observed in the long-term Greek government bond yield. As illustrated in Figure 1 (extracted from Figure 2), the bond yield has significant movements around T_1, T_2 and T_3 with very high levels of the yield before and negative jumps at or slightly after these dates.

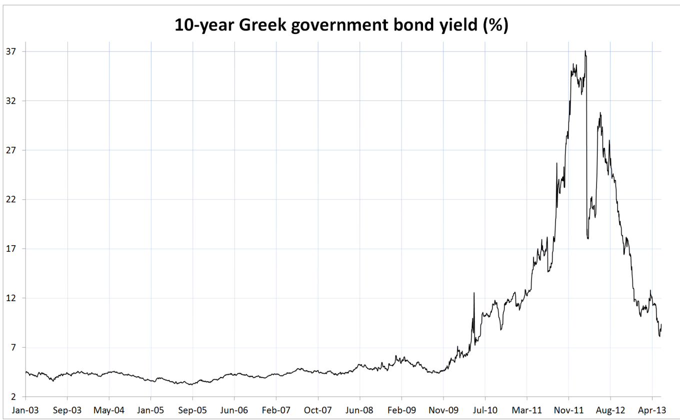
Macroeconomic models of debt crisis emphasize such phenomena by the multiple equilibria in debt markets in presence of credit risk. The prevailing equilibrium depends on the expectations of investors about the probability of default (e.g., Calvo [7]). Before a critical political event,

Figure 1: Greek bond yield around political critical dates (source: Bloomberg)



investors expect a sovereign default with a high probability, in which case the spread of the government bond is very large, and the debt market is in a large-spread equilibrium. Shortly after the critical political event, the expectation of investors about the sovereign default is suddenly discharged to keep a narrow-spread equilibrium. Therefore, one can say that the probability of default at the time of a critical political event is nonzero, characterized by a jump in long-term government bond yield.

Figure 2: Historical 10-year Greek bond yield (source: Bloomberg)



From mathematical point of view, the nonzero probability of default on a predetermined date means that the default time has a predictable component. In the literature on credit risk modelling, two approaches exist: structural approach and reduced-form approach. In a standard structural model, the default time is often a predictable stopping time defined as the first hitting time of a certain default barrier by the asset value process of a firm; while in a reduced-form model, it is usually a totally inaccessible stopping time modeled as the first jump time of a point process with stochastic intensity. Both approaches have been widely used to model corporate credit risks (see the books of Bielecki and Rutkowski [5] and Duffie and Singleton [16] for a detailed description).

In this paper, we propose a hybrid model which is based on both approaches of the classic credit risk models and takes into account the level of the sovereign solvency and the impact of critical political events. We intend to explain the significant movements of the sovereign bond yield during the sovereign debt crisis by the mixed characteristic of the hybrid model. We are inspired by the jump to default CEV (constant elasticity of variance) models in Carr and Linetsky [9] and Campi, Polbennikov and Sbuelz [8], which have been originally proposed for assessing corporate credit risks. In [9], the equity value is a CEV diffusion punctuated by a possible jump to zero which corresponds to default. The default time τ is decomposed into a predictable part, which is the first hitting time of zero by the equity value process, and a totally inaccessible part given by a Cox process model. In [8], the equity value is a CEV process, and the default time is the minimum of the first Poisson jump and the first absorption time of the equity value process by zero in absence of jumps. So the default time can be either predictable according to the CEV process, or totally inaccessible according to a Poisson jump. In the credit risk literature, there exist other hybrid models such as the generalized Cox process model in Bélanger, Shreve and Wong [4] and the credit migration hybrid model in Chen and Filipović [10]. Recently, Gehmlich and Schmidt [20] consider models where the Azéma supermartingale of τ contains jumps (so that the intensity does not exist) and develop the associated term structures. The decomposition of random times also appears in literature on the theory of enlargement of filtrations such as Aksamit, Choulli and Jeanblanc [1] and Coculescu [12].

The hybrid model of sovereign default that we propose in this paper also combines the structural and the reduced-form approaches. The sovereign default time contains an accessible part which describes the macroeconomic and political factors, and a totally inaccessible part as in the Cox process model which describes the idiosyncratic credit risk. we can also recover the jump to default CEV model. We also study this sovereign default model in a general setting which extends the default density approach introduced in El Karoui, Jeanblanc and Jiao [17] and we make the so-called generalized density hypothesis to describe random times whose conditional distribution can admit atoms.

The hybrid model of sovereign default can be applied to the valuation of sovereign defaultable claims. The pricing formula showes that the sovereign bond yield deduced in the model can have jumps at the political critical dates. Furthermore, numerical tests illustrate that the political critical dates and decisions have an important impact on the probability of sovereign default as shown in Figure 2.

The following of the paper is organized as below. In Section 2, we propose the sovereign default model by making precise the different components of risks including the sovereign solvency, the political decision impact and the idiosyncratic credit risk. Section 3 concentrates on the conditional default and survival probabilities, and in particular the probability of default on the political critical dates. In Section 4, we study the sovereign default model in a generalized density framework. We discuss theoretical properties of the random time such as the immersion property and the compensator process, and show that in this model, the intensity does not exist in general. In Section 5, we apply the sovereign default model to the pricing of long-term sovereign zero-coupon bonds and we explain the jumps in the bond yield with numerical results.

2 Hybrid model of sovereign default time

In this section, we present a hybrid model for the sovereign default which takes into account the macroeconomic situations of the country, the impact of political events and the idiosyncratic default intensity.

2.1 Sovereign solvency: a structural model

We start by introducing the notion of solvency. Sovereign default is tightly related to the macroeconomic factors of the country. Notably, the sovereign solvency is an important indicator since it includes several determinant macroeconomic variables related to the sovereign default. Here, we borrow the definition used in [2].

Definition 2.1 *The sovereign solvency S at time t is defined by*

$$\ln S_t = \pi_t - \frac{d_{t-}(r_t - g_t)}{1 + g_t}, \tag{2.1}$$

where d_{t-} denotes the public debt to GDP ratio of the previous observation date, π_t is the primary surplus to GDP ratio, r_t is the real interest rate on government bonds, and g_t is the GDP growth rate. In particular, we say that the government is fiscally sustainable if $S_t \geq 1$, and is insolvent if $S_t < 1$.

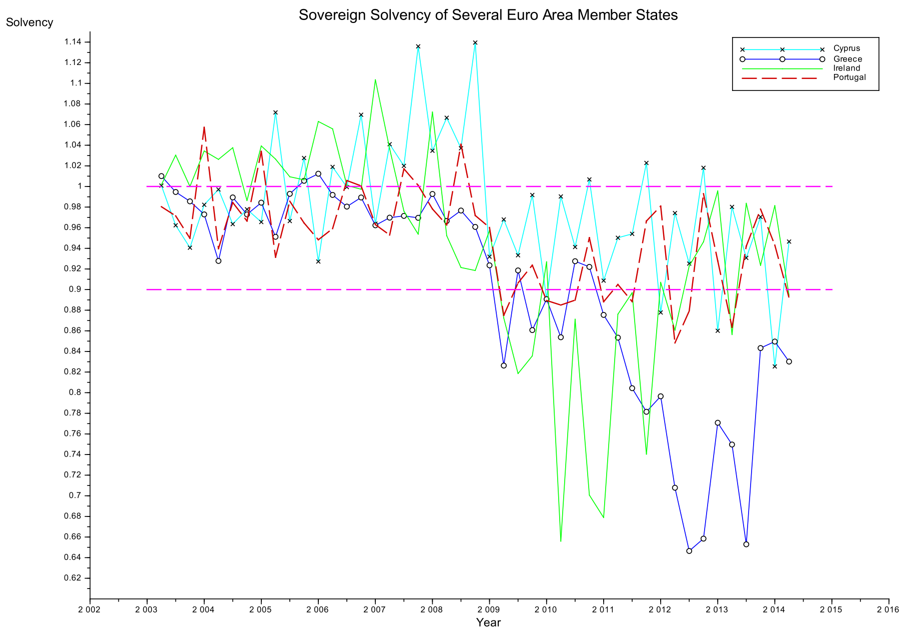
Slightly different from the initial definition, we use the exponential form such that the solvency takes only positive values. By definition, four factors determine whether a government is solvent. The predetermined historical debt is known from the government's balance sheet of the preceding year. The real interest rate on government bonds, which is the cost of debt refinancing, can be deduced from bond yield curves. The GDP growth rate is observable directly from the economic cycle. The primary surplus, which is the measurement of government deficit, can be computed from the government revenue and expenditure, as well as the fiscal dynamics. In practice, these data are available for discrete time observation.

We illustrate in Figure 3 the solvency values computed by using (2.1) for the following four member states of the euro area: Cyprus, Greece, Ireland, and Portugal[1]. We notice that all these countries are insolvent during the crisis (solvency lower than 1 from 2009), with Greece and Ireland in the worst situation. This observation is rather coherent with the reality since Greece and Ireland are the first countries that solicit aid from third parties. Furthermore, the crisis starts at the end of 2009 when the solvency of several countries falls below 0.9, so we can consider 0.9 as an approximate threshold of the debt crisis. Indeed, fears about a debt crisis begin to spread when the solvency of a country hits down a certain threshold.

In a long-term time scale, we model the sovereign solvency by a continuous-time process. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space equipped with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions. Let $W = (W_t, t \geq 0)$ be a Brownian motion which is $\mathbb{F}$-adapted. For a given country,

[1] The data for the interest rates comes from the official website of ECB and that for the other factors from the official website of European Commission.

Figure 3: Solvency of four countries of euro area.



we assume that the solvency is governed by a process $S = (S_t, t \geq 0)$ satisfying the following diffusion:

$$dS_t = S_t(\mu_t\, dt + \sigma_t\, dW_t), \quad S_0 = x, \tag{2.2}$$

where μ and σ are $\mathbb{F}$-predictable processes such that $\int_0^T |\mu_t| dt + \int_0^T |\sigma_t|^2 dt < \infty$ for any $T > 0$. Let L be a real positive constant with $L < S_0$ which represents a threshold of the debt crisis. More precisely, if S falls below L, we consider that the sovereign becomes insolvent. Then, we define a random time τ_0 as the first hitting time of the barrier L by the solvency process S, i.e.,

$$\tau_0 := \inf\{t \geq 0 : S_t \leq L\}, \tag{2.3}$$

with the convention $\inf \emptyset = \infty$. Note that τ_0 is a predictable $\mathbb{F}$-stopping time.

