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$\mathbb{R}^3$ index for four-dimensional $\mathcal{N} = 2$ field theories

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In theories with $\mathcal{N} = 2$ supersymmetry on $\mathbb{R}^{3,1}$, BPS bound states can decay across walls of marginal stability in the space of Coulomb branch parameters, leading to discontinuities in the BPS indices $\Omega(\gamma, u)$. We consider a supersymmetric index $\mathcal{I}$ which receives contributions from 1/2-BPS states, generalizing the familiar Witten index $\mathrm{Tr}\,(-1)^F e^{-\beta H}$. We expect $\mathcal{I}$ to be smooth away from loci where massless particles appear, thanks to contributions from the continuum of multi-particle states. Taking inspiration from a similar phenomenon in the hypermultiplet moduli space of $\mathcal{N} = 2$ string vacua, we conjecture a formula expressing $\mathcal{I}$ in terms of the BPS indices $\Omega(\gamma, u)$, which is continuous across the walls and exhibits the expected contributions from single particle states at large β. This gives a universal prediction for the contributions of multi-particle states to the index $\mathcal{I}$. This index is naturally a function on the moduli space after reduction on a circle, closely related to the canonical hyperkähler metric and hyperholomorphic connection on this space.

It has been clear since the work of Seiberg and Witten [1] that extended supersymmetry gives enough control over four-dimensional quantum field theories to produce exact results on the *dynamics* of the theories, even when these theories are strongly interacting. Remarkably, such results are deeply related to some of the most interesting questions in the mathematics of algebraic geometry and differential geometry. As a significant example, the moduli space of a four-dimensional theory with $\mathcal{N} = 2$ supersymmetry on a circle is a hyperkähler manifold (a special class of manifolds satisfying Einstein's equations), whose metric encodes both instanton corrections to gauge couplings and the spectrum of BPS states in the four-dimensional theory [2]. In this Letter, we reinforce this connection, and construct a canonical function on the afore-mentioned moduli space, which on the one hand generates a solution to the self-dual Yang-Mills equations on this manifold, and on the other hand, purportedly encodes interactions of BPS states in four dimensions.

BPS INDICES AND WITTEN INDEX

In four-dimensional field theories on $\mathbb{R}^{3,1}$ with $\mathcal{N} = 2$ supersymmetry, the spectrum of BPS states in general strongly depends on the value of the Coulomb branch parameters. Part of this dependence can be removed by considering the BPS index

$$\Omega(\gamma, u) = -\tfrac{1}{2}\,\mathrm{Tr}_{\,\mathcal{H}_1(\gamma,u)}(-1)^{2J_3}(2J_3)^2, \qquad (1)$$

where $\mathcal{H}_1(\gamma, u)$ is the Hilbert space of one-particle states with electromagnetic charge $\gamma \in \Gamma$ in the Coulomb vacuum u, J_3 is a component of the rotation group along a fixed axis, and $(-1)^{2J_3}$ is the fermionic parity by virtue of the spin statistics theorem. The BPS index $\Omega(\gamma, u)$, being sensitive only to short multiplets saturating the BPS bound [3], is a locally constant, integer valued function of u, but it is discontinuous across certain walls in moduli space, where some of the BPS bound states with charge γ decay into multi-particle BPS states [1, 4]. The jump of $\Omega(\gamma, u)$ across the walls is governed by a universal wall-crossing formula [5], which can be derived by quantizing the configurational degrees of freedom of multi-centered BPS states near the wall [6–8] (see e.g. [9] for a review).

The present work addresses another apparently protected quantity, the Witten index

$$\mathcal{I}(\beta, u, C) = -\tfrac{1}{2}\,\mathrm{Tr}_{\mathcal{H}(u)}(-1)^{2J_3}(2J_3)^2\sigma\, e^{-\beta H - 2\pi \mathrm{i}\langle\gamma, C\rangle}, \qquad (2)$$

where $\mathcal{H}(u)$ is the full Hilbert space of the four-dimensional theory on $\mathbb{R}^3$. Here, β is the inverse temperature, conjugate to the Hamiltonian H, C are chemical potentials conjugate to the electromagnetic charge γ, and σ is an operator on $\mathcal{H}(u)$ acting by a sign σ_γ in the sector with charge γ, such that $\sigma_\gamma\sigma_{\gamma'} = (-1)^{\langle\gamma,\gamma'\rangle}\sigma_{\gamma+\gamma'}$, where $\langle\gamma, \gamma'\rangle$ is the usual Dirac-Schwinger-Zwanziger product — this sign is crucial in ensuring the consistency of self-dual field theories [10–14]. For simplicity, we restrict to theories without flavor charges. The use of the canonical ensemble with respect to the electromagnetic charges is not essential, but facilitates the geometric interpretation of the index.

Most importantly, unlike the well-studied case of the index on S^3, the spectrum of the Hamiltonian on $\mathbb{R}^3$ is gapless, due to massless gauge bosons and their superpartners, and continuous, as it includes all multi-particle states made out of the discrete states in $\bigoplus_{\gamma\in\Gamma}\mathcal{H}_1(\gamma, u)$. The contribution of the point spectrum to the index (2) is controlled by the BPS indices $\Omega(\gamma, u)$, and is therefore

discontinuous across walls of marginal stability. Multi-particle states, on the other hand, can also contribute despite the fact that they do not saturate the BPS bound, due to a possible spectral asymmetry between bosonic and fermionic states [15]. Our main assumption is that the Witten index (2) is continuous across walls of marginal stability, as a result of cancellations between discontinuities from single and multi-particle state contributions. This assumption is physically reasonable, since the path integral defining (2) suffers no phase transition across the wall. Under this assumption, we propose a formula for expressing (2) in terms of the BPS indices $\Omega(\gamma, u)$.

