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Non standard functional limit laws for the increments of the compound empirical distribution function

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Résumé: Let $(Y_i, Z_i)_{i\geq 1}$ be a sequence of independent, identically distributed (i.i.d.) random vectors taking values in $\mathbb{R}^k \times \mathbb{R}^d$, for some integers k and d. Given $z \in \mathbb{R}^d$, we provide a nonstandard functional limit law for the sequence of functional increments of the compound empirical process, namely

$$\mathbf{\Delta}_{n,\mathfrak{c}}(h_n, z, \cdot) := \frac{1}{nh_n} \sum_{i=1}^{n} 1_{[0,\cdot)}\Big(\frac{Z_i - z}{{h_n}^{1/d}}\Big) Y_i.$$

Provided that $nh_n \sim c \log n$ as $n \to \infty$, we obtain, under some natural conditions on the conditional exponential moments of $Y \mid Z = z$, that

$$\mathbf{\Delta}_{n,\mathfrak{c}}(h_n, z, \cdot) \rightsquigarrow \Gamma \text{ almost surely,}$$

where $\rightsquigarrow$ denotes the clustering process under the sup norm on $[0,1)^d$. Here, Γ is a compact set that is related to the large deviations of certain compound Poisson processes.

Keywords and phrases: Empirical processes, Poisson process, Large deviations.

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$*$. This is an original survey paper

1. Introduction and statement of the results

Let $(Y_i, Z_i)_{i\geq 1}$ be a sequence of independent, identically distributed (i.i.d.) random vectors taking values in $\mathbb{R}^k \times \mathbb{R}^d$, for some integers k and d. Given $s, t \in \mathbb{R}^d$ with respective coordinates $s_1, \ldots, s_d$ and $t_1, \ldots, t_d$, we shall write $[s,t] := [s_1, t_1] \times \ldots \times [s_d, t_d]$, $[s,t) := [s_1, t_1) \times \ldots \times [s_d, t_d)$ and given $a \in \overline{\mathbb{R}}$ we set $[a,t] := [a, t_1] \times \ldots \times [a, t_d]$. For each integer $n \geq 1$, define the compound empirical distribution function as :

$$\mathbb{U}_{n,\mathfrak{c}}(s) := \frac{1}{n}\sum_{i=1}^{n} 1_{(-\infty,s]}(Z_i)Y_i,\ s \in \mathbb{R}^d. \tag{1.1}$$

Here the letter $\mathfrak{c}$ stands for "compound". In this paper, we are concerned with the asymptotic behaviour of the functional increments of $\mathbb{U}_{n,\mathfrak{c}}$, namely, for fixed $h > 0$ and $z \in \mathbb{R}^d$,

$$\mathbf{\Delta}_{n,\mathfrak{c}}(h, z, s) := \frac{1}{nh}\sum_{i=1}^{n} 1_{[0,s)}\Big(\frac{Z_i - z}{h^{1/d}}\Big)Y_i,\ s \in [0,1)^d. \tag{1.2}$$

Note that, in the particular case where $k = 1$ and $Y_1 \equiv 1$, the $\mathbf{\Delta}_{n,\mathfrak{c}}(h, z, \cdot)$ are no more than the functional increments of the empirical distribution function, which have been intensively investigated in the literature (see, e.g., [11, 5, 4, 10]). Among these investigations, Deheuvels and Mason ([5, 4]) have established *nonstandard* functional limit laws for the $\mathbf{\Delta}_{n,\mathfrak{c}}(h, z, \cdot)$ when $k = 1$, $d = 1$, $Y_1 \equiv 1$ and Z_1 is uniformly distributed on $[0,1)$. To cite their results, we need to introduce some further notations. We shall write

$$|| \ s \ ||_k := \max\{| \ s_1 \ |, \ldots, | \ s_k \ |\}$$

for $s \in \mathbb{R}^k$, and we define $B_k([0,1)^d)$ as the space of all mappings from $[0,1)^d$ to $\mathbb{R}^k$ that are bounded. We shall endow $B_k([0,1)^d)$ with the usual sup-norm, namely $| \ g \ |_k := \sup_{s\in[0,1)^d} | \ g(s) \ |_k$. Given a convex real function $\mathfrak{h}$ on $\mathbb{R}^k$, we define the following functional on $B_k([0,1)^d)$: whenever a function g satisfies $g(0) = 0$ and admits a derivative g' with respect to the Lebesgue measure, set

$$J_{\mathfrak{h}}(g) := \int_{[0,1)^d} \mathfrak{h}(g'(s))ds, \tag{1.3}$$

and set $J_{\mathfrak{h}}(g) := \infty$ if it is not the case. We also write, for any $c > 0$,

$$\Gamma_{\mathfrak{h}}(c) := \big\{g \in B_k([0,1)^d),\ J_{\mathfrak{h}}(g) \leq c\big\}. \tag{1.4}$$

Now define the following (Chernoff) function on $[0, \infty)$:

$$\mathfrak{h}_1(x) := \begin{cases} x\log x - x + 1, & \text{for } x > 0\,; \\ 1, & \text{for } x = 0\,; \\ \infty, & \text{for } x < 0. \end{cases} \tag{1.5}$$

A sequence (f_n) in a metric space (E, ρ), is said to be *relatively compact* with limit set equal to K when K is (non void) compact and the following assertions are true

$$\lim_{n\to\infty} \inf_{f\in K} d(f_n, f) = 0, \tag{1.6}$$

$$\forall\, f \in K,\ \liminf_{n\to\infty} d(f_n, f) = 0. \tag{1.7}$$

We shall write this property $x_n \rightsquigarrow K$.
Throughout this article, we shall consider a sequence of constants $(h_n)_{n\geq 1}$ satisfying the so called local nonstandard conditions, namely, as $n \to \infty$,

(HV) $\quad 0 < h_n < 1,\ h_n \downarrow 0,\ nh_n \uparrow \infty,\ nh_n/\log_2 n \to c \in (0, \infty).$

Here we have set $\log_2 n := \log(\log(n \vee 3))$, with the notation $a \vee b := \max\{a, b\}$. In a pioneering work, Deheuvels and Mason [4] established a *nonstandard* functional law of the iterated logarithm for a single functional increment of the empirical distribution function. With the notation of the present paper, their theorem can be stated as follows.

Fact 1.1 (Deheuvels and Mason, 1990)**.** *Let $(h_n)_{n\geq 1}$ be a sequence satisfying (HV) for some constant $c > 0$. Assume that $k = 1$, $d = 1$, $Y_1 \equiv 1$, and that Z_1 is uniformly distributed on $[0, 1)$. Then, given $z \in [0, 1)$, we have almost surely*

$$\mathbf{\Delta}_{n,\mathfrak{c}}(h_n, z, \cdot) \ \rightsquigarrow \Gamma_{\mathfrak{h}_1}(1/c).$$

Later, Deheuvels and Mason [6] extended the just mentioned result to a more general setting, where $d > 1$ and with fewer assumptions on the law of the Z_i, considering the $\mathbf{\Delta}_{n,\mathfrak{c}}(h_n, z, \cdot)$ as random measures indexed by a class of sets. The aim of the present paper is to extend the above mentioned results to the case where the random vectors Y_i are not constant, but do satisfy some assumptions on their conditional exponential moments given $Z = z$. From now on $< \cdot, \cdot >$ will always denote the Euclidian scalar product on $\mathbb{R}^k$ and λ stands for the Lebesgue measure. Define $\mathcal{C}$ as the class of each $C \subset \mathbb{R}^d$ which is the union of d hypercubes of $\mathbb{R}^d$, and with $\lambda(C) > 0$. The two key assumptions that we shall make upon the law of (Y_1, Z_1) are stated as follows.

(HL1) There exists a constant $f(z) > 0$ satisfying, for each $C \in \mathcal{C}$
$\lim_{h\to 0} h^{-1}\mathbb{P}\Big(Z_1 \in z + h^{1/d}C\Big) = \lambda(C)f(z).$

(HL2) There exist two mappings $\mathcal{L}_Y : \mathbb{R}^k \mapsto [0, \infty)$ and
$\mathcal{L}_{|Y|_k} : \mathbb{R} \mapsto [0, \infty)$ such that, for each $t \in \mathbb{R}^k$ and $t' \in \mathbb{R}$ and $C \in \mathcal{C}$, we have
$\lim_{h\to 0} \mathbb{E}\Big(\exp\big(< t, Y_1 > \big)\Big| Z_1 \in z + h^{1/d}C\Big) = \mathcal{L}_Y(t),$
$\lim_{h\to 0} \mathbb{E}\Big(\exp\big(t' \mid Y_1 \mid_k \big)\Big| Z_1 \in z + h^{1/d}C\Big) = \mathcal{L}_{|Y|_k}(t').$

Remark 1.1. *Assumptions (HL1) and (HL2) seem to be the weakest that we can afford in this context, in regard to the methods we make use of in this paper. Note that $(HL2)$ implies that $\mathcal{L}_Y$ is infinitely differentiable on $\mathbb{R}^k$. Some*

straightforward analysis shows that these assumptions are fulfilled when the Y_i are bounded by a constant and when Z_i admit a (version of) density f which is continuous at z. Another interesting case where (HL1) and (HL2) are fulfilled is a general semi parametric setting which appears in the following proposition.

Proposition 1.1. *Assume that there exist a σ-finite measure v on $\mathbb{R}^k$ and an application f_Y from $\mathbb{R}^k \times \mathbb{R}^d$ to $[0, \infty)$, such that*

1. *There exists a neighborhood $\mathcal{V}$ of z such that, for each $z' \in \mathcal{V}$, the law of $Y \mid Z' = z$ is dominated by v, with density $f_Y(\cdot, z')$.*
2. *For v-almost all y, the function $z \to f_Y(y, z)$ is continuous on $\mathcal{V}$.*
3. *For each $z' \in \mathcal{V}$, and each $t \in \mathbb{R}^k$ we have*

$$\int_{\mathbb{R}^k} \exp\big(< t, u > \big) f_Y(u, z') dv(u) < \infty.$$

4. *Z has a version of density (with respect to the Lebesgue measure λ) which is continuous on $\mathcal{V}$.*

Then the random vector (Y, Z) fulfills $(HL1)$ and $(HL2)$.

