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# A Magnetic Flux–Electric Current Volume Integral Formulation Based on Facet Elements for Solving Electromagnetic Problems

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An original and powerful volume integral formulation for solving electromagnetic problems is proposed. It is based on a facet interpolation for representing the magnetic flux in magnetic regions and the current density in conducting regions. This formulation is particularly well adapted for solving electromagnetic problems where air is preponderant, whatever the frequency. First results obtained are very encouraging, in terms of efficiency and accuracy.

*Index Terms*—Electromagnetism, equivalent circuit, volume integral method (VIM), Whitney facet element.

## I. INTRODUCTION

**S**OLVING Maxwell’s equations by Green’s function integral equation method has known a renewed interest in recent years, with the development of matrix compression algorithms (Fast Multipole Method and Adaptive Cross Approximation), which greatly improves storage and resolution of full matrix systems. Zakrzewski and Tomczuk [12] have already shown the interest of using integral method for 3-D magnetic field analysis, the main advantage being that the air region does not need to be meshed. On the other hand, Whitney facet interpolation can be used for representing current density  $\mathbf{J}$  and flux density  $\mathbf{B}$ . With this approach, an electromagnetic problem can be represented by an equivalent circuit using of magnetic fluxes and electric currents through the facets as proposed in [11].

In this paper, we propose a new formulation that combines equivalent network based on Whitney facet interpolation and Green’s function volume integral method (VIM). This approach allows meshing only active materials (ferromagnetic and/or conductive materials). Alotto *et al.* [1] and Nguyen *et al.* [2] have shown very good behaviors and results for solving electric equations with approach combining Whitney circuit representation in active parts and integral volume formulation for considering air interactions. Moreover facet interpolation has been successfully used for linear and nonlinear magnetostatic formulations [3], [4]. In this paper, we propose to couple both approaches for solving low frequency electromagnetic problems (with both eddy currents and magnetic materials).

## II. INTEGRAL EQUATIONS

Let us consider a problem (Fig. 1) with magnetic regions  $\Omega_m$  (presence of magnetization  $\mathbf{M}$ ) and conductive electrical regions  $\Omega_c$  (presence of current density  $\mathbf{J}$ ). Starting from Maxwell equations we can express  $\mathbf{E}$  and  $\mathbf{H}$  through vector

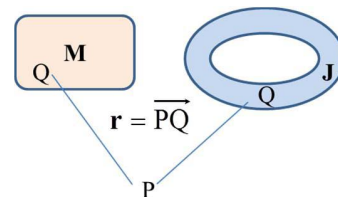


Fig. 1. Electromagnetic problem with electrical and magnetic regions ( $Q$ : integration point in  $\Omega_m$  and  $\Omega_c$ ).

potentials (magnetic  $\mathbf{A}$  and electric  $\mathbf{T}$ ) and scalar potentials (electric  $V$  and magnetic  $\phi$ ) as

$$\mathbf{E} = -\frac{d\mathbf{A}}{dt} - \mathbf{grad}V \quad \mathbf{H} = \mathbf{T} - \mathbf{grad}\phi. \quad (1)$$

By neglecting displacement currents and using adequate gauges, these potentials can be expressed from integrals (2) that are limited to regions  $\Omega_c$  and  $\Omega_m$ . These relations allow computing exact solution of  $\mathbf{E}$  and  $\mathbf{H}$  at any point  $P$  from the knowledge of  $\mathbf{J}$  in conducting regions and  $\mathbf{M}$  in magnetic regions

$$\begin{aligned} \mathbf{A}_{(P)} &= \frac{\mu_0}{4\pi} \int_{\Omega_c} \frac{\mathbf{J}_{(Q)}}{r} d\Omega + \frac{\mu_0}{4\pi} \int_{\Omega_m} \frac{\mathbf{M}_{(Q)} \wedge r}{r^3} d\Omega \\ \frac{dV_{(P)}}{dt} &= \frac{1}{4\pi\epsilon_0} \int_{\Omega_c} \frac{\mathbf{J}_{(Q)} \cdot r}{r^3} d\Omega \\ \mathbf{T}_{(P)} &= \frac{1}{4\pi} \int_{\Omega_c} \frac{\mathbf{J}_{(Q)} \wedge r}{r^3} d\Omega \\ \phi_{(P)} &= \frac{1}{4\pi} \int_{\Omega_m} \frac{\mathbf{M}_{(Q)} \cdot r}{r^3} d\Omega. \end{aligned} \quad (2)$$

