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A New Game Invariant of Graphs: the Game Distinguishing Number

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The distinguishing number of a graph G is a symmetry related graph invariant whose study started two decades ago. The distinguishing number $D(G)$ is the least integer d such that G has a d-distinguishing coloring. A d-distinguishing coloring is a coloring $c : V(G) \to \{1, ..., d\}$ invariant only under the trivial automorphism. In this paper, we introduce a game variant of the distinguishing number. The distinguishing game is a game with two players, the Gentle and the Rascal, with antagonist goals. This game is played on a graph G with a set of $d \in \mathbb{N}^*$ colors. Alternately, the two players choose a vertex of G and color it with one of the d colors. The game ends when all the vertices have been colored. Then the Gentle wins if the coloring is d-distinguishing and the Rascal wins otherwise. This game leads to define two new invariants for a graph G, which are the minimum numbers of colors needed to ensure that the Gentle has a winning strategy, depending on who starts. These invariants could be infinite, thus we start by giving sufficient conditions to have infinite distinguishing numbers, we also show that for graphs with cyclic automorphisms group of prime odd order, both game invariants are finite. After that, we define a class of graphs, the involutive graphs, for which the game distinguishing number can be quadratically bounded above by the classical distinguishing number. The definition of this class is closely related to imprimitive action whose blocks have size 2. Then, we apply results on involutive graphs to compute the exact value of these invariants for hypercubes and even cycles. Finally, we study odd cycles, for which we are able to compute the exact value when their order is not prime. In the prime order case, we give an upper bound of 3.

Keywords: distinguishing number, graph automorphism, combinatorial game, hypercube

1 Background and definition of the game

The distinguishing number of a graph G is a symmetry related graph invariant. Its study started two decades ago in a work by Albertson and Collins [1]. Given a graph G the distinguishing number $D(G)$ is the least integer d such that G has a d-distinguishing coloring. A d-distinguishing coloring is a coloring $c : V(G) \to \{1, ..., d\}$ invariant only under the trivial automorphism. More generally, we say that an automorphism σ of a graph G preserves the coloring c or is a colors preserving automorphism, if for all $u \in V(G)$, $c(u) = c(\sigma(u))$. We denote by $Aut(G)$, the automorphisms group of G. Clearly, for each coloring c of the vertex set of G, the set $Aut_c(G) = \{\sigma \in Aut(G) : c \circ \sigma = c\}$ is a subgroup of $Aut(G)$. A coloring c is distinguishing if $Aut_c(G)$ is trivial. The group $Aut(G)$ acts naturally on the set of its vertices and this action induces a partition of the vertex set $V(G)$ into orbits. If H is a subgroup of $Aut_c(G)$ then the action of H on $V(G)$ is such that each orbit induced by this action is monochromatic.

In particular, an automorphism σ preserves the coloring if and only if the orbits under the action of the subgroup $< \sigma >$ induced by σ, are all monochromatic.

In the last couple of years the study of this invariant was particularly flourishing. See [9, 10] for the work of Imrich, Jerebic and Klavžar on the distinguishing number of cartesian products or [6] for an analog of Brook's theorem. Some variant of the distinguishing number have been already introduced. In [5, 11], an extension of the distinguishing number to any actions of a group H on a set X has been provided and in [6] Collins and Trenk study a distinguishing coloring which must be also a proper coloring. Our goal in this paper is to introduce a game variant of this invariant in the spirit of the game chromatic number χ_G introduced by Brahms in 1981 (see [7]) or of the most recent domination game (see [4]). Even if inventing new game invariant could raise up a lot of promising and interesting questions, it might seem artificial at a first glance. To defend our approach, we recall that game invariants have already proved useful to give a new insight on the classical invariant they are related to. We cite in particular [13], where a greedy like strategy used in the study of the domination game is used to improve several upper bounds on the domination number. See also [15] for an application of coloring game to the graph packing problem.