Proof : The proof is a straightforward application of Schéffé's lemma. □.

Remark 1.2. *Roughly speaking, assumption (HL2) imposes that the Laplace transform of the law $Y \mid Z = z$ is finite on $\mathbb{R}^k$. One could argue that this assumption could be weakened. However, it seems that, when this assumption is dropped, Theorem 1 (see below) does not hold anymore under the strong norm $|| \cdot ||_k$. A close look at the works of Deheuvels [3] and Borovkov [2] on the functional increments of random walks leads to the conjecture that the appropriate topology when $\mathcal{L}$ is finite only on a neighborhood of 0 seems to be the usually called weak star topology (see, e.g. [3]). This topic is however beyond the scope of this article, and shall be investigated in future works.*

Notice that $\mathcal{L}_Y$ and $\mathcal{L}_{|Y|_k}$ are positive convex functions when they exist. We now introduce $\mathfrak{h}_Y$ (resp. $\mathfrak{h}_{|Y|_k}$), which is defined as the Legendre transform of $\mathcal{L}_Y - 1$ (resp. $\mathcal{L}_{|Y|_k} - 1$), namely :

$$\mathfrak{h}_Y(u) := \sup_{t \in \mathbb{R}^k} < t, u > - \big(\mathcal{L}_Y(t) - 1\big), \ u \in \mathbb{R}^k, \tag{1.8}$$

$$\mathfrak{h}_{|Y|_k}(x) := \sup_{t' \in \mathbb{R}} t'x - \big(\mathcal{L}_{|Y|}(t) - 1\big), \ x \in \mathbb{R}. \tag{1.9}$$

Recall that the constant $c > 0$ appears in assumption (HV) and that $\Gamma_{\mathfrak{h}_Y}(1/c(z))$ has been defined by (1.4) and (1.8). Our result can be stated as follows.

Theorem 1. *Under assumptions (HV), (HL1) and (HL2), we have almost surely*

$$f(z)^{-1} \mathbf{\Delta}_{n,\mathfrak{c}}(z, h_n, \cdot) \ \rightsquigarrow \ \Gamma_{\mathfrak{h}_Y}(1/cf(z)).$$

A consequence of this result is the following unconsistency result, for which no proof has been yet provided to the best of our knowledge : let K be real function on $\mathbb{R}^d$ with *bounded variation and compact support*. The Nadaraya-Watson regression estimator of $r(z) := \mathbb{E}\big(Y \mid Z = z\big)$ is defined as :

$$r_n(z) := \sum_{i=1}^{n} \frac{K\left(\frac{Z_i - z}{h_n^{1/d}}\right)}{\sum\limits_{j=1}^{n} K\left(\frac{Z_j - z}{h_n^{1/d}}\right)} Y_j,\ z \in \mathbb{R}^d.$$

Theorem 1 entails that, under (HV), (HL1) and (HL2), the pointwise strong consistency of r_n does not hold.

Corollary 1.1. *Under (HV), (HL1) and (HL2), and assuming that $Y \not\equiv 0$, we have almost surely*

$$\limsup_{n\to\infty} \mid r_n(z) - r(z) \mid_k > 0.$$

Proof : We may assume without loss of generality that K vanishes outside $[0,1)^d$. Consider the random vectors $\widetilde{Y}_i := \big(Y_i, 1\big)$, taking values in $\mathbb{R}^{k+1}$. Some straightforward computations show that $\widetilde{Y}, Z$ satisfy $(HL1)$ and $(HL2)$, and that, writing $m := \nabla\mathcal{L}_{\widetilde{Y}}(0)$, we have

$$\mathfrak{h}_{\widetilde{Y}}(m) = 0. \tag{1.10}$$

Moreover, assuming without loss of generality that Y has a second moment matrix which is strictly positive, we have $\nabla^2\mathcal{L}_{\widetilde{Y}} > 0$ (strictly positive matrix) on $\mathbb{R}^{k+1}$, which ensures that

$$t_0 \mapsto \nabla\mathcal{L}_{\widetilde{Y}}(t)\Big|_{t=t_0}$$

is a C^1 diffeomorphism from $\mathbb{R}^{k+1}$ to an open set $O \ni m$. And hence admits an inverse that we write $\nabla\mathcal{L}_{\widetilde{Y}}^{-1}$. We deduce that

$$\mathfrak{h}_{\widetilde{Y}}(x) =< x, \nabla\mathcal{L}_{\widetilde{Y}}^{-1}(x) > -\Big(\mathcal{L}_{\widetilde{Y}}\big(\nabla\mathcal{L}_{\widetilde{Y}}^{-1}(x)\big) - 1\Big)$$

is continuous in x, which implies, that, for $\epsilon > 0$ small enough we have

$$\sup_{\substack{x\in\mathbb{R}^k \\ ||x-m||_k<\epsilon/f(z)}} \mid \mathfrak{h}_{\widetilde{Y}}(x) \mid_1 < \frac{1}{cf(z)}. \tag{1.11}$$

For $g \in B_{k+1}([0,1)^d)$, we shall write $g = (g_k, g_{k+1})$, where g_{k+1} denotes the last coordinate of g and $g_k \in B_k([0,1)^d)$ is equal to g without its last coordinate. We shall also write, for a Borel set A and for $\ell = (\ell_1, \ldots, \ell_k) \in B_k([0,1)^d)$

$$\ell(A) := \Big(\int_{\mathbb{R}^d} 1_A d\ell_1, \ldots, \int_{\mathbb{R}^d} 1_A d\ell_k\Big). \tag{1.12}$$

which is well defined as soon as either 1_A or each g_i has bounded variations on $\mathbb{R}^d$. Consider the following mappings :

$$\begin{array}{rcl}\Psi:\ \Big(B_{k+1}([0,1)^d),||\cdot||_k\Big) & \mapsto & \mathbb{R}^{k+1}\\ g & \to & \Big(\int_{s\in[0,1)^d} g_k([s,1])dK(s),\int_{s\in[0,1)^d} g_{k+1}([s,1])dK(s)\Big),\end{array}$$

$$\begin{array}{rcl}\Psi':\ \Big(B_{k+1}([0,1)^d),||\cdot||_k\Big) & \mapsto & \mathbb{R}^{k+1}\\ g & \to & \Big(\int_{s\in[0,1)^d}\int_{[s,1]} \dot{g}_k(u)d\lambda(u)dK(s),\int_{s\in[0,1)^d}\int_{[s,1]} \dot{g}_{k+1}(u)d\lambda(u)dK(s)\Big),\end{array}$$

$$\begin{array}{rcl}T:\ \mathbb{R}^k\times(\mathbb{R}-\{0\}) & \mapsto & \mathbb{R}^k\\ (x_1,\ldots,x_k,x_{k+1}) & \to & \frac{1}{x_{k+1}}(x_1,\ldots,x_k).\end{array}$$

Also consider

$$\widetilde{\Gamma}_{\mathfrak{h}_{\widetilde{Y}}}(1/cf(z)):=\Big\{g\in L^2([0,1)^d)^{k+1},\ \int_{[0,1)^d}\mathfrak{h}_{\widetilde{Y}}(g)\leq 1/c\Big\}.$$

We obviously have $\Psi\Big(f(z)\Gamma_{\mathfrak{h}_{\widetilde{Y}}}(1/cf(z))\Big)=\Psi'\Big(f(z)\widetilde{\Gamma}_{\mathfrak{h}_{\widetilde{Y}}}(1/cf(z))\Big)$. As K has bounded variations, Ψ is continuous and so is $T\circ\Psi$. Applying Theorem 1, we then deduce that, almost surely

$$r_n(z)=T\circ\Psi\Big(\mathbf{\Delta_{n,c}}\Big)\ \rightsquigarrow\ T\circ\Psi\Big(f(z)\Gamma_{\mathfrak{h}_{\widetilde{Y}}}(1/cf(z))\Big)\supset T\circ\Psi'\Big(f(z)\widetilde{\Gamma}_{\mathfrak{h}_{\widetilde{Y}}}(1/cf(z))\cap B_{k+1}([0,1)^d)\Big).$$

It hence remains to show that $T\circ\Psi'\Big(f(z)\widetilde{\Gamma}_{\mathfrak{h}_{\widetilde{Y}}}(1/cf(z))\cap B_{k+1}([0,1)^d)\Big)$ has non empty interior, which shall obviously imply that, almost surely, $r_n(z)\not\to r(z)$ as $n\to\infty$. It is well known that, as Ψ' is continuous, surjective and linear from the Banach space $\big(B_{k+1}([0,1)^d),||\cdot||_k\big)$ to $\mathbb{R}^{k+1}$, $\Psi'(O)$ is open for every open set O. Hence, it is sufficient to show that $f(z)\widetilde{\Gamma}_{\mathfrak{h}_{\widetilde{Y}}}(1/cf(z))\cap B_{k+1}([0,1)^d)$ has nonempty interior in $\big(B_{k+1}([0,1)^d),||\cdot||_k\big)$. Consider $\epsilon>0$ that appears in (1.11). Writing $g_m:\equiv m\in B_{k+1}\big([0,1)^d\big)$, we have, by (1.11)

$$||f(z)g-f(z)g_m||_k<\epsilon\ \Rightarrow\ \int_{[0,1)^d}\mathfrak{h}_{\widetilde{Y}}(g)\leq 1/c(z),$$

which concludes the proof. □

The remainder of our paper is organised as follows. In §2, we introduce an almost sure approximation of $\mathbf{\Delta}_{n,\mathfrak{c}}(z,h_n,\cdot)$ by a sum of compound Poisson processes. This approximation is largely inspired by a lemma of Deheuvels and Mason [6]. We then focus on these "poissonised" processes and provide some exponential inequalities on their modulus of continuity. In §3, we establish a Large Deviation Principle (LDP). Then §4 and §5 are devoted to proving points (1.6) and (1.7) of Theorem 1 respectively.

