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INVARIANT MEASURES FOR CARTESIAN POWERS OF CHACON INFINITE TRANSFORMATION

ÉLISE JANVRESSE, EMMANUEL ROY AND THIERRY DE LA RUE

Abstract. We describe all boundedly finite measures which are invariant by Cartesian powers of an infinite measure preserving version of Chacon transformation. All such ergodic measures are products of so-called *diagonal measures*, which are measures generalizing in some way the measures supported on a graph. Unlike what happens in the finite-measure case, this class of diagonal measures is not reduced to measures supported on a graph arising from powers of the transformation: it also contains some weird invariant measures, whose marginals are singular with respect to the measure invariant by the transformation. We derive from these results that the infinite Chacon transformation has trivial centralizer, and has no nontrivial factor.

At the end of the paper, we prove a result of independent interest, providing sufficient conditions for an infinite measure preserving dynamical system defined on a Cartesian product to decompose into a direct product of two dynamical systems.

Keywords: Chacon infinite measure preserving transformation, rank-one transformation, joinings.

MSC classification: 37A40, 37A05.

1. Chacon infinite transformation

1.1. **Introduction.** The classical Chacon transformation, which is a particular case of a finite measure preserving rank-one transformation, is considered as one of the jewels of ergodic theory [11]. It has been formally described in [8], following ideas introduced by Chacon in 1966. Among other properties, it has been proved to have no non trivial factor, and to commute only with its powers [7]. More generally, it has minimal self-joinings [6]. For a symbolic version of this transformation, Del Junco and Keane [5] have also shown that if x and y are not on the same orbit, and at least one of them is outside a countable set of exceptional points, then (x, y) is generic for the product measure.

Adams, Friedman and Silva introduced in 1997 ([2], Section 2) an infinite measure preserving rank-one transformation which can be seen as the analog of the classical Chacon transformation in infinite measure. They proved that all its Cartesian powers are conservative and ergodic.

This transformation, denoted by T throughout the paper, is the main object of the present work. We recall its construction on $\mathbb{R}_+$ by cutting and stacking in the next section. In particular, we are interested in lifting known results about self-joinings of Chacon transformation to the infinite-measure case. This leads us to study all ergodic measures on $(\mathbb{R}_+)^d$ which are boundedly finite and $T^{\times d}$-invariant: we prove in Theorem 2.3 that all such measures are products of so-called *diagonal*

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measures, which are measures generalizing in some way the measures supported on a graph (see Definition 2.2). These diagonal measures are studied in details in Section 3. Surprisingly, besides measures supported on a graph arising from powers of T, we prove the existence of some weird invariant measures whose marginals are singular with respect to the Lebesgue measure. (It may happen that these marginals take only the values 0 or ∞, which is for example the case for a product measure. But even in such a case, it makes sense to consider their absolute continuity.) However, we prove in Theorem 4.1 that, if a $T^{\times d}$-invariant boundedly finite measure has all its marginals absolutely continuous with respect to the Lebesgue measure, then its ergodic components are products of graph joinings arising from powers of T. We derive from these results in Section 5 that the infinite Chacon transformation has trivial centralizer, and has no nontrivial factor. At the end of the paper, we prove in Annex A a result used in the proof of Theorem 2.3 which can be of independent interest: Theorem A.1 provides sufficient conditions for an infinite measure preserving dynamical system defined on a Cartesian product to decompose into a direct product of two dynamical systems.

Another important motivation for the present work comes from the study of T-point processes, which we briefly introduce now. Given an infinite measure preserving dynamical system $(X, \mathscr{A}, \mu, T)$ where X is a complete separable metric space, we consider the space X^* of boundedly finite, simple counting measures on $(X, \mathscr{A})$, which are measures of the form

$$\xi = \sum_{i \in I} \delta_{x_i}$$

where I is at most countable, $x_i \neq x_j$ whenever $i \neq j$, and

$$\xi(A) = |\{i \in I : x_i \in A\}| < \infty$$

for all bounded measurable $A \subset X$. For such a ξ, we define[1]

$$T_*(\xi) := \sum_{i \in I} \delta_{T(x_i)}.$$

It is not true that for any $\xi \in X^*$, $T_*(\xi) \in X^*$. However, we can consider probability measures on X^* which are T_*-invariant. We define a *T-point process* as a T_*-invariant probability measure on X^* which satisfies $\mathbb{E}[\xi(A)] = \mu(A)$ for each $A \in \mathscr{A}$. The canonical example of a T-point process is given by the Poisson process of intensity μ, providing the so-called *Poisson suspension* associated to $(X, \mathscr{A}, \mu, T)$. We show in [10] that, if T satisfies the properties proved in Theorem 4.1, then any T-point process satisfying some integrability condition is a superposition of Poisson processes.

1.2. **Construction of Chacon infinite transformation.** We define the transformation on $X := \mathbb{R}_+$: In the first step we consider the interval $[0, 1)$, which is cut into three subintervals of equal length. We take the extra interval $[1, 4/3)$ and stack it above the middle piece, and 4 other extra intervals of length $1/3$ which we stack above the rightmost piece. Then we stack all intervals left under right, getting a tower of height $h_1 = 8$. The transformation T maps each point to the point exactly above it in the tower. At this step T is yet undefined on the top level of the tower.

[1] In general, if φ is any measurable map from $(X, \mathscr{A})$ to $(Y, \mathscr{B})$, and if m is a measure on $(X, \mathscr{A})$, we denote by $\varphi_*(m)$ the pushforward image of m by φ.

After step n we have a tower of height h_n, called tower n, made of intervals of length $1/3^n$ which are closed to the left and open to the right. At step $(n+1)$, tower n is cut into three subcolumns of equal width. We add an extra interval of length $1/3^{n+1}$ above the middle subcolumn and $3h_n + 1$ other extra intervals above the third one. We pick the extra intervals successively by taking the leftmost interval of desired length in the unused part of $\mathbb{R}_+$. Then we stack the three subcolumns left under right and get tower $n+1$ of height $h_{n+1} = 2(3h_n + 1)$.

Extra intervals used at step $n+1$ are called $(n+1)$-*spacers*, so that tower $(n+1)$ is the union of tower n with $3h_n + 2$ such $(n+1)$-spacers. The total measure of the added spacers being infinite, we get at the end a transformation T defined on $\mathbb{R}_+$, which preserves the Lebesgue measure μ.

For each $n \geq 1$, we define C_n as the bottom half of tower n: C_n is the union of $h_n/2$ intervals of width $1/3^n$, which contains the whole tower $(n-1)$. Notice that $C_n \subset C_{n+1}$, and that $X = \bigcup_n C_n$. We also define a function t_n on tower n, taking values in $\{1, 2, 3\}$, which indicates for each point whether it belongs to the first, the second, or the third subcolumn of tower n.

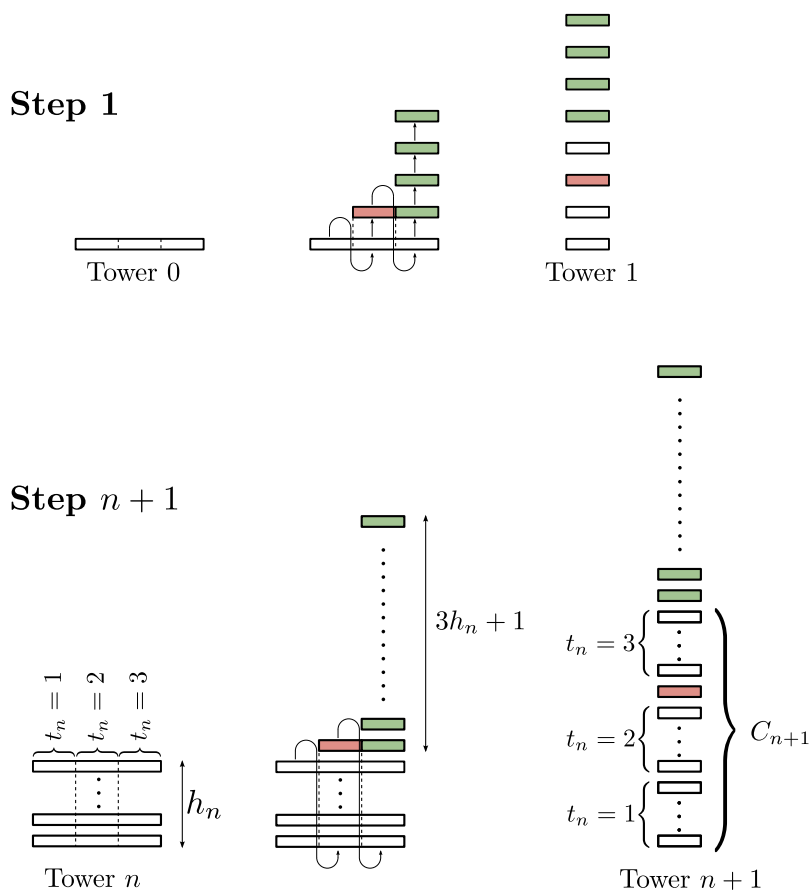

FIGURE 1. Construction of Chacon infinite measure preserving transformation by cutting and stacking

2. Ergodic invariant measures for Cartesian powers of the infinite Chacon transformation

Let $d \geq 1$ be an integer. We consider the d-th Cartesian power of the transformation T:

$$T^{\times d} : X^d \ni (x_1, \dots, x_d) \mapsto (Tx_1, \dots, Tx_d).$$

Definition 2.1. *A measure σ on X^d is said to be* boundedly finite *if $\sigma(A) < \infty$ for all bounded measurable subset $A \subset X^d$.*

Equivalently, σ is boundedly finite if $\sigma(C_n^d) < \infty$ for each n. Obviously, boundedly finite implies σ-finite.

2.1. Products of diagonal measures.

Our purpose in this section is to describe, for each $d \geq 1$, all boundedly finite measures on X^d which are ergodic for the action of $T^{\times d}$. Examples of such measures are given by so-called *graph joinings*: A measure σ on X^d is called a graph joining if there exist some real $\alpha > 0$ and $(d-1)$ μ-preserving transformations $S_2, \dots, S_d$, commuting with T, and such that

$$\sigma(A_1 \times \cdots \times A_d) = \alpha\mu(A_1 \cap S_2^{-1}(A_2) \cap \cdots \cap S_d^{-1}(A_d)).$$

In other words, σ is the pushforward measure of μ by the map $x \mapsto (x, S_2x, \dots, S_dx)$. In the case where the transformations S_j are powers of T, such a graph joining is a particular case of what we call a *diagonal measure*, which we define now.

From the properties of the sets C_n, it follows that $C_n^d \subset C_{n+1}^d$, and that $X^d = \bigcup_n C_n^d$. We call *n-box* a subset of X^d which is a Cartesian product $I_1 \times \cdots \times I_d$, where each I_j is a level of C_n. We call *n-diagonal* a finite family of n-boxes of the form

$$B, T^{\times d}B, \dots, (T^{\times d})^\ell B,$$

which is maximal in the following sense: $(T^{\times d})^{-1}B \not\subset C_n^d$ and $(T^{\times d})^{\ell+1}B \not\subset C_n^d$.

Definition 2.2. *A boundedly finite, $T^{\times d}$-invariant measure σ on X^d is said to be* a diagonal measure *if there exists an integer n_0 such that, for all $n \geq n_0$, $\sigma|_{C_n^d}$ is concentrated on a single n-diagonal.*

Note that, for $d = 1$, there is only one n-diagonal for any n, therefore μ is itself a 1-dimensional diagonal measure. A detailed study of diagonal measures will be presented in Section 3.

Theorem 2.3. *Let $d \geq 1$, and let σ be a nonzero, $T^{\times d}$-invariant, boundedly finite measure on X^d, such that the system $(X^d, \sigma, T^{\times d})$ is ergodic. Then there exists a partition of $\{1, \dots, d\}$ into r subsets $I_1, \dots, I_r$, such that $\sigma = \sigma^{I_1} \otimes \cdots \otimes \sigma^{I_r}$, where σ^{I_j} is a diagonal measure on X^{I_j}.*

If the system $(X^d, \sigma, T^{\times d})$ is totally dissipative, σ is a diagonal measure supported on a single orbit.