2.2 Critical political event

Generally speaking, when a sovereign becomes fiscally vulnerable, a political meeting will be organized at which political decisions need to be made concerning the sovereign default. The meeting date is a critical date for the concerned sovereign and often comes shortly after the solvency barrier hitting time τ_0. In this paper, we assume that the critical date coincides with the time τ_0. We also assume that the result of political decisions depends on some exogenous factor, such as an economic or financial shock: if the shock has occurred before the critical date, then the sovereign can possibly end up in default at τ_0; otherwise, it receives a financial aid package without immediate default at τ_0. Indeed, the term financial aid package is perceived as any assistance from third parties with the aim of improving the solvency of the country in debt crisis, including a bailout loan, a quantitative easing policy, etc. From an economic point of view, when the solvency is below the threshold, an exogenous shock can make things worse so that the aid from third parties will be too costly (for example, austerity policies can do harms to the economy) and the political decisions are in favor of a sovereign default.

We model the exogenous shock by the jump of a Poisson process $N = (N_t, t \geq 0)$ with intensity $\lambda^N > 0$ which is independent of the filtration $\mathbb{F}$. Suppose that the result of political decisions depends on the value of N at the critical date τ_0. More precisely, we define

$$\zeta = \begin{cases} \tau_0, & \text{on } \{N_{\tau_0} \geq 1\}, \\ \infty, & \text{on } \{N_{\tau_0} = 0\}. \end{cases} \tag{2.4}$$

The random time ζ takes into account both the sovereign solvency and the political decisions. Obviously, ζ is not an $\mathbb{F}$-stopping time. However, ζ is an honest time (e.g. Barlow [3]). In fact, for any $t \geq 0$, $\mathbb{1}_{\{\zeta \leq t\}}\zeta = \mathbb{1}_{\{\zeta \leq t\}}\tau_0 = \mathbb{1}_{\{\zeta \leq t\}}(t \wedge \tau_0)$, where $t \wedge \tau_0$ is $\mathcal{F}_t$-measurable.

2.3 Idiosyncratic credit risk: a Cox process model

Besides the macroeconomic and political impact, we also consider the credit risk related to the idiosyncratic financial circumstance of the sovereign, and we adopt the widely-used Cox process model.

Let the idiosyncratic default intensity $\lambda = (\lambda_t, t \geq 0)$ be a positive $\mathbb{F}$-adapted process. In the literature on the corporate credit risks as in [9], the default intensity can depend on the pre-default equity price process. In our case, λ can depend on the solvency S. For example, let $\lambda_t = \lambda(t, S_t)$, where $\lambda : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is assumed to be bounded as $S \to \infty$, which implies that in a healthy situation of solvency, the idiosyncratic default risk is limited.

We introduce the default hazard process $\Lambda = (\Lambda_t, t \geq 0)$ as $\Lambda_t = \int_0^t \lambda_s ds$. Let η be an $\mathcal{A}$-measurable exponentially distributed random variable of unit parameter independent of both $\mathbb{F}$ and the Poisson process N, and ξ be the time of default due to the idiosyncratic financial situations, given by a Cox process model, i.e.,

$$\xi := \inf\{t \geq 0 : \Lambda_t \geq \eta\}. \tag{2.5}$$

As usual, the random time ξ is a totally inaccessible stopping time with respect to the progressive enlargement of the filtration $\mathbb{F}$ by ξ, namely the filtration $\mathbb{F}^\xi = (\mathcal{F}_t^\xi)_{t\geq 0}$, where $\mathcal{F}_t^\xi = \cap_{s>t}(\sigma(\{\xi \leq u\} : u \leq s) \vee \mathcal{F}_s)$.

2.4 Sovereign default time: a hybrid model

We now model the sovereign default by combining the economic and political influences described by ζ and the idiosyncratic credit risk described by ξ. Let the sovereign default time be

$$\tau := \zeta \wedge \xi. \tag{2.6}$$

This sovereign default model has a hybrid nature of both structural and reduced-form approaches, which means that the sovereign default can result either from macroeconomic and political events, or from its own idiosyncratic financial situations..

We make some comparisons with the jump to default extended CEV credit risk model.

1. If $(S_t, t \geq 0)$ follows a CEV model, then the default time τ defined in (2.6) is similar to the jump to default extended CEV model. We refer the readers to [14, 15] for background about the CEV process and we shall discuss the CEV case in more detail in Section 3.2.2. Note that the default time τ in our model is not bounded by its predictable component τ_0. In fact, on the set $\{\tau_0 < \xi\} \cap \{N_{\tau_0} = 0\}$, $\tau = \xi > \tau_0$.

2. We assume some technical hypotheses as in [9] with suitable financial interpretation. Let the idiosyncratic intensity be given in the form $\lambda_t = \lambda(t, S_t)$. When the solvency $S \to \infty$, the sovereign has almost no chance to default, then λ should remain bounded. When $S \to 0$, the solvency is in an unfavorable situation and may trigger a default, so that the idiosyncratic default intensity can explode in this case. In general, the intensity λ is decreasing with respect to the solvency S.

3. If the Poisson intensity of the exogenous shock $\lambda^N \to 0$, then the default never occurs at τ_0, and our model converges to a Cox process model. On the contrary, when $\lambda^N \to \infty$, we have $\tau = \tau_0 \wedge \xi$. In this case, we recover the jump to default extended CEV model.

2.5 Extension to re-adjusted solvency thresholds

In practice, if the debt and deficit situation of the sovereign is excessive and has no improvement, the authorities may be less confident and consequently relax the requirements on the solvency barrier. In this case, other critical political events may be gradually anticipated. This observation motivates us to extend the hybrid model to the case of multiple critical dates where solvency thresholds can be re-adjusted.

Let $n \in \mathbb{N}$, and $L_1, L_2, \ldots, L_n \in \mathbb{R}_+$ such that $S_0 > L_1 > L_2 > \ldots > L_n$, representing different levels of solvency requirements. We define a sequence of solvency barrier hitting times $\{\tau_i\}_{i=1}^n$ as

$$\tau_i = \inf\{t \geq 0 : S_t \leq L_i\}, \quad i \in \{1, \ldots, n\}. \tag{2.7}$$

The sequence $\{\tau_i\}_{i=1}^n$ is increasing since the solvency requirements are decreasing. When the solvency falls below a certain requirement L_i, we assume that a critical political event is organized immediately at τ_i. If an exogenous shock has already arrived before, then the sovereign can possibly default at τ_i and no more critical political events will be planned; if no exogenous shock arrives before τ_i, the sovereign may obtain a financial aid to avoid an immediate default, which makes it possible to predetermine another critical political event when the solvency falls below L_{i+1}, and so on and so forth, until the requirements on the solvency are exhausted.

The exogenous factor is modeled by an inhomogeneous Poisson process N with intensity function $\lambda^N(t)$, and we define a random time ζ^* as

$$\zeta^* = \tau_i, \quad \text{on } \{N_{\tau_{i-1}} = 0\} \cap \{N_{\tau_i} \geq 1\}, \quad i \in \{1, \ldots, n+1\} \tag{2.8}$$

with convention $\tau_0 = 0$ and $\tau_{n+1} = \infty$. Note that for $t \geq 0$, one has

$$\mathbb{1}_{\{\zeta^* > t\}} = \sum_{i=1}^{n+1} \mathbb{1}_{\{\tau_i > t\}} (\mathbb{1}_{\{N_{\tau_{i-1}}=0\}} - \mathbb{1}_{\{N_{\tau_i}=0\}}) = \sum_{i=1}^{n+1} \mathbb{1}_{\{\tau_{i-1} \leq t < \tau_i\}} \mathbb{1}_{\{N_{\tau_{i-1}}=0\}}. \tag{2.9}$$

We have that $\mathbb{P}(\cup_{i=1}^{n} \{\omega : \tau_i(\omega) = \zeta^*(\omega) < \infty\}) = \mathbb{P}(\zeta^* < \infty)$, then ζ^* is an accessible stopping time (c.f. Protter [25, Chapter III.2]).

Similar to the case of single critical date, the sovereign can be caused either by the successive downgrade of solvency or by the idiosyncratic credit risk. Then, the sovereign default time is defined as

$$\tau = \zeta^* \wedge \xi, \tag{2.10}$$

where ξ is still given by (2.5). In this case, the default time τ is decomposed into an accessible part, which has n predictable components, and a totally inaccessible part.

3 Probability of default on multiple critical dates

In this section, we are interested in the probability that the sovereign default occurs immediately at political critical dates. As we show, such default probabilities are nonzero in the hybrid model, which implies atoms in the probability law of default.

3.1 Conditional default and survival probability

We consider the sovereign default given by the hybrid model (2.10). For any $i \in \{1, \ldots, n\}$, let the $\mathcal{F}_t$-conditional probability that the sovereign default τ coincides with τ_i be denoted by $p_t^i := \mathbb{P}(\tau = \tau_i | \mathcal{F}_t)$, $t \geq 0$.

Proposition 3.1 *The process $(p_t^i, t \geq 0)$ is a stopped $\mathbb{F}$-martingale at τ_i and is given by*

$$p_t^i = \mathbb{E}\Big[\big(e^{-\int_0^{\tau_{i-1}} \lambda^N(s)ds} - e^{-\int_0^{\tau_i} \lambda^N(s)ds}\big) e^{-\Lambda_{\tau_i}} | \mathcal{F}_t\Big] \tag{3.1}$$

PROOF: The event $\{\tau = \tau_i\}$ equals $\{\tau_i \leq \xi, N_{\tau_{i-1}} = 0, N_{\tau_i} \geq 1\}$. Since τ_i is $\mathbb{F}$-stopping time, the Poisson process N and the random variable η are mutually independent and in addition independent of $\mathbb{F}$, one has

$$\begin{aligned} \mathbb{P}(\tau = \tau_i | \mathcal{F}_\infty) &= \mathbb{P}(\tau_i \leq \xi | \mathcal{F}_\infty) \mathbb{P}(N_{\tau_{i-1}} = 0, N_{\tau_i} \geq 1 | \mathcal{F}_\infty) \\ &= \mathbb{P}(\Lambda_{\tau_i} \leq \eta | \mathcal{F}_\infty) \left(e^{-\int_0^{\tau_{i-1}} \lambda^N(s)ds} - e^{-\int_0^{\tau_i} \lambda^N(s)ds}\right) \end{aligned} \tag{3.2}$$

which implies (3.1) and the fact that p_t^i is stopped at τ_i. □

We notice from the above proposition that on the set $\{\tau_i \leq t\}$, p_t^i does not depend on t, which means that the information concerning the impact of a political decision neutralizes after the event. In particular, we have

$$\mathbb{P}(\tau = \tau_i) = p_0^i = \mathbb{E}\left[\big(e^{-\int_0^{\tau_{i-1}} \lambda^N(s)ds} - e^{-\int_0^{\tau_i} \lambda^N(s)ds}\big) e^{-\Lambda_{\tau_i}}\right]. \tag{3.3}$$

We now compute the conditional survival probability of the sovereign and study the immersion property.

