Rack-Bialgebras and Star-Products on duals of Leibniz algebras

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Abstract

In this paper we focus on a certain self-distributive multiplication on coalgebras, which leads to so-called rack bialgebra. Inspired by semi-group theory (adapting the Suschkewitsch theorem), we do some structure theory for rack bialgebras and cocommutative Hopf dialgebras. We also construct canonical rack bialgebras (some kind of enveloping algebras) for any Leibniz algebra and compare to the existing constructions.

We are motivated by a differential geometric procedure which we call the Serre functor: To a pointed differentible manifold with multiplication is associated its distribution space supported in the chosen point. For Lie groups, it is well-known that this leads to the universal enveloping algebra of the Lie algebra. For Lie racks, we get rack-bialgebras, for Lie digroups, we obtain cocommutative Hopf dialgebras.

The canonical rack bialgebras we have constructed for any Leibniz algebra lead then to a simple explicit formula of the rack-star-product on the dual of a Leibniz algebra recently constructed by Dherin and Wagemann in 2013. We clarify this framework doing some deformation theory.

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Contents

1 Introduction

All manifolds considered in this manuscript are assumed to be Hausdorff and second countable.

Basic Lie theory relies heavily on the fundamental links between associative algebras, Lie algebras and groups. Some of these links are the passage from an associative algebra A to its underlying Lie algebra A^{Lie} which is the vector space A with the bracket $[a, b] := ab - ba$. On the other hand, to any Lie algebra $\mathfrak{g}$ one may associate its universal enveloping algebra $\mathsf{U}(\mathfrak{g})$ which is associative. Groups arise as groups of units in associative algebras. To any group G, one may associate its group algebra KG which is associative.

The theme of the present article is to investigate links of this kind for more general objects than groups, namely for racks and digroups.

Recall that a *pointed rack* is a pointed set (X, e) together with a binary operation $\triangleright : X \times X \to X$ such that for all $x \in X$, the map $y \mapsto x \triangleright y$ is bijective and such that for all $x, y, z \in X$, the self-distributivity and unit relations

$$x \triangleright (y \triangleright z) \;=\; (x \triangleright y) \triangleright (x \triangleright z), \quad e \triangleright x = x, \quad \text{and} \quad x \triangleright e = e$$

are satisfied. Observe that racks are not algebras over an operad, but the correct algebraic structure is that of a properad. Therefore the standard deformation theory of algebras over an operad does not apply. Imitating the notion of a Lie group, the smooth version of a pointed rack is called *Lie rack*.

An important class of examples of racks are the so-called *augmented racks*, see [11]. An augmented rack is the data of a group G, a G-set X and a map $p : X \to G$ such that for all $x \in X$ and all $g \in G$,

$$p(g \cdot x) \;=\; gp(x)g^{-1}.$$

The set X becomes then a rack by setting $x \triangleright y \;:=\; p(x) \cdot y$. In fact, augmented racks are the Drinfeld center (or the Yetter-Drinfeld modules) in the monoidal category of G-sets over the (set-theoretical) Hopf algebra G, see [16]. Any rack may be augmented in many ways, for example by using the canonical morphism to its associated group (see [11]) or to its group of bijections or to its group of automorphisms.

In order to formalize the notion of a rack, one needs the diagonal map $\mathrm{diag}_X : X \to X \times X$ given by $x \mapsto (x, x)$. Then the self-distributivity relation reads in terms of maps

$$\begin{aligned} &\mathbf{m} \circ (\mathrm{id}_M \times \mathbf{m}) \\ &\quad = \;\; \mathbf{m} \circ (\mathbf{m} \times \mathbf{m}) \circ (\mathrm{id}_M \times \tau_{M,M} \times \mathrm{id}_M) \circ (\mathrm{diag}_M \times \mathrm{id}_M \times \mathrm{id}_M). \end{aligned}$$

Axiomatizing this kind of structure, one may start with a coalgebra C and look for rack operations on this fixed coalgebra, see [4] and [17].

A natural framework where this kind of structure arises (as we show in Section 2) is by taking point-distributions (i.e. applying the *Serre functor*) over (resp. to) the pointed manifold given by a Lie rack. We dub the arising structure as *rack bialgebra*.

Lie racks are intimately related to *Leibniz algebras* $\mathfrak{h}$, i.e. a vector space $\mathfrak{h}$ with a bilinear bracket $[,] : \mathfrak{h} \otimes \mathfrak{h} \to \mathfrak{h}$ such that for all $X, Y, Z \in \mathfrak{h}$, $[X, -]$ acts as a derivation:

$$[X, [Y, Z]] \;=\; [[X, Y], Z] + [Y, [X, Z]]. \tag{1.1}$$

Indeed, Kinyon showed in [14] that the tangent space at $e \in H$ of a Lie rack H carries a natural structure of a Leibniz algebra, generalizing the relation between a Lie group and its tangent Lie algebra. Conversely, every (finite dimensional real or complex) Leibniz algebra $\mathfrak{h}$ may be integrated into a Lie rack $R_{\mathfrak{h}}$ (with underlying manifold $\mathfrak{h}$) using the rack product

$$X \blacktriangleright Y \;:=\; e^{\mathrm{ad}_X}(Y), \tag{1.2}$$

noting that the exponential of the inner derivation ad_X for each $X \in \mathfrak{h}$ is an automorphism.

Another closely related algebraic structure is that of dialgebras. A *dialgebra* is a vector space D with two (bilinear) associative operations $\vdash\colon D \times D \to D$ and $\dashv\colon D \times D \to D$ which satisfy three compatibility relations, namely for $a, b, c \in D$:

$$(a \vdash b) \vdash c \;=\; (a \dashv b) \vdash c, \quad a \dashv (b \vdash c) \;=\; a \dashv (b \dashv c), \quad (a \vdash b) \dashv c \;=\; a \vdash (b \dashv c).$$

A dialgebra D becomes a Leibniz algebras via the formula

$$[a, b] \;=\; a \vdash b - b \dashv a.$$

In this sense $\vdash$ and $\dashv$ are two halves of a Leibniz bracket. Loday and Goichot have defined an enveloping dialgebra of a Leibniz algebra, see [12], [20].

One main point of the first part of this paper is the link between rack bialgebras and cocommutative Hopf dialgebras. In Theorem 2.5, we adapt Suschkewitsch's Theorem in semi-group theory to the present context. The classical result (see Appendix B) treats semi-groups with a left unit e and right inverses (analoguous results in the left-context), called *right groups*. Suschkewitsch shows that such a right group Γ decomposes as a product $\Gamma = \Gamma e \times E$ where E is the set of all idempotent elements.

Its incarnation here shows that a cocommutative right Hopf algebra $\mathcal{H}$ decomposes as a tensor product $\mathcal{H}\mathbf{1} \otimes E_{\mathcal{H}}$ where $E_{\mathcal{H}}$ is the subspace of generalized idempotents.

Furthermore, we will show in Theorem 2.6 how to associate to any augmented rack bialgebra an augmented cocommutative Hopf dialgebra. In Theorem 2.7, we investigate what Suschkewitsch's decomposition gives for a cocommutative Hopf dialgebra A. It turns out that A decomposes as a tensor product $E_A \otimes H_A$ of E_A with H_A which may be identified to the associative quotient A_{ass} of A. This result permits to show that the Leibniz algebra of primitives in A is a hemi-semi-direct product (see [14]), and thus always split. In this way we arrive once again at the result that Lie digroups may serve only to integrate *split* Leibniz algebras which has already been observed by Covez in his master thesis [6].

Given a finite-dimensional real Lie algebra $(\mathfrak{g}, [,])$, its dual vector space $\mathfrak{g}^*$ is a smooth manifold which carries a Poisson bracket on its space of smooth functions, defined for all $f, g \in \mathcal{C}^\infty(\mathfrak{g}^*)$ and all $\xi \in \mathfrak{g}^*$ by the *Kostant-Kirillov-Souriau formula*

$$\{f, g\}(\xi) := \langle \xi, [df(\xi), dg(\xi)] \rangle.$$

Here $df(\xi)$ and $dg(\xi)$ are linear functionals on $\mathfrak{g}^*$, identified with elements of $\mathfrak{g}$.

In the same way, a general finite dimensional Leibniz algebra $\mathfrak{h}$ gives rise to a smooth manifold $\mathfrak{h}^*$, which carries now some kind of generalized Poisson bracket, namely

$$\{f, g\}(\xi) := -\langle \xi, [df(0), dg(\xi)] \rangle,$$

see [9] for an explanation why we believe that this is the correct bracket to consider. In particular, this generalized Poisson bracket need not be skew-symmetric.

The quantization procedure of this generalized Poisson bracket proposed in [9] proceeds as follows: The cotangent lift of the above rack product

$$X \blacktriangleright Y = e^{\mathrm{ad}X}(Y)$$

is interpreted as a symplectic micromorphism. The generating function of this micromorphism serves then as a phase function in a Fourier integral operator, whose asymptotic expansion gives rise to a star-product.

One main goal of the present article is to set up a purely algebraic framework in which one may deformation quantize the dual of a Leibniz algebra. The main feature will be to recover –in a rather explicit algebraic manner– the star-product which has been constructed in [9] by analytic methods.

Our main result in the present paper (Corollary 4.1) is the purely algebraic construction of a star-product on $\mathfrak{h}^*$ based on formula (1.2), and the proof that it is identical to the star-product of [9].

Closing the paper, we set up a general deformation theory framework in which the above star-product appears as a formal deformation, its infinitesimal term defining a cohomology class.

Let us comment on the **content of the paper**:

All our bialgebra notion are based on the standard theory of coalgebras, some features of which as well as our notions are recalled in Appendix A. Rack bialgebras and augmented rack bialgebras are studied in Section 2. Connected, cocommutative Hopf algebras give rise to a special case of rack bialgebras. In Section 2.2, we associate to any Leibniz algebra $\mathfrak{h}$ an augmented rack bialgebra $\mathsf{UAR}^\infty(\mathfrak{h})$ and study the functorial properties of this

association. This rack bialgebra plays the role of an enveloping algebra in our context. It turns out that a truncated, non-augmented version $\mathsf{UR}(\mathfrak{h})$ is a left adjoint of the functor of primitives Prim.

We also study the "group-algebra" functor associating to a rack X its rack bialgebra $K[X]$. Like in the classical framework, $K[-]$ is left adjoint to the functor Slike associating to a track bialgebra its rack of set-like elements. The relation between rack bialgebras and the other algebraic notion discussed in this paper is summarized in the diagram (see the end of Section 2.2) of categories and functors:

$$\begin{array}{ccccc}
\mathrm{Lie} & \underset{\mathrm{Prim}}{\overset{\mathsf{U}}{\rightleftarrows}} & \mathrm{Hopf} & \underset{K[-]}{\overset{\mathrm{Slike}}{\rightleftarrows}} & \mathrm{Grp} \\
\downarrow{\scriptstyle i} & & \downarrow{\scriptstyle j} & & \downarrow \\
\mathrm{Leib} & \underset{\mathrm{Prim}}{\overset{\mathsf{UAR}^\infty}{\rightleftarrows}} & \mathrm{RackBialg} & \underset{K[-]}{\overset{\mathrm{Slike}}{\rightleftarrows}} & \mathrm{Racks}
\end{array}$$

In Section 2.3, we develop the structure theory for rack bialgebras and cocommutative Hopf dialgebras, based on Suschkewitsch's Theorem. Section 2.3 contains Theorem 2.5, Theorem 2.6 and Theorem 2.7 whose content we have described above.

Recollecting basic knowledge about the Serre functor F is the subject of Section 3. In particular, we show in Section 3.2 that F is a strong monoidal functor from the category of pointed manifolds $\mathcal{M}f*$ to the category of coalgebras, based on some standard material on coalgebras (Appendix A). In Section 3.3, we apply F to Lie groups, Lie semi-groups, Lie digroups, and to Lie racks and augmented Lie racks, and study the additional structure which we obtain on the coalgebra. The case of Lie racks motivates the notion of rack bialgebra.

Recall that for a Leibniz algebra $\mathfrak{h}$, the vector space $\mathfrak{h}$ becomes a Lie rack $R_\mathfrak{h}$ with the rack product

$$X \blacktriangleright Y \;=\; e^{\mathrm{ad}_X}(Y).$$

In Theorem 3.8, we show that the rack bialgebra $\mathsf{UAR}^\infty(\mathfrak{h})$ associated to $\mathfrak{h}$ coincides with the rack bialgebra $F(R_\mathfrak{h})$.

The quantization procedure in Section 4 relies on an explicit approach in terms of coordinates on the (finite dimensional real) Leibniz algebra $\mathfrak{h}$. The main theorem (Theorem 4.1) describes the induced rack structure on exponentials from the rack structure which is based on formula (1.2). In Section 4.2, we develop the basics of rack bialgebra deformation theory to show that infinitesimal deformations are classified by H^1 and that the star-product gives a formal deformation of the trivial rack bialgebra.