We will prove the theorem by induction on d. The following proposition deals with the case $d = 1$.

Proposition 2.4. *The Lebesgue measure μ is, up to a multiplicative constant, the only T-invariant, boundedly finite measure on X.*

Proof. Let σ be a T-invariant σ-finite measure. Then for each n, the intervals which are levels of tower n have the same measure. Since the successive towers exhaust $\mathbb{R}_+$, we get that for each n, all intervals of the form $[j/3^n, (j+1)/3^n)$ for integers

$j \geq 0$ have the same measure σ_n. Obviously $\sigma_{n+1} = \sigma_n/3$. Since σ is boundedly finite, $\sigma_0 < \infty$. Hence $\sigma_n < \infty$ and σ is, up to the multiplicative constant σ_0, equal to the Lebesgue measure. □

Observe that assuming only σ-finiteness for the measure σ is not enough: The counting measure on rational points is σ-finite, T-invariant, but singular with respect to Lebesgue measure. Can we have a counterexample where σ is conservative?

2.2. **Technical lemmas.** In the following, d is an integer, $d \geq 2$.

Lemma 2.5. *Let $G_1 \sqcup G_2 = \{1, \dots, d\}$ be a partition of $\{1, \dots, d\}$ into two disjoint sets, one of which is possibly empty. Let us define a transformation $S : X^d \to X^d$ by*

$$S(y_1, \dots, y_d) := (z_1, \dots, z_d), \text{ where } z_i := \begin{cases} Ty_i & \text{if } i \in G_1, \\ y_i & \text{if } i \in G_2. \end{cases}$$

Let $n \geq 1$, let B be an n-box, and let $x = (x_1, \dots, x_d) \in C_n^d$. If $t_n(x_i) = 1$ for $i \in G_1$ and $t_n(x_i) = 2$ for $i \in G_2$, then

$$x \in B \iff (T^{\times d})^{h_n+1} x \in SB.$$

Similarly, if $t_n(x_i) = 2$ for $i \in G_1$ and $t_n(x_i) = 3$ for $i \in G_2$, then

$$x \in SB \iff (T^{\times d})^{-h_n-1} x \in B.$$

Proof. Let $x = (x_1, \dots, x_d) \in C_n^d$ such that $t_n(x_i) = 1$ for $i \in G_1$ and $t_n(x_i) = 2$ for $i \in G_2$. For each $1 \leq i \leq d$, let L_i be the level of C_n containing x_i. If $i \in G_1$, $T^j x_i$, j ranging from 1 to h_n+1, never goes through an $(n+1)$-spacer, hence $T^{h_n+1} x_i \in TL_i$ (see Figure 1). If $i \in G_2$, $T^j x_i$, j ranging from 1 to $h_n + 1$, goes through exactly one $(n+1)$-spacer, hence $T^{h_n+1} x_i \in L_i$. Hence, $(T^{\times d})^{h_n+1} x \in S(L_1 \times \cdots \times L_d)$. Observe that, since B is an n-box, $B \subset C_n^d$, thus both B and SB are Cartesian products of levels of tower n. We then get

$$\begin{aligned} x \in B &\iff B = L_1 \times \cdots \times L_d \\ &\iff SB = S(L_1 \times \cdots \times L_d) \\ &\iff (T^{\times d})^{h_n+1} x \in SB. \end{aligned}$$

The case $t_n(x_i) = 2$ for $i \in G_1$ and $t_n(x_i) = 3$ for $i \in G_2$ is handled in the same way. □

Lemma 2.6. *Let $n \geq 2$, $x = (x_1, \dots, x_d) \in C_{n-1}^d$ and $\ell \geq n$. If $t_\ell(x_i) \in \{1, 2\}$ for each $1 \leq i \leq d$, then $(T^{\times d})^{h_\ell+1} x \in C_n^d$.*

Proof. Let B_ℓ (respectively B_n) be the ℓ-box (respectively the n-box) containing x. Observe that $B_\ell \subset B_n \subset C_{n-1}^d$ because $x \in C_{n-1}^d$. Applying Lemma 2.5, we get $(T^{\times d})^{h_\ell+1} x \in SB_\ell \subset SB_n$, where S is the transformation of X^d acting as T on coordinates i such that $t_\ell(x_i) = 1$ and acting as Id on other coordinates. Since $B_n \subset C_{n-1}^d$, $SB_n \subset C_n^d$, which ends the proof. □

Definition 2.7. *Let $x = (x_1, \dots, x_d) \in X^d$. For each integer $n \geq 1$, we call n-crossing for x a maximal finite set of consecutive integers $j \in \mathbb{Z}$ such that $(T^{\times d})^j x \in C_n^d$.*

Note that, when j ranges over an n-crossing for x, $(T^{\times d})^j\ x$ successively belongs to the n-boxes constituting an n-diagonal, and that for each $1 \le i \le d$, $t_n(T^j x_i)$ remains constant.

Lemma 2.8. *An n-crossing contains at most $h_n/2$ elements. Two distinct n-crossings for the same x are separated by at least $h_n/2$ integers.*

Proof. The first assertion is obvious since C_n is a tower of height $h_n/2$. Consider the maximum element j of an n-crossing for $x = (x_1, \dots, x_d)$. Then there exists $1 \le i \le d$ such that $T^j(x_i) \in C_n$, but $T^{j+1}(x_i) \notin C_n$. By construction, $T^{j+\ell}(x_i) \notin C_n$ for all $1 \le \ell \le h_n/2$, hence $(T^{\times d})^{j+\ell} x \notin C_n^d$. □

Lemma 2.9. *Let $j \ge 0$ and $n \ge 2$ such that $(T^{\times d})^j x \in C_{n-1}^d$. Then $j, j+1, \dots, j+h_{n-1}/2$ belong to the same n-crossing.*

Proof. For all $1 \le i \le d$, $T^j(x_i) \in C_{n-1}$, hence for all $1 \le \ell \le h_{n-1}/2$, $T^{j+\ell}(x_i)$ belongs to tower $(n-1)$, hence to C_n. □

For $x \in X^d$, let us define $n(x)$ as the smallest integer $n \ge 1$ such that $x \in C_n^d$. Observe that $x \in C_n^d$ for each $n \ge n(x)$. In particular, for each $n \ge n(x)$, 0 belongs to an n-crossing for x, which we call the *first n-crossing for x*. Observe also that the first $(n+1)$-crossing for x contains the first n-crossing for x. Since n-crossings for x are naturally ordered, we refer to the next n-crossing for x after the first one (if it exists) as the *second n-crossing for x*.

Lemma 2.10. *Let $x = (x_1, \dots, x_d) \in X^d$ such that, for any $n \ge n(x)$, there exist infinitely many n-crossings for x contained in $\mathbb{Z}_+$. Then there exist infinitely many integers $n \ge n(x) + 1$ such that the first $(n+1)$-crossing for x also contains the second n-crossing for x. Moreover, for such an integer n, $t_n(x_i) \in \{1, 2\}$ for each $i \in \{1, \dots, d\}$, and for j in the second n-crossing, we have $t_n(T^j x_i) = t_n(x_i) + 1$.*

Proof. Let $m \ge n(x) + 1$, and let $\{s, s+1, \dots, s+r\}$ be the second m-crossing for x. Define $n \ge m$ as the smallest integer such that $(T^{\times d})^j x \in C_{n+1}^d$ for each $0 \le j \le s + r$. Then the n-crossing for x containing zero is distinct from the n-crossing for x containing s, and these two n-crossings are contained in the same $(n+1)$-crossing for x. Therefore the first $(n+1)$-crossing for x contains both the first and the second n-crossings for x.

By Lemma 2.8, the first and the second n-crossings are separated by at least $h_n/2$, hence each coordinate has to leave C_n between them. If we had $t_n(x_i) = 3$ for some i, then $T^j(x_i)$ would also leave C_{n+1} before coming back to C_n, which contradicts the fact that both n-crossings are in the same $(n+1)$-crossing. Hence $t_n(x_i) \in \{1, 2\}$ for each i. Moreover, recall that $n \ge m \ge n(x) + 1$, thus $x \in C_{n-1}^d$. Hence x satisfies the assumptions of Lemma 2.6, with $\ell = n$. Therefore, $(T^{\times d})^{h_n+1} x \in C_n^d$, which proves that $h_n + 1$ belongs to the second n-crossing. At time h_n+1, each coordinate has jumped to the following subcolumn: $t_n(T^{h_n+1} x_i) = t_n(x_i) + 1$. The conclusion follows as t_n is constant over an n-crossing. □

2.3. **Proof of Theorem 2.3, conservative case.** Now we consider an integer $d \ge 2$ such that the statement of Theorem 2.3 (in the conservative case) is valid up to $d - 1$. Let σ be a nonzero measure on X^d, which is boundedly finite, $T^{\times d}$-invariant, and such that the system $(X^d, \sigma, T^{\times d})$ is ergodic and conservative. By

Hopf's ergodic theorem, if $A \subset B \subset X^d$ with $0 < \sigma(B) < \infty$, we have for σ-almost every point $x = (x_1, \dots, x_d) \in X^d$

$$\frac{\sum_{j\in I} \mathbb{1}_A((T^{\times d})^j x)}{\sum_{j\in I} \mathbb{1}_B((T^{\times d})^j x)} \xrightarrow[|I|\to\infty]{} \frac{\sigma(A)}{\sigma(B)}, \tag{1}$$

where the sums in the above expression range over an interval I containing 0.

Recall that $C_n^d \subset C_{n+1}^d$, and that $X^d = \bigcup_n C_n^d$. In particular, for n large enough, $\sigma(C_n^d) > 0$ (and $\sigma(C_n^d) < \infty$ because σ is boundedly finite). By conservativity, this implies that almost every $x \in X^d$ returns infinitely often in C_n^d.

We say that $x \in X^d$ is *typical* if, for all n large enough so that $\sigma(C_n^d) > 0$,

(i) Property (1) holds whenever A is an n-box and B is C_n^d,
(ii) $(T^{\times d})^j x \in C_n^d$ for infinitely many integers $j \geq 0$.

We know that σ-almost every $x \in X^d$ is typical. From now on, we consider a fixed typical point $\overline{x} = (\overline{x}_1, \dots, \overline{x}_d)$ and we will estimate the measure σ along its orbit. By (ii), $\overline{x}$ satisfies the assumption of Lemma 2.10. Hence we are in exactly one of the following two complementary cases.

Case 1: There exists n_1 such that, for each $n \geq n_1$ satisfying the condition given in Lemma 2.10, and for each $1 \leq i \leq d$, $t_n(\overline{x}_i) = t_n(\overline{x}_1)$.

Case 2: There exist a partition of $\{1, \dots, d\}$ into two disjoint nonempty sets

$$\{1, \dots, d\} = G_1 \sqcup G_2,$$

and infinitely many integers n satisfying the condition given in Lemma 2.10 such that, for each $i \in G_1$, $t_n(\overline{x}_i) = 1$, and for each $i \in G_2$, $t_n(\overline{x}_i) = 2$.

Theorem 2.3 will be proved by induction on d once we will have shown the following proposition.

Proposition 2.11. *If Case 1 holds, then the measure σ is a diagonal measure.*

If Case 2 holds, then σ is a product measure of the form

$$\sigma = \sigma_{G_1} \otimes \sigma_{G_2},$$

where, for $i = 1, 2$, σ_{G_i} is a measure on X^{G_i} which is boundedly finite, $T^{\times|G_i|}$-invariant, and such that the system $(X^{G_i}, \sigma_{G_i}, T^{\times|G_i|})$ is ergodic and conservative.

Proof. All n-crossings used in this proof are n-crossings for the fixed typical point $\overline{x}$.