Proposition 3.2 *For all $u, t \in \mathbb{R}_+$, the $\mathbb{F}$-conditional survival probability is given by*

$$\mathbb{P}(\tau > u|\mathcal{F}_t) = \mathbb{E}\Big[\exp\Big(-\sum_{i=1}^{n} \mathbb{1}_{\{\tau_i \le u\}} \int_{\tau_{i-1}}^{\tau_i} \lambda^N(s)ds - \Lambda_u\Big)\big|\mathcal{F}_t\Big] \tag{3.4}$$

Proof: For all $u, t \in \mathbb{R}_+$, by (2.9), one has

$$\begin{aligned}\mathbb{P}(\tau > u|\mathcal{F}_t) &= \mathbb{P}(\zeta^* > u, \xi > u|\mathcal{F}_t) \\ &= \mathbb{E}\Big[\big(\sum_{i=1}^{n+1} \mathbb{1}_{\{\tau_{i-1}\le u<\tau_i\}}\mathbb{1}_{\{N_{\tau_{i-1}}=0\}}\big)\mathbb{1}_{\{\xi>u\}}\Big|\mathcal{F}_t\Big].\end{aligned}$$

If $u \le t$, then

$$\begin{aligned}\mathbb{P}(\tau > u|\mathcal{F}_t) &= \sum_{i=1}^{n+1} \mathbb{1}_{\{\tau_{i-1}\le u<\tau_i\}}\mathbb{E}\big[\mathbb{1}_{\{N_{\tau_{i-1}}=0\}}\mathbb{1}_{\{\xi>u\}}\big|\mathcal{F}_t\big] \\ &= \sum_{i=1}^{n+1} \mathbb{1}_{\{\tau_{i-1}\le u<\tau_i\}}\mathbb{E}\big[\mathbb{1}_{\{N_{\tau_{i-1}}=0\}}\mathbb{1}_{\{\eta>\Lambda_u\}}\big|\mathcal{F}_t\big] \\ &= \Big(\sum_{i=1}^{n+1} \mathbb{1}_{\{\tau_{i-1}\le u<\tau_i\}} e^{-\int_0^{\tau_{i-1}} \lambda^N(s)ds}\Big)e^{-\Lambda_u}. \\ &= \exp\Big(-\sum_{i=1}^{n+1} \mathbb{1}_{\{\tau_{i-1}\le u<\tau_i\}} \int_0^{\tau_{i-1}} \lambda^N(s)ds\Big)e^{-\Lambda_u} \\ &= \exp\Big(-\sum_{i=1}^{n} \mathbb{1}_{\{\tau_i\le u\}} \int_{\tau_{i-1}}^{\tau_i} \lambda^N(s)ds - \Lambda_u\Big).\end{aligned}$$

If $u > t$, we calculate $\mathbb{P}(\tau > u|\mathcal{F}_t)$ as the $\mathcal{F}_t$-conditional expectation of $\mathbb{P}(\tau > u|\mathcal{F}_u)$, which implies (3.4). □

Let the global information structure be given as usual by the progressive enlargement of the filtration $\mathbb{F}$ by the sovereign default time τ, that is,

$$\mathcal{G}_t = \bigcap_{s>t} \big(\sigma(\{\tau \le u\} : u \le s) \vee \mathcal{F}_s\big), \quad t \ge 0.$$

Then the couple $(\mathbb{F}, \mathbb{G})$ satisfies the immersion property or the so-called (H)-hypothesis, that is, any $\mathbb{F}$-martingale remains a $\mathbb{G}$-martingale. Indeed, by Proposition 3.2, when $u \le t$, the $\mathbb{F}$-conditional probability does not depend on t, i.e.,

$$\mathbb{P}(\tau > u|\mathcal{F}_t) = \mathbb{P}(\tau > u|\mathcal{F}_u), \quad u \le t.$$

This last equality is equivalent to the (H)-hypothesis (see Elliott, Jeanblanc and Yor [18]). The following result is a direct consequence of [18, Lemma 3.1] and Proposition 3.2.

Corollary 3.3 *For all $t, T \in \mathbb{R}_+$ such that $t \le T$, the $\mathbb{G}$-conditional survival probability is given by*

$$\mathbb{P}(\tau > T|\mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}}\mathbb{E}\Big[\exp\Big(-\sum_{i=1}^{n}\mathbb{1}_{\{t<\tau_i\le T\}}\int_{\tau_{i-1}}^{\tau_i}\lambda^N(s)ds - \int_t^T \lambda_s\, ds\Big)\big|\mathcal{F}_t\Big]. \tag{3.5}$$

3.2 Default probability in a Markovian setting

The general form of the sovereign default probability at critical dates p_t^i is given by Proposition 3.1. We now consider several specific settings when the solvency process is a geometric Brownian motion or a CEV process.

We first make some simplified assumptions. We suppose that the equation (2.2) is homogeneous and that the solvency process is given by

$$dS_t = S_t\big(\mu(S_t)dt + \sigma(S_t)\big)dW_t, \quad S_0 = x$$

where $\mu(\cdot) : \mathbb{R}_+ \to \mathbb{R}$ and $\sigma(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ satisfy regular enough conditions for existence and uniqueness of a strong solution $\{S_t^x, t \ge 0\}$ (see e.g. Revuz and Yor [26, Theorem 3.5] for details). Let $\mathcal{L}$ denote the generator of S, i.e., for any function $f \in C^2 : \mathbb{R}_+ \to \mathbb{R}$,

$$\mathcal{L}f(z) = z\mu(z)f'(z) + \frac{z^2}{2}\sigma^2(z)f''(z).$$

Suppose in addition that the intensity of the exogenous shock is constant, so in the inhomogeneous Poisson process, the intensity function is $\lambda^N(t) = \lambda^N > 0$ for any $t \ge 0$. Furthermore, we specify the idiosyncratic default intensity process $\lambda = (\lambda_t, t \ge 0)$ as a decreasing function of the solvency, i.e., $\lambda_t = \lambda(S_t)$ with $\lambda(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ being decreasing.

We consider the Laplace transform for the $\mathbb{F}$-stopping time

$$\rho_x = \inf\{t \ge 0 : S_t^x \le L\} \quad \text{with} \quad S_0^x = x.$$

For any $k \ge 0$, let

$$Q(x; k, L) := \mathbb{E}\left[\exp\Big(-k\rho_x - \int_0^{\rho_x}\lambda(S_s^x)\, ds\Big)\right] \tag{3.6}$$

One can prove that (3.6) is the representation of the solution to the following Dirichlet problem

$$\begin{aligned} \mathcal{L}u(z) - \big(\lambda(z) + k\big)u(z) = 0 \quad \text{on } \{z > L\} \\ u(L) = 1. \end{aligned} \tag{3.7}$$

Indeed, since ρ_x is a predictable stopping time, there exists an increasing sequence of stopping times $(\rho_m)_{m\ge1}$ such that $\rho_m < \rho_x$ and $\lim_{m\to\infty}\rho_m = \rho_x$, $\mathbb{P}$-a.s.. Let $\beta_t^x = \exp(-\int_0^t(k+\lambda(S_s^x))ds)$ for any $t \ge 0$. Then on the set $\{t < \rho_x\}$, one has

$$d(\beta_t^x u(S_t^x)) = \beta_t^x u'(S_t^x)\sigma(S_t^x)S_t^x dW_t$$

where u is a solution to the Dirichlet problem (3.7). We then have

$$\mathbb{E}[\beta_{\rho_m}^x u(S_{\rho_m}^x)] - u(x) = \mathbb{E}\left[\int_0^{\rho_m}\beta_s^x u'(S_s^x)\sigma(S_s^x)S_s^x dW_s\right],$$

where the right-hand side vanishes thanks to the boundedness of β and the smoothness of u. Thus, when m tends to infinity, $u(x) = \mathbb{E}[\beta^x_{\rho_x} u(L)] = \mathbb{E}[\exp(-\int_0^{\rho_x}(\lambda(S^x_s)+k)ds)]$. We refer the reader to Karatzas and Shreve [23, Chapter 5, Proposition 7.2] and Touzi [27, Theorem 2.8] for a general representation of this kind of Dirichlet problem.

Proposition 3.4 *The $\mathbb{F}$-martingale $(p^i_t, t \geq 0)$, $i \in \{1,\dots,n\}$, is computed as*

$$p^i_t = e^{-\int_0^{t\wedge\tau_i}\lambda(S_u)du} Q(S_{t\wedge\tau_{i-1}}; \lambda^N, L_{i-1}) \\ \cdot \left[e^{-\lambda^N (t\wedge\tau_{i-1})} Q(S_{(t\wedge\tau_i)\vee\tau_{i-1}}; 0, L_i) - e^{-\lambda^N (t\wedge\tau_i)} Q(S_{(t\wedge\tau_i)\vee\tau_{i-1}}; \lambda^N, L_i)\right], \quad t \geq 0,$$

with $L_0 = S_0 = x$.