Our assumption is further supported by analogy with the case of framed BPS indices associated to line defects in $\mathcal{N} = 2$ theories of class S. These indices are defined by a formula almost identical to (2) (without the insertion of $(2J_3)^2$), and are known to be smooth across BPS-walls [16]. The Witten index (2) can be viewed as the extension of the framed index to the case of a trivial line defect.

Another class of examples where a Witten-type index is known to be a smooth function of the moduli arises in $\mathcal{N} = 2$ supersymmetric massive theories in 1+1 dimensions: the BPS indices Ω_{ab}, which count single-particle kinks interpolating between pairs ab of supersymmetric vacua, exhibit similar wall-crossing phenomena as in 3+1 dimensions [17], while the CFIV index $\mathrm{Tr}_{ab}(-1)^F F e^{-\beta H}$ is continuous across the walls, as a result of cancellations between single-particle and multi-particle contributions [18][44].

Yet another way to support our assumption is the general expectation that the Witten index controls quantum corrections to BPS-saturated couplings in the low-energy effective action. In the case of four-dimensional $\mathcal{N} = 2$ theories, an appropriate coupling is the metric on the moduli space of the theory reduced on a spatial circle of radius R. In 2+1 dimensions, Abelian gauge fields can be dualized into scalar fields, and the dynamics on the Coulomb branch can be formulated as a non-linear sigma model. Its target space $\mathcal{M}_3(R)$ is a torus fibration over the Coulomb branch moduli space $\mathcal{M}_4$ in 3+1 dimensions, with the torus fiber parametrizing the holonomies C of the electromagnetic gauge fields around the circle [19]. Supersymmetry requires the metric on $\mathcal{M}_3(R)$ to be hyperkähler (HK). In the limit $R \to \infty$ it is obtained from the special Kähler metric on $\mathcal{M}_4$ via the so-called 'rigid c-map' procedure [20]. For finite radius, however, the metric on $\mathcal{M}_3(R)$ receives instanton corrections of order e^{-R} from BPS states in 3+1 dimensions, whose Euclideanized worldline winds around the circle [2, 19] (a supersymmetric version of a mechanism first envisaged in [21]). Although corrections to the metric components include an infinite series of multi-instanton corrections, they are entirely controlled by the BPS indices $\Omega(\gamma, u)$ counting single-particle states. Furthermore, it is manifest from the twistorial construction of $\mathcal{M}_3(R)$ [2] that the quantum corrected metric is regular across walls of marginal stability, with multi-instanton contributions on one side of the wall replacing the one-instanton correction on the other side (alternatively, the smoothness of the metric on $\mathcal{M}_3(R)$ provides a physical rationale for the wall-crossing formulae of [5]).

Since quantum corrections to the moduli space metric in theories with 8 supercharges are generally saturated by 1/2-BPS contributions, it is natural to expect a connection between the metric on $\mathcal{M}_3(R)$ and the Witten index $\mathcal{I}(\beta, u, C)$ for $\beta = 2\pi R$. The goal of this paper is to construct a natural function on the family of spaces $\mathcal{M}_3(R)$, continuous across the walls, which reproduces the expected contributions of single-particle states to the Witten index in the limit $R \to \infty$. We conjecture that these two functions are equal, which allows us to predict the contributions of the continuum of multi-particle states to $\mathcal{I}(\beta, u, C)$.

The clue for our construction comes from an analogous problem in superstring theory, namely the vector multiplet moduli space $\widetilde{\mathcal{M}}_3$ in type IIA/B string vacua of the form $\mathbb{R}^3 \times S^1(R) \times \mathfrak{Y}$ where $\mathfrak{Y}$ is a Calabi-Yau threefold. After T-duality on the circle and its decompactification, the same moduli space describes the hypermultiplet sector of the dual type IIB/A theory on $\mathbb{R}^4 \times \mathfrak{Y}$ [22]. In contrast to the gauge theory case, $\widetilde{\mathcal{M}}_3$ is a quaternion-Kähler (QK) manifold, where R appears as one of the coordinates. In the limit $R \to \infty$, the metric is obtained by the 'local c-map' procedure from the vector multiplet moduli space $\widetilde{\mathcal{M}}_4$ in type IIA/B on $\mathbb{R}^4 \times \mathfrak{Y}$ [23], whereas for finite radius it receives $\mathcal{O}(e^{-R})$ corrections from four-dimensional BPS states winding around the circle (T-dual to D-instantons). These instanton corrections can be incorporated through the twistor space construction [24, 25] (see [26, 27] for reviews). However, unlike the gauge theory set-up, the instanton series is divergent due to the exponential growth of the BPS indices. Arguably, this is resolved by the existence of further gravitational (or NS5-brane) instanton corrections of order e^{-R^2} [28]. In the sector with zero NUT charge, which is insensitive to these additional instantons, the twistorial construction of [24, 25] is formally isomorphic to the gauge-theoretic one [2], specialized to the case of theories with a non-anomalous $U(1)_R$ symmetry, described by a homogeneous prepotential $F(X)$ of degree 2. This isomorphism was shown to be a particular instance of a general correspondence between QK metrics with quaternionic $U(1)$ action and HK metrics with $U(1)$ isometry rotating the complex structures [29–31] (the correspondence proceeds by lifting the $U(1)$ action on the QK manifold to the Swann bundle and then taking the HK quotient). Through this correspondence, the family of HK metrics $\mathcal{M}_3(R)$ inherits a canonical function, the moment map of the $U(1)_R$ action, which is smooth as long as the metric on $\mathcal{M}_3(R)$ is. On the QK side it appears as the 'contact potential', which relates the $O(2)$-twisted canonical one-form to the holomorphic contact one-form [24, 32]. As we shall see, a 'generalized contact potential' can be defined even when the prepotential F is not homogeneous. Geometrically, it can be understood as the ratio of two Hermitian metrics on the canonical line bundle constructed in [33].