First consider Case 1. Let $m \geq n_1$. We claim that every m-crossing passes through the same m-diagonal as the first m-crossing. Let $J \subset \mathbb{N}$ be an arbitrary m-crossing. Define n as the smallest integer $n \geq m$ such that all integers $j \in \{0, \dots, \sup J\}$ are contained in the same $(n+1)$-crossing. Then n satisfies the conditions of Lemma 2.10: The $(n+1)$-crossing containing 0 contains (at least) two different n-crossings, the one containing 0 and the one containing the m-crossing J. Since we are in Case 1, all coordinates have met the same number of $(n+1)$-spacers between the n-crossing containing 0 and the n-crossing containing J. Hence the n-diagonal where $\overline{x}$ lies is the same as the n-diagonal containing $(T^{\times d})^j \overline{x}$ for $j \in J$. Now we prove the claim by induction on $n - m$. If $n - m = 0$ we have the result. Let $k \geq 0$ such that the claim is true if $n - m \leq k$, and assume that $n - m = k + 1$. We consider the n-crossing containing 0: It may contain several m-crossings, but by the induction hypothesis, all these m-crossings correspond to the same m-diagonal. Now, we know that the n-crossing containing J corresponds to the same n-diagonal

as the n-crossing containing 0, thus all the m-crossings it contains correspond to the same m-diagonal as the m-crossing containing 0. Now, since we have chosen $\overline{x}$ typical, it follows that the m-diagonal containing $\overline{x}$ is the only one which is charged by σ. But this is true for all m large enough, hence σ is a diagonal measure.

Let us turn now to Case 2. Consider the transformation $S : X^d \to X^d$ defined as in Lemma 2.5 by

$$S(y_1, \dots, y_d) = (z_1, \dots, z_d), \text{ where } z_i := \begin{cases} Ty_i \text{ if } i \in G_1, \\ y_i \text{ if } i \in G_2. \end{cases}$$

Let us fix m large enough so that $\sigma(C_{m-1}^d) > 0$. For each m-box B, denote by n_B (respectively n'_B) the number of times the orbit of $\overline{x}$ falls into B along the first n-crossing (respectively the second). We claim that there exists an m-box B such that SB is still an m-box, and $\sigma(B) > 0$. Indeed, it is enough to take any m-box in C_{m-1}^d with positive measure. For such an m-box B, we want now to compare $\sigma(B)$ and $\sigma(SB)$.

Let $n > m$ be a large integer satisfying the condition stated in Case 2. Partition the m-box B into n-boxes: since SB is also an m-box, for each n-box $B' \subset B$, SB' is an n-box contained in SB, and we get in this way all n-boxes contained in SB. Let us fix such an n-box, and apply Lemma 2.5: For each j in the first n-crossing, we have

$$(T^{\times d})^j \overline{x} \in B' \iff (T^{\times d})^{j+h_n+1} \overline{x} \in SB',$$

and in this case, by Lemma 2.8, $j + h_n + 1$ belongs to the second n-crossing. In the same way, for each j in the second n-crossing, we have

$$(T^{\times d})^j \overline{x} \in SB' \iff (T^{\times d})^{j-h_n-1} \overline{x} \in B',$$

and in this case, by Lemma 2.8, $j - h_n - 1$ belongs to the first n-crossing. Summing over all n-boxes B' contained in B, It follows that

$$\text{if both } B \text{ and } SB \text{ are } m\text{-boxes, } n'_{SB} = n_B. \tag{2}$$

Set

$$N := \sum_B n_B, \quad \text{and} \quad N' := \sum_B n'_B,$$

where the two sums range over all m-boxes B. Since we have chosen $\overline{x}$ typical, and since the length of the first n-crossing go to ∞ as $n \to \infty$, we can apply (1) and get, for any m-box B, as $n \to \infty$

$$\frac{n_B}{N} = \frac{\sigma(B)}{\sigma(C_m^d)} + o(1), \quad \text{and} \quad \frac{n_B + n'_B}{N + N'} = \frac{\sigma(B)}{\sigma(C_m^d)} + o(1). \tag{3}$$

Since $N' \geq \sum n'_{SB}$ where the sum ranges over the set $\mathscr{B}_m$ of all m-boxes B such that SB is still an m-box, we get by (2)

$$N' \geq \sum_{B \in \mathscr{B}_m} n_B.$$

Then, applying the left equality in (3) for all $B \in \mathscr{B}_m$, we obtain

$$\frac{N'}{N} \geq \frac{\sum_{B \in \mathscr{B}_m} \sigma(B)}{\sigma(C_m^d)} + o(1).$$

As we know that $\sum_{B\in\mathscr{B}_m}\sigma(B)>0$, it follows that N'/N is larger than some positive constant for n large enough, and we can deduce from (3) that, for all m-box B, we also have as $n\to\infty$

$$\frac{n'_B}{N'}=\frac{\sigma(B)}{\sigma(C_m^d)}+o(1).$$

Let $B\in\mathscr{B}_m$. Applying the above equation for SB and the left equality in (3) for B, and using (2), we get, if $\sigma(B)>0$,

$$\frac{N}{N'}=\frac{\sigma(SB)}{\sigma(B)}+o(1).$$

It follows that the ratio $\sigma(SB)/\sigma(B)$ does not depend on B. We denote it by c_m. Moreover, observe that if $\sigma(B)=0$, we get $n_B/N\to 0$, hence also $n_B/N'=n'_{SB}/N'\to 0$, and $\sigma(SB)=0$. Finally, for all $B\in\mathscr{B}_m$, we have $\sigma(SB)=c_m\sigma(B)$.

Note that any box $B\in\mathscr{B}_m$ is a finite disjoint union of $(m+1)$-boxes in $\mathscr{B}_{m+1}$. This implies that $c_m=c_{m+1}$. Therefore, there exists $c>0$ such that, for all m large enough and all $B\in\mathscr{B}_m$,

$$\sigma(SB)=c\sigma(B).$$

But, as $m\to\infty$, the finite partition of X^d defined by all m-boxes in $\mathscr{B}_m$ increases to the Borel σ-algebra of X^d. Hence, for any measurable subset $B\subset X^d$, the previous equality holds.

A direct application of Theorem A.1 proves that σ has the product form announced in the statement of the proposition. And since σ is boundedly finite, the measures σ_{G_1} and σ_{G_2} are also boundedly finite. □

2.4. **Proof of Theorem 2.3, dissipative case.** We consider now a nonzero measure σ on X^d, which is boundedly finite, $T^{\times d}$-invariant, and such that the system $(X^d,\sigma,T^{\times d})$ is ergodic and totally dissipative. Up to a multiplicative constant, this measure is henceforth of the form

$$\sigma=\sum_{k\in\mathbb{Z}}\delta_{(T^{\times d})^k x}$$

for some $x\in X^d$. And since we assume that σ is boundedly finite, for each n there exist only finitely many n-crossings for x. Now we claim that for n large enough, there is only one n-crossing for x, which will show that σ is a diagonal measure.

Let n be large enough so that $x\in C_{n-1}^d$, and let m be large enough so that all n-crossings for x are contained in a single m-crossing. Assume that there is a second m-crossings for x. Then we consider the smallest integer ℓ such that the first and the second m-crossings are contained in a single $(\ell+1)$-crossing. As in the proof of Lemma 2.10, we have $t_\ell(x_i)\in\{1,2\}$, so we can apply Lemma 2.6. We get $(T^{\times d})^{h_\ell+1}x\in C_n^d$, but $h_\ell+1$ is necessarily in the second m-crossing. This contradicts the fact that all n-crossings for x are contained in a single m-crossing. A similar argument proves that there is no other m-crossing contained in $\mathbb{Z}_-$, and this ends the proof of the theorem.

3. Diagonal measures

The purpose of this section is to provide more information on d-dimensional diagonal measures introduced in Definition 2.2, and which play an important role

in our analysis. We are going to prove that there exist exactly two classes of ergodic diagonal measures:

- graph joinings arising from powers of T, as defined by (8);
- *weird* diagonal measures, whose marginals are singular with respect to μ.

Moreover, we will provide a parametrization of the family of ergodic diagonal measures, and a simple criterion on the parameter to decide to which class a specific measure belongs.

3.1. **Construction of diagonal measures.** Let $d \geq 2$, and let σ be a diagonal measure on X^d. We define $n_0(\sigma)$ as the smallest integer n_0 for which $\sigma(C^d_{n_0-1}) > 0$, and such that, for any $n \geq n_0$, σ gives positive measure to a single n-diagonal, denoted by $D_n(\sigma)$.

Definition 3.1. *Let $n_0 \geq 1$, and for each $n \geq n_0$, let D_n be an n-diagonal. We say that the family $(D_n)_{n\geq n_0}$ is* consistent *if*

- $C^d_{n_0-1} \cap \bigcap_{n\geq n_0} D_n \neq \emptyset$,
- $D_{n+1} \cap C^d_n \subset D_n$ *for each* $n \geq n_0$.

Obviously, the family $(D_n(\sigma))_{n\geq n_0(\sigma)}$ is consistent.

Definition 3.2. *We say that $x \in X^d$ is* seen *by the consistent family of diagonals $(D_n)_{n\geq n_0}$ if, for each $n \geq n_0$, either $x \notin C^d_n$ (which happens only for finitely many integers n), or $x \in D_n$. We say that $x \in X^d$ is* seen *by the diagonal measure σ if it is seen by the family $(D_n(\sigma))_{n\geq n_0(\sigma)}$.*

Observe that, thanks to the first condition in the definition of a consistent family of diagonals, there always exist some $x \in C^d_{n_0-1}$ which is seen by the family. Moreover, if σ is a diagonal measure, then

$$\sigma\Big(\big\{x \in X^d : \ x \text{ is not seen by } \sigma\big\}\Big) = 0. \tag{4}$$

Lemma 3.3. *If x is seen by the consistent family of diagonals $(D_n)_{n\geq n_0}$, then for each $j \in \mathbb{Z}$, $(T^{\times d})^j x$ is also seen by $(D_n)_{n\geq n_0}$.*

Proof. Let $n \geq n_0$. Let $m \geq n$ be large enough so that $(T^{\times d})^i x$ belong to C^d_m for each $0 \leq i \leq j$ (or each $j \leq i \leq 0$). Consider the m-box B containing x: Since x is seen by $(D_n)_{n\geq n_0}$, $B \subset D_m$ and $(T^{\times d})^j B \subset D_m$. Now, observe that an m-box is either contained in an n-box, or it is contained in $X^d \setminus C^d_n$. Hence, either $(T^{\times d})^j x \in (T^{\times d})^j B \subset C^d_n$, or $(T^{\times d})^j x \in (T^{\times d})^j B \subset X^d \setminus C^d_n$. In the former case, $(T^{\times d})^j x \in (T^{\times d})^j B \subset D_n$ because $D_m \cap C^d_n \subset D_n$. This proves that $(T^{\times d})^j x$ is also seen by $(D_n)_{n\geq n_0}$. □

Let $(D_n)_{n\geq n_0}$ be a consistent family of diagonals. We want to describe the relationship between D_n and D_{n+1} for $n \geq n_0$.

Let us consider an n-box B. For each d-tuple $\tau = (\tau(1), \ldots, \tau(d)) \in \{1, 2, 3\}^d$,

$$B(\tau) := \{x \in B : \ t_n(x_i) = \tau(i) \ \forall 1 \leq i \leq d\} \tag{5}$$

is an $(n+1)$-box. Moreover, B is the disjoint union of the 3^d $(n+1)$-boxes $B(\tau)$. Notice that if B and B' are two n-boxes included in the same n-diagonal, then $B(\tau)$ and $B'(\tau)$ are included in the same $(n+1)$-diagonal. Therefore, for each n-diagonal D and each d-tuple $\tau \in \{1, 2, 3\}^d$, we can define the $(n+1)$-diagonal $D(\tau)$ as the unique $(n+1)$-diagonal containing $B(\tau)$ for any n-box B included in D.