The distinguishing game is a game with two players, the Gentle and the Rascal, with antagonist goals. This game is played on a graph G with a set of $d \in \mathbb{N}^*$ colors. Alternately, the two players choose a vertex of G and color it with one of the d colors. The game ends when all the vertices have been colored. Then the Gentle wins if the coloring is d-distinguishing and the Rascal wins otherwise.

This game leads to two new invariants for a graph G. The *G-game distinguishing number* $D_{\mathcal{G}}(G)$ is the minimum of colors needed to ensure that the Gentle has a winning strategy for the game on G, assuming he is playing first. If the Rascal is sure to win whatever the number of colors we allow, then $D_{\mathcal{G}}(G) = \infty$. Similarly, the *R-game distinguishing number* $D_{\mathcal{R}}(G)$ is the minimum number of colors needed to ensure that the Gentle has a winning strategy, assuming the Rascal is playing first.

It arises directly from the definition that for any graph G, $D(G) \leq D_{\mathcal{G}}(G)$ and $D(G) \leq D_{\mathcal{R}}(G)$. Another straight forward remark is that the game distinguishing number of a graph and its complement are equal.

One natural question we start with is to know if having a winning strategy with k colors is a growing property. In other words, has the Gentle a winning strategy if $k \geq D_{\mathcal{G}}$ or $k \geq D_{\mathcal{R}}$ colors are allowed ? The answer is not as obvious as it looks at a first glance. We recall that for the game chromatic number χ_g, it is not known if there is always a winning strategy if we play with $k \geq \chi_g$ colors. However for the game distinguishing numbers we easily show they both have this growing property.

Proposition 1.1 *Let G be a graph and k a positive integer. If $k \geq D_{\mathcal{G}}(G)$ (resp. $k \geq D_{\mathcal{R}}(G)$), then the Gentle has a winning strategy with k colors, if he starts (resp. the Rascal starts).*

Proof: In order to win with k colors, the Gentle plays the same winning strategy he would have played with $D_{\mathcal{G}}(G)$ (resp. $D_{\mathcal{R}}(G)$) colors), except when the Rascal chooses a color strictly greater than $D_{\mathcal{G}}(G)$ (resp. $D_{\mathcal{R}}(G)$). In that case, the Gentle plays the winning move, he would have played if the Rascal had played the colors 1. Let c be the final coloring in the game with k colors with respect to the above strategy. Let $\tilde{c}$ be the coloring defined as follows:

$$\forall v \in V(G) \;\; \tilde{c}(v) = \begin{cases} c(v) & \text{if } c(v) \leq D_{\mathcal{G}}(G) \text{ (resp. } c(v) \leq D_{\mathcal{R}}(G)) \\ 1 & \text{otherwise} \end{cases}.$$

The coloring $\tilde{c}$ is a distinguishing coloring since it is obtained by following a winning strategy for the game with only $D_{\mathcal{G}}(G)$ (resp. $D_{\mathcal{R}}(G)$) colors. Let σ be an automorphism which preserves c. It is clear that σ preserves $\tilde{c}$ too. Since $\tilde{c}$ is distinguishing, σ must be the identity. This shows that c is a k-distinguishing coloring and that the Gentle has a winning strategy with k colors. □

As mentioned in the definition $D_{\mathcal{G}}$ and $D_{\mathcal{R}}$ could be eventually infinite. For example, if the graph has an automorphism of order two, the Rascal can win using a strategy closed to the Tweedledee Tweedledum one for the sum of opposite combinatorial games. He uses the involutive automorphism to copy the move the Gentle just made.

Proposition 1.2 *If G is a graph with a non trivial automorphism of order 2, then:*

1. *$D_{\mathcal{G}}(G) = \infty$ if $|V(G)|$ is even and*
2. *$D_{\mathcal{R}}(G) = \infty$, if $|V(G)|$ is odd.*

Proof: Let A be the set of vertices fixed by σ: $A = \{v \in G : \sigma(v) = v\}$. Note that $|V(G)|$ and $|A|$ have the same parity. We denote by r_i and s_i the i-th vertex played by the Rascal and the Gentle respectively.