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2 Several bialgebras

In the following, let K be an associative commutative unital ring containing all the rational numbers. The symbol $\otimes$ will always denote the tensor product of K-modules over K. For any coalgebra (C, Δ) over K, we shall use Sweedler's notation $\Delta(a) = \sum_{(a)} a^{(1)} \otimes a^{(2)}$ for any $a \in A$. We will feel free to suppress the sum-sign in Sweedler's notation in complicated formulas for typographical reasons. See also Appendix A for a survey on definitions and notations in coalgebra theory.

The following Sections will all deal with the following type of *nonassociative bialgebra*: Let $(B, \Delta, \epsilon, \mathbf{1}, \mu)$ be a K-module such that $(B, \Delta, \epsilon, \mathbf{1})$ is a *coassociative counital coaugmented coalgebra* (a C^3-coalgebra), and such that the linear map $\mu : B \otimes B \to B$ (the multiplication) is a morphism of C^3-coalgebras (it satisfies in particular $\mu(\mathbf{1} \otimes \mathbf{1}) = \mathbf{1}$). We shall call this situation a *nonassociative C^3I-bialgebra* (where I stands for $\mathbf{1}$ being an *idempotent* for the multiplication μ). For another nonassociative C^3I-bialgebra $(B', \Delta', \epsilon', \mathbf{1}', \mu')$ a K-linear map $\phi : B \to B'$ will be called a *morphism of nonassociative C^3I-bialgebras* iff it is a morphism of C^3-coalgebras and is multiplicative in the usual sense $\phi\big(\mu(a \otimes b)\big) = \mu'\big(\phi(a) \otimes \phi(b)\big)$ for all $a, b \in B$. The nonassociative C^3I-bialgebra $(B, \Delta, \epsilon, \mathbf{1})$ is called *left-unital* (resp. *right-unital*) iff for all $a \in B$ $\mu(\mathbf{1} \otimes a) = a$ (resp. $\mu(a \otimes \mathbf{1}) = a$).
Moreover, consider the associative algebra $A := \mathrm{Hom}_K(B, B)$ equipped with the composition of K-linear maps, and the identity map id_B as the unit element. There is an associative convolution multiplication $*$ in the K-module $\mathrm{Hom}_K(B, A)$ of all K-linear maps $B \to \mathrm{Hom}_K(B, B)$, see Appendix A, eqn (A.3) for a definition with $\mathrm{id}_B\epsilon$ as the unit element. For a given nonassociative C^3I-bialgebra $(B, \Delta, \epsilon, \mathbf{1}, \mu)$ we can consider the map μ as a map $B \to \mathrm{Hom}_K(B, B)$ in two ways: as *left multiplication map* $L^\mu : b \mapsto \big(b' \mapsto \mu(b \otimes b')\big)$ or as *right multiplication map* $R^\mu : b \mapsto \big(b' \mapsto \mu(b' \otimes b)\big)$. We call $(B, \Delta, \epsilon, \mathbf{1}, \mu)$ a *left-regular* (resp. *right-regular*) nonassociative C^3I-bialgebra

iff the map L^μ (resp. the map R^μ) has a convolution inverse, i.e. iff there is a K-linear map $\mu' : B \otimes B \to B$ (resp. $\mu'' : B \otimes B \to B$) such that $L^\mu * L^{\mu'} = \mathrm{id}_B \epsilon = L^{\mu'} * L^\mu$ (resp. $R^\mu * R^{\mu''} = \mathrm{id}_B \epsilon = R^{\mu''} * R^\mu$), or on elements $a, b \in B$ for the left regular case

$$\sum_{(a)} \mu\big(a^{(1)} \otimes \mu'(a^{(2)} \otimes b)\big) = \epsilon(a)b = \mu'\big(a^{(1)} \otimes \mu(a^{(2)} \otimes b)\big). \qquad (2.1)$$

Note that every associative unital Hopf algebra $(H, \Delta, \epsilon, \mathbf{1}, \mu, S)$ (where S denotes the antipode, i.e. the convolution inverse of the identity map in $\mathrm{Hom}_K(H, H)$) is right- and left-regular by setting $\mu' = \mu \circ (S \otimes \mathrm{id}_H)$ and $\mu'' = \mu \circ (\mathrm{id}_H \otimes S)$.

Lemma 2.1 *Let $(B, \Delta, \epsilon, \mathbf{1}, \mu)$ be a nonassociative C^3I-bialgebra.*

1. *If B is left-regular (resp. right-regular), then the corresponding K-linear map $\mu' : B \otimes B \to B$ is unique, and in case Δ is cocommutative, μ' is map of C^3-coalgebras.*

2. *If $(B, \Delta, \epsilon, \mathbf{1}, \mu)$ is left-unital (rep. right-unital) and its underlying C^3-coalgebra is connected, then $(B, \Delta, \epsilon, \mathbf{1}, \mu)$ is always left-regular (resp. right-regular).*

Proof: *1.* In any monoid (in particular in the convolution monoid) two-sided inverses are always unique. Moreover, as can easily be checked, a K-linear map $\phi : B \otimes B \to B$ is a morphism of coalgebras iff for each $b \in B$

$$\sum_{(b)} \big(L^\phi(b^{(1)}) \otimes L^\phi(b^{(2)})\big) \circ \Delta = \Delta \circ L^\phi(b). \qquad (2.2)$$

(and analogously for right multiplications). Both sides of the preceding equation, seen as maps of b, are in $\mathrm{Hom}_K\big(B, \mathrm{Hom}_K(B, B \otimes B)\big)$. Since $\mathrm{Hom}_K(B, B \otimes B)$ is an obvious right $\mathrm{Hom}_K(B, B)$-module, the K-module $\mathrm{Hom}_K\big(B, \mathrm{Hom}_K(B, B \otimes B)\big)$ is a right $\mathrm{Hom}_K\big(B, \mathrm{Hom}_K(B, B)\big)$-module with respect to the convolution. Define the K-linear map $F^{\mu'} : B \to \mathrm{Hom}_K(B, B \otimes B)$ by

$$F^{\mu'}(b) := \sum_{(b)} \big(L^{\mu'}(b^{(1)}) \otimes L^{\mu'}(b^{(2)})\big) \circ \Delta - \Delta \circ L^{\mu'}(b).$$

Using eqn (2.2), the fact that $L^{\mu'}$ is a convolution inverse of L^μ, and the cocommutativity of Δ, we get

$$F^{\mu'} * L^\mu = 0, \quad \text{hence} \quad 0 = F^{\mu'} * L^\mu * L^{\mu'} = F^{\mu'} * (\mathrm{id}_B \epsilon) = F^{\mu'}$$

and μ' preserves comultiplications. A similar reasoning where $B \otimes B$ is replaced by K shows that μ' preserves counits. Finally, it is obvious that $L^{\mu'}(\mathbf{1})$ is the inverse

of the K-linear map $L^\mu(\mathbf{1})$, and since the latter fixes $\mathbf{1}$ so does the former. The reasoning for right-regular bialgebras is completely analogous.
2. For left-unital bialgebras we get $L^\mu(\mathbf{1}) = \mathrm{id}_B$, and the generalized Takeuchi-Sweedler argument, see Appendix A, shows that L^μ has a convolution inverse. Right-unital bialgebras are treated in an analogous manner. □

Note that any C^3-coalgebra $(C, \Delta, \epsilon, \mathbf{1})$ becomes a left-unital (resp. right-unital) associative C^3I-bialgebra by equipping with the *left-trivial* (resp. *right-trivial*) multiplication

$$\mu_0(a \otimes b) := \epsilon(a)b \qquad \big(\text{resp. } \mu_0(a \otimes b) := \epsilon(b)a\big). \tag{2.3}$$

We shall call an element $c \in B$ a *generalized idempotent* iff $\sum_{(c)} c^{(1)}c^{(2)} = c$. Moreover $c \in B$ will be called a *generalized left (resp. right) unit element* iff for all $b \in B$ we have $cb = \epsilon(c)b$ (resp. $bc = \epsilon(c)b$).

2.1 Rack bialgebras, (left) Leibniz algebras and structure theory

Definition 2.1 *A **rack bialgebra** $(B, \Delta, \epsilon, \mathbf{1}, \mu)$ is a nonassociative C^3I-bialgebra (where we write for all $a, b \in B$ $\mu(a \otimes b) =: a \triangleright b$) such that the following identities hold for all $a, b, c \in B$*

$$\mathbf{1} \triangleright a = a, \tag{2.4}$$
$$a \triangleright \mathbf{1} = \epsilon(a)\mathbf{1}, \tag{2.5}$$
$$a \triangleright (b \triangleright c) = \sum_{(a)} (a^{(1)} \triangleright b) \triangleright (a^{(2)} \triangleright c). \tag{2.6}$$

*The last condition (2.6) is called the **self-distributivity** condition.*

Note that we do not demand that the C^3-coalgebra B should be cocommutative nor connected.

Remark 2.1
It is shown in Theorem 4.3 of [4] that for a C^4-coalgebra B with a self-distributive map $\lhd = q : B \otimes B \to B$ which is a morphism of coalgebras, the map

$$R_q = (\mathrm{id}_B \otimes q) \circ (\tau \otimes \mathrm{id}_B) \circ (\mathrm{id}_B \otimes \Delta)$$

is a solution of the Yang-Baxter equation. We draw our reader's attention to the fact that Carter-Crans-Elhamdadi-Saito work in [4] with right racks,

while we work here with left racks. The statement of their theorem works also for left racks, but then one has to take

$$\tilde{R}_q \;=\; (\mathrm{id}_B \otimes q) \circ (\Delta \otimes \mathrm{id}_B) \circ \tau.$$

In particular for any rack bialgebra, $\tilde{R}_q$ is a solution of the Yang-Baxter equation. $\diamond$

Example 2.1
Any C^3 coalgebra $(C, \Delta, \epsilon, \mathbf{1})$ carries a *trivial rack bialgebra structure* defined by the left-trivial multiplicaton

$$a \triangleright_0 b := \epsilon(a)b \tag{2.7}$$

which in addition is easily seen to be associative and left-unital, but in general not unital. $\diamond$

Another method of constructing rack bialgebras is *gauging*: Let $(B, \Delta, \epsilon, \mathbf{1}, \mu)$ a rack bialgebra –where we write $\mu(a \otimes b) = a \triangleright b$ for all $a, b \in B$–, and let $f : B \to B$ a morphism of C^3-coalgebras such that for all $a, b \in B$

$$f(a \triangleright b) = a \triangleright \big(f(b)\big), \tag{2.8}$$

i.e. f is μ-equivariant. It is a routine check that $(B, \Delta, \epsilon, \mathbf{1}, \mu_f)$ is a rack bialgebra where for all $a, b \in B$ the multiplication is defined by

$$\mu_f(a \otimes b) := a \triangleright_f b := \big(f(a)\big) \triangleright b. \tag{2.9}$$

We shall call $(B, \Delta, \epsilon, \mathbf{1}, \mu_f)$ the *f-gauge of* $(B, \Delta, \epsilon, \mathbf{1}, \mu)$.

Example 2.2
Let $(H, \Delta_H, \epsilon_H, \mu_H, \mathbf{1}_H, S)$ be a cocommutative Hopf algebra over K. Then it is easy to see (cf. also the particular case $B = H$ and $\Phi = \mathrm{id}_H$ of Proposition 2.1 for a detailed proof) that the new multiplication $\mu : H \otimes H \to H$, written $\mu(h \otimes h') = h \triangleright h'$, defined by the usual *adjoint representation*

$$h \triangleright h' := \mathrm{ad}_h(h') := \sum_{(h)} h^{(1)} h' \big(S(h^{(2)})\big), \tag{2.10}$$

equips the C^4-coalgebra $(H, \Delta_H, \epsilon_H, \mathbf{1}_H)$ with a rack bialgebra structure. $\diamond$

In general, the adjoint representation does not seem to preserve the coalgebra structure if no cocommutativity is assumed.

Example 2.3
Recall that a pointed set (X, e) is a *pointed rack* in case there is a binary operation $\triangleright : X \times X \to X$ such that for all $x \in X$, the map $y \mapsto x \triangleright y$ is bijective and such that for all $x, y, z \in X$, the self-distributivity and unit relations

$$x \triangleright (y \triangleright z) \;=\; (x \triangleright y) \triangleright (x \triangleright z), \quad e \triangleright x = x, \quad \text{and} \quad x \triangleright e = e$$

are satisfied. Then there is a natural rack bialgebra structure on the vector space $K[X]$ which has the elements of X as a basis. $K[X]$ carries the usual coalgebra structure such that all $x \in X$ are set-like: $\triangle(x) = x \otimes x$ for all $x \in X$. The product μ is then induced by the rack product. By functoriality, μ is compatible with $\triangle$ and e.