Let us fix $x \in C_{n_0-1}^d$ which is seen by $(D_n)_{n \geq n_0}$. For each $n \geq n_0$, since $x \in D_n \cap D_{n+1}$, we get

$$D_{n+1} = D_n(t_n(x_1), \ldots, t_n(x_d)).$$

Moreover, we will see that some values for the d-tuple $(t_n(x_1), \ldots, t_n(x_d))$ are forbidden (see Figure 2). As a matter of fact, assume $\{1, 2\} = \{t_n(x_i) : 1 \leq i \leq d\}$. We can apply Lemma 2.5, and observe that the transformation S used in this lemma acts as T on some coordinates and as Id on others. Therefore, x and $(T^{\times d})^{h_n+1}x$ belong to two different n-diagonals, which is impossible by Lemma 3.3. By a similar argument, we prove that the case $\{2, 3\} = \{t_n(x_i) : 1 \leq i \leq d\}$ is also impossible. Eventually, only two cases can arise:

Corner case: $\{1, 3\} \subset \{t_n(x_i) : 1 \leq i \leq d\}$; then the first $(n+1)$-crossing for x contains only one n-crossing for x.

Central case: $t_n(x_1) = t_n(x_2) = \cdots = t_n(x_d)$; then the first $(n+1)$-crossing for x contains three consecutive n-crossings for x, and $D_{n+1} = D_n(1, \ldots, 1) = D_n(2, \ldots, 2) = D_n(3, \ldots, 3)$.

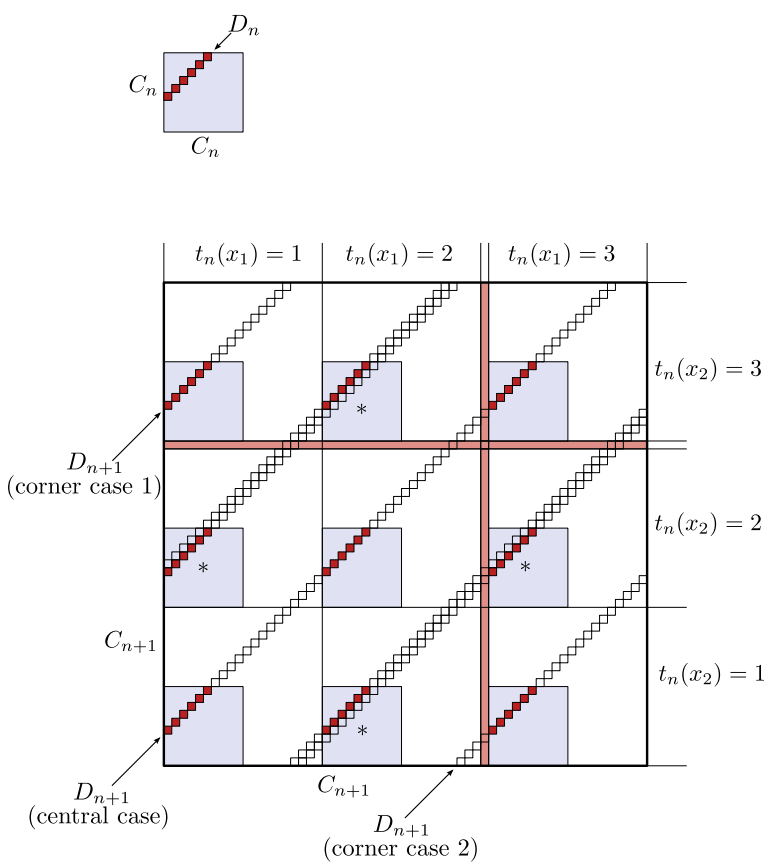

FIGURE 2. Relationship between D_n and D_{n+1} in the case $d = 2$. The 4 positions marked with $*$ are impossible because the corresponding $(n+1)$-diagonal meets another n-diagonal.

It follows from the above analysis that the diagonals D_n, $n \geq n_0$, are completely determined by the knowledge of D_{n_0} and a family of parameters $(\tau_n)_{n\geq n_0}$, where each $\tau_n = (\tau_n(i), 1 \leq i \leq d)$ is a d-tuple in $\{1,2,3\}^d$, satisfying either $\{1,3\} \subset \{\tau_n(i) : \ 1 \leq i \leq d\}$ (corner case), or $\tau_n(1) = \cdots = \tau_n(d)$ (central case).

Lemma 3.4. *If σ is a diagonal measure, and if $(X^d, T^{\times d}, \sigma)$ is conservative, then there are infinitely many integers n such that the transition from $D_n(\sigma)$ to $D_{n+1}(\sigma)$ corresponds to the central case:*

$$D_{n+1}(\sigma) = D_n(\sigma)(1, \ldots, 1).$$

Proof. Since $(X^d, T^{\times d}, \sigma)$ is conservative, for σ-almost all x, for any $n \geq n(x)$, there exist infinitely many n-crossings for x in $\mathbb{Z}_+$. Moreover, σ-almost all x is seen by σ. Applying Lemma 2.10 to such an x, we get that there are infinitely many integers n for which the corner case does not occur, hence such that the transition from $D_n(\sigma)$ to $D_{n+1}(\sigma)$ corresponds to the central case. □

Lemma 3.5. *Let $(\tau_m)_{m\geq m_0}$ be a sequence of d-tuples in $\{1,2,3\}^d$. We define a decreasing sequence of m-boxes by choosing an arbitrary m_0-box B_{m_0} and setting inductively $B_{m+1} := B_m(\tau_m)$. Then*

$$\bigcap_{m\geq m_0} B_m \neq \emptyset$$

if and only if

(6) *for all $1 \leq i \leq d$, there exist infinitely many integers m with $\tau_m(i) \in \{1,2\}$.*

Proof. Recall that the levels of each tower in the construction of T are intervals which are closed to the left and open to the right. If we have a decreasing sequence (I_m) of intervals, where I_m is a level of tower m, then

$$\bigcap_m I_m = \begin{cases} \emptyset, & \text{if } I_{m+1} \text{ is the rightmost subinterval of } I_m \text{ for each large enough } m, \\ \text{a singleton, otherwise.} \end{cases}$$

Since $\tau_m(i)$ indicates the subinterval chosen at step m for the coordinate i, the conclusion follows. □

Lemma 3.6. *Let $n_0 \geq 2$. Let D_{n_0} be an n_0-diagonal such that $D_{n_0} \cap C^d_{n_0-1} \neq \emptyset$. Let $(\tau_n)_{n\geq n_0}$ be a sequence of d-tuples in $\{1,2,3\}^d$ satisfying either $\{1,3\} \subset \{\tau_n(i) : 1 \leq i \leq d\}$, or $\tau_n(1) = \cdots = \tau_n(d)$. Then the inductive relation $D_{n+1} := D_n(\tau_n)$, $n \geq n_0$ defines a consistent family of diagonals if and only if Property (6) holds.*

Proof. Applying Lemma 3.5, the first condition in the definition of a consistent family of diagonals is equivalent to Property (6). The second condition comes from the restrictions made on the d-tuples. □

Proposition 3.7. *Let $n_0 \geq 2$. Let $(D_n)_{n\geq n_0}$ be a consistent family of diagonals. Then there exists a diagonal measure σ, unique up to a multiplicative constant, with $n_0(\sigma) \leq n_0$, and for each $n \geq n_0$, $D_n(\sigma) = D_n$. This measure satisfies $\sigma(X^d) = \infty$.*

If the transition from D_n to D_{n+1} corresponds infinitely often to the central case, then the system $(X^d, T^{\times d}, \sigma)$ is conservative ergodic. Otherwise, it is ergodic and totally dissipative.

Proof. We first define σ on the ring

$$\mathscr{R} := \{B \subset X^d : \ \exists n \geq 1, \ B \text{ is a finite union of } n\text{-boxes}\}.$$

Since we want to determine σ up to a multiplicative constant, we can arbitrarily set $\sigma(C_{n_0}^d) = \sigma(D_{n_0}) := 1$. As we want σ to be invariant under the action of $T^{\times d}$, this fixes the measure of each n_0-box: For each n_0-box B,

$$\sigma(B) := \begin{cases} \dfrac{1}{\text{number of } n_0\text{-boxes in } D_{n_0}} & \text{if } B \subset D_{n_0}, \\ 0 & \text{otherwise.} \end{cases}$$

Now assume that we have already defined $\sigma(B)$ for each n-box, for some $n \geq n_0$, and that we have some constant $\alpha_n > 0$ such that, for any n-box B,

$$\sigma(B) = \begin{cases} \alpha_n & \text{if } B \subset D_n, \\ 0 & \text{otherwise.} \end{cases}$$

We set $\sigma(B') := 0$ for any $(n+1)$-box $B' \not\subset D_{n+1}$, and it remains to define the measure of $(n+1)$-boxes included in D_{n+1}. These boxes must have the same measure, which we denote by α_{n+1}.

- Either the transition from D_n to D_{n+1} corresponds to the corner case. Then each n-box contained in D_n meets only one $(n+1)$-box contained in D_{n+1}, and we set $\alpha_{n+1} := \alpha_n$.
- Or the transition from D_n to D_{n+1} corresponds to the central case. Then each n-box contained in D_n meets three $(n+1)$-boxes contained in D_{n+1}, and we set $\alpha_{n+1} := \alpha_n/3$.

For any $R \in \mathscr{R}$ which is a finite union of n-boxes, we can now define $\sigma(R)$ as the sum of the measures of the n-boxes included in R. At this point, σ is now defined as a finitely additive set function on $\mathscr{R}$.

It remains now to prove that σ can be extended to a measure on the Borel σ-algebra of X^d, which is the σ-algebra generated by $\mathscr{R}$. Using Theorems F p. 39 and A p. 54 (Caratheodory's extension theorem) in [9], we only have to prove the following.

Claim. *If $(R_k)_{k\geq 1}$ is a decreasing sequence in $\mathscr{R}$ such that $\lim_{k\to\infty} \downarrow \sigma(R_k) > 0$, then $\bigcap_k R_k \neq \emptyset$.*

Having fixed such a sequence (R_k), we say that an m-box B is *persistent* if

$$\lim_{k\to\infty} \downarrow \sigma(R_k \cap B) > 0.$$

We are going to construct inductively a decreasing family $(B_m)_{m\geq m_0}$ where B_m is a persistent m-box and

$$\emptyset \neq \bigcap_{m\geq m_0} B_m \subset \bigcap_k R_k.$$

We first consider the case where the transition from D_n to D_{n+1} corresponds infinitely often to the central case. Choose k_0 large enough so that

$$\sigma(R_{k_0}) < \frac{3}{2} \lim_{k\to\infty} \downarrow \sigma(R_k).$$

Then there exists m_0 such that R_{k_0} is a finite union of m_0-boxes, and (choosing a larger m_0 if necessary), the transition from D_{m_0} to D_{m_0+1} corresponds to the central case. Let B be a persistent m_0-box. Then σ on B is concentrated on

the (m_0+1)-boxes $B(1,\dots,1)$, $B(2,\dots,2)$ and $B(3,\dots,3)$. If $B(1,\dots,1)$ is not persistent, we get

$$\begin{aligned}0 < \lim_{k\to\infty} \downarrow \sigma(R_k \cap B) &= \lim_{k\to\infty} \downarrow \sigma(R_k \cap B(2,\dots,2)) + \lim_{k\to\infty} \downarrow \sigma(R_k \cap B(3,\dots,3)) \\ &\le \sigma(R_{k_0} \cap B(2,\dots,2)) + \sigma(R_{k_0} \cap B(3,\dots,3)) \\ &\le \sigma(B(2,\dots,2)) + \sigma(B(3,\dots,3)) \\ &= \frac{2}{3}\sigma(B) = \frac{2}{3}\sigma(R_{k_0} \cap B).\end{aligned}$$

Therefore, there exists some persistent m_0-box B_{m_0} such that $B_{m_0+1} := B_{m_0}(1,\dots,1)$ is also persistent. Indeed, otherwise we would have

$$\begin{aligned}\sigma(R_{k_0}) &\ge \sum_{B \text{persistent } m_0-box} \sigma(R_{k_0} \cap B) \\ &\ge \frac{3}{2} \sum_{B \text{persistent } m_0-box} \lim_{k\to\infty} \downarrow \sigma(R_k \cap B) \\ &= \frac{3}{2} \lim_{k\to\infty} \downarrow \sigma(R_k),\end{aligned}$$

which would contradict the definition of k_0.