2. A Poisson approximation

Recall that $z \in \mathbb{R}^d$ is fixed once for all in our problem. For ease of notation we write

$$\Delta_{n,\mathfrak{c}}(z,h,s) := \frac{1}{nhf(z)} \sum_{i=1}^{n} 1_{[0,s)}\Big(\frac{Z_i - z}{h^{1/d}}\Big) Y_i, \quad s \in [0,1)^d, \ h > 0. \tag{2.1}$$

Throughout this article, we shall refer to a generic stochastic process U, usually called *compound Poisson process*. It is defined as follows : consider an infinite i.i.d array $\big(\mathfrak{Y}_{ij}, \mathfrak{Z}_{ij}\big)_{i\geq 1,\, j\geq 1}$ having the same law as (Y_1, Z_1), as well as a Poisson random variable with expectation equal to 1 fulfilling $\eta \perp\!\!\!\perp \big(\mathfrak{Y}_{ij}, \mathfrak{Z}_{ij}\big)_{i\geq 1,\, j\geq 1}$ (here $\perp\!\!\!\perp$ denotes stochastic independence). Now define

$$U(s) := \sum_{j=1}^{\eta} 1_{[0,s)}\big(\mathfrak{Z}_{ij} - z\big)\mathfrak{Y}_{ij}. \tag{2.2}$$

Note that the law of U is entirely determined by the following property :

For each $p \geq 1$ and for each partition $A_1, \ldots, A_p$ of $[0,1)^d$ we have :

$$\mathbb{E}\Big(\exp\Big(\sum_{j=1}^{p} < t_j, U(A_j) >\Big)\Big) = \exp\Big(\sum_{j=1}^{p} \mathbb{P}(Z - z \in A_j)\big(\mathcal{L}_{Y|A_j}(t_j) - 1\big)\Big), \ \big(t_1, \ldots, t_p\big) \in \big(\mathbb{R}^k\big)^p \tag{2.3}$$

where $\mathcal{L}_{Y|A_j}(t) := \mathbb{E}\Big(\exp\big(< t, Y > \Big| Z - z \in A_j\Big)$, $j = 1, \ldots, p$, $t \in \mathbb{R}^k$. Recall the expression $U(A)$ is understood according to (1.12). The following proposition enables to switch the study of the almost behaviour of the sequence $\big(\Delta_{n,\mathfrak{c}}(z, h_n, \cdot)\big)_{n\geq 1}$ to that of a sequence with the following generic term

$$\Delta\Pi_{n,\mathfrak{c}}(h,s) := \frac{1}{nhf(z)} \sum_{i=1}^{n} U_i(h^{1/d}s), \ s \in [0,1)^d, \ n \geq 1, \tag{2.4}$$

where the U_i are suitably built independent copies of U. This result is in the spirit of Deheuvels and Mason (see [6], Lemma 2.1, or [4], Proposition 2.1).

Proposition 2.1. *On a probability space rich enough $(\Omega, \mathcal{A}, \mathbb{P})$ we can construct an i.i.d. sequence of processes $(U_i)_{i\geq 1}$ having the same law as U and an sequence $(Y_{i1}, Z_{i1})_{i\geq 1}$ having the same law as $(Y_i, Z_i)_{i\geq 1}$ such that, considering the $\Delta_{n,\mathfrak{c}}(z, h_n, \cdot)$ as built with the sequence $(Y_{i1}, Z_{i1})_{i\geq 1}$ we have almost surely*

$$\limsup_{n\to\infty} (nh_n) \ || \ \Delta_{n,\mathfrak{c}}(z, h_n, \cdot) - \Delta\Pi_{n,\mathfrak{c}}\big(h_n, \cdot\big) \ ||_k < \infty, \tag{2.5}$$

with $\Delta\Pi_{n,\mathfrak{c}}(\cdot,\cdot)$ defined in (2.4).

Proof : Denote by U a process having the same law as in (2.2). Set $\mathcal{V}_i := z + h_i^{1/d}[0,1)^d$, $p_i = \mathbb{P}\big(Z_1 \in \mathcal{V}_i\big)$, and let $\big(Y_{ij}^{(1)}, Z_{ij}^{(1)}\big)_{i\geq 1,\, j\geq 1}$, $\big(Y_{ij}^{(2)}, Z_{ij}^{(2)}\big)_{i\geq 1,\, j\geq 1}$, $(\mathbf{b}_i)_{i\geq 1}$, $\big(U_i^*(\cdot)\big)_{i\geq 1}$ and $\big(v_i\big)_{i\geq 1}$ be families of random elements such that

(a) $\mathbb{P}\Big(\big(Y_{ij}^{(1)}, Z_{ij}^{(1)}\big) \in B\Big) = \mathbb{P}\Big((Y_1, Z_1) \in B \Big| Z_1 \in \mathcal{V}_i\Big),\ B$ Borel set, $i, j \geq 1$.

(b) $\mathbb{P}\Big(\big(Y_{ij}^{(2)}, Z_{ij}^{(2)}\big) \in B\Big) = \mathbb{P}\Big((Y_1, Z_1) \in B \Big| Z_1 \notin \mathcal{V}_i\Big),\ B$ Borel set, $i, j \geq 1$.

(c) For each $i \geq 1$ we have $\mathbb{P}\big(v_i = 0\big) = 1 - p_i^{-1}(1 - e^{-p_i})$ and $\mathbb{P}\big(v_i = k\big) = (k!)^{-1} p_i^{k-1} e^{-p_i},\ k = 1, 2, \ldots$.

(d) $\mathbb{P}(\mathbf{b}_i = 1) = 1 - \mathbb{P}(\mathbf{b}_i = 0) = p_i,\ i \geq 1$.

(e) The U_i^* are independent copies of U defined in (2.2)

(f) The union of these five families of random elements is a stochastically independent family.

In (e), equality in law is understood as an equality with respect to the σ-algebra $\mathcal{T}_0$ of $\big(B_k([0,1)^d), ||\cdot||_k\ \big)$ spawned by the open balls. In (f), stochastic independence is understood with respect to a suitably chosen product σ-algebra where each factor is either $\mathcal{T}_0$, the Borel σ-algebra of $\mathbb{R}^k \times \mathbb{R}^d$, or the subsets of $\{0, 1, 2, \ldots\}$. First, notice that $\eta_i^* := v_i \mathbf{b}_i$ is a Poisson random variable with expectation p_i for each $i \geq 1$, and that

$$\forall i \geq 1,\ \mathbb{P}\big(\eta_i^* = \mathbf{b}_i\big) \geq 1 - p_i^2. \tag{2.6}$$

In fact, η_i^* and $\mathbf{b}_i$ are a coupling of a Poisson and Bernouilli random variables $(\eta, \mathbf{b})$ with expectation p_i such that the probability $\mathbb{P}(\eta = \mathbf{b})$ is maximal. Second, notice that the following random vectors

$$\big(Y_{ij}, Z_{ij}\big) := 1_{\mathbf{b}_i = 1}\big(Y_{ij}^{(1)}, Z_{ij}^{(1)}\big) + 1_{\mathbf{b}_i = 0}\big(Y_{ij}^{(2)}, Z_{ij}^{(2)}\big),\ i \geq 1,\ j \geq 1, \tag{2.7}$$

are i.i.d. with common law equal to (Y_1, Z_1). Moreover, the following assertions are true with probability one, for each $i \geq 1$:

$$\forall s \in [0,1)^d,\ 1_{[0,s)}\Big(\frac{Z_{ij} - z}{h_i^{1/d}}\Big) Y_{ij} = \sum_{j=1}^{\mathbf{b}_i} 1_{[0,s)}\Big(\frac{Z_{ij}^{(1)} - z}{h_i^{1/d}}\Big) Y_{ij}^{(1)}. \tag{2.8}$$

We now define, for each $i \geq 1$,

$$U_i(s) := U_i^*\Big([0,s) \cap \{\mathcal{V}_i - z\}^C\Big) + \sum_{j=1}^{\eta_i^*} 1_{[0,s)}\big(Z_{ij}^{(1)} - z\big) Y_{ij}^{(1)}. \tag{2.9}$$

Here, $\mathcal{V}^C$ denotes the complement of a given set $\mathcal{V} \subset \mathbb{R}^d$. Some usual computations on characteristic functions show that the processes $U_i(\cdot)$ fulfill (2.3), and hence are distributed like U. Moreover since $h_{i+p}^{1/d}[0,1)^d \subset h_i^{1/d}[0,1)^d$ for $i \geq 1,\ q \geq 0$, we have almost surely

$$U_i\big(h_{i+q}^{1/d} s\big) = \sum_{j=1}^{\eta_i^*} 1_{[0,s)}\Big(\frac{Z_{ij}^{(1)} - z}{{h_{i+q}}^{1/d}}\Big) Y_{ij}^{(1)},\ s \in [0,1)^d. \tag{2.10}$$

It follows from (2.6), (2.8) and (2.10) that, for each $i \geq 1$,

$$\mathbb{P}\Big(U_i\big(h_{i+q}^{1/d}\,\cdot\,\big) \equiv 1_{[0,\cdot)}\Big(\frac{Z_i - z}{h_{i+q}^{1/d}}\Big) Y_{i1} \text{ for each } q \geq 0\Big) \geq \mathbb{P}\big(\eta_i^* = \mathbf{b}_i\big) \geq 1 - p_i^2. \tag{2.11}$$

Since $p_n = f(z)h_n(1+o(1))$ as $n \to \infty$, and by assumption (HV), we have $\sum p_i^2 < \infty$, which entails, by making use of the Borel-Cantelli lemma, that (2.5) is true with respect to our construction. □

By Proposition 2.1, proving Theorem 1 is equivalent to proving a version of Theorem 1 with the process $\Delta_{n,\mathfrak{c}}(z, h_n, \cdot)$ replaced by their Poisson approxiations $\Delta\Pi_{n,\mathfrak{c}}(h, \cdot)$. This will be the aim of §3, §4 and §5. In each of these three sections, we shall require the following exponential inequality for the *absolute oscillations* of $\Delta\Pi_{n,\mathfrak{c}}$, which are defined as the oscillations of the following process :

$$\overline{\Delta\Pi_{n,\mathfrak{c}}}(h,s) := \frac{1}{nhf(z)} \sum_{i=1}^{n} \sum_{j=1}^{\eta_i} 1_{[0,s)}\Big(\frac{Z_{ij}-z}{h^{1/d}}\Big) \mid Y_{ij} \mid_k,\ s \in [0,1)^d,\ n \geq 1. \tag{2.12}$$

Recall that $\mathfrak{h}_{|Y|_k}$ has been defined by (1.9).