A FAMILY OF SMOOTH FUNCTIONS ON $\mathcal{M}_3(R)$

To define our candidate for the Witten index $\mathcal{I}$, let us first recall the twistorial construction of the HK metric on $\mathcal{M}_3(R)$ [2] (see [34] for a review). The twistor space $\mathcal{Z} = \mathbb{P}_t \times \mathcal{M}_3(R)$ carries a family of functions $\{\mathcal{X}_\gamma(t)\}_{\gamma\in\Gamma}$, holomorphic in complex structure $J(t)$, satisfying the integral equations [45]

$$\frac{\mathcal{X}_\gamma}{\mathcal{X}^{\rm sf}_\gamma} = \exp\left[\sum_{\gamma'} \frac{\Omega(\gamma')}{4\pi\mathrm{i}}\,\langle\gamma,\gamma'\rangle \int_{\ell_{\gamma'}} \frac{\mathrm{d}t'}{t'}\frac{t+t'}{t-t'}\log\left(1-\mathcal{X}_{\gamma'}(t')\right)\right], \tag{3}$$

where $\ell_{\gamma'}$ are the BPS rays $\{t'\in\mathbb{C}^\times :\ Z_{\gamma'}/t'\in \mathrm{i}\mathbb{R}^-\}$ and $\mathcal{X}^{\rm sf}_\gamma$ provide the boundary conditions at $R\to\infty$,

$$\mathcal{X}^{\rm sf}_\gamma = \sigma_\gamma\, e^{-\pi\mathrm{i}R\left(t^{-1}Z_\gamma - t\bar Z_\gamma\right) - 2\pi\mathrm{i}\langle\gamma,C\rangle}. \tag{4}$$

Here $Z_\gamma = \langle\gamma, X\rangle$ is the central charge and $X = (X^\Lambda, F_\Lambda)$ is the holomorphic symplectic section on the special Kähler manifold $\mathcal{M}_4$ with $F_\Lambda = \partial_{X^\Lambda} F$ in special coordinates. In the limit $R\to\infty$, the system (3) can be solved iteratively, generating a sum of products of iterated integrals of $\mathcal{X}^{\rm sf}_\gamma$ interpreted as multi-instanton contributions. Given such a solution, the triplet of Kähler forms on $\mathcal{M}_3(R)$, which define the metric, is read off from the $\mathcal{O}(2)$-twisted holomorphic symplectic form on $\mathcal{Z}$,

$$\omega = \mathrm{i}t^{-1}\omega_+ + \omega_3 + \mathrm{i}t\omega_- = \frac{\epsilon^{ab}}{8\pi^2}\,\frac{\mathrm{d}\mathcal{X}_{\gamma_a}}{\mathcal{X}_{\gamma_a}}\wedge\frac{\mathrm{d}\mathcal{X}_{\gamma_b}}{\mathcal{X}_{\gamma_b}}, \tag{5}$$

where γ_a is a basis of Γ, and ϵ^{ab} is the inverse of $\langle\gamma_a,\gamma_b\rangle$.

With these notations in place, fix any smooth function $F_\gamma(t,u,C)$ on $\Gamma\times\mathcal{Z}$, linear in γ and define

$$\Phi(R,u,C) = \sum_\gamma \Omega(\gamma)\int_{\ell_\gamma}\frac{\mathrm{d}t}{t}\,F_\gamma\,\log\left(1-\mathcal{X}_\gamma\right). \tag{6}$$

We claim that Φ is a smooth function on $\mathcal{M}_3(R)$, provided the BPS indices $\Omega(\gamma)$ jump across walls of marginal stability according to the standard wall-crossing formula [5]. Indeed, on a wall $W(\gamma_1,\gamma_2)$, where the central charges Z_{γ_1}, Z_{γ_2} associated to two primitive charge vectors become aligned in the complex plane, the BPS rays $\ell_{m\gamma_1+n\gamma_2}$ with $m,n\geq 0$ all coalesce into one ray ℓ, across which the potential discontinuity is given by

$$\begin{aligned}\Delta\Phi = \int_\ell \frac{\mathrm{d}t}{t}\sum_\gamma F_\gamma\Big[&\Omega^+(\gamma)\log\left(1-\mathcal{X}^+_\gamma\right)\\ &-\Omega^-(\gamma)\log\left(1-\mathcal{X}^-_\gamma\right)\Big],\end{aligned} \tag{7}$$

where $\Omega^\pm(\gamma)$ and $\mathcal{X}^\pm_\gamma$ are the BPS indices and solutions of the corresponding equations (3) on either side of the wall. Now, recall that the semi-classical limit of the motivic version of the wall-crossing formula implies the functional identity [30]

$$\sum_\gamma \Omega^+(\gamma)\,L_{\sigma_\gamma}(\mathcal{X}^+_\gamma) = \sum_\gamma \Omega^-(\gamma)\,L_{\sigma_\gamma}(\mathcal{X}^-_\gamma) \tag{8}$$

where $L_\varepsilon(z)$ is a variant of the Rogers dilogarithm,

$$L_\varepsilon(z) \equiv \mathrm{Li}_2(z) + \frac{1}{2}\,\log(\varepsilon^{-1}z)\log(1-z). \tag{9}$$

The invariance of (8) under monodromies $M_\gamma : \mathcal{X}_{\gamma'}\mapsto e^{2\pi\mathrm{i}\langle\gamma,\gamma'\rangle}\mathcal{X}_{\gamma'}$ leads to the Γ-valued identity

$$\sum_\gamma \gamma\left[\Omega^+(\gamma)\log\left(1-\mathcal{X}^+_\gamma\right) - \Omega^-(\gamma)\log\left(1-\mathcal{X}^-_\gamma\right)\right] = 0. \tag{10}$$

The vanishing of the discontinuity (7) then follows from (10) and from the linearity of F_γ with respect to γ.