First, assume $|A|$ is even and the Gentle starts. The winning strategy for the Rascal is as follows. If s_i is in A, then the Rascal plays another vertex r_i in A and does not pay attention to the color. This is always possible since $|A|$ is even. Else the Rascal plays such that $r_i = \sigma(s_i)$ and $c(r_i) = c(s_i)$. Since σ has order 2, the vertices outside A can be grouped by pair (u, v) with $\sigma(u) = v$ and $\sigma(v) = u$. Moreover the Gentle will be the first to play outside A. Hence such a move is always available for the Rascal.

Now suppose $|A|$ is odd and the Rascal begins. His first move is to color a vertex in A. The number of uncolored vertices in A are now even and it is the Gentle's turn. Then the Rascal wins with exactly the same strategy as above. □

For example, for cycles C_n or paths P_n on n vertices, we have $D_{\mathcal{G}}(C_n) = D_{\mathcal{G}}(P_n) = \infty$, if n is even, and $D_{\mathcal{R}}(C_n) = D_{\mathcal{R}}(P_n) = \infty$, if n is odd. Note that proposition 1.2 tells nothing about the possible values of $D_{\mathcal{G}}$ (resp. $D_{\mathcal{R}}$) when the number of fixed points is odd (resp. even). For cycles the question is rather complicated. We partially solve it in section 4. But for paths P_n, with $n \geq 2$, it is easy to show that $D_{\mathcal{G}}(P_n) = 2$ (resp. $D_{\mathcal{R}}(P_n) = 2$) if n is odd (resp. even).

There are also graphs with both $D_{\mathcal{G}}$ and $D_{\mathcal{R}}$ infinite, for example graphs with two transpositions moving a common vertex, like the complete graphs on more than 3 vertices. Finding graphs with both invariants finite, is a bit less obvious. Before stating the result, we recall that for a given finite group Γ, there are infinitely many non-isomorphic graphs whose automorphism group is Γ.

Proposition 1.3 *If G is a graph such that $Aut(G) = \mathbb{Z}/p\mathbb{Z}$, where p is prime and $p \geq 3$, then $D_{\mathcal{G}}(G) = D_{\mathcal{R}}(G) = 2$.*

Proof: Since $|Aut(G)|$ is prime all the orbits under the action of the whole group have either cardinal 1 or cardinal p. Let $\mathcal{O}_1$, ...,$\mathcal{O}_l$, with $l > 0$ be the orbits of cardinal p. Since there is at least three distinct vertices in each orbit $\mathcal{O}_i$ and two colored are allowed, the Gentle can always play, whoever starts, in a manner which ensures that at least one of the $\mathcal{O}_i$ is not monochromatic.

Let σ be a non trivial automorphism of $Aut(G)$. Since $|Aut(G)|$ is prime, σ generates $Aut(G)$ and the orbit under $Aut(G)$ and $< \sigma >$ are the same. The automorphism σ cannot preserve the coloring because in that case all the orbits $\mathcal{O}_i$ have to be monochromatic. This shows that the above strategy is winning for the Gentle and that $D_{\mathcal{G}}(G) = D_{\mathcal{R}}(G) = 2$. □

Determining if $D_{\mathcal{R}}$ or $D_{\mathcal{G}}$ is finite is far from being easy in general. However, in the second section of this article, we introduce the class of involutive graphs, for which we prove that the R-game distinguishing number is finite and more precisely, quadratically bounded above by the classical distinguishing number of the graph. This class, closely related to imprimitive action of group with complete block system of size 2, contains a variety of graphs such as hypercubes, even cycles, spherical graphs and even graphs [14, 8]. Then, in section 3, we specie the general result on involutive graphs to compute the exact values of the game invariants for hypercubes.

Theorem 1.4 *Let Q_n be the hypercube of dimension $n \geq 2$.*

1. *We have $D_{\mathcal{G}}(Q_n) = \infty$.*
2. *If $n \geq 5$, then $D_{\mathcal{R}}(Q_n) = 2$. Moreover $D_{\mathcal{R}}(Q_2) = D_{\mathcal{R}}(Q_3) = 3$.*
3. *We have $2 \leq D_{\mathcal{R}}(Q_4) \leq 3$.*

Finally, in section 4, we solve the problem for cycles, except when the number of vertices is prime. For even cycles, it is a straightforward application of a result on involutive graphs, but for odd cycles new ideas are needed.