Proof: By Proposition 3.1 and the section assumptions, we have for any $i \in \{1,\dots,n\}$ and $t \geq 0$ that

$$p^i_t = \mathbb{E}\left[(e^{-\lambda^N\tau_{i-1}} - e^{-\lambda^N\tau_i})e^{-\int_0^{\tau_i}\lambda(S_u)du}\big|\mathcal{F}_t\right].$$

Since p^i is a stopped martingale on τ_i, it suffices to compute p^i_t for $t \leq \tau_i$. On the set $\{\tau_{i-1} \leq t < \tau_i\}$, by the Markovian property of the process S, we obtain

$$p^i_t = e^{-\lambda^N\tau_{i-1}}\mathbb{E}\left[e^{-\int_0^{\tau_i}\lambda(S_u)du}\big|\mathcal{F}_t\right] - \mathbb{E}\left[e^{-\lambda^N\tau_i - \int_0^{\tau_i}\lambda(S_u)du}\big|\mathcal{F}_t\right] \\ = e^{-\int_0^t\lambda(S_u)du}\left[e^{-\lambda^N\tau_{i-1}}Q(S_t; 0, L_i) - e^{-\lambda^N t}Q(S_t; \lambda^N, L_i)\right].$$

In particular, one has

$$p^i_{\tau_{i-1}} = e^{-\lambda^N\tau_{i-1} - \int_0^{\tau_{i-1}}\lambda(S_u)du}\left[Q(L_{i-1}; 0, L_i) - Q(L_{i-1}; \lambda^N, L_i)\right],$$

which yields that on the set $\{t < \tau_{i-1}\}$,

$$p^i_t = e^{-\lambda^N t - \int_0^t\lambda(S_u)du}Q(S_t; \lambda^N, L_{i-1})\left[Q(L_{i-1}; 0, L_i) - Q(L_{i-1}; \lambda^N, L_i)\right].$$

Finally, we note that $Q(S_{\tau_{i-1}}; k, L_{i-1}) = Q(L_{i-1}; k, L_{i-1}) = 1$ for any k, which implies (3.8). $\Box$

The above proposition shows that it is essential to calculate the quantity $Q(x; k, L)$ to obtain explicit form of the atom probability. We present below two cases.

3.2.1 Case of geometric Brownian motion

Let the solvency process S be a geometric Brownian motion which is the solution to $dS_t = S_t(\mu dt + \sigma dW_t)$, $t \geq 0$, where W is a standard Brownian motion, $\mu, \sigma \in \mathbb{R}$ with $\sigma > 0$ and $S_0 = x$. Similar as in [9], we suppose that the idiosyncratic default intensity λ is a decreasing function of the solvency S as

$$\lambda_t = \lambda(S_t) = \frac{a}{S_t^{2\beta}} + b \tag{3.8}$$

where $a > 0$, $b, \beta \geq 0$ represent respectively the scale parameter governing the sensitivity of λ to S, the constant lower bound and the elasticity parameter. Then, by (3.7), $u(x) = Q(x; k, L)$

is the solution to the following Sturm-Liouville equation (see Everitt [19]):

$$\begin{aligned} \frac{1}{2}\sigma^2 x^2 u''(x) + \mu x u'(x) - (ax^{-2\beta} + b + k)u(x) = 0 \quad \text{on } (L, +\infty); \\ u(L) = 1. \end{aligned} \tag{3.9}$$

If $\beta = 0$, let $\hat{k} = a + b + k$. Apart from solving the equation (3.9), one can compute directly $\mathbb{E}[e^{-\hat{k}\rho_x}]$ by noticing that $(\exp(-\sqrt{2\hat{k}}W_t - \hat{k}t), t \geq 0)$ is a martingale. Then the optional sampling theorem yields (see Borodin and Salminen [6, Part II, Chapter 9, 2.0.1])

$$Q(x; k, L) = \left(\frac{L}{x}\right)^{\sqrt{\nu^2 + 2\hat{k}/\sigma^2} + \nu}$$

where $\nu = \mu/\sigma^2 - 1/2$.

If $\beta > 0$, we let $w(z) = Cu(z^{-\frac{1}{\beta}})z^{-\frac{\nu}{\beta}}$, where $C = w(L^{-\beta})L^{-\nu}$. Then, w satisfies the following Bessel equation in a modified form ([19, Chapter 17]):

$$(zw'(z))' - \frac{1}{\beta^2}\left(\nu^2 + 2(k+b)/\sigma^2\right) z^{-1} w'(z) = \frac{2az}{\beta^2\sigma^2} w(z). \tag{3.10}$$

Let $\psi = \frac{1}{\beta}\sqrt{\nu^2 + 2(k+b)/\sigma^2}$, then the above equation admits two basic solutions $I_\psi(z\sqrt{2a}/\sigma\beta)$ and $K_\psi(z\sqrt{2a}/\sigma\beta)$, where I and K are modified Bessel functions with the following properties ([6, Appendix 2.4]):

$$(z^{-\psi} I_\psi(z))' = z^{-\psi} I_{\psi+1}(z), \quad (z^{-\psi} K_\psi(z))' = -z^{-\psi} K_{\psi+1}(z),$$

which implies that $z^{-\psi} I_\psi(z\sqrt{2a}/\sigma\beta)$ is increasing and $z^{-\psi} K_\psi(z\sqrt{2a}/\sigma\beta)$ is decreasing. Moreover, one has $u(z^{-\frac{1}{\beta}}) = (z^{-\psi} w(z)) z^{\psi - \frac{\nu}{\beta}}$. We have, by [24, Theorem 3.1] that (see also [6, Part II, Chapter 9, 2.8.3])

$$Q(x; k, L) = \frac{x^{-\nu} w(x^{-\beta})}{L^{-\nu} w(L^{-\beta})} = \left(\frac{L}{x}\right)^{\nu} \frac{I_\psi(\sqrt{2a}/\sigma\beta x^\beta)}{I_\psi(\sqrt{2a}/\sigma\beta L^\beta)}$$

where I is modified Bessel function of the first kind, defined as

$$I_\psi(x) := \sum_{i=0}^{\infty} \frac{(x/2)^{\psi+2i}}{i!\Gamma(\psi + i + 1)}$$

with Γ being the gamma function.

3.2.2 Case of the CEV process

We now consider the case where the volatility is a monotonic function of the solvency. On the one hand, when the solvency decreases, lower solvency (higher deficit) indicates higher level of government borrowing, leading to lower growth rate, as well as smaller future expenditure to improve the budgetary situation. All these add more uncertainty to the solvency. On the other hand, when the solvency increases, besides higher growth rate, higher solvency (surplus) may imply higher fiscal revenue, which the government is under pressure to disburse for social welfare, also making the solvency become more uncertain. In other words, the volatility has the possibility to be either an increasing or a decreasing function of the solvency. Let the solvency process be a CEV process driven by the following diffusion:

$$dS_t = \mu S_t\,dt + \delta S_t^{\beta+1} dW_t, \qquad S_0 = x, \tag{3.11}$$

where $\beta \in \mathbb{R}$ and $\delta > 0$ are respectively the elasticity parameter and the scale parameter of the volatility. For $\beta > 0$ (respectively $\beta < 0$), the volatility $\sigma(S) = \delta S^{\beta}$ is an increasing (respectively a decreasing) function of S. In particular, the process S is a geometric Brownian motion in the case $\beta = 0$.

The specification of the idiosyncratic default intensity $\lambda(S)$ depends on the sign of the parameter β. More precisely, when $\beta > 0$ (respectively $\beta < 0$), $\lambda(S)$ is an affine function of $\frac{1}{\sigma^2(S)}$ (respectively $\sigma^2(S)$), i.e.,

$$\lambda(S) = \frac{a}{S^{2|\beta|}} + b, \quad a > 0, \quad b \geq 0, \quad \beta \in \mathbb{R}. \tag{3.12}$$

Then, $u(x) = Q(x;k,L)$ is the decreasing solution of the following equation:

$$\begin{aligned} \frac{1}{2}\delta^2 x^{2+2\beta} u'' + \mu x u' - (a x^{-2|\beta|} + b + k)u = 0, \quad &\text{on } (L, +\infty); \\ u(L) = 1.& \end{aligned} \tag{3.13}$$

The fundamental solutions to this last equation are different according to the sign of β.

In literature, another similar equation, called CEV ordinary differential equation (ODE), has been studied by Davydov and Linetsky in [14] for the valuation of path-dependent options, where the coefficient of u is a negative constant. We make use of the knowledge of the solutions to the CEV ODE to solve the equation (3.13). In the following, we consider separately the cases $\beta > 0$ and $\beta < 0$.

Case $\beta > 0$: We let $v(x) = C_1 u(x) e^{\kappa x^{-2\beta}}$, where $\kappa = \frac{1}{2\beta\delta^2}(\sqrt{\mu^2 + 2a\delta^2} - \mu) > 0$ and $C_1 = v(L)e^{-\kappa/L^{2\beta}}$. Then, v satisfies the following CEV ODE:

$$\frac{1}{2}\delta^2 x^{2+2\beta} v'' + \sqrt{\mu^2 + 2a\delta^2}\, x v' - \left(\kappa\beta(2\beta+1)\delta^2 + b + k\right) v = 0. \tag{3.14}$$

The above equation admits an increasing fundamental solution (which we reject since u is decreasing) and a decreasing fundamental solution ([14, Proposition 5]) given by:

$$v(x) = x^{\beta + \frac{1}{2}} e^{\frac{\sqrt{\mu^2+2a\delta^2}}{2\beta\delta^2} x^{-2\beta}} M_{n,m}\left(\frac{\sqrt{\mu^2 + 2a\delta^2}}{\beta\delta^2} x^{-2\beta}\right),$$

where $n = \frac{1}{2} + \frac{1}{4\beta} - \frac{\kappa\beta(2\beta+1)\delta^2+b+k}{2\beta\sqrt{\mu^2+2a\sigma^2}} = \frac{\mu(2\beta+1)+2b+2k}{4\beta\sqrt{\mu^2+2a\delta^2}}$, $m = \frac{1}{4\beta}$ and $M_{n,m}(z) := z^{m+1/2}e^{-z/2}F_1(m - n + 1/2, 2m+1, z)$ is Whittaker function of the first kind with

$$F_1(a,b,z) := 1 + \sum_{j=1}^{\infty} \frac{a(a+1)\dots(a+j-1)z^j}{b(b+1)\dots(b+j-1)j!}$$

being Kummer confluent hypergeometric function of the first kind. This fundamental solution implies that

$$Q(x;k,L) = \frac{v(x)e^{-\kappa/x^{2\beta}}}{C_1} = \frac{x^{\beta+\frac{1}{2}}e^{\frac{\mu}{2\beta\delta^2}x^{-2\beta}}M_{n,m}\left(\frac{\sqrt{\mu^2+2a\delta^2}}{\beta\delta^2}x^{-2\beta}\right)}{L^{\beta+\frac{1}{2}}e^{\frac{\mu}{2\beta\delta^2}L^{-2\beta}}M_{n,m}\left(\frac{\sqrt{\mu^2+2a\delta^2}}{\beta\delta^2}L^{-2\beta}\right)},$$

which is valid for any $\mu \in \mathbb{R}$.