Observe that this construction differs slightly from the construction in [4], Section 3.1. $\diamond$

More generally there is the following structure:

Definition 2.2 *An **augmented rack bialgebra** over K is a quadruple (B, Φ, H, ℓ) consisting of a C^3-coalgebra $(B, \Delta, \epsilon, \mathbf{1})$, of a cocommutative (!) Hopf algebra $(H, \Delta_H, \epsilon_H, \mathbf{1}_H, \mu_H, S)$, of a morphism of C^3-coalgebras $\Phi : B \to H$, and of a left action $\ell : H \otimes B \to B$ of H on B which is a morphism of C^3-coalgebras (i.e. B is a H-module-coalgebra) such that for all $h \in H$ and $a \in B$*

$$h.\mathbf{1} \;=\; \epsilon_H(h)\mathbf{1} \tag{2.11}$$
$$\Phi(h.a) \;=\; \mathrm{ad}_h\big(\Phi(a)\big). \tag{2.12}$$

where ad *denotes the usual adjoint representation for Hopf algebras, see e.g. eqn (2.10).*
*We shall define a **morphism** $(B, \Phi, H, \ell) \to (B', \Phi', H', \ell')$ **of augmented rack bialgebras** to be a pair (ϕ, ψ) of K-linear maps where $\phi : (B, \Delta, \epsilon, \mathbf{1}) \to (B', \Delta', \epsilon', \mathbf{1}')$ is a morphism of C^3-coalgebras, and $\psi : H \to H'$ is a morphism of Hopf algebras such that the obvious diagrams commute:*

$$\Phi' \circ \phi = \psi \circ \Phi, \quad \text{and} \quad \ell' \circ (\psi \otimes \phi) = \phi \circ \ell \tag{2.13}$$

An immediate consequence of this definition is the following

Proposition 2.1 *Let (B, Φ, H, ℓ) be an augmented rack bialgebra. Then the C^3-coalgebra $(B, \epsilon, \mathbf{1})$ will become a left-regular rack bialgebra by means of the multiplication*

$$a \triangleright b := \Phi(a).b \tag{2.14}$$

for all $a, b \in B$. In particular, each Hopf algebra H becomes an augmented rack bialgebra via $(H, \mathrm{id}_H, H, \mathrm{ad})$. In general, for each augmented rack bialgebra the map $\Phi : B \to H$ is a morphism of rack bialgebras.

Proof: We check first that $\triangleright$ is a morphism of C^3-coalgebras $B \otimes B \to B$: Let $a, b \in B$, then –thanks to the fact that the action ℓ and the maps Φ are coalgebra morphisms–

$$\begin{aligned}
\Delta\big(\mu(a \otimes b)\big) &= \Delta(a \triangleright b) = \Delta\big(\Phi(a).b\big) = \sum_{(\Phi(a)),(b)} \Big(\big(\Phi(a)^{(1)}\big).b^{(1)}\Big) \otimes \Big(\big(\Phi(a)^{(2)}\big).b^{(2)}\Big) \\
&= \sum_{(a),(b)} \Big(\big(\Phi(a^{(1)})\big).b^{(1)}\Big) \otimes \Big(\big(\Phi(a^{(2)})\big).b^{(2)}\Big) \\
&= \sum_{(a),(b)} \big(a^{(1)} \triangleright b^{(1)}\big) \otimes \big(a^{(2)} \triangleright b^{(2)}\big)
\end{aligned}$$

whence μ is a morphism of coalgebras. Clearly

$$\epsilon(a \triangleright b) = \epsilon\big(\Phi(a).b\big) = \epsilon_H\big(\Phi(a)\big)\epsilon(b) = \epsilon(a)\epsilon(b)$$

whence μ preserves counits.
We shall next compute both sides of the self-distributivity identity (2.6) to get an idea: For all $a, b, c \in B$

$$a \triangleright (b \triangleright c) = \Phi(a).\big(\Phi(b).c\big) = \big(\Phi(a)\Phi(b)\big).c,$$

and

$$\begin{aligned}
\sum_{(a)} (a^{(1)} \triangleright b) \triangleright (a^{(2)} \triangleright c) &= \sum_{(a)} \big(\Phi(a^{(1)}).b\big) \triangleright \big(\Phi(a^{(2)}).c\big) \\
&= \sum_{(a)} \Big(\Phi\big(\Phi(a^{(1)}).b\big)\Big).\big(\Phi(a^{(2)}).c\big) \\
&= \sum_{(a)} \Big(\Phi\big(\Phi(a^{(1)}).b\big)\Phi(a^{(2)})\Big).c,
\end{aligned}$$

and we compute, using the fact that Φ is a morphism of C^3-coalgebras,

$$\begin{aligned}
\sum_{(a)} \Phi\big(\Phi(a^{(1)}).b\big)\Phi(a^{(2)}) &= \sum_{(a)} \Phi\big((\Phi(a)^{(1)})).b\big)(\Phi(a)^{(2)}) \\
&\overset{(2.12)}{=} \sum_{(a)} \Big(\mathrm{ad}_{\Phi(a)^{(1)}}\big(\Phi(b)\big)\Big)(\Phi(a)^{(2)}) \\
&= \sum_{(a)} \big(\Phi(a)^{(1)}\big)\Phi(b)\Big(S\big(\Phi(a)^{(2)}\big)\Big)(\Phi(a)_{(3)}) \\
&= \sum_{(a)} \big(\Phi(a)^{(1)}\big)\Phi(b)\mathbf{1}_H\epsilon_H\big(\Phi(a)^{(2)}\big) \\
&= \Phi(a)\Phi(b),
\end{aligned}$$

which proves the self-distributivity identity. Moreover we have

$$\mathbf{1}_B \triangleright a = \Phi(\mathbf{1}).a = \mathbf{1}_H.a = a,$$

and

$$a \triangleright \mathbf{1} = \Phi(a).\mathbf{1} \stackrel{(2.11)}{=} \epsilon_H\big(\Phi(a)\big)\mathbf{1} = \epsilon_B(a)\mathbf{1},$$

whence the C^3-coalgebra becomes a rack bialgebra. Finally, if we set

$$\mu'(a \otimes b) := \ell_{S\left(\Phi(a)\right)}(b) = S\big(\Phi(a)\big).b$$

we get

$$\begin{aligned}\sum_{(a)} \mu\big(a^{(1)} \otimes \mu'(a^{(2)} \otimes b)\big) &= \sum_{(a)} \Big(\Phi(a^{(1)})S\big(\Phi(a^{(2)})\big)\Big).b \\ &= \sum_{\left(\Phi(a)\right)} \Big(\Phi(a)^{(1)}S\big(\Phi(a)^{(2)}\big)\Big).b = \epsilon(a)b\end{aligned}$$

and the other equation of the left-regularity condition (2.1) is proved similarly. □

Example 2.4
Exactly in the same way as a pointed rack gives rise to a rack bialgebra $K[X]$, an augmented pointed rack $p : X \to G$ gives rise to an augmented rack bialgebra $p : K[X] \to K[G]$. ◇

Remark 2.2
Motivated by the fact that the augmented racks $p : X \to G$ are exactly the Yetter-Drinfeld modules over the (set-theoretical) Hopf algebra G, we may ask whether augmented rack bialgebras are Yetter-Drinfeld modules.

In fact, any cocommutative augmented rack bialgebra (B, Φ, H, ℓ) gives rise to a Yetter-Drinfeld module over the Hopf algebra H. Indeed, B is a left H-module via ℓ, and becomes a left H-comodule via

$$\rho : B \stackrel{\triangle_B}{\to} B \otimes B \stackrel{\Phi \otimes \mathrm{id}_B}{\to} H \otimes B.$$

Now, in Sweedler notation, the coaction is denoted for all $b \in B$ by

$$\rho(b) \;=\; \sum_{(b)} b_{(-1)} \otimes b_{(0)} \;\in\; H \otimes B.$$

Then the *Yetter-Drinfeld compatibility relation* reads

$$\sum_{(h.b)} (h.b)_{(-1)} \otimes (h.b)_{(0)} \;=\; \sum_{(b),(h)} h^{(1)} b_{(-1)} S(h^{(3)}) \otimes h^{(2)}.b_{(0)}.$$

This relation is true in our case, because ℓ is a morphism of coalgebras and is sent to the adjoint action via Φ.

Conversely, given a Yetter-Drinfeld module C over a Hopf algebra H, together with a linear form $\epsilon_C : C \to K$ satisfying $\epsilon_C(h.c) = \epsilon_H(h)\epsilon_C(c)$, then define a map $\Phi : C \to H$ by

$$\Phi \ := \ (\mathrm{id}_H \otimes \epsilon_C) \circ \rho.$$

The map Φ intertwines the left action on C and the adjoint action on H thanks to the Yetter-Drinfeld condition.

Now define a rack product for all $x, y \in C$ by

$$x \rhd y \ = \ \Phi(x).y,$$

then we obtain a Yetter-Drinfeld version of self-distributivity

$$x \rhd (y \rhd z) \ = \ \sum_{(x)} (x_{(-1)}.y) \rhd (x_{(0)} \rhd z),$$

as there is no comultiplication on C.

The fact that Φ is a morphism of coalgebras is then replaced by the identity

$$(\mathrm{id}_H \otimes \Phi) \circ \rho \ = \ \Delta_H \circ \Phi,$$

which one needs to demand.

Finally, one needs a unit $1_C \in C$ such that for all $h \in H$, $h.1_C = \epsilon_H(h)1_C$, $\epsilon_C(1_C) = 1_K$, $\rho(1_C) = 1_H \otimes 1_C$, and $\Phi(1_C) = 1_H$. This is somehow the closest one can get to a rack bialgebra without having a compatible C^3 coalgebra structure on C. $\diamond$

The link to Leibniz algebras is contained in the following

Proposition 2.2 *Let $(B, \Delta, \epsilon, \mathbf{1}, \mu)$ be a rack bialgebra over K.*

1. *Then its K-submodule of all primitive elements, $\mathsf{Prim}(B) =: \mathfrak{h}$, (see eqn (A.1) of Appendix A) is a subalgebra with respect to μ (written $a \triangleright b$) satisfying the **(left) Leibniz identity***

$$x \triangleright (y \triangleright z) = (x \triangleright y) \triangleright z + y \triangleright (x \triangleright z) \tag{2.15}$$

*for all $x, y, z \in \mathfrak{h} = \mathsf{Prim}(B)$. Hence the pair $(\mathfrak{h}, [\ ,\ ])$ with $[x, y] := x \triangleright y$ for all $x, y \in \mathfrak{h}$ is a **Leibniz algebra** over K. Moreover, every morphism of rack bialgebras maps primitive elements to primitive elements and thus induces a morphism of Leibniz algebras.*

2. More generally, $\mathfrak{h}$ and each subcoalgebra of order $k \in \mathbb{N}$, $B_{(k)}$, (see eqn (A.2) of Appendix A) is stable by left $\triangleright$-multiplications with every $a \in B$. In particular, each $B_{(k)}$ is a rack subbialgebra of $(B, \Delta, \epsilon, \mathbf{1}, \mu)$.