A CANDIDATE FOR THE WITTEN INDEX

Having constructed a family of smooth functions on $\mathcal{M}_3(R)$, we now aim for one that may plausibly be identified with the Witten index (2). For HK manifolds $\mathcal{M}_3(R)$ related to QK manifolds $\widetilde{\mathcal{M}}_3$ by the QK/HK correspondence, a natural candidate is the contact potential on $\widetilde{\mathcal{M}}_3$ [24, 32], which translates on the HK side into

$$\mathcal{I} = \frac{R}{16\mathrm{i}\pi^2}\sum_\gamma \Omega(\gamma)\int_{\ell_\gamma}\frac{\mathrm{d}t}{t}\left(t^{-1}Z_\gamma - t\bar Z_\gamma\right)\log\left(1-\mathcal{X}_\gamma(t)\right). \tag{11}$$

This function is a member of the family (6) with $F_\gamma(t)\propto t^{-1}\,Z_\gamma - t\bar Z_\gamma$, so it is smooth across walls of marginal stability. Its reality follows from the reality property $\overline{\mathcal{X}_\gamma(-1/\bar t)} = \mathcal{X}_{-\gamma}(t)$ and the CPT relation $\Omega(-\gamma)=\Omega(\gamma)$.

In order to assess whether (11) qualifies to represent the Witten index, let us compute the formal multi-instanton expansion of $\mathcal{I}$, by substituting the iterated solution of (3) into (11). Up to second order, $\mathcal{I} = \sum_\gamma \mathcal{I}^{(1)}_\gamma + \sum_{\gamma,\gamma'}\mathcal{I}^{(2)}_{\gamma,\gamma'} + \ldots$ with

$$\begin{aligned}\mathcal{I}^{(1)}_\gamma &= \frac{R}{4\pi^2}\,\sigma_\gamma\,\overline{\Omega}(\gamma)\,|Z_\gamma|K_1(2\pi R|Z_\gamma|)\,e^{-2\pi\mathrm{i}\langle\gamma,C\rangle},\\ \mathcal{I}^{(2)}_{\gamma,\gamma'} &= -\frac{R}{64\pi^3}\,\overline{\Omega}(\gamma)\,\overline{\Omega}(\gamma')\,\langle\gamma,\gamma'\rangle\int_{\ell_\gamma}\frac{\mathrm{d}t}{t}\int_{\ell'_\gamma}\frac{\mathrm{d}t'}{t'}\frac{t+t'}{t-t'}\\ &\quad\times\left(t^{-1}Z_\gamma - t\bar Z_\gamma\right)\mathcal{X}^{\rm sf}_\gamma(t)\mathcal{X}^{\rm sf}_{\gamma'}(t'),\end{aligned} \tag{12}$$

where $\overline{\Omega}(\gamma) = \sum_{d|\gamma}\frac{1}{d^2}\Omega(\gamma/d)$ denotes the rational index. Remarkably, for primitive charge vector γ the one-instanton contribution $\mathcal{I}^{(1)}_\gamma$ agrees with the contribution of a single-particle, relativistic BPS state of charge γ and mass $M=|Z_\gamma|$ to the Witten index (2). To see this, we use a Schwinger time parametrization to linearize the relativistic Hamiltonian $H=\sqrt{-\Delta+M^2}$, and introduce a non-zero chemical potential θ conjugate to J_3 and periodic boundary conditions $\psi(z)=\psi(z+L)$ along the z axis, with $L\gg 1/M$, to regulate infrared divergences. Denoting by $\chi_{\rm spin}(\theta)$ the $SU(2)$ character for the spin degrees

of freedom, we have

$$\begin{aligned}\mathrm{Tr}\, e^{-2\pi RH+\mathrm{i}\theta J_3} &= R\int_0^\infty \frac{\mathrm{d}t}{t^{3/2}}\,\mathrm{Tr}\, e^{-\pi\frac{R^2}{t}-\pi(-\Delta+M^2)t+\mathrm{i}\theta J_3}\\ &= R\int_0^\infty \frac{\mathrm{d}t}{t^{3/2}}\frac{L}{2\pi\sqrt{t}}\,\frac{\chi_{\rm spin}(\theta)}{4\sin^2(\theta/2)}\,e^{-\pi\frac{R^2}{t}-\pi M^2 t}\\ &= \frac{L}{2\pi}\frac{\chi_{\rm spin}(\theta)}{4\sin^2(\theta/2)}\,2M\,K_1(2\pi MR)\,. \quad (13)\end{aligned}$$

For a BPS multiplet of spin j, the spin character is

$$\chi_{\rm spin}(\theta) = \left(2+2\cos\frac{\theta}{2}\right)\frac{\sin\left[(j+\frac12)\theta\right]}{\sin(\theta/2)}\,, \qquad (14)$$

corresponding to a BPS index $\Omega(\gamma)=2\partial_\theta^2\chi_{\rm spin}(\theta)|_{\theta=2\pi} = (-1)^{2j}(2j+1)$. Comparing with the first line in (12) we find

$$\mathcal{I}^{(1)}_\gamma = 2R\lim_{\substack{\theta\to 2\pi\\ L\to\infty}}\partial_\theta^2\left[\frac{\sin^2(\theta/2)}{\pi L}\,\mathrm{Tr}\left(\sigma\, e^{-2\pi RH+\mathrm{i}\theta J_3-2\pi\mathrm{i}\langle\gamma,C\rangle}\right)\right]. \qquad (15)$$

The factor $\sin^2\frac{\theta}{2}/(2\pi L)$ can be understood as dividing by the regularized volume of $\mathbb{R}^3$.

Based on this agreement, and smoothness across walls of marginal stability, we conjecture that (11) in fact computes the Witten index (2), with the specific prescription given in (15) for regulating infrared divergences. If true, this implies that the two-instanton term $\mathcal{I}^{(2)}_{\gamma,\gamma'}$ in (12) should be identified with the contribution of the continuum of two-particle states, and similarly for higher $\mathcal{I}^{(n)}$'s.