Case $\beta < \mathbf{0}$: We let $y(z) = C_2 u(z^{\frac{1}{\gamma}})z^{\frac{1}{2}-\frac{1}{2\gamma}}$, where

$$\gamma = \begin{cases} \sqrt{1+8a/\delta^2}, & \mu > 0, \\ -\sqrt{1+8a/\delta^2}, & \mu \le 0, \end{cases}$$

and $C_2 = y(L^\gamma)L^{\frac{1}{2}-\frac{\gamma}{2}}$. If $\gamma < -1$, y is an increasing function; if $\gamma > 1$, y should be decreasing such that u is decreasing for all $\mu \in \mathbb{R}, \delta, a > 0, b, k \ge 0, \beta < 0$. Then, y satisfies another CEV ODE as follows:

$$\frac{1}{2}\delta^2\gamma^2 z^{2+2\hat{\beta}}y'' + \mu\gamma z y' - \left(b + k + \frac{\mu\gamma - \mu}{2}\right) y = 0, \tag{3.15}$$

where $\hat{\beta} = \frac{\beta}{\gamma}$, the sign of which depends on the sign of μ, and we note that $b + k + \frac{\mu\gamma-\mu}{2} > 0$ for any $\mu \in \mathbb{R}$.

If $\mu > 0$ (respectively $\mu < 0$), one has $\gamma > 1$ and $\hat{\beta} < 0$ (respectively $\gamma < -1$ and $\hat{\beta} > 0$), and we keep the decreasing (respectively increasing) fundamental solution of the equation (3.15). Then, by [14, Proposition 5],

$$y(z) = z^{\frac{\beta}{\gamma}+\frac{1}{2}}e^{\frac{\mu}{2\beta\delta^2}z^{-2\beta/\gamma}}W_{n',m'}\left(-\frac{|\mu|}{\beta\delta^2}z^{-2\beta/\gamma}\right),$$

where $n' = \mathrm{sgn}(\mu\beta)(\frac{1}{2} + \frac{\gamma}{4\beta}) - \frac{2b+2k+\mu\gamma-\mu}{4|\mu\beta|} = \frac{2b+2k-\mu(2\beta+1)}{4|\mu|\beta}$, $m' = -\frac{|\gamma|}{4\beta} = -\frac{\sqrt{1+8a/\delta^2}}{4\beta}$ and the function $W_{n,m}(x) := x^{m+1/2}e^{-x/2}F_2(m-n+1/2, 2m+1, x)$ is Whittaker function of the second kind with

$$F_2(a,b,x) := \frac{\Gamma(1-b)}{\Gamma(1+a-b)}F_1(a,b,x) + \frac{\Gamma(b-1)}{\Gamma(a)}x^{1-b}F_1(1+a-b, 2-b, x)$$

being Kummer confluent hypergeometric function of the second kind.

If $\mu = 0$, then $\gamma < -1$ and $\hat{\beta} > 0$. The increasing fundamental solution to the equation (3.15) is

$$y(z) = \sqrt{z}K_{2m'}\left(-\frac{z^{-\beta/\gamma}}{\delta\beta}\sqrt{2b+2k+\mu\gamma-\mu}\right),$$

where $K_\psi(x)$ is modified Bessel function of the second kind, defined as

$$K_\psi(x) := \frac{\pi}{2\sin(\psi\pi)}\left(I_{-\psi}(x) - I_\psi(x)\right).$$

Thus, the fundamental solution y implies that

$$Q(x;k,L) = \frac{y(x^\gamma)x^{\frac{1}{2}-\frac{\gamma}{2}}}{C_2} = \begin{cases} \frac{x^{\beta+\frac{1}{2}} e^{\frac{\mu}{2\beta\delta^2}x^{-2\beta}} W_{n',m'}\left(-\frac{|\mu|}{\beta\delta^2}x^{-2\beta}\right)}{L^{\beta+\frac{1}{2}} e^{\frac{\mu}{2\beta\delta^2}L^{-2\beta}} W_{n',m'}\left(-\frac{|\mu|}{\beta\delta^2}L^{-2\beta}\right)}, & \mu \neq 0, \\ \frac{\sqrt{x}K_{2m'}\left(-\frac{x^{-\beta}}{\delta\beta}\sqrt{2b+2k+\mu\gamma-\mu}\right)}{\sqrt{L}K_{2m'}\left(-\frac{L^{-\beta}}{\delta\beta}\sqrt{2b+2k+\mu\gamma-\mu}\right)}, & \mu = 0. \end{cases}$$

4 Generalized density framework

In this section, we present a general setting for the sovereign default model we consider. In the literature, the default density approach has been proposed by El Karoui, Jeanblanc and Jiao [17] to study the impact of default events. The key hypothesis is the existence of the conditional density with respect to the reference filtration $\mathbb{F}$ so that the default time is totally inaccessible $\mathbb{G}$-stopping time. We extend this approach to more general random times which have both predictable and totally inaccessible parts.

4.1 Generalized density hypothesis

We first introduce the following assumption, called the generalized density hypothesis, which implies that when avoiding a family of $\mathbb{F}$-stopping times, the random time τ admits a conditional density w.r.t. $\mathbb{F}$.

Assumption 4.1 *We assume that there exists a finite family of $\mathbb{F}$-stopping times $(\tau_i)_{i=1}^n$ satisfying $\mathbb{P}(\tau_i = \tau_j) = 0$ for all $i \neq j$, $(i, j = 1, \cdots, n)$, together with a family of $\mathbb{F} \otimes \mathcal{B}(\mathbb{R}_+)$-adapted processes $\alpha(\cdot)$ such that*

$$\mathbb{E}\left[h(\tau)\prod_{i=1}^n \mathbb{1}_{\{\tau\neq\tau_i\}}\,\middle|\,\mathcal{F}_t\right] = \int_{\mathbb{R}_+} h(u)\alpha_t(u)du \qquad \mathbb{P}\text{-}a.s.$$

for any bounded Borel function h. We call $\alpha(\cdot)$ the generalized $\mathbb{F}$-density of τ.

In the sovereign default model (2.10), $(\tau_i)_{i=1}^n$ correspond to the successive political critical dates which are predictable $\mathbb{F}$-stopping times. In the general case, they can be $\mathbb{F}$-stopping times with both accessible and inaccessible parts. Without loss of generality, $(\tau_i)_{i=1}^n$ can be chosen to be a family of strictly increasing $\mathbb{F}$-stopping times.

There exists a martingale version of the generalized density such that $\alpha(\theta)$ is a càdlàg $\mathbb{F}$-martingale for any $\theta \in \mathbb{R}_+$ (see [22, Proposition 2.3] for details). For each $i \in \{1, \cdots, n\}$, denote by $p^i = (p^i_t, t \geq 0)$ a càdlàg version of the $\mathbb{F}$-martingale where $p^i_t = \mathbb{P}(\tau = \tau_i|\mathcal{F}_t)$. Assumption 4.1 implies that, for any $t \geq 0$,

$$\int_{\mathbb{R}_+} \alpha_t(u)du + \sum_{i=1}^n p^i_t = 1 \qquad \mathbb{P}\text{-a.s.}.$$

Moreover, for any bounded Borel function h, one has

$$\mathbb{E}[h(\tau)|\mathcal{F}_t] = \int_{\mathbb{R}_+} h(u)\alpha_t(u)du + \sum_{i=1}^{n} \mathbb{E}[h(\tau_i)p^i_{\tau_i\vee t}|\mathcal{F}_t]. \tag{4.1}$$

Then the Azéma supermatingale of the random time τ is given by

$$G_t := \mathbb{P}(\tau > t|\mathcal{F}_t) = \int_t^{\infty} \alpha_t(u)du + \sum_{i=1}^{n} \mathbb{1}_{\{\tau_i>t\}}p^i_t. \tag{4.2}$$

4.2 Sovereign default model revisited

The generalized density approach provides a general setting for hybrid default models. In the credit risk literature, the hybrid models such as the generalized Cox process model in [4], the jump to default CEV models in [9] and [8] as well as the credit migration model in [10] satisfy the generalized density hypothesis. In particular, the sovereign default model that we have developed in the previous section is also such a case which we revisit below.

Proposition 4.2 *The random time τ defined in* (2.10) *satisfies Assumption 4.1, and the generalized $\mathbb{F}$-density $\alpha(\cdot)$ is given for all $u,t\in\mathbb{R}_+$, on the set $\cap_{i=1}^n\{\tau_i\neq u\}$, by*

$$\alpha_t(u) = \mathbb{E}\Big[\lambda_u \exp\Big(-\int_0^u \lambda_s\, ds - \sum_{i=1}^{n} \mathbb{1}_{\{\tau_i<u\}}\int_{\tau_{i-1}}^{\tau_i} \lambda^N(s)ds\Big)\Big|\mathcal{F}_t\Big] \tag{4.3}$$

Proof: For any $w\le t$, denote $J_t(w) := \mathbb{P}(\tau\le w|\mathcal{F}_t) = \int_0^w \alpha_t(u)du + \sum_{i=1}^n \mathbb{1}_{\{\tau_i\le w\}}p^i_t$ where we recall that $p^i_t = \mathbb{P}(\tau=\tau_i|\mathcal{F}_t)$, $i\in\{1,\cdots,n\}$, are given by Proposition 3.1 as

$$p^i_t = \Big(e^{-\int_0^{\tau_{i-1}}\lambda^N(s)ds} - e^{-\int_0^{\tau_i}\lambda^N(s)ds}\Big)e^{-\int_0^{\tau_i}\lambda_s\, ds}, \quad \text{on } \{\tau_i\le t\}, \quad i\in\{1,\ldots,n\}.$$