Lemma 2.1. *Given $\delta \in (0, \sqrt{2}-1]$ and $x \geq 0$, there exists $h_x > 0$ such that, for each $0 < h < h_x$ and for each $n \geq 1$, we have*

$$\mathbb{P}\Bigg(\sup_{\substack{s,s' \in [0,1)^d \\ |s'-s|_d \leq \delta}} \Big|\overline{\Delta\Pi_{n,\mathfrak{c}}}(h,s) - \overline{\Delta\Pi_{n,\mathfrak{c}}}(h,s')\Big|_1 \geq 2d\delta x\Bigg) \leq \Big(\frac{10}{\delta}\Big)^d \exp\Big(-d\delta nhf(z)\mathfrak{h}_{|Y|_k}(x)\Big), \tag{2.13}$$

$$\mathbb{P}\Big(\overline{\Delta\Pi_{n,\mathfrak{c}}}(h,1) \geq x\Big) \leq \exp\big(-nhf(z)\mathfrak{h}_{|Y|_k}(x)\big). \tag{2.14}$$

Proof :
Given s and $s' \in [0,1)^d$, we write $s \prec s'$ whenever each coordinate of s is lesser than the corresponding coordinate of s'. Obviously, the $\overline{\Delta\Pi_{n,\mathfrak{c}}}(h,s)$ are almost surely increasing in each coordinate of s. First fix $\delta > 0$ and set

$$M := 1 + \Big[\frac{3}{(\sqrt{2}-1)\delta}\Big]. \tag{2.15}$$

We then discretise $[0,1)^d$ into the following finite grid :

$$s_{\mathbf{i}} := \frac{1}{M}\mathbf{i},\ \mathbf{i} \in \{0,1,\ldots,M-1\}^d. \tag{2.16}$$

By construction, for each s and s' with $\mid s'-s \mid_d \leq \delta$, there exists $\mathbf{i}_s \in \{0,1,\ldots,M-1\}^d$ fulfilling $s_{\mathbf{i}_s} \prec s$ and $\mid s - s_{\mathbf{i}_s} \mid_d \leq 1/M$, which entails

$\mid s' - s_{\mathbf{i}_s} \mid \leq 1/M + \delta$. Hence we can write

$$\mathbb{P}\Bigg(\sup_{\substack{s,s'\in[0,1)^d\\ |s'-s|_d\leq\delta}} \Big|\overline{\Delta\Pi_{n,\mathfrak{c}}}(h,s) - \overline{\Delta\Pi_{n,\mathfrak{c}}}(h,s')\Big|_1 \geq 2d\delta x\Bigg)$$

$$\leq \mathbb{P}\Bigg(\bigcup_{\mathbf{i}\in\{0,1,\ldots,M-1\}^d}\Big\{\sup_{\substack{s_{\mathbf{i}}\prec s',\\ |s'-s_{\mathbf{i}}|_d\leq\delta+1/M}} \Big|\overline{\Delta\Pi_{n,\mathfrak{c}}}(h,s') - \overline{\Delta\Pi_{n,\mathfrak{c}}}(h,s_{\mathbf{i}})\Big|_1 \geq 2d\delta x\Big\}\Bigg)$$

$$\leq M^d \max_{\mathbf{i}\in\{0,1,\ldots,M-1\}^d} \mathbb{P}\Bigg(\sup_{\substack{s_{\mathbf{i}}\prec s',\\ |s'-s_{\mathbf{i}}|_d\leq\delta+1/M}} \Big|\overline{\Delta\Pi_{n,\mathfrak{c}}}(h,s') - \overline{\Delta\Pi_{n,\mathfrak{c}}}(h,s_{\mathbf{i}})\Big|_k \geq 2d\delta x\Bigg).$$

Now notice that, for each $n \geq 1$, we have

$$\big(\overline{\Delta\Pi_{n,\mathfrak{c}}}(h,s)\big)_{s\in[0,1)^d} =_{\mathcal{L}} \left(\frac{1}{nhf(z)}\sum_{i=1}^{\eta_n} 1_{[0,s)}\Big(\frac{Z_i - z}{h^{1/d}}\Big) \mid Y_i \mid_k\right)_{s\in[0,1)^d}, \quad (2.17)$$

where η_n is a Poisson random variable with expectation n and independent of $(Y_i, Z_i)_{i\geq 1}$ (here $=_{\mathcal{L}}$ stands for the equality in law for processes). For a Borel set $B \subset [0,1)^d$, write

$$\begin{aligned}\overline{\Delta\Pi_{n,\mathfrak{c}}}(h,B) &:= \int_{[0,1)^d} 1_B(s) d\overline{\Delta\Pi_{n,\mathfrak{c}}}(h,s)\\ &= \frac{1}{nhf(z)}\sum_{i=1}^{\eta_n} 1_B\Big(\frac{Z_i - z}{h^{1/d}}\Big) \mid Y_i \mid_k . \end{aligned} \quad (2.18)$$

By the triangle inequality we have almost surely

$$\begin{aligned}&\sup_{\substack{s_{\mathbf{i}}\prec s',\\ |s'-s_{\mathbf{i}}|_d\leq\delta+1/M}} \Big|\overline{\Delta\Pi_{n,\mathfrak{c}}}(h,s') - \overline{\Delta\Pi_{n,\mathfrak{c}}}(h,s_{\mathbf{i}})\Big|_1\\ \leq &\sup_{\substack{s_{\mathbf{i}}\prec s',\\ |s'-s_{\mathbf{i}}|_d\leq\delta+1/M}} \frac{1}{nhf(z)}\sum_{i=1}^{\eta_n}\big(1_{[0,s')} - 1_{[0,s_{\mathbf{i}})}\big)\Big(\frac{Z_i - z}{h^{1/d}}\Big) \mid Y_i \mid_k\\ \leq &\frac{1}{nhf(z)}\sum_{i=1}^{\eta_n}\big(1_{[0,s_{\mathbf{i}}^+)} - 1_{[0,s_{\mathbf{i}})}\big)\Big(\frac{Z_i - z}{h^{1/d}}\Big) \mid Y_i \mid_k, \end{aligned} \quad (2.19)$$

where $s_{\mathbf{i}}^+$ is defined by adding $M^{-1}(\big[M\delta\big] + 2)$ to each coordinate of s_i. Line (2.19) is a consequence of the fact that, if $s_{\mathbf{i}} \prec s'$ and $\mid s' - s \mid_d \leq \delta + 1/M$, then $s_{\mathbf{i}} \prec s' \prec s_{\mathbf{i}^+}$. We shall now write $B_{\mathbf{i}} := [0, s_{\mathbf{i}}^+) - [0, s_{\mathbf{i}})$. Now choose $t = t(x)$ fulfilling

$$tx - \big(\mathcal{L}_{|Y|}(t) - 1\big) \geq \frac{1}{2}\mathfrak{h}_{|Y|_k}(x). \quad (2.20)$$

By Markov's inequality we have

$$\begin{aligned}
&\mathbb{P}\Big(\frac{1}{nhf(z)}\sum_{i=1}^{\eta_n}\big(1_{[0,s_{\mathbf{i}}^+)}-1_{[0,s_{\mathbf{i}})}\big)\Big(\frac{Z_i-z}{h^{1/d}}\Big)\mid Y_i\mid_k\geq 2d\delta x\Big)\\
&\leq \exp\big(-2d\delta nhf(z)tx\big)\mathbb{E}\Big[\exp\Big(t\sum_{i=1}^{\eta_n}1_{B_{\mathbf{i}}}\Big(\frac{Z_i-z}{h^{1/d}}\Big)\mid Y_i\mid_k\Big)\Big]\\
&\leq \exp\big(-2d\delta nhf(z)tx\big)\exp\Big(n\big(\mathcal{L}_{h,\mathbf{i}}(t)-1\big)\Big),
\end{aligned}\tag{2.21}$$

with

$$\mathcal{L}_{h,\mathbf{i}}(t):=\mathbb{E}\Big[\exp\Big(t1_{B_{\mathbf{i}}}\Big(\frac{Z_1-z}{h^{1/d}}\Big)\mid Y_1\mid_k\Big)\Big].$$

Note that (2.21) has been obtained by conditioning with respect to η_n. Now, by conditioning with respect to $E_{\mathbf{i},h}:=\{h^{-1/d}(Z_1-z)\in B_{\mathbf{i}}\}$, and writing

$$\mathcal{L}'_{h,\mathbf{i}}(t):=\mathbb{E}\Big[\exp\big(t\mid Y_1\mid_k\big)\Big|E_{\mathbf{i},h}\Big],\tag{2.22}$$

we obtain

$$\begin{aligned}
\mathcal{L}_{h,\mathbf{i}}(t)-1=&\mathbb{P}\big(E_{\mathbf{i},h}\big)\mathcal{L}'_{h,\mathbf{i}}(t)+\Big(1-\mathbb{P}\big(E_{\mathbf{i},h}\big)\Big)-1\\
=&\mathbb{P}(E_{\mathbf{i},h})\Big(\mathcal{L}'_{h,\mathbf{i}}(t)-1\Big).
\end{aligned}\tag{2.23}$$