GEOMETRIC NATURE OF $\mathcal{I}$

While the Ansatz (11) was suggested by the QK/HK correspondence, we now wish to elucidate its nature from the viewpoint of the HK geometry of $\mathcal{M}_3(R)$, without assuming that the prepotential $F(X)$ is homogeneous. For this purpose, we need to recall an additional construct on the HK space $\mathcal{M}_3(R)$, namely the canonical hyperholomorphic line bundle $\mathscr{L}$ introduced in [33], generalizing the one afforded by the QK/HK correspondence in the homogeneous case [29, 30]. In twistorial terms, $\mathscr{L}$ descends from a line bundle $\mathscr{L}_\mathcal{Z}$ on $\mathcal{Z}$ determined by the local holomorphic section $\hat\Upsilon = \Upsilon_{\rm sf}\Upsilon_{\rm inst}$, given in unitary gauge by

$$\begin{aligned}\Upsilon_{\rm sf} &= e^{-\mathrm{i}\pi\left[\frac{R^2}{4}\left(\frac{2F-X^\Lambda F_\Lambda}{t^2}+t^2(2\bar F-\bar X^\Lambda\bar F_\Lambda)\right)+\frac{R}{2}\langle t^{-1}X-t\bar X,C\rangle\right]},\\ \Upsilon_{\rm inst} &= \exp\left[\frac{1}{8\pi^2}\sum_\gamma\Omega(\gamma)\int_{\ell_\gamma}\frac{\mathrm{d}t'}{t'}\,\frac{t+t'}{t-t'}\left(L_{\sigma_\gamma}(\mathcal{X}_\gamma(t'))\right.\right.\\ &\qquad \left.\left.-\frac12\,\log\frac{\mathcal{X}^{\rm sf}_\gamma(t)}{\mathcal{X}^{\rm sf}_\gamma(t')}\log\left(1-\mathcal{X}_\gamma(t')\right)\right)\right]. \qquad (16)\end{aligned}$$

The line bundle $\mathscr{L}_\mathcal{Z}$ is equipped with a family of connections, represented by the one-forms

$$\hat\lambda(t) = \frac{\varepsilon^{ab}}{8\pi^2}\,\log\mathcal{X}_{\gamma_a}\,\mathrm{d}\log\mathcal{X}_{\gamma_b} - \frac{1}{2\pi\mathrm{i}}\,\mathrm{d}\log\hat\Upsilon, \qquad (17)$$

whose curvature is given by the holomorphic symplectic form (5). We define the "generalized contact potential" as the contraction of $\hat\lambda(t)$ with the vector field $\kappa = t\partial_t$ on $\mathcal{Z}=\mathbb{P}_t\times\mathcal{M}$ generating the $\mathbb{C}^\times$ action on the first factor and leaving the second factor invariant,

$$\hat\mu(t) = -\mathrm{i}\left(\iota_\kappa\hat\lambda(t)\right). \qquad (18)$$

This is a function on $\mathcal{Z}$, meromorphic on each fiber. A computation shows that, upon adding a suitable "classical" term,

$$\mathcal{I}_{\rm tot} \equiv \mathcal{I} - \frac{R^2}{2}\,\mathrm{Im}\,(\bar X^\Lambda F_\Lambda) \qquad (19)$$

the completed (purported) Witten index $\mathcal{I}_{\rm tot}$ is equal to the constant term in the Laurent expansion of $\hat\mu(t)$ around $t=0$, or equivalently, $t=\infty$.

By the twistor correspondence [35], the complex line bundle $\mathscr{L}_\mathcal{Z}$ on $\mathcal{Z}$ descends to a line bundle with a hyperholomorphic connection on $\mathcal{M}_3(R)$. In the same unitary trivialization as above, it can be represented by the one-form $\hat\lambda(t)+\check\lambda(t)$ where

$$\begin{aligned}\check\lambda(t) = \frac{R}{8\pi^2}\Bigg[&2\pi\mathrm{i}\,\epsilon^{ab}\left(t^{-1}Z_{\gamma_a}-t\bar Z_{\gamma_a}\right)\mathrm{d}\log\mathcal{X}_{\gamma_b} \qquad (20)\\ &+\sum_\gamma\Omega(\gamma)\int_{\ell_\gamma}\frac{\mathrm{d}t'}{t'}\Big(\frac{Z_\gamma}{t'}-t'\bar Z_\gamma\Big)\frac{t'\partial^{(0)}+t\bar\partial^{(0)}}{t'-t}\log\left(1-\mathcal{X}_\gamma(t')\right)\Bigg]\end{aligned}$$

is a (1,0)-form in complex structure $J(t)$, smooth across BPS rays, such that $\hat\lambda(t)+\check\lambda(t)$ is independent of t [46]. The curvature $\mathscr{F} = \mathrm{d}(\hat\lambda+\check\lambda)$ is then hyperholomorphic, i.e. of type (1,1) in any complex structure $J(t)$ [30, 33], and is given by

$$\mathscr{F} = \frac{\mathrm{i}}{\pi}\,\partial^{(t)}\bar\partial^{(t)}\,\mathrm{Re}\,\log\hat\Upsilon. \qquad (21)$$

Our second claim is that the constant term $\check\lambda_0$ in the Laurent expansion of $\check\lambda(t)$ around $t=0$ (or equivalently $t=\infty$) satisfies

$$\check\lambda_0 = \mathrm{i}\big(\partial^{(0)}-\bar\partial^{(0)}\big)\mathcal{I}_{\rm tot}. \qquad (22)$$

This implies that the completed Witten index determines the difference between the hyperholomorphic curvature $\mathscr{F}$ and the Kähler form for the instanton corrected HK metric on $\mathcal{M}_3(R)$ in complex structure $J(0)$ via

$$\omega_3 - \mathscr{F} = 2\mathrm{i}\partial^{(0)}\bar\partial^{(0)}\mathcal{I}_{\rm tot}. \qquad (23)$$

This equation generalizes the well-known statement in the homogeneous case that the moment map μ of the $U(1)_R$ action with respect to ω_3 (which coincides with the contact potential on the QK side) is such that $\omega_3 - 2\mathrm{i}\partial^{(0)}\bar\partial^{(0)}\mu$ is a hyperholomorphic two-form [29, 31]. One way to express (23) geometrically is to say that $\mathcal{I}_{\rm tot}$ is equal to the logarithm of the ratio of two different Hermitian metrics on the line bundle $\mathscr{L}_\mathcal{Z}$, whose curvatures are equal to the

Kähler form ω_3 and the hyperholomorphic curvature $\mathscr{F}$, respectively.