Assume that we have already defined B_{m_i} and $B_{m_i+1} = B_{m_i}(1,\dots,1)$ for some $i \ge 0$. Then we choose k_{i+1} large enough so that

$$\sigma(R_{k_{i+1}} \cap B_{m_i+1}) < \frac{3}{2} \lim_{k\to\infty} \downarrow \sigma(R_k \cap B_{m_i+1}).$$

We choose $m_{i+1} > m_i + 1$ such that $R_{k_{i+1}}$ is a finite union of m_{i+1}-boxes, and the transition from $D_{m_{i+1}}$ to $D_{m_{i+1}+1}$ corresponds to the central case. Then the same argument as above, replacing R_k by $R_k \cap B_{m_i+1}$, proves that there exists a persistent m_{i+1}-box $B_{m_{i+1}} \subset B_{m_i+1}$ such that $B_{m_{i+1}+1} := B_{m_{i+1}}(1,\dots,1)$ is also persistent.

Now we can complete in a unique way our sequence to get a decreasing sequence $(B_m)_{m\ge m_0}$ of persistent boxes. Since we have $B_{m_i+1} = B_{m_i}(1,\dots,1)$ for each $i \ge 0$, Lemma 3.5 ensures that

$$\bigcap_m B_m \neq \emptyset.$$

It only remains to prove that $\bigcap_m B_m \subset \bigcap_k R_k$. Indeed, let us fix k and let $\overline{m}$ be such that R_k is a finite union of $\overline{m}$-boxes. In particular, R_k contains all persistent $\overline{m}$-boxes, which implies

$$\bigcap_m B_m \subset B_{\overline{m}} \subset R_k.$$

Now we consider the case where there exists $m_0 \ge n_0$ such that, for $n \ge m_0$, the transition from D_n to D_{n+1} always correspond to the corner case. That is, there exists a family $(\tau_n)_{n\ge m_0}$ of d-tuples in $\{1,2,3\}$, with $\{1,3\} \subset \{\tau_n(i), 1 \le i \le d\}$ for each $n \ge m_0$, such that $D_{n+1} = D_n(\tau_n)$. By Lemma 3.6, property (6) holds for $(\tau_n)_{n\ge m_0}$. We will now construct the family $(B_m)_{m\ge m_0}$ of m-boxes satisfying the required conditions. Start with B_{m_0} which is a persistent m_0-box (such a box always exists). Since the transition from D_{m_0} to D_{m_0+1} corresponds to the corner case, there is only one (m_0+1)-box contained in $D_{m_0+1} \cap B_{m_0}$, and this box is precisely $B_{m_0}(\tau_{m_0})$. Therefore this box is itself persistent, and defining inductively $B_{m+1} := B_m(\tau_m)$ gives a decreasing family of persistent boxes. By Lemma 3.5,

$bigcap_{m\geq m_0} B_m \neq \emptyset$. We prove as in the preceding case that $\bigcap_m B_m \subset \bigcap_k R_k$. This ends the proof of the claim.

This proves that σ can be extended to a $T^{\times d}$-invariant measure, whose restriction to each C_n^d, $n \geq n_0$, is by construction concentrated on the single diagonal D_n. And since $C_{n_0-1}^d \cap D_{n_0} \neq \emptyset$, we get $n_0(\sigma) \leq n_0$. If B is an n-box, then $(T^{\times d})^{h_n/2} B \subset C_{n+1}^d$. Moreover, by Lemma 2.8, $(T^{\times d})^{h_n/2} B \not\subset C_n^d$. It follows that $(T^{\times d})^{h_n/2} D_n \subset C_{n+1}^d \setminus C_n^d$. But $\sigma\big((T^{\times d})^{h_n/2} D_n\big) = \sigma(D_n)$, hence $\sigma(C_{n+1}^d) \geq 2\sigma(C_n^d)$. We conclude that $\sigma(X^d) = \infty$.

Now we want to show the ergodicity of the system $(X^d, T^{\times d}, \sigma)$. Let $A \subset X^d$ be a $T^{\times d}$-invariant measurable set, with $\sigma(A) \neq 0$. Let n be such that $\sigma(A \cap C_n^d) > 0$. Given $\varepsilon > 0$, we can find $m > n$ large enough such that there exists $\tilde{A}$, a finite union of m-boxes, with

$$\sigma\left((A \triangle \tilde{A}) \cap C_n^d\right) < \varepsilon\, \sigma(A \cap C_n^d).$$

Let B be an m-box in D_m, and set $s_m := \sigma(A \cap B)$: By invariance of A under the action of $T^{\times d}$, s_m does not depend on the choice of B. We have

$$\sigma(A \cap C_n^d) = \sum_{\substack{B \ m\text{-box in } D_m \\ B \subset C_n^d}} \sigma(A \cap B) = s_m \cdot \left|\{B \ m\text{-box}: \ B \subset D_m \cap C_n^d\}\right|.$$

On the other hand, we can write

$$s_m \cdot \left|\{B \ m\text{-box}: \ B \subset D_m \cap C_n^d \setminus \tilde{A}\}\right| \leq \sigma\left((A \triangle \tilde{A}) \cap C_n^d\right) < \varepsilon\, \sigma(A \cap C_n^d).$$

It follows that

$$\frac{\left|\{B \ m\text{-box}: \ B \subset D_m \cap C_n^d \setminus \tilde{A}\}\right|}{\left|\{B \ m\text{-box}: \ B \subset D_m \cap C_n^d\}\right|} < \varepsilon,$$

hence

$$\sigma(\tilde{A} \cap C_n^d) > (1-\varepsilon)\sigma(C_n^d),$$

and finally

$$\sigma(A \cap C_n^d) > (1-2\varepsilon)\sigma(C_n^d).$$

But this holds for any $\varepsilon > 0$, which proves that $\sigma(A \cap C_n^d) = \sigma(C_n^d)$. Again, this holds for any large enough n, thus $\sigma(X^d \setminus A) = 0$, and the system is ergodic.

We can observe that, if the central case occurs infinitely often, the measure α_n of each n-box on D_n decreases to 0 as n goes to infinity, which ensures that σ is continuous. Therefore the conservativity of $(X^d, T^{\times d}, \sigma)$ is a consequence of the ergodicity of this system. On the other hand, if the central case occurs only finitely many times, there exists m_0 such that for each $m \geq m_0$, $\alpha_m = \alpha_{m_0} > 0$. It follows that σ is purely atomic, and by ergodicity of $(X^d, T^{\times d}, \sigma)$, σ is concentrated on a single orbit. □

3.2. **A parametrization of the family of diagonal measures.** If σ is a diagonal measure, by definition of $n_0(\sigma)$, the diagonal $D_{n_0(\sigma)}(\sigma)$ is initial in the sense given by the following definition.

Definition 3.8. *Let $n_0 \geq 1$, and D an n_0-diagonal. We say that D is an* initial *diagonal if*

- *Either there exist at least two (n_0-1)-diagonals which have non-empty intersection with D;*
- *Or D has non-empty intersection with exactly one (n_0-1)-diagonal, but does not intersect $C^d_{n_0-2}$ (with the convention that $C^d_0=\emptyset$).*

In Proposition 3.7, it is clear that $n_0=n_0(\sigma)$ if and only if D_{n_0} is initial.

Now we are able to provide a canonical parametrization of the family of diagonal measures: we consider the set of parameters

$$\mathscr{D} := \Big\{(n_0, D, \tau)\Big\},$$

where

- $n_0 \geq 1$,
- D is an initial n_0-diagonal;
- $\tau = (\tau_n)_{n\geq n_0}$, where for each $n \geq n_0$, $\tau_n \in \{1,2,3\}^d$ and satisfies either $\{1,3\} \subset \{\tau_n(i),\ 1 \leq i \leq d\}$ (corner case), or $\tau_n(i)=1$ for each $1 \leq i \leq d$ (central case);
- Property (6) holds for (τ_n).

To each $(n_0, D, \tau) \in \mathscr{D}$, by Proposition 3.7 we can canonically associate an ergodic diagonal measure $\sigma_{(n_0,D,\tau)}$, setting $\sigma_{(n_0,D,\tau)}(C^d_{n_0}) := 1$. Conversely, any ergodic diagonal measure σ can be written as

$$\sigma = \lambda\, \sigma_{(n_0,D,\tau)}$$

for some $(n_0, D, \tau) \in \mathscr{D}$, where $\lambda := \sigma(C^d_{n_0(\sigma)})$, $n_0 := n_0(\sigma)$, and $D := D_{n_0(\sigma)}(\sigma)$.

Note that, by construction, for each $n \geq 1$, each $(n_0, D, \tau) \in \mathscr{D}$, and each n-box B, we have $\sigma_{(n_0,D,\tau)}(B) \leq 1$. Thus,

$$\forall n \geq 1,\ \forall (n_0, D, \tau) \in \mathscr{D},\ \sigma_{(n_0,D,\tau)}(C^d_n) \leq \left(\frac{h_n}{2}\right)^d. \tag{7}$$

3.3. **Identification of graph joinings.**

Proposition 3.9. *Graph joinings of the form*

$$\sigma(A_1 \times \cdots \times A_d) = \alpha\, \mu(A_1 \cap T^{-k_2}(A_2) \cap \cdots \cap T^{-k_d}(A_d)) \tag{8}$$

for some real $\alpha > 0$ and some integers $k_2, \ldots, k_d$, are the diagonal measures $\sigma_{(n_0,D,\tau)}$ for which there exists $n_1 \geq n_0$ such that, for $n \geq n_1$, $\tau_n(i) = 1$ for each $1 \leq i \leq d$.

Proof. Let $\sigma := \sigma_{(n_0,D,\tau)}$, and assume that for $n \geq n_1$, $\tau_n(i) = 1$ for each $1 \leq i \leq d$. Consider $n \geq n_1$, and let B be an n-box in $D_n(\sigma)$. Then B is of the form $B_1 \times T^{k_2}B_1 \times \cdots \times T^{k_d}B_1$ for some level B_1 of tower n and some integers $k_2, \ldots, k_d$. Moreover, $k_2, \ldots, k_d$ do not depend on the choice of B in $D_n(\sigma)$. Let us also write B as $T^{\ell_1}F_n \times \cdots \times T^{\ell_d}F_n$, where F_n is the bottom level of tower n. Then $k_i = \ell_i - \ell_1$ for each $2 \leq i \leq d$. Now, recalling notation (5), consider $B(1,\ldots,1)$, which is an $(n+1)$-box in $D_{n+1}(\sigma)$. Then $B(1,\ldots,1) = T^{\ell_1}F_{n+1} \times \cdots \times T^{\ell_d}F_{n+1}$, thus this $(n+1)$-box is of the form $B'_1 \times T^{k_2}B'_1 \times \cdots \times T^{k_d}B'_1$, for some level B'_1 in tower $(n+1)$, and the *same* integers $k_2, \ldots, k_d$ as above. By induction, this is true for any n-box in $D_n(\sigma)$ for any $n \geq n_1$. As in the proof of Proposition 3.7, let us denote by α_n the measure of each n-box in $D_n(\sigma)$. By hypothesis, all transitions from n_1 correspond to the central case hence for each $n \geq n_1$, $\alpha_n = \alpha_{n_1}/3^{n-n_1}$.

Fix $n \geq n_1$ and consider some n-box B, of the form $B = A_1 \times A_2 \times \cdots \times A_d$ for sets A_i which are levels of tower n. We have

$$\sigma(B) = \begin{cases} \alpha_{n_1}/3^{n-n_1} & \text{if } A_1 \cap T^{-k_2}(A_2) \cap \cdots \cap T^{-k_d}(A_d) = A_1 \\ 0 & \text{otherwise, that is if } A_1 \cap T^{-k_2}(A_2) \cap \cdots \cap T^{-k_d}(A_d) = \emptyset. \end{cases}$$

Observing that $\mu(A_1) = \mu(F_{n_1})/3^{n-n_1}$, we get

$$\sigma(A_1 \times A_2 \times \cdots \times A_d) = \alpha\mu(A_1 \cap T^{-k_2}(A_2) \cap \cdots \cap T^{-k_d}(A_d)),$$

with $\alpha := \alpha_{n_1}/\mu(F_{n_1})$. Finally, the above formula remains valid if the sets A_i are finite unions of levels of tower n, then for any choice of these sets.