Proof: *2.* Let $x \in \mathfrak{h}$ and $a \in B$. Since μ is a morphism of C^3-coalgebras and x is primitive, we get

$$\begin{aligned}
\Delta(a \triangleright x) &= \sum_{(a)} (a^{(1)} \triangleright x) \otimes (a^{(2)} \triangleright \mathbf{1}) + \sum_{(a)} (a^{(1)} \triangleright \mathbf{1}) \otimes (a^{(2)} \triangleright x) \\
&\overset{(2.5)}{=} \sum_{(a)} \big((a^{(1)}\epsilon(a^{(2)})) \triangleright x\big) \otimes \mathbf{1} + \sum_{(a)} \mathbf{1} \otimes \big((\epsilon(a^{(1)})a^{(2)}) \triangleright x\big) \\
&= (a \triangleright x) \otimes \mathbf{1} \ + \ \mathbf{1} \otimes (a \triangleright x),
\end{aligned}$$

whence $a \triangleright x$ is primitive. For the statement on the $B_{(k)}$, we proceed by induction: For $k = 0$, this is clear. Suppose the statement is true until $k \in \mathbb{N}$, and let $x \in B_{(k+1)}$. Then

$$\begin{aligned}
&\Delta(a \triangleright x) - (a \triangleright x) \otimes \mathbf{1} - \mathbf{1} \otimes (a \triangleright x) \\
&= \sum_{(a)(x)} \Big((a^{(1)} \triangleright x^{(1)}) \otimes (a^{(2)} \triangleright x^{(2)}) - (a^{(1)} \triangleright x) \otimes (a^{(2)} \triangleright \mathbf{1}) \\
&\qquad\qquad - (a^{(1)} \triangleright \mathbf{1}) \otimes (a^{(2)} \triangleright x) \Big) \\
&= \big(\Delta(a)\big) \triangleright \big(\Delta(x) - x \otimes \mathbf{1} - \mathbf{1} \otimes x\big) \\
&= \sum_{(a)(x)} (a^{(1)} \triangleright x^{(1)'}) \otimes (a^{(2)} \triangleright x^{(2)'})
\end{aligned}$$

where we have used the extended multiplication (still denoted $\triangleright$) $\triangleright : (B \otimes B) \otimes (B \otimes B) \to (B \otimes B)$ and set

$$\Delta(x) - x \otimes \mathbf{1} - \mathbf{1} \otimes x =: \sum_{(x)} x^{(1)'} \otimes x^{(2)'} \quad \in \ B_{(k)} \otimes B_{(k)}$$

by the definition of $B_{(k+1)}$, see Appendix A. By the induction hypothesis, all the terms $a^{(1)} \triangleright x^{(1)'}$ and $a^{(2)} \triangleright x^{(2)'}$ are in $B_{(k)}$, whence $\Delta(a \triangleright x) - (a \triangleright x) \otimes \mathbf{1} - \mathbf{1} \otimes (a \triangleright x)$ is in $B_{(k)} \otimes B_{(k)}$, implying that $a \triangleright x$ is in $B_{(k+1)}$.
1. It follows from *2.* that $\mathfrak{h}$ is a subalgebra with respect to μ. Let $x, y, z \in \mathfrak{h}$. Then since x is primitive, it follows from $\Delta(x) = x \otimes \mathbf{1} + \mathbf{1} \otimes x$ and the self-distributivity identity (2.6) that

$$x \triangleright (y \triangleright z) = (x \triangleright y) \triangleright (\mathbf{1} \triangleright z) + (\mathbf{1} \triangleright y) \triangleright (x \triangleright z) \overset{(2.4)}{=} (x \triangleright y) \triangleright z + y \triangleright (x \triangleright z).$$

proving the left Leibniz identity. The morphism statement is clear, since each morphism of rack bialgebras is a morphism of C^3-coalgebras and preserves primitives.

□

Leibniz algebras have been invented by A. M. Blokh [2] in 1965, and then rediscovered by J.-L. Loday in 1992 in the search of an explanation for the absence of periodicity in algebraic K-Theory [19, p.323, eqn (10.6.1.1)'].

As an immediate consequence, we get that the functor Prim induces a functor from the category of all rack bialgebras over K to the category of all Leibniz algebras over K.

Remark 2.3
Define *set-like elements* to be elements a in a rack bialgebra B such that $\Delta(a) = a \otimes a$. Thanks to the fact that $\triangleright$ is a morphism of coalgebras, the set of set-like elements Slike(B) is closed under $\triangleright$. In fact, Slike(B) is a rack, and one obtains in this way a functor Slike : RackBialg $\to$ Racks. ◇

Proposition 2.3 *The functor of set-likes* Slike : RackBialg $\to$ Racks *has the functor* $K[-]$: Racks $\to$ RackBialg *(see Example 2.3) as its left-adjoint.*

Proof: This follows from the adjointness of the same functors, seen as functors between the categories of pointed sets and of C^4-coalgebras, observing that the C^4-coalgebra morphism induced by a morphism of racks respects the rack product.
□

Observe that the restriction of Slike : RackBialg $\to$ Racks to the subcategory of connected, cocommutative Hopf algebras Hopf (where the Hopf algebra is given the rack product defined in eqn (2.10)) gives the usual functor of group-like elements.

Remark 2.4
Compare the self-distributivity identity

$$a \triangleright (b \triangleright c) = \sum_{(a)} (a^{(1)} \triangleright b) \triangleright (a^{(2)} \triangleright c)$$

to the *braided Leibniz identity*

$$a \triangleright (b \triangleright c) = (a \triangleright b) \triangleright c + (b_{\langle 1 \rangle} \triangleright (a_{\langle 2 \rangle} \triangleright c),$$

discussed in [16] or [17]. While the first identity takes place in a coalgebra, the second one takes place in a vector space V with a braiding τ, i.e. a bilinear map $\tau : V \otimes V \to V \otimes V$ whose components we denote by $\tau(a \otimes b) = b_{\langle 1 \rangle} \otimes a_{\langle 2 \rangle}$ (in generalized Sweedler notation). The braided Leibniz identity gives back the (left) Leibniz identity for the tensor flip $\tau(a \otimes b) = b \otimes a$. ◇

2.2 (Augmented) rack bialgebras for any Leibniz algebra

Let $(\mathfrak{h}, [\ ,\ ])$ be a *Leibniz algebra over* K, i.e. $\mathfrak{h}$ is a K-module equipped with a K-linear map $[\ ,\ ] : \mathfrak{h} \otimes \mathfrak{h} \to \mathfrak{h}$ satisfying the (left) Leibniz identity (1.1).

Recall first that each Lie algebra over K is a Leibniz algebra giving rise to a functor from the category of all Lie algebras to the category of all Leibniz algebras.

Furthermore, recall that each Leibniz algebra has two canonical K-submodules

$$\begin{aligned} Q(\mathfrak{h}) &:= \big\{x \in \mathfrak{h} \mid \exists\ N \in \mathbb{N} \setminus \{0\},\ \exists\ \lambda_1, \ldots, \lambda_N \in K,\ \exists\ x_1, \ldots, x_N \\ &\qquad \text{such that } x = \sum_{r=1}^{N} \lambda_r [x_r, x_r]\big\}, \qquad (2.16) \\ \mathfrak{z}(\mathfrak{h}) &:= \big\{x \in \mathfrak{h} \mid \forall\ y \in \mathfrak{h} :\ [x, y] = 0\big\}. \qquad (2.17) \end{aligned}$$

It is well-known and not hard to deduce from the Leibniz identity that both $Q(\mathfrak{h})$ and $\mathfrak{z}(\mathfrak{h})$ are two-sided abelian ideals of $(\mathfrak{h}, [\ ,\ ])$, that $Q(\mathfrak{h}) \subset \mathfrak{z}(\mathfrak{h})$, and that the quotient Leibniz algebras

$$\overline{\mathfrak{h}} := \mathfrak{h}/Q(\mathfrak{h}) \quad \text{and} \quad \mathfrak{g}(\mathfrak{h}) := \mathfrak{h}/\mathfrak{z}(\mathfrak{h}) \qquad (2.18)$$

are Lie algebras. Since the ideal $Q(\mathfrak{h})$ is clearly mapped into the ideal $Q(\mathfrak{h}')$ by any morphism of Leibniz algebras $\mathfrak{h} \to \mathfrak{h}'$ (which is a priori not the case for $\mathfrak{z}(\mathfrak{h})$!), there is an obvious functor $\mathfrak{h} \to \overline{\mathfrak{h}}$ from the category of all Leibniz algebras to the category of all Lie algebras.

In order to perform the following constructions of rack bialgebras for any given Leibniz algebra $(\mathfrak{h}, [\ ,\ ])$, choose first a two-sided ideal $\mathfrak{z} \subset \mathfrak{h}$ such that

$$Q(\mathfrak{h}) \subset \mathfrak{z} \subset \mathfrak{z}(\mathfrak{h}), \qquad (2.19)$$

let $\mathfrak{g}$ denote the quotient Lie algebra $\mathfrak{h}/\mathfrak{z}$, and let $p : \mathfrak{h} \to \mathfrak{g}$ be the natural projection. The data of $\mathfrak{z} \subset \mathfrak{h}$, i.e. of a Leibniz algebra $\mathfrak{h}$ together with an ideal $\mathfrak{z}$ such that $Q(\mathfrak{h}) \subset \mathfrak{z} \subset \mathfrak{z}(\mathfrak{h})$, could be called an *augmented Leibniz algebra*. Thus we are actually associating an augmented rack bialgebra to every augmented Leibniz algebra. In fact, we will see that this augmented rack bialgebra does not depend on the choice of the ideal $\mathfrak{z}$ and therefore refrain from introducing augmented Leibniz algebras in a more formal way.

The Lie algebra $\mathfrak{g}$ naturally acts as derivations on $\mathfrak{h}$ by means of (for all $x, y \in \mathfrak{h}$)

$$p(x).y := [x, y] =: \mathrm{ad}_x(y) \qquad (2.20)$$

because $\mathfrak{z} \subset \mathfrak{z}(\mathfrak{h})$. Note that

$$\mathfrak{h}/\mathfrak{z}(\mathfrak{h}) \cong \big\{\mathrm{ad}_x \in \mathrm{Hom}_K(\mathfrak{h},\mathfrak{h}) \mid x \in \mathfrak{h}\big\}. \tag{2.21}$$

as Lie algebras.

Consider now the C^5-coalgebra $(B = \mathsf{S}(\mathfrak{h}), \Delta, \epsilon, \mathbf{1})$ which is actually a commutative cocommutative Hopf algebra over K with respect to the symmetric multiplication $\bullet$. The linear map $p : \mathfrak{h} \to \mathfrak{g}$ induces a unique morphism of Hopf algebras

$$\tilde{\Phi} = \mathsf{S}(p) : \mathsf{S}(\mathfrak{h}) \to \mathsf{S}(\mathfrak{g}) \tag{2.22}$$

satisfying

$$\tilde{\Phi}(x_1 \bullet \cdots \bullet x_k) = p(x_1) \bullet \cdots \bullet p(x_k) \tag{2.23}$$

for any nonnegative integer k and $x_1, \ldots, x_k \in \mathfrak{h}$. In other words, the association $\mathsf{S} : V \to \mathsf{S}(V)$ is a functor from the category of all K-modules to the category of all commutative unital C^5-coalgebras. Consider now the universal enveloping algebra $\mathsf{U}(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$. Since $\mathbb{Q} \subset K$ by assumption, the Poincaré-Birkhoff-Witt Theorem (in short: PBW) holds (see e.g. [22, Appendix]). More precisely, the symmetrisation map $\omega : \mathsf{S}(\mathfrak{g}) \to \mathsf{U}(\mathfrak{g})$, defined by

$$\omega(\mathbf{1}_{\mathsf{S}(\mathfrak{g})}) = \mathbf{1}_{\mathsf{U}(\mathfrak{g})}, \quad \text{and} \quad \omega(\xi_1 \bullet \cdots \bullet \xi_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \xi_{\sigma(1)} \cdots \xi_{\sigma(k)}, \tag{2.24}$$

see e.g. [10, p.80, eqn (3)], is an isomorphism of C^5-coalgebras (in general not of associative algebras). We now need an action of the Hopf algebra $H = \mathsf{U}(\mathfrak{g})$ on B, and an intertwining map $\Phi : B \to \mathsf{U}(\mathfrak{g})$. In order to get this, we first look at $\mathfrak{g}$-modules: The K-module $\mathfrak{h}$ is a $\mathfrak{g}$-module by means of eqn (2.20), the Lie algebra $\mathfrak{g}$ is a $\mathfrak{g}$-module via its adjoint representation, and the linear map $p : \mathfrak{h} \to \mathfrak{g}$ is a morphism of $\mathfrak{g}$-modules since p is a morphism of Leibniz algebras. Now $\mathsf{S}(\mathfrak{h})$ and $\mathsf{S}(\mathfrak{g})$ are $\mathfrak{g}$-modules in the usual way, i.e. for all $k \in \mathbb{N} \setminus \{0\}$, $\xi, \xi_1, \ldots, \xi_k \in \mathfrak{g}$, and $x_1 \ldots, x_k \in \mathfrak{h}$

$$\xi.(x_1 \bullet \cdots \bullet x_k) := \sum_{r=1}^{k} x_1 \bullet \cdots \bullet (\xi.x_r) \bullet \cdots \bullet x_k, \tag{2.25}$$

$$\xi.(\xi_1 \bullet \cdots \bullet \xi_k) := \sum_{r=1}^{k} \xi_1 \bullet \cdots \bullet [\xi.\xi_r] \bullet \cdots \bullet \xi_k, \tag{2.26}$$

and of course $\xi.\mathbf{1}_{\mathsf{S}(\mathfrak{h})} = 0$ and $\xi.\mathbf{1}_{\mathsf{S}(\mathfrak{g})} = 0$. Recall that $\mathsf{U}(\mathfrak{g})$ is a $\mathfrak{g}$-module via the adjoint representation $\mathrm{ad}_\xi(u) = \xi.u = \xi u - u\xi$ (for all $\xi \in \mathfrak{g}$ and all $u \in \mathsf{U}(\mathfrak{g})$).