Another interesting consequence of (23) is that $\mathcal{I}_{\rm tot}$ is a quasi-harmonic function on $\mathcal{M}_3(R)$,

$$\Delta \mathcal{I}_{\rm tot} = -4r, \tag{24}$$

where Δ is the Laplace-Beltrami operator for the HK metric on $\mathcal{M}_3(R)$ and the right-hand side is minus the real dimension of $\mathcal{M}_3(R)$. The condition (24) is reminiscent of the second order differential equations which typically constrain BPS saturated amplitudes.

In the Appendix we show that the geometrical objects introduced in this section can be defined for a general class of HK manifolds, and obtain a generalization of Eq. (23) valid in any complex structure $J(t)$.

DISCUSSION

In this article we conjectured a formula (11) for the generalized Witten index (2) in four-dimensional $\mathcal{N} = 2$ gauge theories. The formula is manifestly smooth across walls of marginal stability, and correctly reproduces the expected BPS bound states contributions. The evidence for this conjecture is admittedly weak, since within the class (6) of smooth functions on the Coulomb branch $\mathcal{M}_3(R)$ in three dimensions, one could easily find other functions which would differ only at higher order in the multi-particle expansion. Our proposal however is distinguished by the fact that a completed version (19) of $\mathcal{I}$ is related to the Kähler form and hyperholomorphic curvature on $\mathcal{M}_3(R)$ via (23), in accordance with the general slogan that corrections to the moduli space metric in theories with 8 supercharges are saturated by 1/2-BPS contributions. It would be interesting to derive the "classical term" in (19) from the contribution of massless gauge bosons and supersymmetric partners, and extend our construction to gauge theories with massive flavors [47]. We note that the function $\mathcal{I}$ has already appeared in the context of the analogy of the system (3) with TBA equations [2], where it is identified with the free energy of the corresponding integrable system [36], and in the context of minimal surfaces in AdS_5 [37, 38].

If correct, our conjecture predicts that multi-particle state contributions to the Witten index are universal functions of the BPS indices $\Omega(\gamma)$ associated to the constituents. The predicted contribution of the continuum of two-particle states can be found in (12), while higher orders can be easily obtained by combining (11) with the iterated solution to the TBA-like system (3). It is a challenge to check these predictions from a direct computation of the difference of densities of bosonic and fermionic states of a system of n dyons. While the result near a wall of marginal stability can actually be deduced by analyzing the non-relativistic electron-monopole system [39], the result (12) should hold throughout moduli space, where the constituents are relativistic.

Note also that our conjecture naturally extends to the case of $\mathcal{N} = 2$ string vacua, where the formula (11) computes instanton corrections to the contact potential on the QK moduli space $\widetilde{\mathcal{M}}_3$ generated by multi-dyonic BPS black holes. Therefore, another check would be to reproduce the smooth, duality invariant partition function for two-centered D4-D2-D0 black holes constructed in [40], extending the arguments in [41] beyond the one-instanton level.

As we have mentioned, the generalized Witten index (2) may be viewed as the analog of the framed BPS index for a trivial line defect. One possible way to derive (2) would then be to study the fusion of two line defects whose OPE contain the trivial line defect. This analogy also suggests the existence of a refined Witten index, which would arise in the fusion of framed protected spin characters. It is natural to conjecture that this refined index might be related to the CFIV index of the two-dimensional theory obtained by placing the four-dimensional theory on an Ω-background with $\epsilon_1 \neq 0, \epsilon_2 = 0$ [48].

Finally, our conjecture – if true – could reveal interesting and nontrivial information on BPS spectra which is not easily accessible by other means. For example, consider a theory of class S where the ultraviolet curve C is a compact Riemann surface with negative curvature. In this case $\mathcal{I}_{\rm tot}$ is just the moment map for the natural $U(1)$ action on Hitchin data and hence proportional to the L^2 norm-square of the Higgs field [31]. The expression (11) is highly nontrivial already in the A_1 case. In this case one may be able to give a systematic large R expansion of the norm-square of the Higgs field by solving the classical sinh-Gordon theory on C. Using the parametrization of [42, Eq. (13.14)], it is easy to show that, on a real slice of moduli space one needs to expand

$$\mathcal{I}_{\rm tot} = \frac{\mathrm{i}R^2}{4} \int_C \lambda\bar\lambda \cosh(2h) \tag{25}$$

at large R for solutions to the sinh-Gordon equation

$$\partial\bar\partial h - 2R^2 \lambda\bar\lambda \sinh(2h) = 0 \tag{26}$$

with boundary condition $h \sim -\frac{1}{2}\log|z - z_a| + \cdots$ at the first order zeros $z = z_a$ of the quadratic differential λ^2. We hope to return to this problem in a future publication.

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APPENDIX

In this section we elaborate on the geometric origin of the function $\mathcal{I}_{\rm tot}$ and establish (23) as a special case of a more general formula derived from the twistorial construction of HK manifolds developed in [43].