Theorem 1.5 *Let C_n be a cycle of order $n \geq 3$.*

1. *If n is even, then $D_{\mathcal{G}}(C_n) = \infty$. If n is odd, then $D_{\mathcal{R}}(C_n) = \infty$.*
2. *If n is even and $n \geq 8$, then $D_{\mathcal{R}}(C_n) = 2$. Moreover $D_{\mathcal{R}}(C_4) = D_{\mathcal{R}}(C_6) = 3$.*
3. *If n is odd, not prime and $n \geq 9$, then $D_{\mathcal{G}}(C_n) = 2$.*
4. *If n is prime and $n \geq 5$, then $D_{\mathcal{G}}(C_n) \leq 3$. Moreover $D_{\mathcal{G}}(C_3) = \infty$ and $D_{\mathcal{G}}(C_5) = D_{\mathcal{G}}(C_7) = 3$.*

2 Involutive graphs

In this section, we introduce the new class of involutive graphs. This class contains well known graphs such as hypercubes, even cycles, even toroidal grids, spherical graphs and even graphs [14, 8]. For graphs in this class, we are going to prove that their R-game distinguishing number is quadratically bounded above by the distinguishing number.

An *involutive graph* G is a graph together with an involution, $bar : V(G) \to V(G)$, which commutes with all automorphisms and have no fixed point. In others words:

- $\forall u \in V(G)\ \overline{\overline{u}} = u$ and $\overline{u} \neq u$,
- $\forall \sigma \in Aut(G)\ \forall u \in V(G)\ \sigma(\overline{u}) = \overline{\sigma(u)}$.

The set $\{u, \overline{u}\}$ will be called a *block* and $\bar{u}$ will be called the *opposite* of u. An important remark is that, since bar commutes with all automorphism, the image of a block under an automorphism is also a block. The block terminology comes from imprimitive group action theory. Indeed, if we add the condition that the automorphism group of the graph acts transitively, then an involutive graph is a graph such that this action has a complete block system, whose blocks have size 2.

Before stating the results on involutive graphs, we want to emphasize that the no fixed point condition is not necessary but only convenient to write these results. With fixed points, we would have to permute who starts, depending on the parity of their number.

Let G be an involutive graph on n vertices. We define G' as the graph obtained from G by deleting all the edges and putting a new edge between all vertices and their opposites. The resulting graph is a disjoint union of $n/2$ copies of K_2. Moreover, it is also an involutive graph and it has the same blocks as G. The next proposition states that these graphs are in a sense the worst involutive graphs for the Gentle.

Proposition 2.1 *If G is an involutive graph then $D(G) \leq D(G')$ and $D_{\mathcal{R}}(G) \leq D_{\mathcal{R}}(G')$.*

Proof: It is enough to prove that any distinguishing coloring c of G' is also a distinguishing coloring of G. Let σ be a non trivial automorphism of G. Assume first, there are two disjoint blocks $\{u, \bar{u}\}$ and $\{v, \bar{v}\}$, with $u, v \in V(G)$, such that $\sigma(\{u, \bar{u}\}) = \{v, \bar{v}\}$. Since the coloring c is distinguishing for G', we have $c(\{u, \bar{u}\}) \neq c(\{v, \bar{v}\})$. Hence, σ does not preserve c. Suppose now that there is not two such blocks. Since σ is not trivial, there exists $u \in V(G)$ such that $\sigma(u) = \bar{u}$. But u and $\bar{u}$ must have distinct colors, otherwise the transposition which switch them would be a non trivial colors preserving automorphism of G'. Therefore, σ is not a colors preserving automorphism. We conclude that c is also a distinguishing coloring for G. □

For graphs which are disjoint union of K_2, we are able to compute the exact value of $D_{\mathcal{G}}$ and $D_{\mathcal{R}}$.