Conversely, assume that σ is a graph joining of the form given by (8). Observe that if A is a level of C_n, and if $|k| \leq h_n$, then

$$A \cap T^k A = \begin{cases} A & \text{if } k = 0, \\ \emptyset & \text{otherwise.} \end{cases}$$

Take n large enough so that for all $1 \leq i \leq d$, $h_n/2 > |k_i|$. Let B be an n-box, which can always be written as $B = A \times T^{k'_2} A_2 \times \cdots \times T^{k'_d} A_d$ for some level A of C_n and some integers $k'_2, \ldots, k'_d$ satisfying $|k'_i| \leq h_n/2$. Then

$$\sigma(B) = \alpha\mu(A \cap T^{k'2-k_2}(A) \cap \cdots \cap T^{k'd-k_d}(A)),$$

which is positive if and only if for each $1 \leq i \leq d$, $k_i = k'_i$. Hence $\sigma|_{C_n^d}$ is concentrated on a single diagonal, which is constituted by n-boxes of the form $A \times T^{k_2}(A) \times \cdots \times T^{k_d}(A)$. This already proves that σ is a diagonal measure. Moreover, if B is such an n-box, then $B(1, \ldots, 1)$ is an $(n+1)$-box of the same form, hence the transition from n to $n+1$ corresponds to the central case. □

Definition 3.10. *We say that $x_1 \in X$ is* compatible *with the diagonal measure $\sigma_{(n_0,D,\tau)}$ if there exists $(x_2, \ldots, x_d) \in X^{d-1}$ such that $(x_1, \ldots, x_d)$ is seen by $\sigma_{(n_0,D,\tau)}$.*

Proposition 3.11. *Let $\sigma_{(n_0,D,\tau)}$ be a diagonal measure. If the set of $x_1 \in X$ which are compatible with $\sigma_{(n_0,D,\tau)}$ is of positive measure μ, then $\sigma_{(n_0,D,\tau)}$ is a graph joining arising from powers of T, as defined by (8).*

Proof. Let x_1 be compatible with the diagonal measure $\sigma := \sigma_{(n_0,D,\tau)}$, and let $(x_2, \ldots, x_d) \in X^{d-1}$ be such that $(x_1, \ldots, x_d)$ is seen by σ. Let $n \geq n_0$ be large enough so that $(x_1, \ldots, x_d) \in C_n^d$. Then

$$(x_1, \ldots, x_d) \in D_{n+1}(\sigma) = D_n(\sigma)(\tau_n(1), \ldots, \tau_n(d)).$$

If we further assume that $(\tau_n(1), \ldots, \tau_n(d)) \neq (1, \ldots, 1)$, then the transition from $D_n(\sigma)$ to $D_{n+1}(\sigma)$ corresponds to the corner case, and there is only one occurrence of $D_n(\sigma)$ inside $D_{n+1}(\sigma)$. Since also $(x_1, \ldots, x_d) \in D_n(\sigma)$, it follows that $t_n(x_1) = \tau_n(1)$. Therefore, if there exist infinitely many integers n such that $(\tau_n(1), \ldots, \tau_n(d)) \neq (1, \ldots, 1)$, then the compatibility of x_1 with the diagonal measure σ forces the value of $t_n(x_1)$ for infinitely many integers n. This implies that x_1 belongs to a fixed set which is μ-negligible.

To conclude the proof, it is enough to apply Proposition 3.9. □

Remark 3.12. *Taking $(n_0, D, \tau) \in \mathscr{D}$ for which the corner case occurs infinitely often, and considering the corresponding diagonal measure $\sigma_{(n_0,D,\tau)}$, we see that there exist ergodic diagonal measures which are not graph joinings. By Proposition 3.11, these measures are concentrated on sets $N_1 \times N_2 \times \cdots \times N_d$, where each N_i, $1 \le i \le d$, is a μ-negligible set. We call such a measure a* weird *measure. It is conservative whenever the central case occurs infinitely often.*

4. Ergodic decomposition with absolute continuity of the marginals

Let σ be a boundedly finite measure on X^d which is $T^{\times d}$-invariant. We recall that by Hopf's decomposition, σ can be written as $\sigma = \sigma_{\text{diss}} + \sigma_{\text{cons}}$, where σ_{diss} and σ_{cons} are mutually singular, boundedly finite, $T^{\times d}$-invariant, the system $(X^d, \sigma_{\text{diss}}, T^{\times d})$ is totally dissipative, and $(X^d, \sigma_{\text{cons}}, T^{\times d})$ is conservative. The conservative part σ_{cons} admits an *ergodic decomposition* (see [1], Section 2.2.9). By Theorem 2.3, its ergodic components are all products of diagonal measures. Note that the dissipative part σ_{diss} can also be written as

$$\sigma_{\text{diss}} = \int_W \omega_x \, d\sigma_{\text{diss}}(x),$$

where W is a wandering set satisfying $\bigsqcup_{k\in\mathbb{Z}} (T^{\times d})^k W = X^d \mod \sigma_{\text{diss}}$, and ω_x is defined by

$$\omega_x := \sum_{k\in\mathbb{Z}} \delta_{(T^{\times d})^k x}.$$

By Theorem 2.3, these measures ω_x are (weird) d-dimensional diagonal measures.

Observe that, even if the σ-algebra generated by one coordinate is not σ-finite, we can always define the marginals of σ as the respective pushforward measures of σ by the projections on each coordinate. Note that these marginal measures may take only the values 0 or ∞, which is for example the case when σ is the product measure $\mu^{\otimes d}$ with $d \ge 2$. But even in such a case, it makes sense to consider the absolute continuity of the marginal with respect to μ.

The purpose of the present section is to show that, with an assumption of absolute continuity of the marginals of σ, no weird measure can appear in the decomposition of σ. More precisely, we will prove the following theorem.

Theorem 4.1. *Let σ be a boundedly finite, $T^{\times d}$-invariant measure on X^d. Assume that all its marginals are absolutely continuous with respect to μ. Then the system $(X^d, \sigma, T^{\times d})$ is conservative, and the ergodic components of σ are products of graph joinings arising from powers of T.*

4.1. Contribution of d-dimensional diagonal measures.

In this section we consider the contribution of d-dimensional diagonal measures to the decomposition of σ, which takes into account all d-dimensional diagonal measures appearing as ergodic components of σ_{cons}, and all γ_x appearing in the decomposition of σ_{diss}. More precisely, using the parametrization of ergodic diagonal measures presented in Section 3.2, this contributions takes the form

$$\sigma_\Delta = \int_{\mathscr{D}} \sigma_{(n_0,D,\tau)} \, dm(n_0, D, \tau), \tag{9}$$

where m is a σ-finite measure on $\mathscr{D}$ (this measure takes into account the multiplicative constants up to which the diagonal measures are defined).

Proposition 4.2. *Let σ be a boundedly finite, $T^{\times d}$-invariant measure on X^d, whose first marginal is absolutely continuous with respect to μ. Then the system $(X^d, \sigma, T^{\times d})$ is conservative, and the ergodic components of σ which are d-dimensional diagonal measures are graph joinings arising from powers of T.*

Before proving the proposition, we need some additional technical tools.

Definition 4.3. *We say that $(x_2, \dots, x_d) \in X^{d-1}$ is* compatible *with $x_1 \in X$ if there exists a diagonal measure $\sigma_{(n_0,D,\tau)}$ such that $(x_1, \dots, x_d)$ is seen by $\sigma_{(n_0,D,\tau)}$. We set*

$$\overline{x_1} := \{(x_2, \dots, x_d) \in X^{d-1} : (x_2, \dots, x_d) \text{ is compatible with } x_1\}.$$

Remark 4.4. *It follows from the definition of $\mathscr{D}$ and from Proposition 3.7 that $\overline{x_1}$ is the set of $(x_2, \dots, x_d) \in X^{d-1}$ satisfying, for all large enough n:*

- *either $t_n(x_i) = t_n(x_1)$ for each $2 \le i \le d$,*
- *or $\{1, 3\} \subset \{t_n(x_i) : 1 \le i \le d\}$.*

Lemma 4.5. *For each $x_1 \in X$ and each $(x_2, \dots, x_d) \in \overline{x_1}$, there exists a unique $(n_0, D, \tau) \in \mathscr{D}$ such that $(x_1, \dots, x_d)$ is seen by $\sigma_{(n_0,D,\tau)}$.*

Proof. If $(x_1, \dots, x_d)$ is seen by two diagonal measures σ and σ', then for all n large enough, $(x_1, \dots, x_d) \in D_n(\sigma)$ and $(x_1, \dots, x_d) \in D_n(\sigma')$. It follows that $D_n(\sigma) = D_n(\sigma')$ for all large enough n, hence σ and σ' are proportional. □

The preceding lemma enables us to define, for any $x_1 \in X$, the measurable function $\varphi_{x_1} : \overline{x_1} \to \mathscr{D}$, by

$$\varphi_{x_1}(x_2, \dots, x_d) := \text{ the unique } (n_0, D, \tau) \in \mathscr{D} \text{ such that } (x_1, \dots, x_d) \text{ is seen by } \sigma_{(n_0,D,\tau)}.$$

Obviously, for any $(x_2, \dots, x_d) \in \overline{x_1}$,

$$x_1 \text{ is compatible with the diagonal measure } \sigma_{\varphi_{x_1}(x_2,\dots,x_d)}. \tag{10}$$

Lemma 4.6. *For each $x_1 \in X$, we have $\overline{Tx_1} = T^{\times(d-1)}(\overline{x_1})$. Moreover,*

$$\varphi_{Tx_1} \circ T^{\times(d-1)} = \varphi_{x_1}.$$

Proof. This follows from the fact that $(x_1, \dots, x_d)$ is seen by the diagonal measure $\sigma_{(n_0,D,\tau)}$ if and only if $(Tx_1, \dots, Tx_d)$ is seen by $\sigma_{(n_0,D,\tau)}$. □

Remark 4.7. *Using Remark 4.4 and the fact that $t_n(x_1) = t_n(Tx_1)$ if n is large enough, we also get $\overline{Tx_1} = \overline{x_1}$.*

Proof of Proposition 4.2. Since σ_Δ is absolutely continuous with respect to σ, its first marginal is absolutely continuous with respect to μ. Therefore, we can disintegrate σ_Δ with respect to μ (see e.g. [3], Theorem 1): There exists a family $(\nu_{x_1})_{x_1 \in X}$ of σ-finite measures on X^{d-1}, such that for each measurable $B \subset X^{d-1}$, $x_1 \mapsto \nu_{x_1}(B)$ is measurable, and for each measurable $A \subset X$,

$$\sigma_\Delta(A \times B) = \int_A \nu_{x_1}(B) d\mu(x_1). \tag{11}$$

Let us consider the following measurable subset of X^d:

$$C := \{(x_1, \dots, x_d) : \ (x_2, \dots, x_d) \text{ is compatible with } x_1\} = \bigcup_{x_1 \in X} \{x_1\} \times \overline{x_1}.$$

Recalling (4), for any diagonal measure $\sigma_{(n_0,D,\tau)}$, we have

$$\sigma_{(n_0,D,\tau)}(X^d \setminus C) = 0.$$

Thus, by (9), $\sigma_\Delta(X^d \setminus C) = 0$. It follows that, for μ-almost all $x_1 \in X$, ν_{x_1} is concentrated on $\overline{x_1}$. This enables us to define, for μ-almost all $x_1 \in X$, the measure γ_{x_1} on $\mathscr{D}$ as the pushforward of ν_{x_1} by the map φ_{x_1} introduced after Lemma 4.5. By (10), γ_{x_1} is concentrated on the set of $(n_0, D, \tau) \in \mathscr{D}$ such that x_1 is compatible with $\sigma_{(n_0,D,\tau)}$.