Note that assumptions (HL1) and (HL2) readily entail

$$\lim_{h\to 0}\max_{\mathbf{i}\in\{0,\dots,M-1\}^d}\Big|\frac{\mathbb{P}(E_{\mathbf{i},h})\big(\mathcal{L}_{2,h,\mathbf{i}}(t)-1\big)}{f(z)\lambda(B_{\mathbf{i}})h\big(\mathcal{L}_{|Y|_k}(t)-1\big)}-1\Big|=0.\tag{2.24}$$

Choose $h_x>0$ small enough so that the quantity involved in (2.24) is lesser that $\sqrt{2}-1$ and notice that for each $\mathbf{i}$ we have $\lambda(B_{\mathbf{i}})\leq d(\delta+1/M)\leq\sqrt{2}d\delta$ by (2.15). By combining (2.19), (2.21) and (2.23), we conclude that, for all $0<h<h_x$,

$$\begin{aligned}
&\max_{\mathbf{i}\in\{0,1,\dots,M-1\}^d}\mathbb{P}\Bigg(\sup_{\substack{s_{\mathbf{i}}\prec s',\\ |s'-s_{\mathbf{i}}|_d\leq\delta+1/M}}\Big|\overline{\Delta\Pi_{n,\mathfrak{c}}}(h,s')-\overline{\Delta\Pi_{n,\mathfrak{c}}}(h,s_{\mathbf{i}})\Big|_k\geq 2d\delta x\Bigg)\\
&\leq\exp\Big(-2d\delta nhf(z)\big(tx-\mathcal{L}_{|Y|_k}(t)+1\big)\Big),
\end{aligned}\tag{2.25}$$

whence, by (2.20) we get

$$\begin{aligned}
&\mathbb{P}\Big(\sup_{\substack{s,s'\in[0,1)^d\\ |s'-s|_d\leq\delta}}\Big|\overline{\Delta\Pi_{n,\mathfrak{c}}}(h,s)-\overline{\Delta\Pi_{n,\mathfrak{c}}}(h,s')\Big|_k\geq 2d\delta x\Big)\\
&\leq M^d\exp\Big(-d\delta nhf(z)\mathfrak{h}_{|Y|_k}(x)\Big)\\
&\leq\Big(1+\frac{3}{(\sqrt{2}-1)\delta}\Big)^d\exp\Big(-d\delta nhf(z)\mathfrak{h}_{|Y|_k}(x)\Big)\\
&\leq\Big(\frac{10}{\delta}\Big)^d\exp\Big(-d\delta nhf(z)\mathfrak{h}_{|Y|_k}(x)\Big).
\end{aligned}$$

This concludes the proof of Lemma 2.1. □

3. Large deviations for $\Delta\Pi_{n,\mathfrak{c}}(h_n,\cdot)$

In this section, we establish a Large Deviation Principle (LDP) for the sequence of processes $\Delta\Pi_{n,\mathfrak{c}}(h_n,\cdot)$. For the definition of large deviations for sequences for bounded stochastic processes and of a (good) rate function, we refer to Arcones [1].

3.1. Some tools in large deviation theory

We begin this subsection with some well known properties (see, e.g., [3], Lemma 2.1, or Borovkov [2] just above the main Theorem) of $\mathfrak{h}_Y$ and $\mathfrak{h}_{|Y|_k}$ given in (1.8) and (1.9) respectively.

Fact 3.1. *The functions $\mathfrak{h}_Y$ and $\mathfrak{h}_{|Y|_k}$ are positive convex. Moreover, since $\mathcal{L}_Y$ is finite on $\mathbb{R}^k$, we have*

$$\lim_{|u|_k\to\infty}\frac{\mathfrak{h}_Y(u)}{\mid u\mid_k}=\infty,$$
$$\lim_{|x|\to\infty}\frac{\mathfrak{h}_{|Y_k|}(x)}{\mid x\mid}=\infty.$$

Arcones (see [1], Theorem 3.1) has established a very useful criterion to establish a LDP for processes in $B_k([0,1)^d)$ (actually only with $k=1$ but the extension of his results to $k>1$ is straightforward). We cannot make a direct use of his Theorem 3.1 and shall make use of a slight modification of it. To state this modification, we shall introduce some more notations. For each integer $p\geq 1$, consider a finite grid

$$\begin{aligned}S_p=&\Big\{s_{\mathbf{j},p},\mathbf{j}\in\{1,\ldots,2^p\}^d\Big\}\\ :=&\Big\{2^{-p}\mathbf{j},\ \mathbf{j}\in\{1,\ldots,2^p\}^d\Big\}.\end{aligned}\tag{3.1}$$

and consider its associated partition of $[0,1)^d$ into hypercubes, namely

$$C_{\mathbf{j},p}:=\Big[2^{-p}(\mathbf{j}-1),2^{-p}\mathbf{j}\Big),\ \mathbf{j}\in\{1,\ldots,2^p\}^d.\tag{3.2}$$

Here we have written $\mathbf{j}-1=(j_1-1,\ldots,j_d-1)$. Now for each integer $p\geq 1$ and for each $g\in B_k([0,1)^d)$ write

$$g^{(p)}(s):=g(s_{\mathbf{j},p}),\ s\in C_{\mathbf{j},p},\ \mathbf{j}\in\{1,\ldots,2^p\}^d.\tag{3.3}$$

The following proposition is a straightforward variation of Theorem 1 of Arcones [1], and is written according to the notation of that theorem (in particular, we refer to [1] for a definition of the outer probability $\mathbb{P}^*$).

Proposition 3.1. *Let $(X_n)_{n\geq 1}$ be a sequence of stochastic processes and let $(\epsilon_n)_{n\geq 1}$ be a sequence of constants fulfilling $\epsilon_n>0,\ n\geq 1$ and $\epsilon_n\to 0$ as $n\to\infty$. Assume that the following conditions are satisfied.*

1. *The sequence of stochastic processes* $(X_n^{(p)})_{n\geq 1}$ *satisfies the LDP for* $(\epsilon_n^{-1})_{n\geq 1}$ *and for a rate function* J_p *on* $\big(B_k([0,1)^d), ||\cdot||_k\big)$.
2. *For each* $\tau > 0$ *and* $M > 0$ *there exists an integer* $p \geq 1$ *satisfying*

$$\limsup_{n\to\infty} \epsilon_n \log\Big(\mathbb{P}^*\Big(\max_{\mathbf{j}\in\{1,\ldots,2^p\}^d} \sup_{s\in C_{\mathbf{j},p}} \mid X_n(s) - X_n(s_{\mathbf{j},p}) \mid_k \geq \tau\Big)\Big) \leq -M.$$

Then $(X_n)_{n\geq 1}$ *satisfies the LDP for* $(\epsilon_n^{-1})_{n\geq 1}$ *and for the following rate function.*

$$J(g) := \sup_{p\geq 1} J_p\Big(g^{(p)}\Big),\ g \in B_k([0,1)^d).$$

Proof : The proof follows the same lines as in the proof of Theorem 3.1 of Arcones [1]. We omit details for sake of briefness. □

For $g = (g_1, \ldots, g_k) \in B_k([0,1)^d)$ and A Borel set, we shall write

$$g(A) := \Big(\int_{[0,1)^d} 1_A(s)dg_1(s), \ldots, \int_{[0,1)^d} 1_A(s)dg_k(s)\Big), \tag{3.4}$$

which is valid as long as 1_A or each g_l have bounded variations. We shall now consider the following (rate) functions on $\big(B_k([0,1)^d), ||\cdot||_k\big)$ that will play the role of successive approximations of $J_{\mathfrak{h}_Y}$: given $p \geq 1$ and $g \in B_k([0,1)^d)$ we set

$$J_{\mathfrak{h}_Y}^{(p)}(g) := \sum_{\mathbf{j}\in\{1,\ldots,2^p\}^d} \lambda(C_{\mathbf{j},p})\mathfrak{h}_Y\Big(\lambda(C_{\mathbf{j},p})^{-1} g(C_{\mathbf{j},p})\Big). \tag{3.5}$$

The following fact is a straightforward extension to the multivariate case of Proposition 2.1 in [12]. Recall that $J_{\mathfrak{h}_Y}$ has been defined through (1.3) and (1.8).

Fact 3.2. *For any* $g \in B_k([0,1)^d)$ *we have*

$$J_{\mathfrak{h}_Y}(g) = \lim_{p\to\infty} J_{\mathfrak{h}_Y}^{(p)}(g). \tag{3.6}$$

As a consequence, $J_{\mathfrak{h}_Y}$ *is lower semicontinuous on* $B_k([0,1)^d)$.

Our next lemma states that the function $J_{\mathfrak{h}_Y}$ (recall (1.3)) is a "rate" function.

Lemma 3.1. *The sets* $\Gamma_{J_{\mathfrak{h}_Y}}(a)$, $a \geq 0$ *are compact subsets of* $\big(B_k([0,1)^d), ||\cdot||_k\big)$. *In other words,* $J_{\mathfrak{h}_Y}$ *is a rate function in* $\big(B_k([0,1)^d), ||\cdot||_k\big)$.