Proposition 2.2 *If G is the disjoint union of $n \geq 1$ copies of K_2, then $D_{\mathcal{G}}(G) = \infty$ and $D_{\mathcal{R}}(G) = n+1$.*

Proof: Such a graph G has an automorphism of order 2, with no fixed point, the one which exchanges the two vertices in each copy of K_2. By Proposition 1.2, we directly conclude that $D_{\mathcal{G}}(G) = \infty$.

We are now going to prove that $D_{\mathcal{R}}(G) = n + 1$. First, assume that only $k < n + 1$ colors are allowed during the game. The Rascal's strategy is to play always the color 1 in a way that at least one vertex of each copy of K_2 is colored with 1. Since there is strictly less than $n + 1$ different colors, whatever the Gentle plays there will be two distinct K_2 colored with the same couple of colors or there will be a monochromatic K_2. If there is a monochromatic K_2, the transposition which permutes its two vertices is clearly a colors preserving automorphism. In the other case, there are four distinct vertices $u_1, u_2, v_1, v_2 \in V(G)$, such that u_1 and u_2 (resp. v_1 and v_2) are in the same K_2. Moreover, u_1 and v_1 (resp. u_2 and v_2) share the same color. The automorphism σ, defined by $\sigma(u_i) = v_i$, $\sigma(v_i) = u_i$, with $i \in \{1, 2\}$, and $\sigma(w) = w$ for $w \in V(G) \setminus \{u_1, u_2, v_1, v_2\}$ preserves the coloring. Hence, the strategy described is winning for the Rascal. In conclusion, $D_{\mathcal{R}}(G) \geq n + 1$.

We now allow $n + 1$ colors. We recall that the Rascal plays first. The winning strategy for the Gentle is as follows. He always colors the remaining uncolored vertex in the copy of K_2, where the Rascal has just played before. He chooses his color such that the couple of colors for this K_2 is different from all the couples of colors of the previously totally colored copies of K_2. Moreover, he ensures that no K_2 is monochromatic. This is possible because, even if the first color in a K_2 is fixed by the Rascal, the Gentle can make n different no monochromatic couples of colors. Let σ be a non trivial automorphism of G and let $u \in V(G)$ be such that $\sigma(u) \neq u$. If u and $\sigma(u)$ belong to the same copy of K_2, σ does not preserve the coloring. Indeed, there is no monochromatic copy of K_2. On the other hand, if u and $\sigma(u)$ do not belong to the same copy, it means that the copy which contains u is sent by σ to the disjoint copy which contains $\sigma(u)$. But each copy of K_2 has a unique couple of colors. Hence, σ is not a colors preserving automorphism. This shows that the Gentle has a winning strategy with $n + 1$ colors. In conclusion, $D_{\mathcal{R}}(G) = n + 1$. □

It is shown in [6], that if G consists in $n \geq 1$ disjoint copies of K_2, then $D(G) = \lceil \frac{1+\sqrt{8n+1}}{2} \rceil$. Hence, we get examples of graphs for which the R-game distinguishing number is quadratic in the classical one. The following results are straightforward application of Proposition 2.1 and Proposition 2.2.

Proposition 2.3 *If G is an involutive graph on n vertices, then:*

- $D(G) \leq \lceil \frac{1+\sqrt{4n+1}}{2} \rceil$ *and*
- $D_{\mathcal{R}}(G) \leq \frac{n}{2} + 1$.

This result shows that involutive graphs have finite R-game distinguishing number. But the bound we obtained does not depend on the classical distinguishing number of the graph. If the Gentle knows this number, having in mind a distinguishing coloring, he can use it to play in a smarter way. In other words, for an involutive graph G, we can bound the game distinguishing number $D_{\mathcal{R}}$ using the classical one. It turns out that $D_{\mathcal{R}}(G)$ is in this case at most of order $D(G)^2$.

Theorem 2.4 *If G is an involutive graph with $D(G) \geq 2$, then $D_{\mathcal{R}}(G) \leq D(G)^2 + D(G) - 2$.*

Proof: We set $d = D(G)$. We are going to give a winning strategy for the Gentle with $d^2 + d - 2$ colors. Note that, since the Rascal starts the game, the Gentle can play in a way he is never the first to color a vertex in a block. When the Rascal colors a vertex $u \in V(G)$, the Gentle's strategy is to always answer by coloring the vertex $\overline{u}$ in same block.