Indeed, for any $w\le t$,

$$\begin{aligned}
J_t(w) &= \int_0^w \lambda_u e^{-\int_0^u \lambda_s\, ds - \sum_{i=1}^n \mathbb{1}_{\{\tau_i<u\}}\int_{\tau_{i-1}}^{\tau_i}\lambda^N(s)ds} du + \sum_{i=1}^n \mathbb{1}_{\{\tau_i\le w\}}p^i_t \\
&= -\int_0^w e^{-\sum_{i=1}^n \mathbb{1}_{\{\tau_i<u\}}\int_{\tau_{i-1}}^{\tau_i}\lambda^N(s)ds} d(e^{-\int_0^u\lambda_s\, ds}) + \sum_{i=1}^n \mathbb{1}_{\{\tau_i\le w\}}p^i_t \\
&= -\int_0^w \Big(\sum_{i=1}^{n+1}\mathbb{1}_{\{\tau_{i-1}<u\le\tau_i\}}e^{-\int_0^{\tau_{i-1}}\lambda^N(s)ds}\Big)d(e^{-\int_0^u\lambda_s\, ds}) + \sum_{i=1}^n \mathbb{1}_{\{\tau_i\le w\}}p^i_t \\
&= -\sum_{i=1}^{n+1} e^{-\int_0^{\tau_{i-1}}\lambda^N(s)ds}\int_0^w \mathbb{1}_{\{\tau_{i-1}<u\le\tau_i\}}d(e^{-\int_0^u\lambda_s\, ds}) + \sum_{i=1}^n \mathbb{1}_{\{\tau_i\le w\}}p^i_t \\
&= -\sum_{i=1}^{n+1}\mathbb{1}_{\{\tau_{i-1}\le w\}}e^{-\int_0^{\tau_{i-1}}\lambda^N(s)ds}\Big(e^{-\int_0^{w\wedge\tau_i}\lambda_s\, ds} - e^{-\int_0^{\tau_{i-1}}\lambda_s\, ds}\Big) + \sum_{i=1}^n \mathbb{1}_{\{\tau_i\le w\}}p^i_t
\end{aligned}$$

By rewriting explicitly p^i_t, one has

$$\begin{aligned}
J_t(w) &= 1 - \sum_{i=1}^{n+1} \mathbb{1}_{\{\tau_{i-1} \le w\}} e^{-\int_0^{\tau_{i-1}} \lambda^N(s)ds - \int_0^{w\wedge\tau_i} \lambda_s\, ds} + \sum_{i=1}^{n} \mathbb{1}_{\{\tau_i \le w\}} e^{-\int_0^{\tau_{i-1}} \lambda^N(s)ds - \int_0^{\tau_i} \lambda_s\, ds} \\
&= 1 - \sum_{i=1}^{n+1} \mathbb{1}_{\{\tau_{i-1} \le w < \tau_i\}} e^{-\int_0^{w} \lambda_s\, ds - \int_0^{\tau_{i-1}} \lambda^N(s)ds} \\
&= 1 - e^{-\int_0^w \lambda_s\, ds - \sum_{i=1}^n \mathbb{1}_{\{\tau_i \le w\}} \int_{\tau_{i-1}}^{\tau_i} \lambda^N(s)ds} = 1 - \mathbb{P}(\tau > w|\mathcal{F}_t) = \mathbb{P}(\tau \le w|\mathcal{F}_t).
\end{aligned}$$

When $u > t$, we have $\alpha_t(u) = \mathbb{E}[\alpha_u(u)|\mathcal{F}_t]$ by martingale property, which finishes the proof. □

Remark 4.3 We note that in the sovereign default model (2.10), the following equalities are satisfied: $\alpha_t(u) = \alpha_u(u)$ for $0 \le u \le t$ on $\bigcap_{i=1}^n \{\tau_i \neq u\}$ (see the proposition above) and $p^i_t = p^i_{\tau_i \wedge t}$ for any $i \in \{1, \cdots, n\}$ (see Proposition 3.1). In the generalized density framework, these two equalities imply $\mathbb{P}(\tau > u|\mathcal{F}_t) = \mathbb{P}(\tau > u|\mathcal{F}_u)$, for $u \le t$ and hence the immersion property holds (c.f. [22, Proposition 5.1]), as already mentioned previously.

4.3 Compensator process

The compensator and the intensity processes of default play an important role in the reduced-form approach of credit risk modelling. Under the generalized density hypothesis, we show that the compensator process of τ is in general discontinuous and the intensity does not necessarily exist.

Recall that an increasing càdlàg $\mathbb{F}$-predictable process $\Lambda^{\mathbb{F}}$ is called $\mathbb{F}$-compensator process of a random time τ if the process $(\mathbb{1}_{\{\tau \le t\}} - \Lambda^{\mathbb{F}}_{t\wedge\tau}, t \ge 0)$ is a $\mathbb{G}$-martingale. The process $\Lambda^{\mathbb{G}} = (\Lambda^{\mathbb{F}}_{t\wedge\tau}, t \ge 0)$ is called $\mathbb{G}$-compensator of τ. The general method for computing the compensator is given in Jeulin and Yor [21, Proposition 2] and [18] by using the Doob-Meyer decomposition of the Azéma supermartingale G.

When the immersion property holds, G is the unique solution of the following stochastic differential equation:

$$dG_t = -G_{t^-} d\Lambda^{\mathbb{F}}_t, \quad t > 0, \quad G_0 = 0.$$

Then one has $G = \mathcal{E}(-\Lambda^{\mathbb{F}})$ where $\mathcal{E}$ denotes the Doléan-Dade exponential. Under the generalized density hypothesis, in terms of $\alpha(\cdot)$ and $(p^i)_{i=1}^n$, the $\mathbb{F}$-compensator process $\Lambda^{\mathbb{F}}$ of τ is then given as

$$\Lambda^{\mathbb{F}}_t = \int_0^t \frac{\alpha_s(s)ds}{G_{s^-}} + \sum_{i=1}^n \mathbb{1}_{\{\tau_i \le t\}} \frac{p^i_{\tau_i}}{G_{\tau_i^-}}, \quad t \in \mathbb{R}_+. \tag{4.4}$$

In the sovereign default model (2.10), the Azéma supermartingale (4.2) has an explicit form given by Proposition 3.2 as

$$G_t = \mathbb{P}(\tau > t|\mathcal{F}_t) = \exp\Big(- \int_0^t \lambda_s\, ds - \sum_{i=1}^n \mathbb{1}_{\{\tau_i \le t\}} \int_{\tau_{i-1}}^{\tau_i} \lambda^N(s)ds \Big), \quad t \in \mathbb{R}_+, \tag{4.5}$$

which is a decreasing process due to the immersion property. By consequence, we obtain

$$\Lambda_t^{\mathbb{F}} = \int_0^t \lambda_s \, ds + \sum_{i=1}^n \mathbb{1}_{\{\tau_i \le t\}} \Big(1 - e^{-\int_{\tau_{i-1}}^{\tau_i} \lambda^N(s)ds}\Big), \quad t \in \mathbb{R}_+. \tag{4.6}$$

We underline that the intensity of sovereign default does not exist because of the discontinuity of the compensator process at the $\mathbb{F}$-stopping times $(\tau_i)_{i=1}^n$. In literature, Gehmlich and Schmidt [20] provide a class of models where the Azéma supermartingale contains jumps and propose a generalization of this class with an additional stochastic integral containing atoms at predictable stopping times.

We can also compute the hazard process Γ ([5, Chapter 5]) of the sovereign default τ as

$$\Gamma_t = -\ln G_t = \int_0^t \lambda_s \, ds + \sum_{i=1}^n \mathbb{1}_{\{\tau_i \le t\}} \int_{\tau_{i-1}}^{\tau_i} \lambda^N(s) ds, \quad t \in \mathbb{R}_+, \tag{4.7}$$

Thus, the processes $\Lambda^{\mathbb{F}}$ and Γ have the following relationship

$$\Lambda_t^{\mathbb{F}} = \Gamma_t^c + \sum_{0 < s \le t} \left(1 - e^{-\Delta \Gamma_s}\right), \quad t \in \mathbb{R}_+,$$

where Γ^c is the continuous part of Γ and $\Delta\Gamma_t = \Gamma_t - \Gamma_{t-}$. We observe that the absolutely continuous part of $\Lambda^{\mathbb{F}}$ and Γ are identical and depends on the idiosyncratic default intensity λ. Their jump parts depend on the solvency (through the political critical dates) and the exogenous shock.

Remark 4.4 It is known that if the compensator process is continuous, then the hazard process is also continuous and coincides with the compensator ([5, Proposition 6.2.2]). In the sovereign default model above, we provide a counterexample where the hazard process does not equal to the compensator.

5 Applications to sovereign defaultable claims

In this section, we apply the sovereign default model to financial assets which are subject to sovereign risk such as the government bonds. We are particularly interested in the behavior of long term bond yield during the sovereign crisis and we show that the hybrid model provides an explanation to the jump behaviors of the bond yield around critical dates.

5.1 Sovereign bond and credit spread

We consider a defaultable sovereign zero-coupon bond of maturity T. The recovery payment at default is represented by an $\mathbb{F}$-predictable process R valued in $[0, 1)$ if the sovereign default τ occurs prior to the maturity. In a financial market with credit risk, when the immersion property holds, the risk neutral probability in $\mathbb{F}$ is also a risk neutral probability in $\mathbb{G}$ (c.f. Coculescu, Jeanblanc and Nikeghbali [13]). Let $\mathbb{Q}$ be a risk-neutral probability and assume that all the dynamics of the sovereign default model are given under $\mathbb{Q}$. The generalized density hypothesis remains valid under an equivalent probability change. We denote by $r = (r_t, t \ge 0)$ the default-free interest rate process and by $D(t, T)$ the value at $t < T$ of the zero-coupon bond.

Proposition 5.1 *The value of the defaultable zero-coupon bond is given by*

$$D(t,T) = D^0(t,T) + D^1(t,T), \tag{5.1}$$

where D^0 is the pre-default price related to the payment at maturity, computed as

$$D^0(t,T) = \mathbb{1}_{\{\tau>t\}}\mathbb{E}_{\mathbb{Q}}\Big[\exp\big(-\int_t^T (r_s+\lambda_s)ds - \sum_{i=1}^n \mathbb{1}_{\{t<\tau_i\leq T\}}\int_{\tau_{i-1}}^{\tau_i}\lambda^N(s)ds\big)\big|\mathcal{F}_t\Big], \tag{5.2}$$

and D^1 is related to the recovery payment, given by

$$D^1(t,T) = \frac{\mathbb{1}_{\{\tau>t\}}}{G_t}\,\mathbb{E}_{\mathbb{Q}}\Big[\int_t^T e^{-\int_t^u r_s\,ds}R_u\alpha_u(u)du + \sum_{i=1}^n \mathbb{1}_{\{t<\tau_i\leq T\}}e^{-\int_t^{\tau_i} r_s\,ds}R_{\tau_i}p^i_{\tau_i}\big|\mathcal{F}_t\Big], \tag{5.3}$$

where $G_t = \mathbb{Q}(\tau > t|\mathcal{F}_t)$.