Proof : By Fact 3.1 we have $\mid x \mid_k \leq \mid x \mid_k 1_{|x|_k\leq M} \wedge J_{\mathfrak{h}_Y}$ for some $M > 0$ and for each x. Hence, for any $g \in \Gamma_{J_{\mathfrak{h}_Y}}(a)$ we have (recall that λ stands for the Lebesgue measure)

$$\int_{[0,1)^d} \mid g' \mid_k d\lambda = \int_{|g'|_k\leq M} \mid g' \mid_k d\lambda + \int_{|g'|_k>M} J_{\mathfrak{h}_Y}(g')d\lambda \tag{3.7}$$

$$\leq M + a, \tag{3.8}$$

from where we conclude that $\Gamma_{J_{\mathfrak{h}_Y}}$ is relatively compact in $B_k([0,1)^d)$. It is also closed in $B_k([0,1)^d)$ by a combination of Fact 3.2 and (3.8), which proves Lemma 3.1.□

3.2. A large deviation principle

In this subsection, we state and prove a large deviation principle that will play a crucial role in the sequel of our proof of Theorem 1. This LDP is stated as follows :

Proposition 3.2. *Under assumptions* (HV), $(HL1)-(HL2)$, *the sequence* $\Big(\Delta\Pi_{n,\mathfrak{c}}(h_n,\cdot)\Big)_{n\geq 1}$ *satisfies the LDP in* $B_k([0,1)^d)$ *for* $(\epsilon_n^{-1})_{n\geq 1}=\big((nh_nf(z))^{-1}\big)_{n\geq 1}$ *and for the rate function* $J_{\mathfrak{h}_Y}$.

Proof : As we shall make use of Proposition 3.1, we have to check conditions 1 and 2 of that proposition, which will be the aim of the following lemmas. Notice that, almost surely, we have

$$\Delta\Pi_{n,\mathfrak{c}}(h,C):=\frac{1}{nhf(z)}\sum_{i=1}^{n}\sum_{j=1}^{\eta_i}1_C\Big(\frac{Z_{i,j}-z}{h^{1/d}}\Big)Y_{i,j},\ C \text{ Borel},$$

with $\Delta\Pi_{n,\mathfrak{c}}(h,C)$ defined according to (1.12). Our proof is divided in two steps, where we shall respectively verify conditions 1 and 2 of Proposition 3.1.

Step 1 : To check condition 2 of Proposition 3.1, we shall make use of Lemma 2.1, which readily entails, for fixed $p\geq 1$ and $\tau>0$, and for all $n\geq n(p,\tau)$:

$$\begin{aligned}&\mathbb{P}\Big(\max_{\mathbf{j}\in\{1,\dots,2^p\}^d}\sup_{s\in C_{\mathbf{j},p}}\mid\Delta\Pi_{n,\mathfrak{c}}(h_n,s)-\Delta\Pi_{n,\mathfrak{c}}(h_n,s_{\mathbf{j},p})\mid_k\geq\tau\Big)\\ \leq&10^d2^{pd}\exp\Big(-d2^{-p}nh_nf(z)\mathfrak{h}_{|Y|_k}\big(d^{-1}2^{p-1}\tau\big)\Big).\end{aligned}$$

Now fix $M>0$ and $\tau>0$. By Fact (3.1), we have, for all large p :

$$\frac{\mathfrak{h}_{|Y|_k}\big(d^{-1}2^{p-1}\tau\big)}{d^{-1}2^{p-1}\tau}>4M\tau,$$

which implies that condition 2 of Proposition 3.1 is verified.
Step 2 : To check condition 1 of Proposition 3.1, we shall require the following preliminary lemma.

Lemma 3.2. *For any sequence* $(h_n)_{n\geq 1}$ *fulfilling* $h_n\to 0$ *and* $nh_n\to\infty$, *and for any fixed* $p\geq 1$, *the sequence of random vectors of* $\mathbb{R}^{k^{2^p}}$

$$\Big(\Delta\Pi_{n,\mathfrak{c}}(h_n,C_{\mathbf{j}}),\ \mathbf{j}\in\{1,\dots,2^p\}^d\Big)_{n\geq 1}\tag{3.9}$$

satisfies the LDP for the sequence $(\epsilon_n^{-1})_{n\geq 1} := ((nh_n f(z))^{-1})_{n\geq 1}$ *and the following rate function*

$$\begin{aligned} J^{(p)} : \left(\mathbb{R}^k\right)^{2^p} &\mapsto \mathbb{R} \\ x &\mapsto \sum_{\mathbf{j}\in\{1,\ldots,2^p\}^d}^{p} \lambda(C_{\mathbf{j},p})\mathfrak{h}_Y\Big(\lambda(C_{\mathbf{j},p})^{-1}x_{\mathbf{j}}\Big). \end{aligned}$$

Here we write $x = (x_{\mathbf{j}},\ \mathbf{j}\in\{1,\ldots,2^p\}^d)$, *with* $x_{\mathbf{j}}\in\mathbb{R}^k$ *for each* $\mathbf{j}\in\{1,\ldots,2^p\}^d$.

Proof : The proof of Lemma 3.2 is divided into three steps. The two first steps deal with a single component of the random vectors written in (3.9).
Step 1 :In our *first step*, we make an additional assumption on $\mathcal{L}_Y$, which allows us to make a full use of the Gärtner-Ellis theorem (see, e.g., [7], p. 44).

$$(H_0):\quad \forall x\in\mathbb{R}^k \text{ fulfilling } \mathfrak{h}_Y(x)<\infty,\ \exists\eta\in\mathbb{R}^k,\ x=\nabla\mathcal{L}_Y(\eta).$$

Lemma 3.3. *Assume that* (H_0) *is true in addition to the assumptions of Theorem 1. Then, for each* $p\geq 1$ *and* $\mathbf{j}\in\{1,\ldots,2^p\}^d$, *the sequence*

$$\Big(\Delta\Pi_{n,\mathfrak{c}}(h_n, C_{\mathbf{j},p})\Big)_{n\geq 1}$$

satisfies the LDP for the sequence $(nh_n f(z))^{-1}$ *and the rate function* $\lambda(C_{\mathbf{j},p})\mathfrak{h}_Y\big(\lambda(C_{\mathbf{j},p})^{-1}\cdot$ $)$.

Proof of Lemma 3.3 :
We shall first show that, for each $t\in\mathbb{R}^k$, we have

$$\lim_{n\to\infty}\frac{1}{nh_n f(z)}\log\Big(\mathbb{E}\Big(\exp<t, nh_n f(z)\Delta\Pi_{n,\mathfrak{c}}(C_{\mathbf{j},p},h_n)>\Big)\Big) = \lambda\big(C_{\mathbf{j},p}\big)\Big(\mathcal{L}_Y\Big(\lambda\big(C_{\mathbf{j},p}\big)^{-1}t\Big)\Big). \tag{3.10}$$

To show this, we start from the equality (2.17) to obtain by convolution :

$$\log\Big(\mathbb{E}\Big(\exp<t, nh_n f(z)\Delta\Pi_{n,\mathfrak{c}}(C_{\mathbf{j},p},h_n)>\Big)\Big) = n\log\Big(\mathbb{E}\Big(\exp<t, nh_n f(z)U\big(h_n^{1/d}C_{\mathbf{j},p}\big)>\Big)\Big).$$

Recall that U has been defined in (2.2). Next, we use the characterisation (2.3), which is applied to the simple partition $\Big(h_n^{1/d}C_{\mathbf{j},p},\ [0,1)^d - h_n^{1/d}C_{\mathbf{j},p}\Big)$. Using that relation with $t_1 = t$ and $t_2 = 0$, we obtain

$$\begin{aligned} &\log\Big(\mathbb{E}\Big(\exp<t, nh_n f(z)\Delta\Pi_{n,\mathfrak{c}}(C_{\mathbf{j},p},h_n)>\Big)\Big) \\ =&n\mathbb{P}\Big(Z-z\in h_n^{1/d}C_{\mathbf{j},p}\Big)\mathbb{E}\Big(\exp\big(<t,Y>\big)\Big|Z-z\in h_n\in z+h_n^{1/d}C_{\mathbf{j},p})\Big)-1\Big). \end{aligned}$$

Hence (3.10) follows from assumptions $(HL1)-(HL2)$.
B Lemma 2.3.9 in [7], p 46, we know that (H_0) implies that the set of *exposed points* of $\mathfrak{h}_Y$ is equal to $\{x\in\mathbb{R}^k,\ \mathfrak{h}(x)<\infty\}$, from where the proof of Lemma 3.3 is concluded by an application of the Gärtner-Ellis theorem (see, e.g., [7], p.

44). □

Step 2 : In our *second step*, we shall get rid of assumption (H_0), which is unfortunately not verified in all situations (for example, take $k = 1$, $Y \equiv 1$, which leads to $\mathcal{L}_Y(t) = \exp(t)$, $t \in \mathbb{R}$ and $\mathfrak{h}_Y(0) = 1$, but (H_0) is not satisfied for $x = 0$).

Lemma 3.4. *Lemma 3.3 is true without making assumption* (H_0).