Before stating how he chooses the color, we need some definitions. Let c be a distinguishing coloring of G with $d \geq 2$ colors. We set $k = d^2 + d - 2$. For all $i, j \in \{1, ..., d\}$, we define the following sets of vertices: $V_{ij} = \{u \in V(G) | c(u) = i \text{ and } c(\overline{u}) = j\}$. There are $\frac{d(d-1)}{2}$ sets V_{ij}, with $1 \leq i < j \leq d$, and d sets V_{ii}, with $i \in \{1, ...d\}$. To each set V_{ij}, with $1 \leq i < j \leq d$, we associate a specific number δ_{ij} in $\{1, ..., \frac{d(d-1)}{2}\}$. For each V_{ij}, with $i, j \in \{1, ...d\}$, we defined r_{ij} as follows:

$$r_{ij} = \begin{cases} \delta_{ij} & \text{if } 1 \leq i < j \leq d \\ k - r_{ji} & \text{if } 1 \leq j < i \leq d \\ \frac{d(d-1)}{2} + i & \text{if } i = j \text{ and } 1 \leq i \leq d-1 \\ 0 & \text{if } i = j = d \end{cases}$$

This defines only d^2 distinct numbers. We need despite $d^2 + d - 2$ colors to ensure the second following properties:

- $(*)$ $\forall i, i', j, j' \in \{1, ..., d\}$ $r_{ij} \equiv r_{i',j'}[k]$ if and only if $i = i'$ and $j = j'$,
- $(**)$ $\forall i, j \in \{1, ..., d\}$ if $i \neq j$, $r_{ii} \not\equiv -r_{jj}[k]$.

Let now $c' : V(G) \to \{1, ..., k\}$ be the coloring built during the game. The Gentle chooses the color as follows. When the Rascal colors $u \in V_{ij}$ with $c'(u)$, he colors $\overline{u}$ such that $c'(\overline{u}) - c'(u) \equiv r_{ij}[k]$.

To prove this strategy is winning, it is enough to show that for any automorphism σ which preserves c', we have $\sigma(V_{ij}) = V_{ij}$. Since all V_{ij} are monochromatic for the distinguishing coloring c, this will show that σ also preserves c and σ must therefore be the identity.

Let u be a vertex of G and $i, j \in \{1, ..., d\}$. We show that $u \in V_{ij}$, with $i \neq j$, if and only if $c'(\overline{u})-c'(u) \equiv r_{ij}[k]$ and $u \in V_{ii}$ if and only if $c'(\overline{u}) - c'(u) \equiv \pm r_{ii}[k]$. First, assume that $u \in V_{ij}$, with $i \neq j$. If u was colored by the Rascal, we have $c'(\overline{u}) - c'(u) \equiv r_{ij}[k]$. Else, $\overline{u}$ was colored by the Rascal. Since $\overline{u}$ is in V_{ji}, we have $c'(\overline{\overline{u}}) - c'(\overline{u}) \equiv r_{ji} \equiv -r_{ij}[k]$. Since $r_{ij} \equiv -r_{ji}[k]$, we also conclude that $c'(\overline{u}) - c'(u) \equiv r_{ij}[k]$. Now, if u is in V_{ii}, we only get that $c'(u) - c'(\overline{u}) \equiv \pm r_{ii}[k]$. Reciprocally, assume that $c'(\overline{u}) - c'(u) \equiv \pm r_{ij}[k]$. We have to prove that u belongs to V_{ij}. The vertex u belongs to some set V_{lm}, with $l, m \in \{1, ..., d\}$. By the previous part of the proof, we know that $c'(\overline{u}) - c'(u) \equiv \pm r_{lm}[k]$. Assume first that $c'(\overline{u}) - c'(u) \equiv r_{lm}[k]$. By property $(*)$, we conclude that $i = l$ and $j = m$. In other words, u is in V_{ij}. Second, if $c'(\overline{u}) - c'(u) \equiv -r_{lm}[k]$, we now that $l = m$. Hence, $r_{ij} \equiv -r_{ll}[k]$. It could happen only in two cases: $i = j = l = d$ or $i = j = l = d - 1$. In both cases, it shows that u is in V_{ij}. Finally, let σ be an automorphism which preserves c'. For any vertex $u \in V_{ij}$, with $i, j \in \{1, ..., d\}$, we have $c'(\overline{u}) - c'(u) \equiv c'(\overline{\sigma(u)}) - c'(\sigma(u))[k]$. Hence, $c'(\overline{\sigma(u)}) - c'(\sigma(u)) \equiv r_{ik}[k]$ and we conclude that $\sigma(u)$ is in the same set V_{ij} as u. □