In the general case,  $\mathbf{M}$  and  $\mathbf{J}$  are not known (in the presence of eddy currents or magnetization for instance) and we have to add constitutive laws of materials to obtain a complete solution. In our case, we consider two constitutive laws linking, respectively, the density  $\mathbf{J}$  to the electric field  $\mathbf{E}$  (in  $\Omega_c$ ) and the magnetization  $\mathbf{M}$  to the magnetic field  $\mathbf{H}$  (in  $\Omega_m$ )

$$\mathbf{J} = \sigma \mathbf{E} \quad (3)$$

$$\mathbf{M} = \chi \mathbf{H} = (\nu_0 - \nu) \mathbf{B}. \quad (4)$$

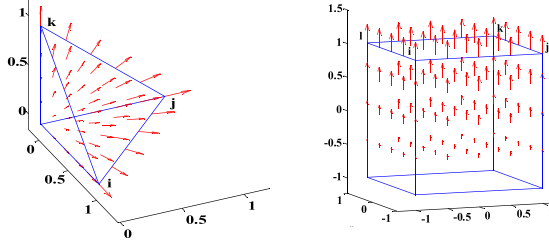


Fig. 2. Representation of facet shape functions for reference tetrahedra and hexahedra.

In practice, an integral volume formulation consists to write the following.

- 1) At any point P of the conducting regions, the electric field obtained by integral relations (2) matching the electric local law (3).
- 2) At any point P of the magnetic regions, the magnetic field obtained by integral relations (2) matching the magnetic local law (4).

It can be achieved by applying two Galerkin procedures, respectively, associated with  $\Omega_c$  and  $\Omega_m$

$$\int_{\Omega_c} \mathbf{w}_i \cdot \left( \frac{\mathbf{J}}{\sigma} - \mathbf{E} \right) d\Omega = 0 \quad \int_{\Omega_m} \mathbf{w}_f \cdot \left( \frac{\mathbf{M}}{\chi} - \mathbf{H} \right) d\Omega = 0 \quad (5)$$

where  $\mathbf{E}$  and  $\mathbf{H}$  are obtained thanks to (1) and (2) and where  $\mathbf{w}_i$  and  $\mathbf{w}_f$  are adequate projection functions of the Galerkin procedure. Relations (5) link  $\mathbf{J}$  and  $\mathbf{M}$ , which can be naturally chosen as unknowns. A numerical solution can then be obtained by interpolating  $\mathbf{J}$  and  $\mathbf{M}$ , respectively, on  $\Omega_c$  and  $\Omega_m$  [5], [6]. However, other unknowns can be chosen: electric vector potential  $\mathbf{T}$  ( $\mathbf{J} = \text{curl } \mathbf{T}$ ) can be used in electric region [7], and magnetic scalar potential, for instance, can be used in magnetic region [8]. In this paper, we propose to use facet interpolations for  $\mathbf{J}$  and  $\mathbf{B}$  in conducting regions and magnetic regions, respectively. This choice avoids any connexity problem and strongly imposes the solenoidality of  $\mathbf{J}$  and  $\mathbf{B}$  thanks to a circuit representation.

### III. FACET ELEMENT FORMULATION

Magnetic and current regions are discretized with Whitney first-order facet finite element functions. The current and flux density are interpolated as

$$\mathbf{J} = \sum_j \mathbf{w}_j I_j \quad \mathbf{B} = \sum_g \mathbf{w}_g \Psi_g \quad (6)$$

where  $\mathbf{w}_j$  and  $\mathbf{w}_g$  are facet shape functions (Fig. 2) and  $I_j$  and  $\Psi_g$  the fluxes across the facets. Facet interpolation is well adapted to  $\mathbf{J}$  and  $\mathbf{B}$  unknowns since it imposes the continuity of the fluxes into the finite element mesh.