Proof of Lemma 3.4 : First notice that the "closed sets" part of the LDP stated in Lemma 3.3 can be proved by making use of the Gärtner-Ellis theorem, without making assumption (H_0). Only the "open sets" part of Lemma 3.3 needs assumption (H_0), since it implies that the set of *exposed points* of $\mathfrak{h}_Y$ is equal to $\{x \in \mathbb{R}^k,\ \mathfrak{h}_Y(x) < \infty\}$. We only need to prove that, without assumption (H_0), for any open set $O \subset \mathbb{R}^k$ with $\mathfrak{h}_Y(O) < \infty$ (nontrivial case), we have

$$\liminf_{n\to\infty} \frac{1}{nh_n f(z)} \log\Big(\mathbb{P}\Big(\Delta\Pi_{n,\mathfrak{c}}\big(C_{\mathbf{j},p}, h_n\big) \in O\Big)\Big) \geq -\mathfrak{h}_Y(O). \tag{3.11}$$

To achieve this goal, we shall slightly modify the $Y_{i,j}$ by adding small Gaussian random vectors. Fix $O \subset \mathbb{R}^k$ open, with $\mathfrak{h}_Y(O) < \infty$, and $\delta > 0$. There exists $x \in O$ and $\delta_1 \in (0, \delta)$ such that $B(x, 2\delta_1) \subset O$ and $\mathfrak{h}_Y(O) \leq \mathfrak{h}_Y(B(x, 2\delta_1)) \leq \mathfrak{h}_Y(x) \leq \mathfrak{h}_Y(O) + \delta < \delta_1^{-1}$. Here $B(x, \epsilon)$ denotes the open ball with centre x and radius ϵ. Now introduce an array $\big(\zeta_{ij}\big)_{i,j\geq 1}$ of $\mathbb{R}^k$ valued standard random vectors, that are independent of the array $\big(Y_{i,j}, Z_{i,j}\big)_{i,j\in\mathbb{N}}$. Also define

$$\begin{aligned}
\Delta\Pi'_{n,\mathfrak{c}}\big(C_{\mathbf{j},p}, h_n\big) :=& \frac{1}{nh_n f(z)} \sum_{i=1}^{n} \sum_{j=1}^{\eta_i} 1_{C_{\mathbf{j},p}}\Big(\frac{Z_{ij} - z}{h_n^{1/d}}\Big)\zeta_{ij},\\
\Delta\Pi''_{n,\mathfrak{c}}\big(C_{\mathbf{j},p}, h_n\big) :=& \frac{1}{nh_n f(z)} \sum_{i=1}^{n} \sum_{j=1}^{\eta_i} 1_{C_{\mathbf{j},p}}\Big(\frac{Z_{ij} - z}{h_n^{1/d}}\Big)\big(Y_{ij} + \delta_1^2\zeta_{ij}\big)\\
=& \Delta\Pi_{n,\mathfrak{c}}\big(C_{\mathbf{j},p}, h_n\big) + \delta_1^2 \Delta\Pi'_{n,\mathfrak{c}}\big(C_{\mathbf{j},p}, h_n\big).
\end{aligned}$$

We shall first show that the vector $Y + \zeta$ fulfills assumptions (H_0). To prove this first notice that $\mathcal{L}_{Y+\zeta} = \mathcal{L}_Y \mathcal{L}_\zeta$, which holds since Y and ζ are independent conditionally to Z. Obviously we have, since $\zeta \perp\!\!\!\perp Z$,

$$\mathcal{L}_\zeta(t) = \exp\Big(\frac{1}{2} \mid t \mid_k^2 \Big),$$

which shows that ζ fulfills (H_0). Moreover, by Jensen's inequality we have

$$\mathcal{L}_Y(t) \geq \exp\big(< m_Y, t > \big),\ t \in \mathbb{R}^k,$$

where $m_Y = \mathbb{E}\big(Y \big| Z = z\big)$, which leads to

$$\mathcal{L}_{Y+\delta_1^2\zeta}(t) \geq \exp\Big(< m_Y, t > + \frac{\delta_1^4}{2} \mid t \mid_k^2 \Big). \tag{3.12}$$

Now consider $x \in \mathbb{R}^k$, and define the function $g(t) =< x, t > -\big(\mathcal{L}_{Y+\delta_1^2\zeta}(t) - 1\big)$. By (3.12) we have $g(t) \to -\infty$ as $\mid t \mid_k \to \infty$. Hence, the continuous and differentiable function g admits a maximum at some $\eta \in \mathbb{R}^k$ fulfilling $0 = \nabla g(\eta) = y - \nabla \mathcal{L}_{Y+\zeta}(x)$. This proves that the vectorζ fulfills (H_0) and hence, by Lemma 3.3 we have :

$$\liminf_{n\to\infty} \frac{1}{nh_n f(z)} \log\Big(\mathbb{P}\Big(\Delta\Pi''_{n,\mathfrak{c}}(C_{\mathbf{j},p}, h_n) \in O\Big)\Big) \geq -\mathfrak{h}_{Y+\delta_1^2\zeta}(O), \quad (3.13)$$

$$\limsup_{n\to\infty} \frac{1}{nh_n f(z)} \log\Big(\mathbb{P}\Big(\Big|\Big|\Delta\Pi'_{n,\mathfrak{c}}(C_{\mathbf{j},p}, h_n)\Big|\Big|_k \geq \delta_1^{-1}\Big)\Big) \leq -\inf_{||x||_k \geq \delta_1^{-1}} \mathfrak{h}_\zeta(x) \quad (3.14)$$

$$\leq -\delta_1^{-1}.$$

The last inequality holds for $\delta_1 > 0$ small enough, by Fact 3.1, replacing Y by ζ. Hence, by the triangle inequality, we have for all large n :

$$\begin{aligned}
\mathbb{P}\Big(\Delta\Pi_{n,\mathfrak{c}}(C_{\mathbf{j},p}, h_n) \in O\Big) \geq& \mathbb{P}\Big(\Big|\Big|\Delta\Pi_{n,\mathfrak{c}}(C_{\mathbf{j},p}, h_n) - x\Big|\Big|_k < 2\delta_1\Big) \\
\geq& \mathbb{P}\Big(\Big(\Big|\Big|\Delta\Pi''_{n,\mathfrak{c}}(C_{\mathbf{j},p}, h_n) - x\Big|\Big|_k < \delta_1\Big) - \mathbb{P}\Big(\delta_1^2\Big|\Big|\Delta\Pi'_{n,\mathfrak{c}}(C_{\mathbf{j},p}, h_n)\Big|\Big|_k > \delta_1\Big) \\
\geq& \exp\Big(-nh_n f(z)\Big(\delta + \mathfrak{h}_{Y+\zeta}\big(B(x,\delta_1)\big)\Big)\Big) - \exp\Big(-nh_n f(z)\delta_1^{-1}\Big) \\
\geq& \exp\Big(-nh_n f(z)\Big(\delta + \mathfrak{h}_Y(x)\Big)\Big) - \exp\Big(-nh_n f(z)\delta_1^{-1}\Big) \quad (3.15) \\
\geq& \frac{1}{2}\exp\Big(-nh_n f(z)\Big(2\delta + \mathfrak{h}_Y(O)\Big)\Big). \quad (3.16)
\end{aligned}$$

Note that (3.15) is a consequence $\mathfrak{h}_{Y+\zeta} \leq \mathfrak{h}_Y$, which follows directly from $\mathcal{L}_\zeta \geq 1$. Also, (3.16) is a consequence of $\mathfrak{h}_Y(x) \leq \mathfrak{h}_Y(O) + \delta$ together with $\delta_1^{-1} > \mathfrak{h}_Y(O) + 2\delta$, which is true by the choice of δ_1. The proof of Lemma 3.4 is then concluded since O and δ are arbitrary.□

Step 3 : The proof of Lemma 3.2 by a tensorisation argument brought by Lynch an Sethuraman. Since, for each n, the collection

$$\Delta\Pi_{n,\mathfrak{c}}(h_n, C_{\mathbf{j}}),\ \mathbf{j} \in \{1, \ldots, 2^p\}^d$$

is independent, and since each sequence $\Big(\Delta\Pi_{n,\mathfrak{c}}(h_n, C_{\mathbf{j}})\Big)_{n\geq 1}$ satisfies the LDP with the rate function $\lambda(C_{\mathbf{j},p})\mathfrak{h}_Y\big(\lambda(C_{\mathbf{j},p})^{-1}\cdot\big)$. Then Lemma 3.2 is proved by applying Lemma 2.8 in [9]. □

A direct consequence of Lemma 3.2 is that condition 1 of Proposition 3.1 is satisfied, as shows our next lemma.

Lemma 3.5. *If $h_n \to 0$ and $nh_n \to \infty$, then the sequence of processes*

$$\left(\Delta\Pi^{(p)}_{n,\mathfrak{c}}(h_n,\cdot)\right)_{n\geq 1}$$

satisfies the LDP for $\epsilon_n := (nh_n f(z))^{-1}$. and for the rate function $J^{(p)}_{\mathfrak{h}_Y}$.

Proof : The proof is a straightforward application of the contraction principle (see, e.g., [1], Theorem 2.1), considering, for fixed p, the following application, from $\mathbb{R}^{k^{2^{pd}}}$ to $\left(B_k([0,1)^d); || \cdot ||_k\right)$ (here we write $x = (x_{\mathbf{j}},\ \mathbf{j} \in \{1,\ldots,2^p\}^d)$, with each $x_{\mathbf{j}}$ belonging to $\mathbb{R}^k$)

$$\begin{array}{rcl} \mathcal{R}_p(x) : [0,1)^d & \mapsto & [0,\infty) \\ s & \to & \displaystyle\sum_{C_{\mathbf{j},p}\subset[0,s_1]\times\ldots\times[0,s_p]} x_{\mathbf{i}}.\square \end{array}$$

We conclude the proof of Proposition 3.2 by combining Step 1 and Step 2 with Proposition 3.1.□

4. Proof of point (1.6) of Theorem 1

We shall make use of some usual blocking arguments along the following subsequence :

$$n_k := \left[\exp\left(k\exp\left(-\sqrt{\log k}\right)\right)\right], \tag{4.1}$$

with associated blocks $N_k := \{n_{k-1}+1,\ldots,n_k\}$. Here, $[u]$ denotes the only integer fulfilling $[u] \leq u \leq [u]+1$. We point out two key properties of $(n_k)_{k\geq 1}$:

$$\lim_{k\to\infty}\frac{n_k}{n_{k-1}} = 1, \quad \lim_{k\to\infty}\frac{\log_2 n_k}{\log k} = 1. \tag{4.2}$$

For any $\epsilon > 0$ and $A \subset B_k([0,1)^d)$, we shall write :

$$A^\epsilon := \left\{g \in B_k([0,1)^d),\ \inf_{g'\in A} || g - g' ||_k < \epsilon\right\}. \tag{4.3}$$