Proof: The pre-default value of the bond is given by

$$D(t,T) = \mathbb{E}_{\mathbb{Q}}\big[e^{-\int_t^T r_s\,ds}\mathbb{1}_{\{\tau>T\}}|\mathcal{G}_t\big] + \mathbb{E}_{\mathbb{Q}}\big[e^{\int_t^\tau r_s\,ds}\mathbb{1}_{\{t<\tau\leq T\}}R_\tau|\mathcal{G}_t\big] =: D^0(t,T) + D^1(t,T).$$

The first term $D^0(t,T)$ is obtained by

$$D^0(t,T) = \mathbb{1}_{\{\tau>t\}}\mathbb{E}_{\mathbb{Q}}\Big[\frac{G_T}{G_t}e^{-\int_t^T r_s\,ds}|\mathcal{F}_t\Big]$$

together with (4.5), and the second term results from [5, Proposition 5.1.1] as

$$\begin{aligned}D^1(t,T) &= \frac{\mathbb{1}_{\{\tau>t\}}}{G_t}\mathbb{E}_{\mathbb{Q}}\Big[\mathbb{E}_{\mathbb{Q}}\big[\mathbb{1}_{\{t<\tau\leq T\}}\exp(-\int_t^\tau r_s\,ds)R_\tau|\mathcal{F}_T\big]\big|\mathcal{F}_t\Big]\\ &= \frac{\mathbb{1}_{\{\tau>t\}}}{G_t}\mathbb{E}_{\mathbb{Q}}\Big[\int_t^T e^{-\int_t^u r_s\,ds}R_u\alpha_T(u)du + \sum_{i=1}^n \mathbb{1}_{\{t<\tau_i\leq T\}}e^{-\int_t^{\tau_i} r_s\,ds}R_{\tau_i}p^i_T\Big|\mathcal{F}_t\Big].\end{aligned}$$

We complete the proof by using the equality (4.5) and the following properties: $\alpha_T(u) = \alpha_u(u)$ for $t \leq u \leq T$ on $\bigcap_{i=1}^n\{\tau_i \neq u\}$ and $p^i_T = p^i_{\tau_i}$ on $\{t < \tau_i \leq T\}$ for any $i \in \{1, \cdots, n\}$ (see Remark 4.3). □

We are interested in the bond prices at the political critical dates $(\tau_i)_{i=1}^n$ and in particular the jump behavior. Let

$$\Delta D(t,T) := D(t,T) - D(t-,T), \quad t \leq T$$

which is the sum of $\Delta D^0(t,T)$ and $\Delta D^1(t,T)$ that we compute respectively. We assume that the filtration $\mathbb{F}$ only supports continuous martingales. On the one hand,

$$\begin{aligned}D^0(t,T) = \mathbb{1}_{\{\tau>t\}}\mathbb{E}_{\mathbb{Q}}\Big[\exp\Big(-\int_0^T (r_s+\lambda_s)\,ds - \sum_{i=1}^n \mathbb{1}_{\{\tau_i\leq T\}}\int_{\tau_{i-1}}^{\tau_i}\lambda^N(s)\,ds\,\Big|\,\mathcal{F}_t\Big]\\ \cdot\exp\Big(\int_0^t (r_s+\lambda_s)\,ds + \sum_{i=1}^n \mathbb{1}_{\{\tau_i\leq t\}}\int_{\tau_{i-1}}^{\tau_i}\lambda^N(s)\,ds\Big)\end{aligned}$$

which implies

$$\Delta D^0(t,T) = D^0(t,T)\sum_{i=1}^{n}\mathbb{1}_{\{\tau_i=t\}}\Big(1-e^{-\int_{\tau_{i-1}}^{\tau_i}\lambda^N(s)\,ds}\Big), \quad \text{on } \{\tau>t\}. \tag{5.4}$$

On the other hand, by (5.3),

$$\Delta D^1(t,T) = \Delta(G_t^{-1})\mathbb{E}_{\mathbb{Q}}\Big[\int_t^T e^{-\int_t^u r_s ds}R_u\alpha_T(u)du + \sum_{i=1}^{n}\mathbb{1}_{\{t<\tau_i\le T\}}e^{-\int_t^{\tau_i} r_s\,ds}R_{\tau_i}p^i_{\tau_i}\Big|\mathcal{F}_t\Big]$$
$$-\frac{1}{G_{t-}}\sum_{i=1}^{n}\mathbb{1}_{\{\tau_i=t\}}R_{\tau_i}p^i_{\tau_i}, \quad \text{on } \{\tau>t\}.$$

Moreover, by (4.5) one has

$$\Delta(G_t^{-1}) = \frac{1}{G_t}\sum_{i=1}^{n}\mathbb{1}_{\{\tau_i=t\}}\Big(1-e^{-\int_{\tau_{i-1}}^{\tau_i}\lambda^N(s)\,ds}\Big).$$

We then deduce

$$\Delta D^1(t,T) = D^1(t,T)\sum_{i=1}^{n}\mathbb{1}_{\{\tau_i=t\}}\Big(1-e^{-\int_{\tau_{i-1}}^{\tau_i}\lambda^N(s)\,ds}\Big) - \frac{1}{G_{t^-}}\sum_{i=1}^{n}\mathbb{1}_{\{\tau_i=t\}}R_{\tau_i}p^i_{\tau_i} \quad \text{on } \{\tau>t\}, \tag{5.5}$$

which implies, combining (5.4) and (5.5), that

$$\Delta D(t,T) = \sum_{i=1}^{n}\mathbb{1}_{\{\tau_i=t\}}D(\tau_i,T)\Big(1-e^{-\int_{\tau_{i-1}}^{\tau_i}\lambda^N(s)\,ds}\Big) - \sum_{i=1}^{n}\mathbb{1}_{\{\tau_i=t\}}\frac{1}{G_{\tau_i^-}}R_{\tau_i}p^i_{\tau_i} \quad \text{on } \{\tau>t\}.$$

By using the relation

$$\frac{p^i_{\tau_i}}{G_{\tau_i-}} = 1-e^{-\int_{\tau_{i-1}}^{\tau_i}\lambda^N(s)\,ds},$$

we obtain finally

$$\Delta D(t,T) = \sum_{i=1}^{n}\mathbb{1}_{\{\tau_i=t\}}\big(D(\tau_i,T)-R_{\tau_i}\big)\Big(1-e^{-\int_{\tau_{i-1}}^{\tau_i}\lambda^N(s)\,ds}\Big), \tag{5.6}$$

and in particular,

$$\Delta D(\tau_i,T) = \big(D(\tau_i,T)-R_{\tau_i}\big)\Big(1-e^{-\int_{\tau_{i-1}}^{\tau_i}\lambda^N(s)\,ds}\Big) \quad \text{on} \quad \{\tau_i\le T\}. \tag{5.7}$$

Let the pre-default yield to maturity of the defaultable bond on $\{t<\tau\}$ be

$$Y^d(t,T) = -\frac{\ln D(t,T)}{T-t}. \tag{5.8}$$

Similarly, the yield to maturity of a classical default-free zero coupon bond is given as

$$Y(t,T) = -\frac{\ln B(t,T)}{T-t}.$$

where $B(t,T) = \mathbb{E}_{\mathbb{Q}}[e^{-\int_t^T r_s\,ds}|\mathcal{F}_t]$ denotes the price at $t \leq T$ of the default-free zero coupon bond of maturity T. Let the pre-default credit spread, noted $S(t,T)$, be defined as the difference between the two yields to maturity, i.e.,

$$S(t,T) := Y^d(t,T) - Y(t,T) = -\frac{1}{T-t}\ln\frac{D(t,T)}{B(t,T)}.$$

Then,

$$\Delta S(t,T) = S(t,T) - S(t-,T) = -\frac{\Delta \ln D(t,T)}{T-t} = -\frac{1}{T-t}\ln\left(1+\frac{\Delta D(t,T)}{D(t^-,T)}\right). \tag{5.9}$$

which implies by (5.7) that the jump of the bond yield at a critical date τ_i is negative if and only if $\Delta D(\tau_i,T)$ is positive. More precisely, $\Delta S(\tau_i,T) < 0$ on $\{\tau_i < T \wedge \tau\}$ if and only if

$$D(\tau_i,T) > R_{\tau_i} \quad \text{a.s.}. \tag{5.10}$$

In particular, if $R \equiv 0$, one has $D^1 = 0$ and $\Delta \ln D^0(t,T) = \sum_{i=1}^n \mathbb{1}_{\{\tau_i=t\}} \int_{\tau_{i-1}}^{\tau_i} \lambda^N(s)ds$. Then, we can compute the jump in the credit spread at τ_i as

$$\Delta S(\tau_i,T) = -\mathbb{1}_{\{\tau_i<T\}}\frac{\Delta \ln D^0(\tau_i,T)}{T-\tau_i} = -\mathbb{1}_{\{\tau_i<T\}}\frac{\int_{\tau_{i-1}}^{\tau_i}\lambda^N(s)ds}{T-\tau_i}. \tag{5.11}$$

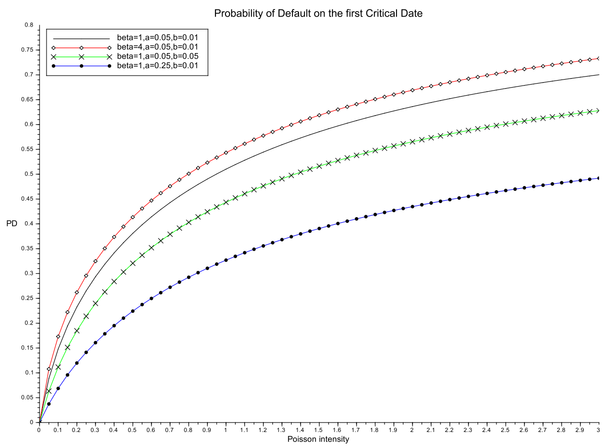
We notice that whether the jump of sovereign bond yield at a critical date τ_i is negative or not depends on the intensity of the exogenous shock, the elapsed time between τ_{i-1} and τ_i (the solvency indirectly), and the value of the recovery payment at τ_i. When the recovery payment is small enough, the condition in (5.10) can be satisfied. Moreover, if no recovery payment is made, the size of the jumps only depends on the solvency and the exogenous shock.

5.2 Numerical illustrations

We now present numerical examples to illustrate the results obtained previously concerning the sovereign default probability and the defaultable bond yield.