#### A. Facet Shape Functions

Expressions for first-order shape function  $\mathbf{w}_i$  can be easily computed for different kind of elements [8]. The main property of such element family is that the normal component of  $\mathbf{w}_i$  is conserved through each facet.

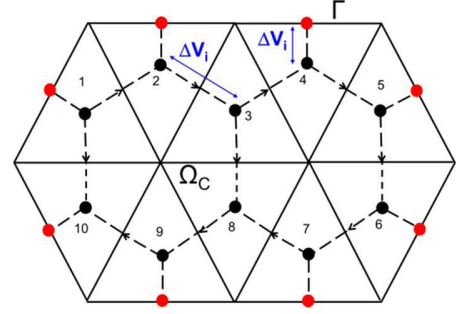


Fig. 3. Circuit representation (dual mesh).

For first-order shape functions, we have

$$\mathbf{w}_i \cdot \mathbf{n} = \pm \frac{1}{s_i} \text{ on facet } i; \quad \mathbf{w}_i \cdot \mathbf{n} = 0 \text{ on other facets} \quad (7)$$

$$\text{div } \mathbf{w}_i = \pm \frac{1}{v} \quad (8)$$

where  $s_i$  is the surface of facet  $i$  and  $v$  is the volume of element  $e$  that contains the facet  $j$ . The sign ( $\pm$ ) depends on the facet orientation in the global mesh. Consider now any potential  $\varphi$  and the integral of  $\mathbf{w}_i \cdot \text{grad } \varphi$  on element  $e$  containing the facet  $i$ . Using Green and divergence theorems, we get

$$\int_{\Omega_e} \mathbf{w}_i \cdot \text{grad } \varphi d\Omega = \int_{\Omega_e} \text{div } \mathbf{w}_i \varphi d\Omega - \int_{\Gamma_e} \mathbf{w}_i \cdot \mathbf{n} \varphi d\Gamma. \quad (9)$$

Let us consider an element  $e$  sharing a face  $i$  with the boundary of  $\Omega_c$  or  $\Omega_m$ . Using relations (7) and (8), we can then write (by considering that the normal vector of the facet  $i$  is oriented outward of the element  $e$ )

$$\int_{\Omega} \mathbf{w}_i \cdot \text{grad } \varphi d\Omega = \Delta V_i = V_e - V_{\Gamma_i} \quad (10)$$

where  $V_e$  is the averaged potential on the volume element  $e$  and  $V_{\Gamma}$  the averaged potential on the facet  $i$ . For a facet  $i$  between two adjacent elements  $e$  and  $f$ , we get

$$\int_{\Omega} \mathbf{w}_i \cdot \text{grad } \varphi d\Omega = \Delta V_i = V_e - V_f. \quad (11)$$

From previous considerations, an equivalent circuit can be generated. Each branch of this circuit is associated with a facet of the initial mesh. Each element of the finite element mesh can be seen as a node of this circuit (Fig. 3).

#### B. Galerkin Formulation

From relations (2) and noting that  $\mathbf{B} = (v_0 - v) \mathbf{M}$ , we can express  $\mathbf{J}$ ,  $\mathbf{A}$ ,  $\mathbf{B}$ , and  $\mathbf{T}$  in terms of currents  $I_j$  through the facets of the electrical regions and magnetic fluxes  $\Psi_j$  through the facets of magnetic regions. After integration of (5) on  $\Omega_c$  and  $\Omega_m$ , we obtain a system of equations, which links the difference of averaged potentials between two adjacent elements ( $\Delta V$  and  $\Delta \Phi$ ) with the fluxes through the facets ( $I_j$  and  $\Psi_j$ )

$$\begin{bmatrix} [\mathbf{R}] + d/dt[\mathbf{L}] \\ [\mathbf{D}] \end{bmatrix} \begin{bmatrix} d/dt[\mathbf{C}] \\ [\mathbf{E}] \end{bmatrix} \begin{bmatrix} I \\ \Psi \end{bmatrix} = \begin{bmatrix} \Delta V \\ \Delta \Phi \end{bmatrix} \quad (12)$$