It is easy to see that the map $\tilde{\Phi}$ (2.23) is a morphism of $\mathfrak{g}$-modules, and it is well-known that the symmetrization map ω (2.24) is also a morphism of $\mathfrak{g}$-modules, see e.g. [10, p.82, Prop. 2.4.10]. Define the K-linear map $\Phi : \mathsf{S}(\mathfrak{h}) \to \mathsf{U}(\mathfrak{g})$ by the composition

$$\Phi := \omega \circ \tilde{\Phi}. \tag{2.27}$$

Then Φ is a map of C^5-coalgebras and a map of $\mathfrak{g}$-modules. Thanks to the universal property of the universal enveloping algebra, it follows that $\mathsf{S}(\mathfrak{h})$ and $\mathsf{U}(\mathfrak{g})$ are left $\mathsf{U}(\mathfrak{g})$-modules, via (for all $\xi_1, \ldots, \xi_k \in \mathfrak{g}$, and for all $a \in \mathsf{S}(\mathfrak{h})$)

$$(\xi_1 \cdots \xi_k).a = \xi_1.(\xi_2.(\cdots \xi_k.a)\cdots) \tag{2.28}$$

and the usual adjoint representation (2.10) (for all $u \in \mathsf{U}(\mathfrak{g})$)

$$\mathrm{ad}_{\xi_1 \cdots \xi_k}(u) = \big(\mathrm{ad}_{\xi_1} \circ \cdots \circ \mathrm{ad}_{\xi_k}\big)(u), \tag{2.29}$$

and that Φ intertwines the $\mathsf{U}(\mathfrak{g})$-action on $C = \mathsf{S}(\mathfrak{h})$ with the adjoint action of $\mathsf{U}(\mathfrak{g})$ on itself.
Finally it is a routine check using the above identities (2.25) and (2.10) that $\mathsf{S}(\mathfrak{h})$ becomes a module coalgebra.
We can resume the preceding considerations in the following

Theorem 2.1 *Let $(\mathfrak{h}, [\ ,\ ])$ be a Leibniz algebra over K, let $\mathfrak{z}$ be a two-sided ideal of $\mathfrak{h}$ such that $Q(\mathfrak{h}) \subset \mathfrak{z} \subset \mathfrak{z}(\mathfrak{h})$, let $\mathfrak{g}$ denote the quotient Lie algebra $\mathfrak{h}/\mathfrak{z}$ by $\mathfrak{g}$, and let $p : \mathfrak{h} \to \mathfrak{g}$ be the canonical projection.*

1. *Then there is a canonical $\mathsf{U}(\mathfrak{g})$-action ℓ on the C^5-coalgebra $B := \mathsf{S}(\mathfrak{h})$ (making it into a module coalgebra leaving invariant $\mathbf{1}$) and a canonical lift of p to a map of C^5-coalgebras, $\Phi : \mathsf{S}(\mathfrak{h}) \to \mathsf{U}(\mathfrak{g})$ such that eqn (2.12) holds.*
   Hence the quadruple $(\mathsf{S}(\mathfrak{h}), \Phi, \mathsf{U}(\mathfrak{g}), \ell)$ is an augmented rack bialgebra whose associated Leibniz algebra is equal to $(\mathfrak{h}, [\ ,\ ])$ (independently of the choice of $\mathfrak{z}$).
   The resulting rack multiplication μ of $\mathsf{S}(\mathfrak{h})$ (written $\mu(a \otimes b) = a \triangleright b$) is also independent on the choice of $\mathfrak{z}$ and is explicitly given as follows for all positive integers k, l and $x_1, \ldots, x_k$, $y_1, \ldots, y_l \in \mathfrak{h}$:

$$\big(x_1 \bullet \cdots \bullet x_k\big) \triangleright \big(y_1 \bullet \cdots \bullet y_l\big) = \frac{1}{k!} \sum_{\sigma \in S_k} \big(\mathrm{ad}^s_{x_{\sigma(1)}} \circ \cdots \circ \mathrm{ad}^s_{x_{\sigma(k)}}\big)\big(y_1 \bullet \cdots \bullet y_l\big) \tag{2.30}$$

   where ad^s_x denotes the action of the Lie algebra $\mathfrak{h}/\mathfrak{z}(\mathfrak{h})$ (see eqn (2.21)) on $\mathsf{S}(\mathfrak{h})$ according to eqn (2.25).

2. *In case $\mathfrak{z} = Q(\mathfrak{h})$, the construction mentioned in 1. is a functor $\mathfrak{h} \to \mathsf{UAR}^\infty(\mathfrak{h})$ from the category of all Leibniz algebras to the category of all augmented rack bialgebras associating to $\mathfrak{h}$ the rack bialgebra*

$$UAR^\infty(\mathfrak{h}) := (\mathsf{S}(\mathfrak{h}), \Phi, \mathsf{U}(\mathfrak{g}), \ell)$$

and to each morphism f of Leibniz algebras the pair $\big(\mathsf{S}(f), \mathsf{U}(\overline{f})\big)$ where $\overline{f}$ is the induced Lie algebra morphism.

3. *For each nonnegative integer k, the above construction restricts to each subcoalgebra of order k, $\mathsf{S}(\mathfrak{h})_{(k)} = \oplus_{r=0}^{k} \mathsf{S}^r(\mathfrak{h})$, to define an augmented rack bialgebra $(\mathsf{S}(\mathfrak{h})_{(k)}, \Phi_{(k)}, \mathsf{U}(\mathfrak{g}), \ell|_{\mathsf{U}(\mathfrak{g})\otimes\mathsf{S}(\mathfrak{h})_{(k)}})$ which in case $\mathfrak{z} = Q(\mathfrak{h})$ defines a functor $\mathfrak{h} \to \mathsf{UAR}_{(k)}(\mathfrak{h}) := \big(\mathsf{UAR}^\infty(\mathfrak{h})\big)_{(k)}$ from the category of all Leibniz algebras to the category of all augmented rack bialgebras.*

Proof: *1.* All the statements except the last two ones have already been proven. Note that for all $x, y \in \mathfrak{h}$ we have by definition

$$[x, y] = p(x).y = x \triangleright y,$$

independently of the chosen ideal $\mathfrak{z}$. Moreover we compute

$$\begin{aligned}
&\big(x_1 \bullet \cdots \bullet x_k\big) \triangleright \big(y_1 \bullet \cdots \bullet y_l\big) \\
&\quad = \big((\omega \circ \tilde{\Phi})(x_1 \bullet \cdots \bullet x_k)\big).\big(y_1 \bullet \cdots \bullet y_l\big) \\
&\quad = \frac{1}{k!} \sum_{\sigma \in S_k} \big(p(x_{\sigma(1)}) \cdots p(x_{\sigma(k)})\big).\big(y_1 \bullet \cdots \bullet y_l\big),
\end{aligned}$$

which gives the desired formula since for all $x \in \mathfrak{h}$ and $a \in \mathsf{S}(\mathfrak{h})$, we have

$$p(x).a = \mathrm{ad}^s_x(a).$$

2. Let $f : \mathfrak{h} \to \mathfrak{h}'$ be a morphism of Leibniz algebras, and let $\overline{f} : \overline{\mathfrak{h}} \to \overline{\mathfrak{h}'}$ be the induced morphism of Lie algebras. Hence we get

$$p' \circ f = \overline{f} \circ p \tag{2.31}$$

where $p' : \mathfrak{h}' \to \overline{\mathfrak{h}'}$ denotes the corresponding projection modulo $Q(\mathfrak{h}')$. Let $\mathsf{S}(f) : \mathsf{S}(\mathfrak{h}) \to \mathsf{S}(\mathfrak{h}')$, $\mathsf{S}(\overline{f}) : \mathsf{S}(\overline{\mathfrak{h}}) \to \mathsf{S}(\overline{\mathfrak{h}'})$, and $\mathsf{U}(\overline{f}) : \mathsf{U}(\overline{\mathfrak{h}}) \to \mathsf{U}(\overline{\mathfrak{h}'})$ the induced maps of Hopf algebras, i.e. $\mathsf{S}(f)$ (resp. $\mathsf{S}(\overline{f})$) satisfies eqn (2.23) (with p replaced by f (resp. by $\overline{f}$)), and $\mathsf{U}(\overline{f})$ satisfies

$$\mathsf{U}(\overline{f})\big(\xi_1 \cdots \xi_k\big) = \overline{f}(\xi_1) \cdots \overline{f}(\xi_k)$$

for all positive integers k and $\xi_1, \ldots, \xi_k \in \overline{\mathfrak{h}}$. If $\omega : \mathsf{S}(\overline{\mathfrak{h}}) \to \mathsf{U}(\overline{\mathfrak{h}})$ and $\omega' : \mathsf{S}(\overline{\mathfrak{h}'}) \to \mathsf{U}(\overline{\mathfrak{h}'})$ denote the corresponding symmetrisation maps (2.24) then it is easy to see from the definitions that

$$\omega' \circ \mathsf{S}(\overline{f}) = \mathsf{U}(\overline{f}) \circ \omega.$$

Equation (2.31) implies

$$\tilde{\Phi}' \circ \mathsf{S}(f) = \mathsf{S}(p') \circ \mathsf{S}(f) = \mathsf{S}(\overline{f}) \circ \mathsf{S}(p) = \mathsf{S}(\overline{f}) \circ \tilde{\Phi},$$

and composing from the left with ω' yields the equation

$$\Phi' \circ \mathsf{S}(f) = \mathsf{U}(\overline{f}) \circ \Phi. \tag{2.32}$$

Moreover for all $x, y \in \mathfrak{h}$ we have, since f is a morphism of Leibniz algebras,

$$f\big(p(x).y\big) = f\big([x,y]\big) = \big[f(x), f(y)\big]' = p'\big(f(x)\big).f(y) = \overline{f}\big(p(x)\big).f(y),$$

hence for all $\xi \in \overline{h}$

$$f(\xi.y) = \big(\overline{f}(\xi)\big).\big(f(y)\big),$$

and upon using eqn (2.25) we get for all $a \in \mathsf{S}(\mathfrak{h})$

$$\mathsf{S}(f)\big(\xi.a\big) = \big(\overline{f}(\xi)\big).\Big(\mathsf{S}(f)\big(a\big)\Big),$$

showing finally for all $u \in \mathsf{U}(\mathfrak{h})$ and all $a \in \mathsf{S}(\mathfrak{h})$

$$\mathsf{S}(f)\big(u.a\big) = \big(\mathsf{U}(\overline{f})(u)\big).\Big(\mathsf{S}(f)\big(a\big)\Big). \tag{2.33}$$

Associating to every Leibniz algebra $(\mathfrak{h}, [\ ,\ ])$ the above defined augmented rack bialgebra $(\mathsf{S}(\mathfrak{h}), \Phi, \mathsf{U}(\overline{\mathfrak{h}}), \ell)$, and to every morphism $\psi : \mathfrak{h} \to \mathfrak{h}'$ of Leibniz algebras the pair of K-linear maps $\big(\Psi = \mathsf{S}(\psi), \overline{\Psi} = \mathsf{U}(\overline{\psi})\big)$, we can easily check that Ψ is a morphism of C^5-coalgebras, $\overline{\Psi}$ is a morphism of Hopf algebras, such that the two relevant diagrams (2.13) commute which easily follows from (2.32) and (2.33). The rest of the functorial properties is a routine check.

3. By definition, the $\mathsf{U}(\mathfrak{g})$-action on $\mathsf{S}(\mathfrak{h})$ (cf. eqs (2.25) and (2.28)) leaves invariant each K-submodule $\mathsf{S}^r(\mathfrak{h})$ for each nonnegative integer r whence it leaves invariant each subcoalgebra of order k, $\mathsf{S}(\mathfrak{h})_{(k)}$. It follows that the construction restricts well. □

Remark 2.5

This theorem should be compared to Proposition 3.5 in [4]. In [4], the authors work with the vector space $N := K \oplus \mathfrak{h}$, while we work with the whole symmetric algebra on the Leibniz algebra. In some sense, we extend their Proposition 3.5 "to all orders". However, as we shall see below, N is already enough to obtain a left-adjoint to the functor of primitives. ◇

The above rack bialgebra associated to a Leibniz algebra $\mathfrak{h}$ can be seen as one version of an *enveloping algebra* of $\mathfrak{h}$.