According to Lemma 4.6, the following diagram commutes:

$$\begin{tikzcd}
\overline{x_1} \arrow[rr, "T^{\times(d-1)}"] \arrow[dr, "\varphi_{x_1}"'] & & \overline{Tx_1} = \overline{x_1} \arrow[dl, "\varphi_{Tx_1}"] \\
& \mathscr{D} &
\end{tikzcd}$$

Using the invariance of σ_Δ by $T^{\times d}$ and the invariance of μ by T, we get that $(T^{\times(d-1)})_*(\nu_{x_1}) = \nu_{Tx_1}$ for μ-almost all $x_1 \in X$. Therefore, for μ-almost all $x_1 \in X$, we obtain

$$\begin{aligned}
\gamma_{x_1} &= (\varphi_{x_1})_*(\nu_{x_1}) \\
&= (\varphi_{Tx_1})_*(T^{\times(d-1)})_*(\nu_{x_1}) \\
&= (\varphi_{Tx_1})_*(\nu_{Tx_1}) \\
&= \gamma_{Tx_1}.
\end{aligned}$$

By ergodicity of T, it follows that there exists some measure γ on $\mathscr{D}$ such that $\gamma_{x_1} = \gamma$ for μ-almost all $x_1 \in X$. Moreover, γ is concentrated on the set of parameters (n_0, D, τ) such that μ-almost every x_1 is compatible with $\sigma_{(n_0,D,\tau)}$. From Proposition 3.11, it follows that γ is concentrated on the set of parameters corresponding to graph joinings arising from powers of T. For $k_2, \dots, k_d \in \mathbb{Z}$, let us denote by $\pi(k_2, \dots, k_d) \in \mathscr{D}$ the parameter corresponding to the graph joining given by (8). Then there exist non-negative coefficients $c_{k_2,\dots,k_d}$, $k_2, \dots, k_d \in \mathbb{Z}$, such that

$$\gamma = \sum_{k_2,\dots,k_d \in \mathbb{Z}} c_{k_2,\dots,k_d} \delta_{\pi(k_2,\dots,k_d)}.$$

Observe now that, for any $x_1 \in X$, the only point $(x_2, \dots, x_d) \in \overline{x_1}$ such that $(x_1, x_2, \dots, x_d)$ is seen by the graph joining $\sigma_{\pi(k_2,\dots,k_d)}$ is given by $x_i = T^{k_i}(x_1)$, $2 \le i \le d$. Therefore, for μ-almost every $x_1 \in X$,

$$\nu_{x_1} = \sum_{k_2,\dots,k_d \in \mathbb{Z}} c_{k_2,\dots,k_d} \delta_{(T^{k_2}x_1,\dots,T^{k_d}x_1)}.$$

Coming back to formula (11), we obtain

$$\sigma_\Delta(A \times B) = \sum_{k_2,\dots,k_d \in \mathbb{Z}} c_{k_2,\dots,k_d}\, \mu\Big(A \cap (T^{k_2} \times \cdots \times T^{k_d})^{-1}(B)\Big).$$

In particular, we see that no measure of the form ω_x appear in the decomposition of σ, hence $\sigma_{\text{diss}} = 0$ and the system $(X^d, \sigma, T^{\times d})$ is conservative. $\square$

4.2. **Proof of Theorem 4.1.** We already know by Proposition 4.2 that, under the assumptions of the theorem, the system $(X^d, \sigma, T^{\times d})$ is conservative. We can therefore consider the ergodic decomposition of σ, which by Theorem 2.3 and the parametrization of the set of boundedly finite diagonal measures can be described as follows.

For each nonempty $I \subset \{1, \dots, d\}$, let $\mathscr{D}^I$ be the set of parameters for the boundedly finite, $T^{\times|I|}$-invariant, diagonal measures on X^I ($\mathscr{D}^I$ is the exact analog of $\mathscr{D}$, which corresponds precisely to $I = \{1, \dots, d\}$). For each $\omega \in \mathscr{D}^I$, we thus have a canonical diagonal measure σ^I_ω on X^I, and each diagonal measure on X^I is of the form $c\,\sigma^I_\omega$ for some $c > 0$ and some $\omega \in \mathscr{D}^I$. Let $\mathscr{P}_d$ be the set of all partitions of $\{1, \dots, d\}$. For any $\pi = \{I_1, \dots, I_r\} \in \mathscr{P}_d$, let

$$\mathscr{D}^\pi := \mathscr{D}^{I_1} \times \cdots \times \mathscr{D}^{I_r}.$$

$\mathscr{D}^\pi$ can be viewed as a natural set of parameters for boundedly finite, $T^{\times d}$-invariant measures, which are of the form $\sigma^{I_1} \otimes \cdots \otimes \sigma^{I_r}$, where each σ^{I_j} is a diagonal measure on X^{I_j}. From Theorem 2.3, it follows that the ergodic decomposition of σ can be written as

$$\sigma = \sum_{\pi = \{I_1, \dots, I_r\} \in \mathscr{P}_d} p_\pi \int_{\omega = (\omega_1, \dots, \omega_r) \in \mathscr{D}^\pi} c(\omega) \sigma^{I_1}_{\omega_1} \otimes \cdots \otimes \sigma^{I_r}_{\omega_r} \, dm_\pi(\omega),$$

where $p_\pi \geq 0$, $\sum_\pi p_\pi = 1$, m_π is a probability measure on $\mathscr{D}^\pi$, and $c(\omega) > 0$ m_π-a.s.

Let us fix $\pi = \{I_1, \dots, I_r\} \in \mathscr{P}_d$ such that $p_\pi > 0$, and assume that $1 \in I_1$. Set

$$\sigma_1 := \int_{\mathscr{D}^\pi} \sigma^{I_1}_{\omega_1} \, dm_\pi(\omega).$$

This is a measure on X^{I_1}, which is $T^{\times|I_1|}$-invariant, and boundedly finite by (7). We want to show that Proposition 4.2 can be applied to σ_1, and for this we only need to check that its first marginal is absolutely continuous with respect to μ. Let $N \subset X$ be a μ-negligible set. We know that $\sigma(N \times X^{d-1}) = 0$, thus

$$\int_{\omega = (\omega_1, \dots, \omega_r) \in \mathscr{D}^\pi} c(\omega) \sigma^{I_1}_{\omega_1}(N \times X^{I_1 \setminus 1}) \sigma^{I_2}_{\omega_2}(X^{I_2}) \cdots \sigma^{I_r}_{\omega_r}(X^{I_r}) \, dm_\pi(\omega) = 0.$$

By Proposition 3.7, we know that

$$\sigma^{I_2}_{\omega_2}(X^{I_2}) \cdots \sigma^{I_r}_{\omega_r}(X^{I_r}) = \infty,$$

and since $c(\omega) > 0$ m_π-a.s., we deduce that

$$c(\omega) \sigma^{I_2}_{\omega_2}(X^{I_2}) \cdots \sigma^{I_r}_{\omega_r}(X^{I_r}) = \infty \quad m_\pi\text{-a.s.}$$

It follows that

$$\sigma_1(N \times X^{I_1 \setminus 1}) = \int_{\omega = (\omega_1, \dots, \omega_r) \in \mathscr{D}^\pi} \sigma^{I_1}_{\omega_1}(N \times X^{I_1 \setminus 1}) \, dm_\pi(\omega) = 0.$$

Then Proposition 4.2 gives that, for m_π-almost all $\omega \in \mathscr{D}^\pi$, $\sigma^{I_1}_{\omega_1}$ is a graph joining arising from powers of T. Since we assumed that all marginals of σ are absolutely continuous with respect to μ, the same argument applies for each $\sigma^{I_j}_{\omega_j}$, $1 \leq j \leq r$, and this ends the proof.

5. Consequences for Chacon infinite transformation

5.1. Commutant of T.

Proposition 5.1. *The centralizer of T is reduced to the powers of T.*

Proof. Let S be a μ-preserving transformation commuting with T. Then the graph joining defined on $X \times X$ by

$$\sigma_S(A \times B) := \mu(A \cap S^{-1}(B))$$

is a conservative ergodic $T \times T$-invariant measure which is supported on the graph of S. This measure is also boundedly finite. Since it is not proportional to the product measure, by Theorem 2.3 it has to be a 2-dimensional diagonal measure. Moreover, its marginals are absolutely continuous with respect to μ, hence by Proposition 4.2, σ is supported by the graph of a power of T. □

5.2. Joinings and factors.

Let $(Y_i, \mathscr{B}_i, \nu_i, S_i)$, $i = 1, 2$, be two infinite measure preserving dynamical systems. We recall that a *joining* between them is any $S_1 \times S_2$-invariant measure m on the Cartesian product $Y_1 \times Y_2$, whose marginals are respectively ν_1 and ν_2. In particular, in the dynamical system $(Y_1 \times Z, \mathscr{B} \otimes \mathcal{Z}, m)$, the sub-$\sigma$-algebra generated by the projection on coordinate i is σ-finite if and only if $(Y_i, \mathscr{B}_i, \nu_i)$ is a σ-finite measure space.

Proposition 5.2. *For any $\alpha \in (0, \infty]$, $\alpha \neq 1$, there is no joining between $(X, \mathscr{A}, \mu, T)$ and $(X, \mathscr{A}, \alpha\mu, T)$.*

Proof. Assume that there exists a joining m between $(X, \mathscr{A}, \mu, T)$ and $(X, \mathscr{A}, \alpha\mu, T)$. Then m is $T \times T$-invariant, and its marginals are absolutely continuous with respect to μ. Moreover, since the first marginal is μ, m is boundedly finite. By Proposition 4.2, the system it defines is conservative, and any ergodic component of m which is a diagonal measure is the graph joining supported on the graph of T^k for some $k \in \mathbb{Z}$. But $\mu \otimes \mu$ cannot appear as an ergodic component of m (otherwise the σ-algebra generated by the first coordinate would not be σ-finite). Therefore, there exist nonnegative numbers $a_k \in \mathbb{Z}$ with $\sum_{k \in \mathbb{Z}} a_k = 1$ such that the ergodic decomposition of m writes

$$m\left(A_1 \times A_2\right) = \sum_{k \in \mathbb{Z}} a_k \, \mu\left(A_1 \cap T^{-k} A_2\right).$$

But then, both marginals of m are equal to μ, thus $\alpha = 1$. □

This proposition leads to a nice corollary, for which we need to recall from [1] the following definition.

Definition 5.3. *A* law of large numbers *for a conservative, ergodic, measure preserving dynamical system $(Y, \mathscr{B}, \nu, S)$ is a function $L : \{0,1\}^{\mathbb{N}} \to [0, \infty]$ such that for all $B \in \mathscr{B}$, for ν-almost every $y \in Y$,*

$$L\left(\mathbb{1}_B(y), \mathbb{1}_B(Sy), \ldots\right) = \nu(B).$$

Theorem 3.2.5 in [1] provides a sufficient condition for S to have a law of large numbers, which is exactly the conclusion of Proposition 5.2.