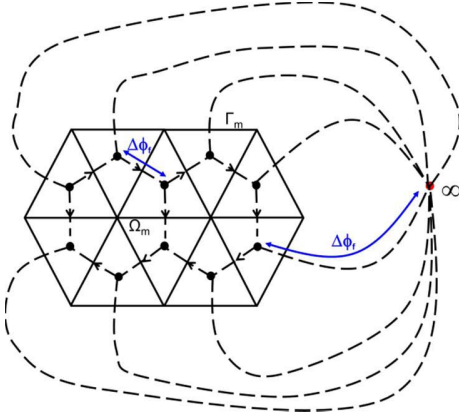


Fig. 4. Circuit representation for magnetic regions.

with

$$\begin{aligned}
 \mathbf{R}_{ij} &= \int_{\Omega_c} \mathbf{w}_i \cdot \frac{\mathbf{w}_j}{\sigma} d\Omega & \mathbf{L}_{ij} &= \frac{\mu_0}{4\pi} \int_{\Omega_c} \mathbf{w}_i \cdot \int_{\Omega_c} \frac{\mathbf{w}_j}{r} d\Omega d\Omega \\
 \mathbf{C}_{ig} &= \frac{\mu_0}{4\pi} \int_{\Omega_c} \mathbf{w}_i \cdot \int_{\Omega_m} (v_0 - v) \frac{\mathbf{w}_g \wedge \mathbf{r}}{r^3} d\Omega d\Omega \\
 \mathbf{E}_{fg} &= \int_{\Omega_m} \mathbf{w}_f v \mathbf{w}_g d\Omega \\
 \mathbf{D}_{fj} &= -\frac{1}{4\pi} \int_{\Omega_c} \mathbf{w}_f \cdot \int_{\Omega_m} \frac{\mathbf{w}_j \wedge \mathbf{r}}{r^3} d\Omega d\Omega.
 \end{aligned} \quad (13)$$

### C. Equivalent Circuit Representation

Electrical and magnetic circuits defined by (12) being open have to be completed. For low-frequency applications, by neglecting capacitive effects, the electrical circuit associated with  $\Omega_c$  can be simply completed by suppressing all the branches linked to boundary faces (the normal component of  $\mathbf{J}$  being null). Thus, the circuit becomes closed. For magnetic regions, the flux generally leak out of  $\Omega_m$ . Finally, (2) allows us to express the average potential  $\phi$  on a facet  $f$  of  $\Gamma_m$

$$\phi_f = \int_{\Gamma_f} \frac{1}{S_f} \phi d\Gamma = \frac{1}{4\pi} \int_{\Gamma_f} \frac{1}{S_f} \int_{\Omega_m} (v_0 - v) \frac{\mathbf{B} \cdot \mathbf{r}}{r^3} d\Omega d\Omega. \quad (14)$$

Consequently, we can extend branches associated with the facets on  $\Gamma_m$  to an infinity point (Fig. 4)

$$(\Delta\phi_f)_{\text{ext}} = \phi_f - \phi_\infty. \quad (15)$$

Including these new contributions, (12) becomes

$$\begin{bmatrix} [\mathbf{R}] + d/dt[\mathbf{L}] & d/dt[\mathbf{C}] \\ [\mathbf{D}] & [\mathbf{E}] + [\mathbf{F}] \end{bmatrix} \begin{bmatrix} I \\ \Psi \end{bmatrix} = \begin{bmatrix} \Delta V \\ \Delta \Phi \end{bmatrix} \quad (16)$$

with

$$\mathbf{F}_{fg} = \frac{1}{4\pi} \int_{\Gamma_f} \frac{1}{S_f} \int_{\Omega_m} (v - v_0) \frac{\mathbf{w}_g \cdot \mathbf{r}}{r^3} d\Omega d\Omega. \quad (17)$$

Finally, the electromagnetic problem has been changed into two coupled equivalent circuits.