In the above proof, decreasing the number of sets V_{ij} tightens the bound. In particular, we have the following corollary.

Corollary 2.5 *Let G be an involutive graph. If there is a coloring with $d \geq 2$ colors, such that bar is the only non trivial automorphism which preserves this coloring, then $D_{\mathcal{R}}(G) \leq 2d - 2$.*

Proof: We use the same notations as in the proof of Theorem 2.4. As in this proof the Gentle uses the distinguishing coloring c with d colors to build his strategy. Since we ask that bar is preserved, all the set V_{ij}, with $i \neq j$ are empty. So the Gentle needs only $2d - 2$ colors to apply his strategy. Indeed, he plays in a way that for all $u \in V(G)$, $c'(\bar{u}) - c'(u) \equiv i - 1[2d - 2]$, where $i \in \{1..., d\}$ is such that $u \in V_{ii}$. As for the proof of the previous theorem, we can prove that an automorphism which preserves the coloring c' must also preserve the coloring c. Thus, it is either the identity or the bar involution. But, it is clear that the Gentle's strategy leads to at least one block which is not monochromatic. Hence the bar involution cannot preserve the coloring c'. In conclusion, c' is a distinguishing coloring and the Gentle has a winning strategy with $2d - 2$ colors. □

3 Hypercubes

In this section, we use Corollary 2.5 to study the game distinguishing number of Q_n, the hypercube of dimension $n \geq 2$ and prove theorem 1.4. The vertices of Q_n will be denoted by words of length n on the binary alphabet $\{0, 1\}$. For a vertex u, $u(i)$ denotes its i-th letter. For the classical distinguishing number the question is solved in [2]: $D(Q_3) = D(Q_2) = 3$ and $D(Q_n) = 2$ for $n \geq 4$. As mention before, hypercubes are involutive graphs. The bar involution can actually be defined as the application which switch all the letters 0 to 1 and all the letters 1 to 0. Moreover, this application is an automorphism. Hence, by Proposition 1.2, we know that $D_{\mathcal{G}}(Q_n) = \infty$, for all $n \geq 2$. If we apply directly Theorem 2.4, we obtain that $D_{\mathcal{R}}(Q_n) \leq 4$, for $n \geq 2$. To improve this bound in order to prove the first item of Theorem 1.4, we are going to use Corollary 2.5. We will have to design nice distinguishing coloring of the hypercube with 2 colors. For this purpose, we introduce determining sets.

A subset S of vertices of a graph G is a *determining set* if the identity is the only automorphism of G whose restriction to S is the identity on S. Determining sets are introduced and studied in [3]. In the following lemma, we give a sufficient condition for a subset of Q_n to be determining.