Corollary 5.4. *The dynamical system $(X, \mathscr{A}, \mu, T)$ has a law of large numbers.*

Proposition 5.5. *Let $(Z, \mathcal{Z}, \rho, R)$ be any dynamical system, and assume that there exists a joining $(X \times Z, \mathcal{A} \otimes \mathcal{Z}, m, T \times R)$. Then $(X, \mathscr{A}, \mu, T)$ is a factor of $(Z, \mathcal{Z}, \rho, R)$.*

Proof. Since the marginal of m on the second coordinate is ρ, there exists a family $(\mu_z)_{z \in Z}$ of probability measures on X (defined ρ-almost everywhere), such that we have the following disintegration of m: for all $A \in \mathscr{A}$ and all $B \in \mathcal{Z}$,

$$m(A \times B) = \int_B \mu_z(A)\, d\rho(z).$$

Since m is $T \times R$-invariant, we have ρ-almost everywhere

$$\mu_{Rz} = T_*(\mu_z). \tag{12}$$

We can then form the relatively independent joining of $(X \times Z, \mathcal{A} \otimes \mathcal{Z}, m, T \times R)$ over $(Z, \mathcal{Z}, \rho, R)$, that is:

$$(X \times Z \times X, \mathcal{A} \otimes \mathcal{Z} \otimes \mathcal{A}, m \otimes_{\mathcal{Z}} m, T \times R \times T)\,,$$

where

$$m \otimes_{\mathcal{Z}} m\,(A_1 \times B \times A_2) = \int_B \mu_z \otimes \mu_z\,(A_1 \times A_2)\,\rho\,(dz)\,,$$

and extract from it a self-joining $(X \times X, \mathcal{A} \otimes \mathcal{A}, \widetilde{m}, T \times T)$ where

$$\widetilde{m}\,(A_1 \times A_2) = \int_Z \mu_z \otimes \mu_z\,(A_1 \times A_2)\,\rho\,(dz)\,.$$

Then $\widetilde{m}$ is $T \times T$-invariant, and its marginals are both equal to μ. As in the proof of Proposition 5.2, we deduce that there exist nonnegative numbers $a_k \in \mathbb{Z}$ with $\sum_{k \in \mathbb{Z}} a_k = 1$ such that the ergodic decomposition of $\widetilde{m}$ writes

$$\int_Z \mu_z \otimes \mu_z\,(A_1 \times A_2)\,\rho\,(dz) = \sum_{k \in \mathbb{Z}} a_k\, \mu\left(A_1 \cap T^{-k} A_2\right).$$

For ρ-a.e. $z \in Z$, the probability measure $\mu_z \otimes \mu_z$ is therefore supported by the graphs of T^k, $k \in \mathbb{Z}$. In particular, μ_z is a discrete probability measure, and its support is necessarily contained in a single T-orbit. This support can be totally ordered according to the place on the orbit, thus we can measurably choose one point $\varphi(z)$ on the support of μ_z by looking at the point with the highest weight and the lowest place in the orbit (this is well defined as the number of such points is finite). Since μ_z is supported by the T-orbit of $\varphi(z)$, we have a family $(w_i)_{i \in \mathbb{Z}}$ of measurable functions from Z to $[0, 1]$ such that, for ρ-almost every z,

$$\mu_z = \sum_{i \in \mathbb{Z}} w_i(z)\, \delta_{T^i \varphi(z)}.$$

Then, the disintegration of m becomes

$$m(A \times B) = \sum_{i \in \mathbb{Z}} \int_B w_i(z) \mathbb{1}_A\big(T^i \varphi(z)\big)\, d\rho(z). \tag{13}$$

Of course, since μ_z is a probability, we have $\sum_{i \in \mathbb{Z}} w_i(z) = 1$, ρ-almost everywhere. Moreover, from (12), we deduce that $\varphi \circ R = T \circ \varphi$, and that each function w_i is R-invariant. To show that φ is a homomorphism between the dynamical systems

$(Z, \mathcal{Z}, \rho, R)$ and $(X, \mathscr{A}, \mu, T)$, it only remains to check that $\varphi_*(\rho) = \mu$. But this comes from the following computation: for each $A \in \mathscr{A}$, we have

$$\begin{aligned}
\rho\big(\varphi^{-1}(A)\big) &= \int_Z \mathbb{1}_A\big(\varphi(z)\big)\, d\rho(z) \\
&= \int_Z \sum_{i\in\mathbb{Z}} w_i(z)\mathbb{1}_A\big(\varphi(z)\big)\, d\rho(z) \\
&= \sum_{i\in\mathbb{Z}} \int_Z w_i(R^i z)\mathbb{1}_A\big(\varphi(R^i z)\big)\, d\rho(z) \quad \text{(by } R\text{-invariance of } \rho) \\
&= \sum_{i\in\mathbb{Z}} \int_Z w_i(z)\mathbb{1}_A\big(T^i\varphi(z)\big)\, d\rho(z) \\
&= m(A \times Z) \quad \text{(by (13))} \\
&= \mu(A).
\end{aligned}$$

□

Proposition 5.6 (T has no non-trivial factor)**.** *Assume that $(Z, \mathcal{Z}, \rho, R)$ is a factor of $(X, \mathscr{A}, \mu, T)$. Then any homomorphism $\pi : X \to Z$ between the two systems is in fact an isomorphism.*

Proof. To any homomorphism $\pi : X \to Z$, we can associate the joining Δ_π of the two systems defined by

$$\Delta_\pi(A \times B) := \mu(A \cap \pi^{-1}B)$$

for any $A \in \mathscr{A}$, $B \in \mathcal{Z}$. Let us repeat the construction made in the proof of Proposition 5.5 with $m = \Delta_\pi$, and use the same notations as in this proof. Since T is ergodic, R is also ergodic, hence the weights $w_i(z)$, $i \in \mathbb{Z}$, which are R-invariant, are ρ-almost everywhere constant. By construction, $w_0 > 0$, and we claim that for $i \neq 0$, $w_i = 0$. Indeed, otherwise we would have, for ρ-almost all z, $z = \pi(\varphi(z)) = \pi(T^i\varphi(z))$. This would imply that, for μ-almost all x, $\pi(x) = \pi(T^i x)$, hence π would be constant as T^i is ergodic. This is impossible because $(Z, \mathcal{Z}, \rho, R)$ cannot be reduced to a single point system (since ρ is σ-finite).

We conclude that the conditional measure μ_z is ρ-almost everywhere the Dirac mass at $\varphi(z)$. Therefore, π is inversible, and its inverse is φ. □

Remark 5.7. *It is easily seen that all the results proved in Section 5 are valid for any dynamical system $(X, \mathscr{A}, \mu, T)$ for which the conclusion of Theorem 4.1 holds. Concerning Corollary 5.4, it is known in fact that Chacon infinite transformation admits a* measurable *law of large numbers: this is a consequence of Theorem 3.3.1 in* [1]*, and the fact that Chacon infinite transformation is rationally ergodic* [4]*. We do not know whether the conclusion of Theorem 4.1 alone implies the existence of a measurable law of large numbers.*

Appendix A. Product theorem

Theorem A.1. *Let X and Y be two standard Borel measurable spaces. Let $T : X \to X$ and $S : Y \to Y$ be invertible, bi-measurable transformations. Let σ be a σ-finite measure on $X \times Y$ satisfying*

- *there exist $X_0 \subset X$ and $Y_0 \subset Y$ with $0 < \sigma(X_0 \times Y_0) < \infty$,*
- *σ is $T \times S$-invariant,*

- *the dynamical system* $(X \times Y, T \times S, \sigma)$ *is conservative and ergodic,*
- $\mathrm{Id} \times S$ *is non-singular with respect to* σ.

Then, σ *is in fact* $\mathrm{Id} \times S$*-invariant, and there exist two measures* μ *and* ν *respectively on* X *and* Y*, invariant by* T *and* S*, such that* $\sigma = \mu \otimes \nu$*. Moreover, the dynamical systems* (X, μ, T) *and* (Y, ν, S) *are conservative and ergodic.*

Proof. Since $\mathrm{Id} \times S$ commutes with $T \times S$, the density

$$\frac{d(\mathrm{Id} \times S)_* \sigma}{d\sigma}(x, y)$$

is $T \times S$-invariant. Hence, by ergodicity, it is σ-almost everywhere equal to some constant c, $0 < c < \infty$.

Set, for each $n \in \mathbb{Z}$, $X_n := T^n X_0$, and $Y_n := S^n Y_0$, where X_0 and Y_0 are given in the assumptions of the theorem. As σ is invariant by $T \times S$, we deduce that, for all $(m, n) \in \mathbb{Z}^2$,

$$\sigma\left(X_n \times Y_m\right) = \sigma\left(X_0 \times Y_{m-n}\right) = c^{n-m} \sigma\left(X_0 \times Y_0\right).$$

Choose two sequences of positive numbers $(k_n)_{n \in \mathbb{Z}}$ and $(\ell_n)_{n \in \mathbb{Z}}$ such that

$$\sum_{(n,m) \in \mathbb{Z}^2} k_n \ell_m c^{n-m} = \left(\sum_{n \in \mathbb{Z}} k_n c^n\right)\left(\sum_{m \in \mathbb{Z}} \ell_m c^{-m}\right) < \infty.$$

Define $f := \sum_{n \in \mathbb{Z}} k_n \mathbb{1}_{X_n}$ and $g := \sum_{n \in \mathbb{Z}} \ell_n \mathbb{1}_{Y_n}$. As $f \otimes g$ is supported on $\cup_{(n,m) \in \mathbb{Z}^2} \left(X_n \times Y_m\right)$ which contains $\cup_{n \in \mathbb{Z}} \left(X_n \times Y_n\right) = X \times Y \bmod \sigma$ (by ergodicity of $T \times S$), we deduce that $f \otimes g > 0$ σ-a.e. Moreover,

$$\int_{X \times Y} f \otimes g \, d\sigma = \sigma\left(X_0 \times Y_0\right)\left(\sum_{n \in \mathbb{Z}} k_n c^n\right)\left(\sum_{m \in \mathbb{Z}} \ell_m c^{-m}\right) < \infty.$$

So we can assume that $\int_{X \times Y} f \otimes g \, d\sigma = 1$, and we can define the probability measure ρ whose density with respect to σ is equal to $f \otimes g$. We denote its respective projections on X and Y by ρ_X and ρ_Y.

Let us compute the density of $(\mathrm{Id} \times S)_*(\rho)$ with respect to ρ. For any measurable non-negative functions h on X and k on Y, we have

$$\begin{aligned}
&\int_{X \times Y} h \otimes k \circ (\mathrm{Id} \times S)(x, y) \, d\rho(x, y) \\
&= \int_{X \times Y} h(x) \, k(Sy) \, f(x) \, g(y) \, d\sigma(x, y) \\
&= c \int_{X \times Y} h(x) \, k(y) \, f(x) \, g\left(S^{-1} y\right) d\sigma(x, y) \\
&= c \int_{X \times Y} h(x) \, k(y) \, \frac{g\left(S^{-1} y\right)}{g(y)} d\rho(x, y).
\end{aligned}$$

This proves that the sought-after density is equal to $c \frac{g\left(S^{-1} y\right)}{g(y)}$. In particular, it only depends on y, and by taking $h = 1$ in the above computation, we get that S is non-singular with respect to ρ_Y, with the same density.

Now we wish to prove that the non-singular dynamical system (Y, ρ_Y, S) is ergodic and conservative. Indeed, if A is an S-invariant set with $\rho_Y(A) > 0$, then $X \times A$ is $T \times S$-invariant with $\rho(X \times A) > 0$. By ergodicity of $T \times S$, $\rho(X \times A) = 1$ and $\rho_Y(A) = 1$. In the same vein, if W is a wandering set for S, then $X \times W$ is a

wandering set for $T \times S$, therefore $\rho_Y(W) = \rho(X \times W) = 0$, by conservativity of $T \times S$.

Consider the measure ν on Y whose density with respect to ρ_Y is equal to $1/g(y)$. It is straightforward to check that the density of $S_*(\nu)$ with respect to ν is constant equal to c. We claim that $c = 1$. Indeed, we consider the Maharam extension of S defined on $(Y \times \mathbb{R}_+^*, \nu \otimes dt)$ by

$$\tilde{S}(y,t) := (Sy, t/c) \in Y \times \mathbb{R}_+^*.$$

Observe that if $c \neq 1$, $\tilde{S}$ is totally dissipative. But we know that (Y, S, ν) is conservative, hence $\tilde{S}$ is also conservative by Theorem 2 in [12], and we conclude that $c = 1$. This proves that σ is in fact invariant by $\mathrm{Id} \times S$.

The same arguments applied on the first coordinate lead to similar results: If μ is the measure on X whose density with respect to ρ_X is equal to $1/f(x)$, then μ is invariant by T, and the measure-preserving dynamical system (X, μ, T) is conservative and ergodic.

The end of the proof is an application of Lemma 3.1.1 in [13] to the measure ρ: This lemma proves that ρ is the product of its marginals ρ_X and ρ_Y, thus $\sigma = \mu \otimes \nu$. □

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