Definition 2.3 *Let $\mathfrak{h}$ be a Leibniz algebra. We will call the augmented rack bialgebra $(\mathsf{S}(\mathfrak{h}), \Phi, \mathsf{U}(\mathfrak{g}), \ell)$ the* ***enveloping algebra of infinite order of*** *$\mathfrak{h}$. As such, it will be denoted by $\mathsf{UAR}^\infty(\mathfrak{h})$.*

This terminology is justified, for example, by the fact that $\mathfrak{h}$ is identified to the primitives in $\mathsf{S}(\mathfrak{h})$ (cf Proposition 2.2). This is also justified by the following theorem the goal of which is to show that the (functorial version of the above) enveloping algebra $\mathsf{UAR}^\infty(\mathfrak{h})$ fits into the following diagram of functors:

$$\begin{array}{ccc} \mathrm{Lie} & \xrightarrow{\ \mathsf{U}\ } & \mathrm{Hopf} \\ \Big\downarrow{\scriptstyle i} & & \Big\downarrow{\scriptstyle j} \\ \mathrm{Leib} & \xrightarrow{\ \mathsf{UAR}^\infty\ } & \mathrm{RackBialg} \end{array}$$

Here, i is the embedding functor of Lie algebras into Leibniz algebras, and j is the embedding functor of the category of connected, cocommutative Hopf algebras into the category of rack bialgebras, using the adjoint action (see eqn (2.10)) as a rack product.

Theorem 2.2 *Let $\mathfrak{g}$ be a Lie algebra. The PBW isomorphism $\mathsf{U}(\mathfrak{g}) \cong \mathsf{S}(\mathfrak{g})$ induces an isomorphism of functors*

$$j \circ \mathsf{U} \ \cong\ \mathsf{UAR}^\infty \circ i.$$

Proof: The enveloping algebra $\mathsf{UAR}^\infty(\mathfrak{h})$ is by definition the functorial version of the rack bialgebra $\mathsf{S}(\mathfrak{h})$, i.e. associated to the ideal $Q(\mathfrak{h})$. But in case $\mathfrak{h}$ is a Lie algebra, $Q(\mathfrak{h}) = \{0\}$. Then the map p is simply the identity, and $\mathsf{UAR}^\infty(\mathfrak{h}) = j\big(U(\mathfrak{h})\big)$. □

As a relatively easy corollary we obtain from the preceding construction the computation of *universal rack bialgebras*. More precisely, we look for a left adjoint functor for the functor Prim, seen as a functor from the category of all rack bialgebras to the category of all Leibniz algebras. For a given Leibniz algebra $\big(\mathfrak{h}, [\ ,\ ]\big)$ define the subcoalgebra of order 1 of the first component of $\mathsf{UAR}_{(1)}(\mathfrak{h})$ (see the third statement of Theorem 2.1), i.e.

$$\mathsf{UR}(\mathfrak{h}) := \mathsf{UAR}^\infty(\mathfrak{h})_{(1)} := K \oplus \mathfrak{h} \tag{2.34}$$

with $\mathbf{1} := 1 = 1_K$ which is rack subbialgebra according to Proposition 2.2. Its structure reads for all $\lambda, \lambda' \in K$ and for all $x, y \in \mathfrak{h}$

$$\begin{aligned} \Delta(\lambda\mathbf{1} + x) &= \lambda\mathbf{1} \otimes \mathbf{1} + x \otimes \mathbf{1} + \mathbf{1} \otimes x, && (2.35)\\ \epsilon(\lambda\mathbf{1} + x) &= \lambda, && (2.36)\\ \mu\big((\lambda\mathbf{1} + x) \otimes (\lambda'\mathbf{1} + x')\big) &= \lambda\lambda'\mathbf{1} + \lambda x' + [x, x']. && (2.37) \end{aligned}$$

For the particular case of a Lie algebra $\big(\mathfrak{h}, [\ ,\ ]\big)$, the above construction can be found in [4, Section 3.2]. Moreover, for any other Leibniz algebra $\big(\mathfrak{h}', [\ ,\ ]'\big)$ and any morphism of Leibniz algebras $f : \mathfrak{h} \to \mathfrak{h}'$ define the K-linear map $\mathsf{UR}(f) : \mathsf{UR}(\mathfrak{h}) \to \mathsf{UR}(\mathfrak{h}')$ as the first component of $\mathsf{UAR}_{(1)}(f)$ (cf. the third statement of Theorem 2.1) by

$$\mathsf{UR}(f)\big(\lambda\mathbf{1} + x\big) = \lambda\mathbf{1} + f(x), \tag{2.38}$$

which is clearly is a morphism of rack bialgebras. Hence UR is a functor from the categry of all Leibniz algebras to the category of all rack bialgebras. Now let $(C, \Delta_C, \epsilon_C, \mathbf{1}_C, \mu_C)$ be a rack bialgebra, and let $f : \mathfrak{h} \to \mathsf{Prim}(C)$ be a morphism of Leibniz algebras. Define the K-linear map $\hat{f} : \mathsf{UR}(\mathfrak{h}) \to C$ by

$$\hat{f}(\lambda\mathbf{1} + x) = \lambda\mathbf{1}_C + f(x), \tag{2.39}$$

and it is again a routine check that it defines a morphism of rack bialgebras. Moreover, due to the almost trivial coalgebra structure of $\mathsf{UR}(\mathfrak{h})$ it is clear that any morphism of rack bialgebras $\mathsf{UR}(\mathfrak{h}) \to C$ is of the above form and is uniquely determined by $\mathsf{Prim}(\hat{f}) = f$. Hence we have shown the following

Theorem 2.3 *There is a left adjoint functor,* UR*, for the functor* Prim *(associating to each Rack bialgebra its Leibniz algebra of all primitive elements). For a given Leibniz algebra* $\big(\mathfrak{h}, [\ ,\ ]\big)$*, the object* $\mathsf{UR}(\mathfrak{h})$ *–which we shall call the* ***Universal Rack Bialgebra of the Leibniz algebra*** $\big(\mathfrak{h}, [\ ,\ ]\big)$*– has the usual universal properties.*

Next we can refine the above universal construction by taking into account the augmented rack bialgebra structure of $\mathsf{UAR}^{\infty}(\mathfrak{h})$ to define another universal object. Consider the more detailed category of all augmented rack bialgebras. Again, the functor Prim applied to the coalgebra B (and not to the Hopf algebra H) gives a functor from the first category to the category of all Leibniz algebras, and we seek again a left adjoint of this functor, called UAR. Hence, a natural candidate for a *universal augmented rack bialgebra associated to a given Leibniz algebra* $\mathfrak{h}$ is

$$\mathsf{UAR}(\mathfrak{h}) := \mathsf{UAR}_{(1)}(\mathfrak{h}) = \big(K \oplus \mathfrak{h}, \Phi_{(1)}, \mathsf{U}(\overline{h}), \mathrm{ad}^s|_{\mathsf{U}(\overline{h})\otimes(K\oplus\mathfrak{h})}\big). \tag{2.40}$$

The third statement of Theorem 2.1 tells us that this is a well-defined augmented rack bialgebra, and that UAR is a functor from the category of all Leibniz algebras to the category of all augmented rack bialgebras. Now let (B', Φ', H', ℓ') be an augmented rack bialgebra, and let $f : \mathfrak{h} \to \mathsf{Prim}(B')$ be a morphism of Leibniz algebras. Clearly, as has been shown in Theorem 2.3, the map $\hat{f} : \mathsf{UR}(\mathfrak{h}) \to B'$ given by eqn (2.38) is a morphism of rack

bialgebras. Observe that the morphism of C^3-coalgebras Φ' sends the Leibniz subalgebra $\mathsf{Prim}(B')$ of B' into K-submodule of all primitive elements of the Hopf algebra H', $\mathsf{Prim}(H')$, which is known to be a Lie subalgebra of H' equipped with the commutator Lie bracket $[\ ,\ ]_{H'}$. Moreover this restriction is a morphism of Leibniz algebras. Indeed, for any $x', y' \in \mathsf{Prim}(B')$ we have

$$\begin{aligned}\Phi'\big([x',y']'\big) &= \Phi'(x' \triangleright y') = \Phi'\big((\Phi'(x')).y'\big) = \mathrm{ad}_{\Phi'(x')}\big(\Phi'(y')\big)\\ &= \Phi'(x')\Phi'(y') - \Phi'(y')\Phi'(x') = \big[\Phi'(x'),\Phi'(y')\big]_{H'}.\end{aligned}$$

It folloys that the two-sided ideal $Q\big(\mathsf{Prim}(B')\big)$ of the Leibniz algebra $\mathsf{Prim}(B')$ is in the kernel of the restriction of Φ' to $\mathsf{Prim}(B')$, whence the map Φ' induces a well-defined K-linear morphism of Lie algebras $\overline{\Phi'|} : \overline{\mathsf{Prim}(B')} \to \mathsf{Prim}(H')$. It follows that the composition $\overline{\Phi'|} \circ \overline{f} : \overline{\mathfrak{h}} \to \mathsf{Prim}(H') \subset H'$ is a morphism of Lie algebras, and by the universal property of universal envelopping algebras there is a unique morphism of associative unital algebras $\psi := \mathsf{U}\big(\overline{\Phi'|} \circ \overline{f}\big) : \mathsf{U}(\overline{\mathfrak{h}}) \to H'$. But we have for all $\xi_1, \ldots, \xi_k \in \overline{\mathfrak{h}}$

$$\begin{aligned}&\Delta_{H'}\big(\psi(\xi_1 \cdots \xi_k)\big) = \Delta_{H'}\big(\psi(\xi_1)\cdots\psi(\xi_k)\big) = \Delta_{H'}\big(\psi(\xi_1)\big)\cdots\Delta\big(\psi(\xi_k)\big)\\ &= \big(\psi(\xi_1)\otimes\mathbf{1}_{H'} + \mathbf{1}_{H'}\otimes\psi(\xi_1)\big)\cdots\big(\psi(\xi_k)\otimes\mathbf{1}_{H'} + \mathbf{1}_{H'}\otimes\psi(\xi_k)\big)\\ &= \big(\psi\otimes\psi\big)(\xi_1\otimes\mathbf{1}_{\mathsf{U}(\overline{\mathfrak{h}})} + \mathbf{1}_{\mathsf{U}(\overline{\mathfrak{h}})}\otimes\xi_1)\cdots\big(\psi\otimes\psi\big)(\xi_k\otimes\mathbf{1}_{\mathsf{U}(\overline{\mathfrak{h}})} + \mathbf{1}_{\mathsf{U}(\overline{\mathfrak{h}})}\otimes\xi_k)\\ &= \big(\psi\otimes\psi\big)\big(\Delta_{\mathsf{U}(\overline{\mathfrak{h}})}(\xi_1)\big)\cdots\big(\psi\otimes\psi\big)\big(\Delta_{\mathsf{U}(\overline{\mathfrak{h}})}(\xi_k)\big)\\ &= \big(\psi\otimes\psi\big)\Big(\Delta_{\mathsf{U}(\overline{\mathfrak{h}})}(\xi_1\cdots\xi_k)\Big)\end{aligned}$$

since ψ maps primitives to primitives whence ψ is a morphism of coalgebras. It is easy to check that ψ preserves counits, whence ψ is a morphism of C^5-Hopf-algebras. For all $\lambda \in K$ and $x \in \mathfrak{h}$ we get

$$\begin{aligned}\big(\psi\circ\Phi_{(1)}\big)(\lambda\mathbf{1}+x) &= \psi\big(\lambda\mathbf{1}_{\mathsf{U}(\overline{\mathfrak{h}})} + p(x)\big) = \lambda\mathbf{1}_{H'} + \big(\overline{\Phi'|}\circ\overline{f}\big)\big(p(x)\big)\\ &= \lambda\mathbf{1}_{H'} + \Phi'\big(f(x)\big) = \Phi'\big(\hat{f}(\lambda\mathbf{1}+x)\big),\end{aligned}$$

showing the first equation $\psi\circ\Phi_{(1)} = \Phi'\circ\hat{f}$ of the morphism equation (2.13). Moreover for all $\lambda \in K$, $x \in \mathfrak{h}$, and $u \in \mathsf{U}(\overline{\mathfrak{h}})$ we get

$$\hat{f}\big(u.(\lambda\mathbf{1}+x)\big) = \hat{f}\big(\lambda\epsilon_{\mathsf{U}(\overline{\mathfrak{h}})}(u)\mathbf{1} + u.x\big) = \lambda\epsilon_{\mathsf{U}(\overline{\mathfrak{h}})}(u)\mathbf{1}_{C'} + f(u.x)$$