### D. Application: Magnetodynamic Formulation With Coils

We apply the previous formulation in the case of low-frequency magneto-harmonic problems, in the presence of source coils, with known current density  $\mathbf{J}_0$ . The formulation is adapted by adding potential and field source terms. In this case,  $\mathbf{A}$  and  $\mathbf{T}$  are written as

$$\begin{aligned}
 \mathbf{A} &= \frac{\mu_0}{4\pi} \int_{\Omega_c} \frac{\mathbf{J}}{r} d\Omega + \frac{\mu_0}{4\pi} \int_{\Omega_0} \frac{\mathbf{J}_0}{r} d\Omega + \frac{\mu_0}{4\pi} \int_{\Omega_m} \frac{\mathbf{M} \wedge \mathbf{r}}{r^3} d\Omega \\
 \mathbf{T} &= \frac{1}{4\pi} \int_{\Omega_c} \frac{\mathbf{J} \wedge \mathbf{r}}{r^3} d\Omega + \frac{1}{4\pi} \int_{\Omega_0} \frac{\mathbf{J}_0 \wedge \mathbf{r}}{r^3} d\Omega.
 \end{aligned} \quad (18)$$

Therefore, (17) becomes

$$\begin{bmatrix} [\mathbf{R}] + j\omega[\mathbf{L}] & [\mathbf{C}] \\ [\mathbf{D}] & [\mathbf{E}] + [\mathbf{F}] \end{bmatrix} \begin{bmatrix} I \\ \Psi \end{bmatrix} = \begin{bmatrix} \Delta V \\ \Delta \Phi \end{bmatrix} + \begin{bmatrix} \mathbf{U} \\ \mathbf{W} \end{bmatrix} \quad (19)$$

with

$$\begin{aligned}
 U_i &= -\frac{\mu_0}{4\pi} \int_{\Omega_c} \mathbf{w}_i \cdot \left( \int_{\Omega_0} \frac{\mathbf{J}_0}{r} d\Omega \right) d\Omega \\
 W_f &= \frac{1}{4\pi} \int_{\Omega_m} \mathbf{w}_f \cdot \left( \int_{\Omega_0} \frac{\mathbf{J}_0 \wedge \mathbf{r}}{r^3} d\Omega \right) d\Omega.
 \end{aligned} \quad (20)$$

To solve electrical and magnetic circuit equations (19), we use a similar technique than in [2] using independent loops technique [10]. Fundamental circuit equations to be solved are

$$\begin{aligned}
 [\mathbf{M}_c] \Delta V &= 0 \\
 [\mathbf{M}_m] \Delta \Phi &= 0
 \end{aligned} \quad (21)$$

with  $[\mathbf{M}_c]$  and  $[\mathbf{M}_m]$  are the branch-fundamental independent loop matrixes. The unknowns of the system to be solved are the mesh currents  $I_M$  and mesh flux density  $\Psi_M$

$$\begin{aligned}
 I &= [\mathbf{M}_c]^t I_M \\
 \Psi &= [\mathbf{M}_m]^t \Psi_M.
 \end{aligned} \quad (22)$$

The final system to be solved is

$$\begin{bmatrix} [\mathbf{M}_c] ([\mathbf{R}] + j\omega[\mathbf{L}]) [\mathbf{M}_c]^t & j\omega[\mathbf{M}_c] [\mathbf{C}] [\mathbf{M}_m]^t \\ [\mathbf{M}_m] [\mathbf{D}] [\mathbf{M}_c]^t & [\mathbf{M}_m] ([\mathbf{E}] + [\mathbf{F}]) [\mathbf{M}_m]^t \end{bmatrix} \begin{bmatrix} I_M \\ \Psi_M \end{bmatrix} = \begin{bmatrix} [\mathbf{M}_c] \mathbf{U} \\ [\mathbf{M}_m] \mathbf{W} \end{bmatrix}.$$

Let us note that we have used Gauss quadrature for computing the different integrals of the linear system. In the cases of double integration with self-interaction, analytical integration (if exist), or Gauss point shift has been used to treat the singularity of the Green kernel.