In the first example, we are interested in the default probability p_0^i on a political critical date τ_i, $(i = 1, 2, 3)$, given by (3.3). We assume that the solvency process S is modelled by a geometric Brownian motion as in Section 3.2.1, and we use the solvency data of Greece during the period from 2003 to 2013 to estimate the parameters and obtain $S_0 = 1.01$, $\mu = -0.01$ and $\sigma = 0.14$. Let the idiosyncratic default intensity process λ be specified by $\lambda(S) = \frac{a}{S^{2\beta}} + b$ as in (3.8) and the Poisson intensity be a constant λ^N. The solvency barrier is re-adjustable with three values $L_1 = 0.9$, $L_2 = 0.8$ and $L_3 = 0.7$. Figure 4 and Figure 5 plots the probability that the sovereign default occurs on τ_1 and respectively on τ_2 and τ_3 as functions of the Poisson intensity λ^N for different parameters a, b and β, and we show in particular the impact of the exogenous shock intensity λ^N. We observe that the probability of default on τ_1 is an increasing function of λ^N since it is more probable for the exogenous shock to occur when λ^N is larger, in which case the sovereign has higher possibility to default at the first critical date τ_1 due to an unfavorable political decision. However, when λ^N is large, the probability of default on other critical dates τ_2 and τ_3 after τ_1 is reduced because the exogenous shock has more chance to occur before τ_1. As a result, the probabilities of default on τ_2 and τ_3 are increasing functions of λ^N for small λ^N and decreasing for large λ^N. For comparison concerning the parameters of the idiosyncratic intensity process, we set the parameters $a = 0.05$, $b = 0.01$ and $\beta = 1$ and examine the impact of each parameter by considering also values $a = 0.25$, $b = 0.05$ and $\beta = 4$ respectively. Other things being equal, the probabilities of default on τ_1, τ_2 and τ_3 are smaller for bigger a (respectively bigger b) because the sovereign is more probable to default due to the idiosyncratic credit risk when $\lambda(S)$ is bigger. The impact of the elasticity parameter β depends on the level of solvency, more precisely, $\lambda(S)$ is decreasing (respectively increasing) when $S \geq 1$ (respectively $S < 1$). Consequently, the probability of default on τ_1 (respectively τ_2, τ_3) is smaller for smaller β (respectively bigger β).

Figure 4: Probability of sovereign default on τ_1



In the second example, we consider the sovereign default probability $\mathbb{P}(\tau \leq T)$, which can be computed by Proposition 3.2. The solvency process S is given as a geometric Brownian motion with the same parameters as in the previous example. We fix the values of $a = 0.05$, $b = 0.01$ and $\beta = 1$ for idiosyncratic default intensity. Figure 6 plots the probability of default from 1 to 30 years for different values of the Poisson intensity: $\lambda^N = 0$ (the Cox process model), $\lambda^N = 0.05$ and 0.2 respectively. We note that unsurprisingly, an exogenous shock with larger intensity value increases the sovereign default probability.

In the third example, we illustrate the bond yield and its jump at a critical date for a sovereign defaultable bond. The solvency is described by a CEV process as in (3.11) and we set the parameters to be $S_0 = 1.01$, $\mu = -0.01$, $\delta = 0.03$ and $\beta = 1$. The idiosyncratic default intensity process λ is specified by $\lambda(S) = \frac{a}{S^{2|\beta|}} + b$ as in (3.12) with coefficients $a = 0.005$ and $b = 0.01$. We assume that there is only one critical date with the solvency barrier $L = 0.9$ and that the risk-free interest rate is 0. Figure 7 plots the time-varying bond yield (5.8) of a defaultable zero-coupon bond of maturity 5 years without recovery payment, as well as the corresponding simulation scenario of the solvency process. We present two different exogenous shock intensities: $\lambda^N = 0.05$ and $\lambda^N = 0.2$. We observe in this example that when the solvency

Figure 5: Probability of sovereign default on τ_2 and τ_3 respectively.

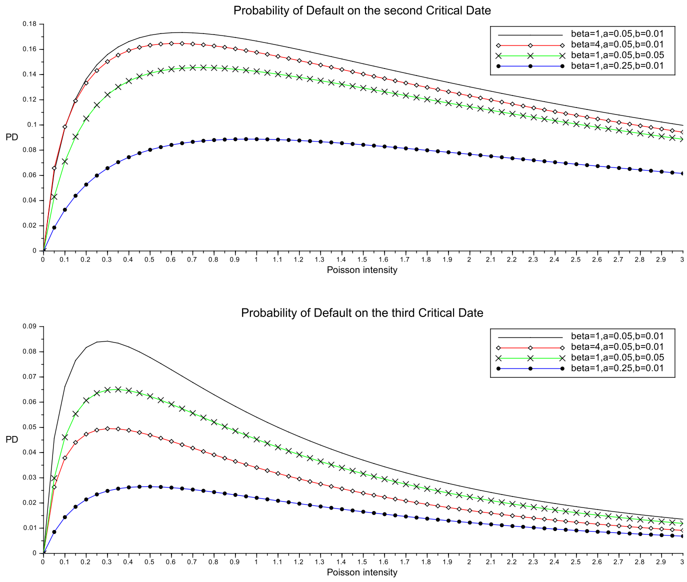
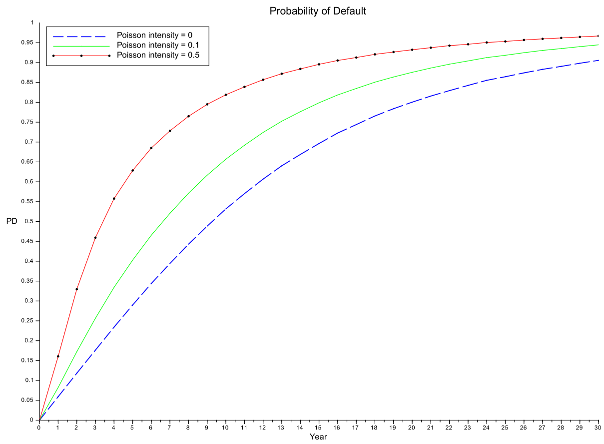


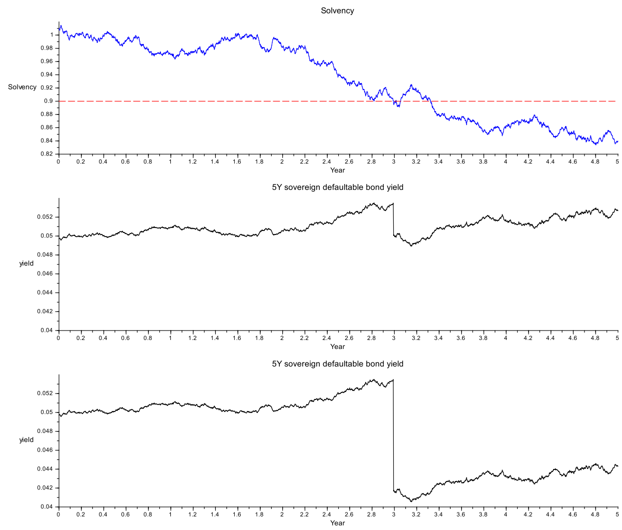
Figure 6: Sovereign default probability.



hits the threshold 0.9, the bond yield has a negative jump, the size of which depends on the value of λ^N. More precisely, a larger value of the exogenous shock intensity λ^N results in a larger jump in the bond yield.

In the last example, we consider the long term Greece government bond yield of maturity 10 years. The solvency of Greece is described by a CEV process. We estimate the parameters

Figure 7: Jump at a critical date in the sovereign defaultable bond yield (with the corresponding solvency sample path): $\lambda^N = 0.05$ and 0.2 respectively.



by using the solvency data as in Figure 3 where δ and β are jointly calibrated (c.f. Chesney, Elliott, Madan and Yang [11] and Yuen, Yang and Chu [28]) and obtain $S_0 = 1.01$, $\mu = -0.01$, $\delta = 0.03$ and $\beta = -4.92$. The coefficients of the idiosyncratic default intensity (as in (3.12)) are $a = 0.013$ and $b = 0.035$, estimated from the 3-month Greek bond yield. The solvency barrier is re-adjustable with three values $L_1 = 0.9$, $L_2 = 0.8$ and $L_3 = 0.7$. We suppose that the intensity of the inhomogeneous Poisson process for the exogenous shock is a piecewise constant function which change its value at each critical date. By Figure 2, given the sizes of the three jumps, we let $\lambda^N(t) = 0.07$ for $t \in [0, \tau_1]$, $\lambda^N(t) = 0.16$ for $t \in (\tau_1, \tau_2]$ and $\lambda^N(t) = 3.15$ for $t \in (\tau_2, \tau_3]$, which are computed using (5.11). Figure 8 plots the time-varying bond yield of a 10-year Greek government zero-coupon bond, as well as a sample path of the solvency of Greece which corresponds to the period of 2003-2013. We observe that the solvency of Greece tends to fall gradually through time and hits the three thresholds successively. The bond yield has three negative jumps at the barrier hitting times: in particular, there is a large negative jump when the solvency falls below 0.7 since the exogenous shock intensity is at a high level, while the first two values of exogenous intensity are relatively small. This looks like the full view of the historical data in Figure 2 where the three jumps correspond respectively to the critical dates in Figure 1.

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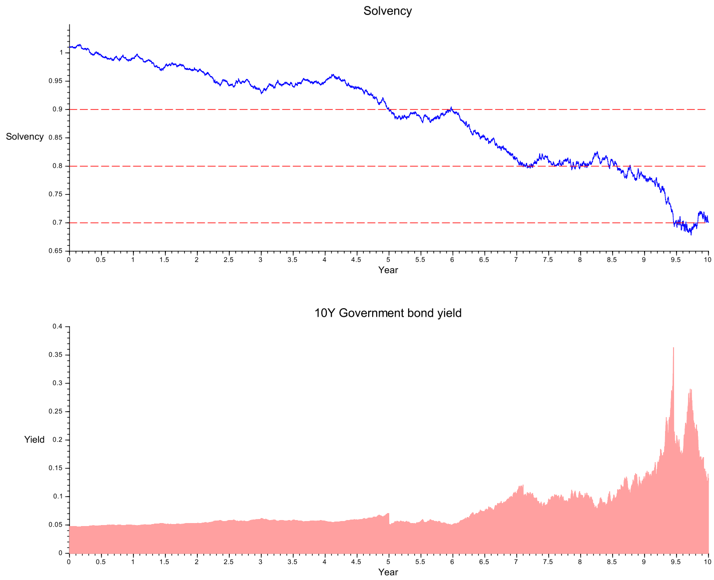
Figure 8: Simulated 10-year Greek government bond yield with re-adjustable Poisson intensity and the corresponding solvency sample path.



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