For a HK manifold $\mathcal{M}$, let us choose an atlas $\cup\mathcal{U}_i$ covering the twistor space $\mathcal{Z}=\mathbb{P}_t\times\mathcal{M}$ and a set of Darboux coordinate systems $\Xi_a^{[i]}=(\xi^\Lambda_{[i]},\tilde\xi^{[i]}_\Lambda)$, $\Lambda=1\dots n$, regular everywhere on $\mathcal{U}_i$ except for $\xi^\Lambda_{[+]}$ and $\xi^\Lambda_{[-]}$, which are allowed to have simple poles at two special points $t=0$ and $t=\infty$. The real structure is assumed to map $\Xi_a^{[i]}(t)$ to $\overline{\Xi_a^{[\bar\imath]}}(-1/\bar t)$, where $\mathcal{U}_{\bar\imath}$ is the patch opposite to $\mathcal{U}_i$ under the antipodal map. The holomorphic symplectic structure on $\mathcal{Z}$ is specified by a set of local holomorphic functions $H^{[ij]}(\xi_{[i]},\tilde\xi^{[j]},t)$ which generate symplectomorphisms on overlaps $\mathcal{U}_i\cap\mathcal{U}_j$ of two patches,

$$\xi^\Lambda_{[j]}=\xi^\Lambda_{[i]}-\partial_{\tilde\xi^{[j]}_\Lambda}H^{[ij]},\qquad \tilde\xi^{[j]}_\Lambda=\tilde\xi^{[i]}_\Lambda+\partial_{\xi^\Lambda_{[i]}}H^{[ij]},\tag{27}$$

subject to the obvious reality and cocycle conditions on triple overlaps $\mathcal{U}_i\cap\mathcal{U}_j\cap\mathcal{U}_k$. The HK manifold $\mathcal{M}$ parametrizes the space of solutions $\Xi_a^{[i]}(t)$ to the gluing conditions (27). Substituting $\Xi_a^{[i]}(t)$ into $\omega^{[i]}=\mathrm{d}\tilde\xi^{[i]}_\Lambda\wedge\mathrm{d}\xi^\Lambda_{[i]}$ and equating with (5) gives access to the triplet of Kähler forms, hence to the HK metric.

The set of transition functions $H^{[ij]}$ naturally defines a holomorphic affine bundle $\mathscr{L}_\mathcal{Z}$ on $\mathcal{Z}$, whose sections satisfy the following gluing conditions

$$\alpha^{[j]}=\alpha^{[i]}+H^{[ij]}-\xi^\Lambda_{[i]}\partial_{\xi^\Lambda_{[i]}}H^{[ij]}.\tag{28}$$

The consistency of (28) on triple overlaps is ensured by the cocycle condition on $H^{[ij]}$. The affine bundle $\mathscr{L}_\mathcal{Z}$ carries a connection which gives rise to a holomorphic one-form

$$\lambda^{[i]}\equiv-\mathrm{d}\alpha^{[i]}-\xi^\Lambda_{[i]}\mathrm{d}\tilde\xi^{[i]}_\Lambda.\tag{29}$$

We use it to define the "generalized contact potential" as in (18)

$$\mu^{[i]}(t)\equiv-\mathrm{i}\left(\iota_\kappa\lambda^{[i]}\right).\tag{30}$$

Using the system of integral equations equivalent to the gluing conditions (27), (28) (see [26, Eq.(2,11)]), one obtains [49]

$$\begin{aligned}\mu^{[i]}(t)=\frac{1}{4\pi}\sum_j\oint_{C_j}\frac{\mathrm{d}t'}{t'}\Big[&\left(t'^{-1}Y^\Lambda-t'\bar Y^\Lambda\right)\partial_{\xi^\Lambda}H^{[ij]}\\&+\frac{t'+t}{t'-t}\,t'\partial_{t'}H^{[ij]}\Big],\end{aligned}\tag{31}$$

where C_j surround the patches and Y^Λ ($\bar Y^\Lambda$) is the residue of $\xi^\Lambda_{[\pm]}$ at $t=0$ (∞). Moreover, using again the integral expression for $\alpha^{[i]}$, one can prove that

$$K^{[i]}(t)=2\,\mathrm{Re}\,\mu^{[i]}(t)-2\,\frac{1-|t|^2}{1+|t|^2}\,\mathrm{Im}\,\alpha^{[i]}\tag{32}$$

provides a Kähler potential for the HK metric in complex structure $J(t)$. In the presence of a $U(1)_R$ isometric action, the transition functions $H^{[ij]}$ are independent of t, so the last term in (31) disappears and $\mu^{[i]}$ becomes independent of t as well (and hence on the patch index i). Under the QK/HK correspondence, μ is identified with the contact potential on the dual QK manifold [30].

Let us now explain the relation between this construction and the one given in the main text as well as why (32) generalizes (23) to arbitrary complex structure. For this purpose, we need to specify the atlas and the transitions functions $H^{[ij]}$ relevant for the HK manifold $\mathcal{M}_3(R)$. The atlas consists of the patches $\mathcal{U}_\pm$ around the north and south poles, and the patches $\mathcal{U}_\gamma$ lying in between two consecutive BPS rays, the one on the right being ℓ_γ (see [26, Fig. 4.2]). Then we take

$$H^{[+\gamma]}=\frac{R^2}{4t^2}F\left(\frac{2t\xi}{R}\right),\quad H^{[-\gamma]}=\frac{(Rt)^2}{4}\bar F\left(-\frac{2\xi}{tR}\right)\tag{33}$$

as transition functions from $\mathcal{U}_\pm$ to $\mathcal{U}_\gamma$, and

$$\begin{aligned}H_\gamma&=G_\gamma(\mathcal{X}_\gamma)-\frac{1}{2}\,p^\Lambda q_\Lambda[G'_\gamma(\mathcal{X}_\gamma)]^2,\\G_\gamma(\mathcal{X})&=\frac{\Omega(\gamma)}{(2\pi)^2}\,\mathrm{Li}_2(\mathcal{X}),\end{aligned}\tag{34}$$