Let $x_1, \ldots, x_k \in \mathfrak{h}$ such that $u = p(x_1)\cdots p(x_k)$. Then

$$\begin{aligned}f\big(u.x\big) &= f\big([x_1,[x_2,\ldots[x_k,x]\cdots]\big) = f(x_1)\triangleright\big(f(x_2)\triangleright\cdots\triangleright\big(f(x_k)\triangleright f(x)\big)\cdots\big)\\ &= \Big(\Phi'\big(f(x_1)\big)\cdots\Phi'\big(f(x_k)\big)\Big).\big(f(x)\big)\\ &= \Big(\psi\big(p(x_1)\big)\cdots\psi\big(p(x_k)\big)\Big).\big(f(x)\big) = \big(\psi(u)\big).\big(f(x)\big),\end{aligned}$$

and therefore

$$\hat{f}\big(u.(\lambda\mathbf{1}+x)\big) = \big(\psi(u)\big).\big(\hat{f}(\lambda\mathbf{1}+x)\big)$$

showing the second equation $\psi\circ\Phi_{(1)} = \Phi'\circ\hat{f}$ of the morphism equation (2.13). It follows that the pair $(\hat{f},\psi)$ is a morphism of augmented rack bialgebras. We therefore have the following

Theorem 2.4 *There is a left adjoint functor,* UAR*, for the functor* Prim *(associating to each augmented rack bialgebra its Leibniz algebra of all primitive elements). For a given Leibniz algebra* $\big(\mathfrak{h},[\ ,\ ]\big)$*, the object* $\mathsf{UAR}(\mathfrak{h})$ *–which we shall call the* ***Universal Augmented Rack Bialgebra of the Leibniz algebra*** $\big(\mathfrak{h},[\ ,\ ]\big)$*– has the usual universal properties.*

The relationship between the different notions (taking into account also Remark (2.3)) is resumed in the following diagram:

$$\begin{array}{ccccc}
\mathrm{Lie} & \underset{\mathrm{Prim}}{\overset{\mathsf{U}}{\rightleftarrows}} & \mathrm{Hopf} & \underset{K[-]}{\overset{\mathrm{Slike}}{\rightleftarrows}} & \mathrm{Grp} \\
\downarrow i & & \downarrow j & & \downarrow \\
\mathrm{Leib} & \underset{\mathrm{Prim}}{\overset{\mathsf{UAR}^\infty}{\rightleftarrows}} & \mathrm{RackBialg} & \underset{K[-]}{\overset{\mathrm{Slike}}{\rightleftarrows}} & \mathrm{Racks}
\end{array}$$

where UAR^∞ is not left-adjoint to Prim, while UR is, but does not render the square commutative. There is a similar diagram for augmented notions.

2.3 Relation with bar-unital di(co)algebras

In the beginning of the nineties the 'enveloping structure' associated to Leibniz algebras has been the structure of *dialgebras*, see e.g. [20]. We shall show in this section that rack bialgebras and certain cocommutative Hopf dialgebras are strongly related.

2.3.1 Left-unital bialgebras and right Hopf algebras

Let $(B,\Delta,\epsilon,\mathbf{1},\mu)$ be a nonassociative left-unital C^3-bialgebra. It will be called *left-unital* C^3*-bialgebra* iff μ is associative. In general, $(B,\Delta,\epsilon,\mathbf{1},\mu)$ need not be unital, i.e. we do not have in general $a\mathbf{1} = a$. However, it is easy to see that the sub-module $B\mathbf{1}$ of B is a C^3-subcoalgebra of $(B,\Delta,\epsilon,\mathbf{1})$, and a subalgebra of (B,μ) such that $(B\mathbf{1},\Delta',\epsilon',\mathbf{1},\mu')$ is a unital (i.e. left-unital and right-unital) bialgebra. Here Δ', ϵ', and μ' denote the obvious restrictions and corestrictions.
In a completely analogous way right-unital C^3-bialgebras are defined.

A left-unital (resp. right unital) *cocommutative* C^3-bialgebra $(B, \Delta, \epsilon, \mathbf{1}, \mu)$ will be called a *cocommutative right Hopf algebra* (resp. a *cocommutative left Hopf algebra*), $(B, \Delta, \epsilon, \mathbf{1}, \mu, S)$, iff there is a *right antipode* S (resp. *left antipode* S), i.e. there is a K-linear map $S : B \to B$ which is a morphism of C^3-coalgebras $(B, \Delta, \epsilon, \mathbf{1})$ to itself such that

$$\mathrm{id} * S = \mathbf{1}\epsilon \qquad \big(\text{resp. } \; S * \mathrm{id} = \mathbf{1}\epsilon\big) \tag{2.41}$$

where $*$ denotes the convolution product (see Appendix A for definitions). It will become clear *a posteriori* that right or left antipodes are always unique, see Lemma 2.2.
A first class of examples is of course the well-known class of all *cocommutative Hopf algebras* $(H, \Delta, \epsilon, \mathbf{1}, \mu, S)$ for which $\mathbf{1}$ is a unit element, and S is a right and left antipode.
Secondly it is easy to check that every C^4-coalgebra $(C, \Delta, \epsilon, \mathbf{1})$ equipped with the left-trivial multiplication (resp. right trivial multiplication) μ_0 (see eqn (2.3)) and *trivial right antipode* (resp. *trivial left antipode*) S_0 defined by $S_0(x) = \epsilon(x)\mathbf{1}$ for all $x \in C$ (in both cases) is a cocommutative right Hopf algebra (resp. cocommutative left Hopf algebra) called the cocommutative *left-trivial right Hopf algebra* (resp. right-trivial left Hopf algebra) defined by the C^4-coalgebra $(C, \Delta, \epsilon, \mathbf{1})$.

We have the following elementary properties showing in particular that each right (resp. left) antipode is unique:

Lemma 2.2 *Let $(\mathcal{H}, \Delta, \epsilon, \mathbf{1}, \mu, S)$ be a cocommutative right Hopf algebra.*

1. *$S * (S \circ S) = \mathbf{1}\epsilon$, $S \circ S = \mathrm{id} * \mathbf{1}\epsilon$, $S * \mathbf{1}\epsilon = S$, and $S \circ S \circ S = S$, which for each $a \in \mathcal{H}$ implies $\sum_{(a)} a^{(1)} S(a^{(2)}) = \mathbf{1}\epsilon(a) = \sum_{(a)} S(a^{(1)}) a^{(2)}\mathbf{1}$. It follows that right antipodes are unique.*

2. *For all $a, b \in \mathcal{H}$: $S(ab) = S(b)S(a)$.*

3. *For any element $c \in \mathcal{H}$, c is a generalized idempotent if and only if $c = \sum_{(c)} S(c^{(1)})c^{(2)}$ iff there is $x \in \mathcal{H}$ with $c = \sum_{(x)} S(x^{(1)})x^{(2)}$, and all these three statements imply that c is a generalized left unit element.*

Proof: *1.* Since S is a coalgebra morphism, it preserves convolutions when composing from the right. This gives the first equation from statement *1.* Hence the elements id, S, and $S \circ S$ satisfy the hypotheses of the elements a, b, c of Lemma B.1 in the left-unital convolution semigroup $\big(\mathrm{Hom}_K(\mathcal{H}, \mathcal{H}), *, \mathbf{1}\epsilon\big)$, whence the second and third equation of statement *1.* are immediate, and the fourth follows from composing the second from the right with S and using the third. Clearly S is unique according to Lemma B.1.

2. Again the elements μ, $S \circ \mu$ and $(\mathrm{id} * 1\epsilon) \circ \mu$ satisfy the hypotheses on the elements a, b, c of Lemma B.1 in the left-unital convolution semigroup $\big(\mathrm{Hom}_K(\mathcal{H} \otimes \mathcal{H}, \mathcal{H}), *, \mathbf{1}(\epsilon \otimes \epsilon)\big)$ (using the fact that μ is a morphism of coalgebras) whence $S \circ \mu$ is the unique right inverse of μ. A computation shows that also $\mu \circ \tau \circ (S \otimes S)$ is a right inverse of μ, whence we get statement *2.* by uniqueness of right inverses (Lemma B.1).
3. The second statement obviously implies the third, and it is easy to see by straight-forward computations that the third statement implies the first and the second. Conversely, if $c \in \mathcal{H}$ is a generalized idempotent, i.e. $c = (\mu \circ \Delta)(c)$, we get –since $\mu \circ \Delta$ is a morphism of C^3-coalgebras– that

$$\begin{aligned} \sum_{(c)} S(c^{(1)})c^{(2)} &= \sum_{(c)} S(c^{(1)})\big((\mu \circ \Delta)(c)\big)^{(2)} = \sum_{(c)} S(c^{(1)})(\mu \circ \Delta)\big(c^{(2)}\big) \\ &= \sum_{(c)} S(c^{(1)})c^{(2)}c^{(3)} = \sum_{(c)} S(c^{(1)})c^{(2)}\mathbf{1}c^{(3)} \overset{\text{Lemma 2.2,1.}}{=} c, \end{aligned}$$

and all the three statements are equivalent. In order to see that every such element c is a generalized left unit element pick $y \in \mathcal{H}$ and

$$cy = \sum_{(x)} S(x^{(1)})x^{(2)}y = \sum_{(x)} S(x^{(1)})x^{(2)}\mathbf{1}y \overset{\text{Lemma 2.2,1.}}{=} \epsilon(x)y = \epsilon(c)y$$

since obviously $\epsilon(c) = \epsilon(x)$, so c is a generalized left unit element. □

There is the following right Hopf algebra analogue of the *Suschkewitsch decomposition theorem* for right groups (see Appendix B):

Theorem 2.5 *Let $(\mathcal{H}, \Delta, \epsilon, \mathbf{1}, \mu, S)$ a cocommutative right Hopf algebra. Then the following holds:*

1. *The K-submodule $(\mathcal{H}\mathbf{1}, \Delta|_{\mathcal{H}\mathbf{1}}, \epsilon|_{\mathcal{H}\mathbf{1}}, \mu_{\mathcal{H}\mathbf{1}\otimes\mathcal{H}\mathbf{1}}, S|_{\mathcal{H}\mathbf{1}})$ is a unital Hopf subalgebra of $(\mathcal{H}, \Delta, \epsilon, \mathbf{1}, \mu, S)$.*

2. *The K-submodule $E_{\mathcal{H}} := \{x \in \mathcal{H} \mid x \text{ is a generalized idempotent}\}$ is a right Hopf subalgebra of $\mathcal{H}$ equal to the left-trivial right Hopf algebra defined by the C^4-coalgebra $(E_{\mathcal{H}}, \Delta|_{E_{\mathcal{H}}}, \epsilon|_{E_{\mathcal{H}}}, \mathbf{1}\epsilon|_{E_{\mathcal{H}}})$.*

3. *The map*

$$\Psi : \mathcal{H} \to \mathcal{H}\mathbf{1} \otimes E_{\mathcal{H}} : x \mapsto \sum_{(x)} x^{(1)}\mathbf{1} \otimes S(x^{(2)})x^{(3)}$$

is an isomorphism of right Hopf algebras whose inverse Ψ^{-1} is the restriction of the multiplication map.

Proof: *1.* It is easy to see that $\mathcal{H}\mathbf{1}$ equipped with all the restrictions is a unital bialgebra. Note that for all $a \in \mathcal{H}$

$$\begin{aligned}\big(S|_{\mathcal{H}\mathbf{1}} * \mathrm{id}_{\mathcal{H}\mathbf{1}}\big)(a\mathbf{1}) &= \Big(\big(S * \mathrm{id}_{\mathcal{H}}\big)(a)\Big)\mathbf{1} = \big(S * \mathrm{id}_{\mathcal{H}} * (\mathbf{1}\epsilon)\big)(a) \\ &\overset{\text{Lemma 2.2}}{=} (\mathbf{1}\epsilon)(a) = (\mathbf{1}\epsilon)(a\mathbf{1}) = (\mathbf{1}\epsilon|_{\mathcal{H}\mathbf{1}})(a\mathbf{1})\end{aligned}$$

whence $S|_{\mathcal{H}\mathbf{1}}$ is also a left antipode. It follows that $\mathcal{H}\mathbf{1}$ is a Hopf algebra.
2. Since the property of being an generalized idempotent is a K-linear condition, it follows that $E_{\mathcal{H}}$ is a K-submodule. Moreover since each $c \in E_{\mathcal{H}}$ is of the general form $c = \sum_{(x)} S(x^{(1)})x^{(2)}$, $x \in \mathcal{H}$, and since the map $\iota : \mathcal{H} \to \mathcal{H}$ defined by $\iota(x) = S(x^{(1)})x^{(2)}$ is a idempotent morphism of C^3-coalgebras, we get

$$\Delta(c) = \Delta\big(\iota(c)\big) = (\iota \otimes \iota)\big(\Delta(x)\big)$$

showing that $E_{\mathcal{H}}$ is a C^3-subcoalgebra of $\mathcal{H}$. Furthermore, since every element of $E_{\mathcal{H}}$ is a generalized left unit element (Lemma 2.2, *3.*), the restriction of the multiplication of μ of $\mathcal{H}$ to $E_{\mathcal{H}} \otimes E_{\mathcal{H}}$ is left trivial. Finally,