Lemma 3.1 *Let S be a subset of vertices of Q_n, with $n \geq 2$. If for all $i \in \{1, ..., n\}$, there are two vertices v_0^i and v_1^i such that $v_0^i(i) = 0$, $v_1^i(i) = 1$ and all the other letters are the same, then S is a determining set of Q_n.*

Proof: Let $\sigma \in Aut(G)$, which is the identity on S. By absurd, suppose there exists a vertex u in $Q_n \setminus S$ such that $\sigma(u) \neq u$. Then, there exists $k \in \{1, ..., n\}$ such that $u(k) \neq \sigma(u)(k)$. Without lost of generality, we suppose that $u(k) = 0$. The vertices u and v_0^k differ on l letters, with $l \in \{1, ...n\}$. This means that the distance between u and v_0^k is equal to l. Since σ fixes v_0^k the distance between $\sigma(u)$ and v_0^k must be l too. Therefore, the vertices u and v_1^k have $l+1$ distinct letters, whereas the vertices $\sigma(u)$ and v_1^k have only $l-1$ distinct letters. This is impossible because the distance between u and v_1^k must be the same as the distance between $\sigma(u)$ and v_1^k. Therefore $\sigma(u) = u$ for all $u \in V(Q_n)$. In conclusion S is a determining set. □

Proposition 3.2 *Let Q_n be the n-dimensional hypercube with $n \geq 5$. Then $D_{\mathcal{R}}(Q_n) = 2$.*

Proof: For $i \in \{0, ..., n-1\}$, let v_i be the vertex with the i first letters equal to 1 and the $n-i$ others letter equal to 0. We define also the following vertices in Q_n: $f = 10010...0$, $c_1 = 010...01$ and $c_2 = 110...01$. The subgraph S will be the graph induced by $\{v_i, \overline{v_i} | 0 \leq i < n\} \cup \{f, \overline{f}, c_1, \overline{c_1}, c_2, \overline{c_2}\}$ (see Fig. 1). Let c be the coloring with two colors defined by: $c(u) = 1$ if $u \in V(S)$ and $c(u) = 2$ otherwise. We will show that this coloring fits hypothesis of Corollary 2.5. This would imply that $D_{\mathcal{R}}(Q_n) = 2 \times 2 - 2 = 2$.

Clearly, bar is preserved and we prove now this is the only non trivial colors preserving automorphism. An automorphisms σ which preserves this coloring lets S stable: $\sigma(S) = S$. The restriction of σ to S, say $\sigma|_S$, is an automorphism of S. The vertices v_0, ...,v_{n-1}, and $\overline{v_0}$ show that the subgraph S verifies Lemma 3.1 hypothesis. Hence σ is totally determined by the images of the elements of S. The two vertices f and $\overline{f}$ are the only vertices of degree 1 in S. So, either $\sigma(f) = \overline{f}$ and $\sigma(\overline{f}) = f$ or the both are fixed. In the first case, this implies that $\sigma(v_1) = \overline{v_1}$ and $\sigma(\overline{v_1}) = v_1$. In the second case v_1 and $\overline{v_1}$ are also fixed by σ. This is because they are respectively the only neighbors of f and $\overline{f}$ in S. The vertices v_0 and v_2 are the two remaining neighbors of v_1. Since they have not the same degree in S, they can't be switched by σ. Hence, in the first case, $\sigma(v_i) = \overline{v_i}$ and $\sigma(\overline{v_i}) = v_i$, with $i \in \{0, 2\}$. In the second case, v_0 and v_2 are fixed by σ. After that, it is easy to show that $\sigma|_S$ is either the identity or the bar involution. □

The above proof fails for Q_4 because the subgraph S will have other automorphisms than the bar involution (see Fig. 2). We can despite decrease the upper bound from 4 to 3.

Proposition 3.3 *We have $D_{\mathcal{R}}(Q_4) \leq 3$.*

Proof: Let S be the subgraph of Q_4 induced by the five vertices 0000, 1000, 1100, 1110 and 1011. Let $\overline{S}$ be the subgraphs induced by the opposite vertices of those in S. Note that S and $\overline{S}$ are disjoint. The Gentle's strategy is as follows. When the Rascal colors a vertex u in $V(G)$, the Gentle colors $\overline{u}$. He chooses the coloring c according to these rules:

- if u is in $V(S)$, $c(\overline{u}) - c(u) \equiv 1[3]$,
- if u is in $V(\overline{S})$ $c(\overline{u}) - c(u) \equiv -1[3]$,
- $c(\overline{u}) - c(u) \equiv 0[3]$ otherwise.