## IV. APPLICATION

In this section, we propose to validate our formulation through the solving of an academic example in the frequency domain. The test problem is composed of a source coil, a magnetic region, and a conducting region (Fig. 5). A reference solution is obtained with a  $T$ - $\phi$  finite element method (FEM), using a mesh that insures the convergence of the solution at any frequency (from 0 to 1000 Hz). In practice, only 3700 elements are used with integral  $B$ - $J$  formulation that

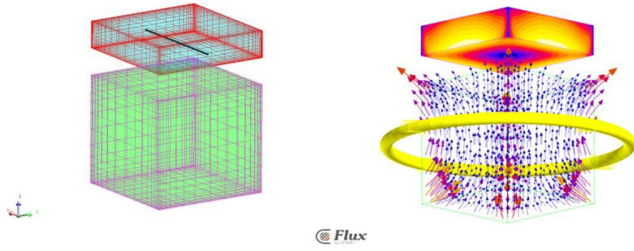


Fig. 5. Geometry and mesh of an academic example containing a coil, a magnetic region ( $\mu r = 100$ ), and a conductive region ( $\sigma = 6 E + 7 S \cdot m^{-1}$ ). On the right, flux density in magnetic region (arrows) and modulus of the current density in conducting region (solution obtained for 400 Hz).

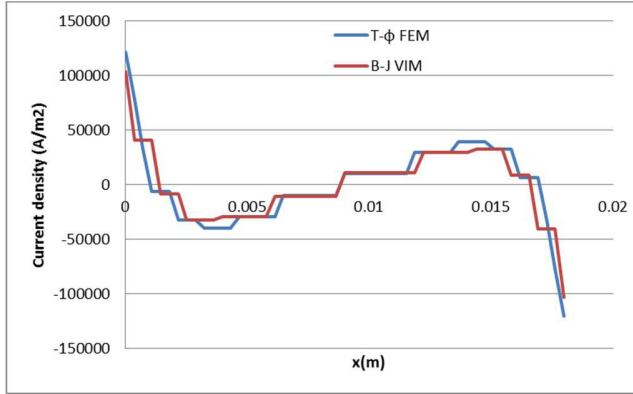


Fig. 6. Imaginary part of the current density (y component) at 1000 Hz along a line (Fig. 5), obtained by FEM and VIM.

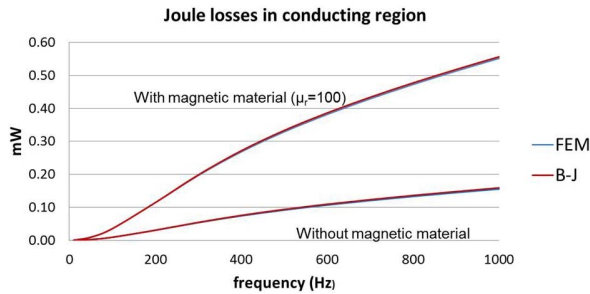


Fig. 7. Eddy current losses obtained for 3-D  $T-\phi$  FEM and  $B-J$  integral volume formulation.

allows to obtain a solution very close to FEM. Fig. 6 compares the current density obtained along a line while Fig. 7 compares eddy currents losses that are very close to the reference values for any frequency (differences of less than 1.5%).

## V. CONCLUSION

The results obtained by the proposed method present a very good quality, and basically show that a light facet element mesh provides a good solution. Ultimately, this approach mixes the accuracy of VIMs, the generality of the Whitney finite element discretization and the efficiency of electrical circuit solvers. The method is particularly well suited to model all electromagnetic devices where air is predominant, in a wide frequency range.

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