Now, recalling the definition of $\Delta\Pi_{n,\mathfrak{c}}$ in (2.4), we define the following normalised Poisson processes that will play a crucial role in our blocking arguments.

$$\mathcal{H}_n(s) := \frac{1}{n_k h_{n_k} f(z)}\sum_{i=1}^{n} U_i(h_{n_k}s),\ k\geq 1,\ n\in N_k,\ s\in[0,1)^d. \tag{4.4}$$

Fix $\epsilon > 0$. We shall proceed in two steps : first, we will prove that, we have almost surely, ultimately as $n \to \infty$,

$$\mathcal{H}_n \in \Gamma_{J_{\mathfrak{h}_Y}}(1/cf(z))^{2\epsilon}, \tag{4.5}$$

then we shall show that almost surely :

$$\lim_{k\to\infty}\max_{n\in N_k} || \mathcal{H}_n(\cdot) - \Delta\Pi_{n_k,\mathfrak{c}}(h_{n_k},\cdot) ||_k \leq 3\epsilon. \tag{4.6}$$

Step 1 : We first prove (4.5). In order to make use of usual blocking arguments along the blocks N_k we shall first show that

$$\lim_{k\to\infty} \max_{n\in N_k} \mathbb{P}\Big(\,||\, \mathcal{H}_n(\cdot) - \Delta\Pi_{n_k,\mathfrak{c}}(h_{n_k},\cdot) \,||_k > \epsilon \Big) = 0. \tag{4.7}$$

To prove this, choose $k \geq 1$ and $n \in N_k$ arbitrarily. A rough upper bound gives (excluding the trivial case where $n = n_k$).

$$\begin{aligned}\mathbb{P}_{n,1} :=& \mathbb{P}\Big(\,||\, \mathcal{H}_n(\cdot) - \Delta\Pi_{n_k,\mathfrak{c}}(h_{n_k},\cdot) \,||_k > \epsilon \Big)\\ \leq& \mathbb{P}\Big(\sum_{i=1}^{n_k-n} \sum_{j=1}^{\eta_i} |\, Y_{i,j} \,|_k > \epsilon \frac{n_k}{n_k-n}(n_k-n) h_{n_k} f(z) \Big)\\ =& \mathbb{P}\Big(\overline{\Delta\Pi}_{n_k-n,\mathfrak{c}}(h_{n_k},\cdot) > \epsilon \frac{n_k}{n_k-n} \Big)\end{aligned} \tag{4.8}$$

Now making use of point (2.14) of Proposition 2.1 with $x := \epsilon n_k/(n_k - n)$ we get, for all large k and for each $n \in N_k$ with $n \neq n_k$,

$$\mathbb{P}_n \leq \exp\Big(-\epsilon n_k h_{n_k} f(z) \frac{n_k - n}{\epsilon n_k} \mathfrak{h}_{|Y|_k}\big(\frac{\epsilon n_k}{n_k - n}\big)\Big). \tag{4.9}$$

Now, as $n_k - n \geq n_k - n_{k-1}$, $n_k/(n_k - n_{k-1}) \to \infty$ and by Fact 3.1 we readily infer (4.7).

We are now able to make use of a well known maximal inequality (see, e.g., Deheuvels and Mason [5], Lemma 3.4) to conclude that, for all large k,

$$\begin{aligned}\mathbb{P}_{k,2} :=& \mathbb{P}\Big(\bigcup_{n\in N_k} \mathcal{H}_n \notin \Gamma^{2\epsilon}_{J_{\mathfrak{h}_Y}} \Big)\\ \leq& 2\mathbb{P}\Big(\mathcal{H}_{n_k} \notin \Gamma^{\epsilon}_{J_{\mathfrak{h}_Y}} \Big)\\ =& 2\mathbb{P}\Big(\Delta\Pi_{n_k,\mathfrak{c}}(h_{n_k},\cdot) \notin \Gamma^{\epsilon}_{J_{\mathfrak{h}_Y}} \Big).\end{aligned} \tag{4.10}$$

Applying proposition 3.2 to the closed set $F := B_k([0,1)^d) - \big(\Gamma_{J_{\mathfrak{h}_Y}}\big)^{\epsilon}$, which satisfies $J_{\mathfrak{h}_Y}(F) \geq (1+3\alpha)/cf(z)$ for some $\alpha > 0$ (by lower semi continuity of $J_{\mathfrak{h}_Y}$) we get, ultimately as $k \to \infty$,

$$\begin{aligned}\mathbb{P}_{k,2} \leq& 2\exp\Big(-\frac{n_k h_{n_k}(1+2\alpha)}{cf(z)} \Big)\\ \leq& \exp\big(-(1+\alpha)\log\log n_k \big),\end{aligned} \tag{4.11}$$

where (4.11) is a consequence of assumption (HV). By (4.2), we conclude that $(\mathbb{P}_{k,2})_{k\geq 1}$ is summable, which proves (4.5) by making use of the Borel-Cantelli lemma.

Step 2 To prove (4.6) we shall make use of the following almost sure equality

$$\Delta\Pi_{n,\mathfrak{c}}(h_n, s) := \frac{n_k h_{n_k}}{n h_n} \mathcal{H}_n\big(\frac{h_{n_k}}{h_n} s\big). \tag{4.12}$$

By (4.2) together with (HV) we straightforwardly infer that

$$\lim_{k\to\infty}\max_{n\in N_k}\Big|\frac{n_k h_{n_k}}{nh_n}-1\Big|=0,\ \lim_{k\to\infty}\max_{n\in N_k}\frac{h_{n_k}}{h_n}=1. \tag{4.13}$$

Moreover, making use of (3.8), we infer that

$$\lim_{T\to 1,\ \rho\uparrow 1}\sup_{g\in\Gamma_{J_{\mathfrak{h}_Y}}}\ ||\ Tg(\rho\cdot)-g(\cdot)\ ||_k=0. \tag{4.14}$$

Hence, (4.6) follows from a combination of (4.13), (4.14) and (4.5) together with the triangle inequality.
The proof of point (1.6) of Theorem 1 is concluded by combining (4.5) and (4.6) and recalling that $\epsilon>0$ was arbitrary.□

5. Proof of point (1.7) of Theorem 1

We introduce the following subsequence

$$\overline{n}_k:=k^{2k},\ k\geq 1.$$

Obviously, $\overline{n}_k$ satisfies the following properties :

$$\log_2\overline{n}_k=\log k+\log_2 k+\log 2,\ \ \overline{n}_k/\overline{n}_{k-1}=e^{-2}k^{-2}(1+o(1)). \tag{5.1}$$

we also shall write $v_k:=\overline{n}_k-\overline{n}_{k-1}$. Now define the sequence

$$\mathcal{H}'_k:=\frac{1}{v_k h_{\overline{n}_k}f(z)}\sum_{\overline{n}_{k-1}+1}^{\overline{n}_k}1_{[0,\cdot]}\Big(\frac{Z_i-z}{h_{\overline{n}_k}^{1/d}}\Big)Y_i.$$

Now choose $\epsilon>0$ and $g\in\Gamma_{\mathfrak{h}_Y}(1/cf(z))$ arbitrarily. We shall prove that, with probability one

$$\limsup_{n\to\infty}\Big|\Big|\mathcal{H}'_k-g\Big|\Big|\leq 2\epsilon, \tag{5.2}$$

which would conclude the proof of point (1.7) of Theorem 1 by a classical compactness argument. Obviously g satisfies

$$\lim_{\rho\to 1}\ ||\ g(\rho\cdot)-g(\cdot)\ ||_k=0. \tag{5.3}$$

Some routine analysis also shows that, for some $\alpha>0$ we have $J\big(g^\epsilon\big)<(1-2\alpha)/cf(z)$. By (5.1) we have $v_k h_{\overline{n}_k}\to\infty$ as $k\to\infty$. Hence, by Proposition (3.2), which we apply to the open ball g^ϵ we obtain, for all large k

$$\begin{aligned}\mathbb{P}\Big(\mathcal{H}'_k\in g^\epsilon\Big)&\geq\exp\Big(-\frac{v_k h_{\overline{n}_k}f(z)(1-2\alpha)}{cf(z)}\Big)\\&\geq\exp\Big(-\log k+\log_2 k+\log 2\Big),\end{aligned}$$

where the last inequality is a consequence of (5.1). As the $(\mathcal{H}'_k)_{k\geq 1}$ are independent, the Borel-Cantelli lemma entails, almost surely,

$$|| \mathcal{H}'_k - g ||_k \leq \epsilon \text{ for all large } k. \tag{5.4}$$

To conclude the proof, notice that

$$\begin{aligned}\mathcal{H}'_k =& \frac{v_k}{\overline{n}_k}\mathcal{H}'_k + \frac{1}{\overline{n}_k h_{\overline{n}_k} f(z)} \sum_{i=1}^{\overline{n}_{k-1}} 1_{[0,\cdot]}\Big(\frac{Z_i - z}{h_{\overline{n}_k}^{1/d}}\Big) Y_i \\ =:& \frac{v_k}{\overline{n}_k}\mathcal{H}'_k + \zeta_k. \end{aligned} \tag{5.5}$$

Hence, if we show that $|| \zeta_k ||_k \to 0$ almost surely, then (5.2) will follow by noticing that $v_k/\overline{n}_k \to 1$ and applying both (5.3) and (5.4). Noticing that

$$|| \zeta_k ||_k \leq \frac{\overline{n}_{k-1}}{\overline{n}_k} \overline{\Delta\Pi}_{\overline{n}_{k-1},\mathfrak{c}}(h_{\overline{n}_k}, 1),$$

we readily infer, by (5.1) and point (2.14) of Lemma 2.1, that $\mathbb{P}\big(|| \zeta_k ||_k > \delta \big) = O(k^{-2})$ for any $\delta > 0$. This concludes the proof of point (1.7) of Theorem 1.

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