for transition functions across ℓ_γ. Here $\gamma=(p^\Lambda,q_\Lambda)$ and $\mathcal{X}_\gamma=\sigma_\gamma\,e^{-2\pi\mathrm{i}\langle\gamma,\Xi\rangle}$. The generalized contact potential (31) is then given in any patch $\mathcal{U}_\gamma$ by

$$\mu^{[\gamma]}(t)=\mathcal{I}_{\rm tot}+\frac{R^2}{4\mathrm{i}}\left(t^{-2}f-t^2\bar f\right)+\frac{R}{2\mathrm{i}}\left(t^{-1}f_\Lambda v^\Lambda+t\bar f_\Lambda\bar v^\Lambda\right),\tag{35}$$

where $f=2F-X^\Lambda F_\Lambda$ and v^Λ is the constant term in the Laurent expansion of $\xi^\Lambda_{[+]}$. Thus, like $\hat\mu$ in (18), the constant term in the Laurent expansion $\mu^{[\gamma]}$ generates the (purported, completed) Witten index.

To understand the relation between $\mu^{[\gamma]}$ and $\hat\mu$, and between $\alpha^{[\gamma]}$ and $\hat\Upsilon$, let us introduce a variant of α,

$$\hat\alpha^{[i]}=\alpha^{[i]}+\frac{1}{2}\sum_j\oint_{C_j}\frac{\mathrm{d}t'}{2\pi\mathrm{i}t'}\,\frac{t'+t}{t'-t}\frac{1-t'^2}{1+t'^2}\,t'\partial_{t'}H^{[ij]},\tag{36}$$

which satisfies (compare with (28))

$$\hat\alpha^{[j]}=\hat\alpha^{[i]}+H^{[ij]}-\xi^\Lambda_{[i]}\partial_{\xi^\Lambda_{[i]}}H^{[ij]}+\frac{t^{-1}-t}{t^{-1}+t}\,t\partial_tH^{[ij]}.\tag{37}$$

Similarly we define $\hat\lambda^{[i]}$ as in (29) with $\alpha^{[i]}\to\hat\alpha^{[i]}$. One can check that $\hat\Upsilon$ in (16) and $\hat\lambda$ in (17) are related to hatted quantities in the patches $\mathcal{U}_\gamma$ via

$$\hat\Upsilon=e^{\pi\mathrm{i}\left(2\hat\alpha^{[\gamma]}+\xi^\Lambda_{[\gamma]}\tilde\xi^{[\gamma]}_\Lambda\right)},\qquad\hat\lambda=\hat\lambda^{[\gamma]}.\tag{38}$$

In particular, the relation $\lambda^{[\gamma]}-\hat\lambda^{[\gamma]}=\mathrm{d}\left(\hat\alpha^{[\gamma]}-\alpha^{[\gamma]}\right)$ explains why the constant terms in $\mu^{[\gamma]}$ and $\hat\mu$ coincide. It is worth noting that for a homogeneous prepotential, $\hat\alpha^{[i]}$ coincides with $\alpha^{[i]}$, and the construction of [30] is recovered. If F is not homogeneous, the unhatted counterpart

of (38) formally produces another hyperholomorphic curvature on the universal cover of $\mathcal{M}_3(R)$, albeit one which is inconsistent with symplectic invariance and periodicity under integral shifts of C.

Finally, let us derive (23) from the general property (32). In the patch around the north pole, one obtains

$$\mu^{[+]}(0) = \mathcal{I}_{\rm tot} + \frac{R^2}{4\mathrm{i}}\,\bar{X}^\Lambda f_\Lambda + \frac{1}{2\mathrm{i}}\,F_{\Lambda\Sigma\Xi}X^\Lambda v^\Sigma v^\Xi. \quad (39)$$

$$\alpha^{[+]}(0) = -\frac{\mathrm{i}}{2\pi}\lim_{t\to 0}\Big[\log\hat{\Upsilon} + \text{hol.}\Big] + \frac{\mathrm{i}}{2}\,\mathrm{Im}\left[R^2 f + \frac{R^2}{2}\,\bar{X}^\Lambda f_\Lambda + F_{\Lambda\Sigma\Xi}X^\Lambda v^\Sigma v^\Xi\right], \quad (40)$$

where hol. denotes a holomorphic function needed to cancel the pole of $\log\hat{\Upsilon}$ at $t=0$. Substituting (39) and (40) into (32) one finds

$$K(0) = 2\,\mathcal{I}_{\rm tot} + \frac{1}{\pi}\,\mathrm{Re}\lim_{t\to 0}\Big[\log\hat{\Upsilon} + \text{hol.}\Big] - R^2\,\mathrm{Im}\,f. \quad (41)$$

Eq. (32) then follows by applying the operator $\mathrm{i}\partial^{(0)}\bar{\partial}^{(0)}$ and taking into account (21).

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[44] In fact, the authors of [18] proposed a variant of (2) in 3+1 dimensions, investigated later by S. Sethi and C. Vafa (unpublished).
[45] For brevity, we omit the dependence of $\Omega(\gamma,u)$ on the Coulomb branch parameters. Since the BPS indices are piecewise constant, this dependence does not spoil holomorphicity away from the walls.
[46] In (20), $\partial^{(t)}$ denotes the Dolbeault operator in complex structure $J(t)$. The first term in (20) is manifestly of type $(1,0)_t$. So is the second one, due to the fact that the operator $t'\partial^{(0)} + t\bar{\partial}^{(0)}$ maps $J(t')$-holomorphic functions into $(1,0)_t$ forms.
[47] Massless flavors do not pose any difficulty, but non-zero masses disturb the integrality of the periods of the curvature of $\mathscr{L}$.
[48] We thank D. Gaiotto and N. Nekrasov for suggesting this interpretation.
[49] In this formula the operator $t\partial_t$ should not be confused with the vector field κ appearing in (30). Here it just acts on the last argument of the function $H^{[ij]}(\xi_{[i]},\tilde{\xi}^{[j]},t)$.