$$\begin{aligned}S(c) = S\big(\iota(c)\big) &= \sum_{(c)} S\big(S(c^{(1)})c^{(2)}\big) \overset{\text{Lemma 2.2, 2.}}{=} \sum_{(c)} S(c^{(1)})S\big(S(c^{(2)})\big) \\ &\overset{\text{Lemma 2.2, 1.}}{=} \sum_{(c)} S(c^{(1)})c^{(2)}\mathbf{1} \overset{\text{Lemma 2.2, 1.}}{=} \epsilon(c)\mathbf{1} = S_0(c),\end{aligned}$$

showing that the the restriction of S to $E_{\mathcal{H}}$ is the trivial right antipode.
3. It is clear from the two preceding statements that Ψ is a well-defined linear map into the tensor product of two cocommutative right Hopf algebras. We have for all $x \in \mathcal{H}$

$$(\mu \circ \Psi)(x) = \sum_{(x)} x^{(1)}\mathbf{1}S(x^{(2)})x^{(3)} \overset{\text{Lemma 2.2, 1.}}{=} \sum_{(x)} \epsilon(x^{(1)})x^{(2)} = x$$

and for all $a \in \mathcal{H}$, $c \in E_{\mathcal{H}}$ the term $(\Psi \circ \mu)(a\mathbf{1} \otimes c)$ is equal to

$$\begin{aligned}&\sum_{(a)(c)} \big(a^{(1)}c^{(1)}\mathbf{1}\big) \otimes \big(S(c^{(2)})S(a^{(2)})a^{(3)}c^{(3)}\big) \\ &= \sum_{(c)} \big(ac^{(1)}\mathbf{1}\big) \otimes \big(S(c^{(2)})c^{(3)}\big) = \sum_{(c)} (a\mathbf{1}) \otimes \big(S(c^{(1)})c^{(2)}\big) = (a\mathbf{1}) \otimes c\end{aligned}$$

because all the terms $S(a^{(2)})a^{(3)}$ and the components $c^{(1)}, \ldots$ of iterated comultiplications of generalized idempotents can be chosen in $E_{\mathcal{H}}$ (since the latter has been shown to be a subcoalgebra), and are thus generalized left unit elements (Lemma 2.2, *3.*). Hence Ψ is a K-linear isomorphism. Moreover, it is easy to see from its definition that Ψ is a morphism of C^3-coalgebras.
Next we compute for all $a, a' \in \mathcal{H}$ and $c, c' \in E_H$:

$$\Psi^{-1}\Big(\big((a\mathbf{1}) \otimes c\big)\big((c'\mathbf{1}) \otimes f'\big)\Big) = \Psi^{-1}\big((a\mathbf{1}a'\mathbf{1}) \otimes \epsilon(c)c'\big) = \epsilon(c)aa'c',$$

and -since c is a generalized left unit element–

$$\Psi^{-1}\big((a\mathbf{1})\otimes c\big)\Psi^{-1}\big((a'\mathbf{1})\otimes c'\big) \;=\; acac' = \epsilon(c)aa'c',$$

showing that Ψ^{-1} and hence Ψ is a morphism of left-unital algebras. Finally we obtain

$$\begin{aligned}(S|_{\mathcal{H}\mathbf{1}}\otimes S_0)\big(\Psi(x)\big) &= \sum_{(x)}\big(S(x^{(1)})\mathbf{1}\big)\otimes\big(\epsilon\big(S(x^{(2)})x^{(3)}\big)\mathbf{1}\big) = S(x)\mathbf{1}\otimes\mathbf{1},\\ \Psi\big(S(x)\big) &= \sum_{(x)}\big(S(x^{(1)})\mathbf{1}\big)\otimes\big(S(S(x^{(2)}))S(x^{(3)})\big) = S(x)\mathbf{1}\otimes\mathbf{1},\end{aligned}$$

thanks to Lemma 2.2, and Ψ intertwines right antipodes. □

Note that the K-submodule of all *generalized left unit elements* of a right Hopf algebra $\mathcal{H}$ is given by $K\mathbf{1}\oplus\big(\Phi^{-1}(\mathcal{H}\mathbf{1}\otimes E_{\mathcal{H}}^{+})\big)$ and thus in general much bigger than the submodule $E_{\mathcal{H}}$ of all generalized idempotents.

As it is easy to see that every tensor product $H\otimes C$ of a unital cocommutative Hopf algebra H and a C^4-coalgebra C (equipped with the left-trivial multiplication and the trivial right antipode) is a right Hopf algebra, it is a fairly routine check –using the preceding Theorem 2.5– that the category of all cocommutative right Hopf algebras is equivalent to the product category of all cocommutative Hopf algebras and of all C^4-coalgebras.
In the sequel, we shall need the dual *left Hopf algebra version* where all the formulas have to be put in reverse order: Here every left Hopf algebra is isomorphic to $C\otimes H$.

2.3.2 Dialgebras and Rack Bialgebras

Recall (cf. e.g. [20]) that a *dialgebra* over K is a K-module D equipped with two associative multiplications $\vdash,\dashv\colon A\otimes A\to A$ (written $a\otimes b\mapsto a\vdash b$ and $a\otimes b\mapsto a\dashv b$) satisfying for all $a,b,c\in A$:

$$(a\vdash b)\vdash c \;=\; (a\dashv b)\vdash c, \tag{2.42}$$
$$a\dashv(b\dashv c) \;=\; a\dashv(b\vdash c), \tag{2.43}$$
$$(a\vdash b)\dashv c \;=\; a\vdash(b\dashv c). \tag{2.44}$$

An element $\mathbf{1}$ of A is called a *bar-unit element* of the dialgebra $(A,\vdash,\dashv)$ and $(A,\mathbf{1},\vdash,\dashv)$ is called a *bar-unital dialgebra* iff in addition the following holds

$$\mathbf{1}\vdash a \;=\; a, \tag{2.45}$$
$$a\dashv\mathbf{1} \;=\; a, \tag{2.46}$$

for all $a \in A$. Moreover, we shall call a bar-unital dialgebra $(A, \mathbf{1}, \vdash, \dashv)$ *balanced* iff in addition for all $a \in A$

$$a \vdash \mathbf{1} = \mathbf{1} \dashv a. \tag{2.47}$$

Clearly each associative algebra is a dialgebra upon setting $\vdash = \dashv$ equal to the given multiplication. The class of all (bar-unital and balanced) dialgebras forms a category where morphisms preserve both multiplications and map the initial bar-unit to the target bar-unit.

These algebras had been introduced to have a sort of 'associative analogue' for Leibniz algebras. More precisely, there is the following important fact, which can easily be checked, see e.g. [20]:

Proposition 2.4 *Let $(A, \vdash, \dashv)$ be a dialgebra. Then the K-module A equipped with the bracket $[\ ,\ ] : A \otimes A \to A$, written $[a, b]$,*

$$[a, b] := a \vdash b - b \dashv a \tag{2.48}$$

is a Leibniz algebra, denoted by A^-.

In fact this construction is well-known to give rise to a functor $A \to A^-$ from the category of all dialgebras to the category of all Leibniz algebras in complete analogy to the obvious functor from the category of all associative algebras to the category of all Lie algebras.

An important construction of (bar-unital) dialgebras is the following:

Example 2.5
Let $(B, \mathbf{1}_B)$ be a unital associative algebra over K, and let A be a K-module which is a B-*bimodule*, i.e. there are K-linear maps $B \otimes A \to A$ and $A \otimes B \to A$ (written $(b \otimes x) \mapsto bx$ and $(x \otimes b) \mapsto xb$) equipping A with the structure of a left B-module and a right B-module such that $(bx)b' = b(xb')$ for all $b, b' \in B$ and for all $x \in A$. Suppose in addition that there is a bimodule map $\Phi : A \to B$, i.e. $\Phi(bxb') = b\Phi(x)b'$ for all $b, b' \in B$ and for all $x \in A$. Then it is not hard to check that the two multiplications $\vdash, \dashv$: $A \otimes A \to A$ defined by

$$x \vdash y := \Phi(x)y \quad \text{and} \quad x \dashv y := x\Phi(y) \tag{2.49}$$

equip A with the structure of a dialgebra. If in addition there is an element $\mathbf{1} \in A$ such that $\Phi(\mathbf{1}) = \mathbf{1}_B$, then $(A, \mathbf{1}, \vdash, \dashv)$ will be a bar-unital dialgebra. We shall call this structure (A, Φ, B) an *augmented dialgebra*. $\diamond$

In fact, every dialgebra $(A, \vdash, \dashv)$ arises in that fashion: Consider the K-submodule $I \subset A$ whose elements are linear combinations of arbitrary product expressions

$$p\big(a_1, \ldots, a_{r-1}, (a_r \vdash b_r - a_r \dashv b_r), a_{r+1}, \ldots, a_n\big)$$

(where all reasonable parentheses and symbols $\vdash$ and $\dashv$ are allowed) for any two strictly positive integer $r \leq n$, and $a, \ldots, a_n, b_r \in A$. It follows that the quotient module A/I is equipped with an associative multiplication induced by both $\vdash$ and $\dashv$. Let $A^1_{\rm ass}$ be equal to A/I if A is bar-unital: In that case, the bar-unit $\mathbf{1}$ of A projects on the unit element of A/I; and let $A^1_{\rm ass}$ be equal to $A/I \oplus K$ (adjoining a unit element) in case A does not have a bar-unit. Thanks to the defining equations (2.42), (2.43), (2.44), it can be shown by induction that for any strictly positive integer n, any $a_1, \ldots, a_n, a \in A$, and any product expression made of the preceding elements upon using $\vdash$ or $\dashv$

$$\begin{aligned} p(a_1, \ldots, a_n) \vdash a &= a_1 \vdash \cdots \vdash a_n \vdash a = (a_1 \dashv \cdots \dashv a_n) \vdash a, \\ a \dashv p(a_1, \ldots, a_n) &= a \dashv a_1 \dashv \cdots \dashv a_n = a \dashv (a_1 \vdash \cdots \vdash a_n), \end{aligned}$$

proving in particular that I acts trivially from the left (via $\vdash$) and from the right (via $\dashv$) on A such that there is a well-defined $A^1_{\rm ass}$-bimodule structure on A such that the natural map $\Phi_A : A \to A^1_{\rm ass}$ is a bimodule morphism. Hence $(A, \Phi_A, A^1_{\rm ass})$ is always an augmented dialgebra, and the assignment $A \to (A, \Phi_A, A^1_{\rm ass})$ is known to be a faithful functor.
Note also that this construction allows to *adjoin a bar-unit* to a dialgebra $(A, \vdash, \dashv)$: Consider the K-module $\tilde{A} := A \oplus A^1_{\rm ass}$ with the obvious $A^1_{\rm ass}$-bimodule structure $\alpha.(b + \beta) = \alpha.b + \alpha\beta$ and $(b + \beta).\alpha = b.\alpha + \beta\alpha$ for all $\alpha, \beta \in A^1_{\rm ass}$ and $b \in A$. Observe that the obvious map $\Phi_{\tilde{A}} : \tilde{A} \to A^1_{\rm ass}$ defined by $\Phi_{\tilde{A}}(b + \beta) = \Phi_A(b) + \beta$ is an $A^1_{\rm ass}$-bimodule map, and that $\mathbf{1} = \mathbf{1}_{A^1_{\rm ass}}$ is a bar-unit. The bar-unital augmented dialgebra $(\tilde{A}, \Phi_{\tilde{A}}, A^1_{\rm ass})$ is easily seen to be balanced. There are nonbalanced bar-unital dialgebras as can be seen from the augmented bar-unital dialgebra example $(B \otimes B, \mathbf{1}_B \otimes \mathbf{1}_B, \mu_B, B)$ where $(B, \mathbf{1}, \mu_B)$ is any unital associative algebra and the bimodule action is defined by $b.(b_1 \otimes b_2).b' := (bb_1) \otimes (b_2 b')$ for all $b, b', b_1, b_2 \in B$.
Again, in case the dialgebra $(A, \vdash, \dashv, \mathbf{1})$ is bar-unital and balanced, note that $A \vdash \mathbf{1} = \mathbf{1} \dashv A$ is an associative unital subalgebra A' of A whose multiplication is induced by both $\vdash$ and $\dashv$, i.e. $a' \vdash b' = a' \dashv b'$ for all $a', b' \in A'$. Since the K-linear map $\pi_A : A \to A : a \mapsto a \vdash \mathbf{1} = \mathbf{1} \dashv a$ descends to a surjective morphism of associative algebras $A^1_{\rm ass} \to A'$ by the above, it is clear that the ideal I contains the kernel of π_A. On the other hand, if $a \in \mathrm{Ker}(\pi_A)$ then $0 = \pi(a) = \mathbf{1} \vdash a$, and obviously $a = \mathbf{1} \vdash a - \mathbf{1} \dashv a \in I$, thus inducing a useful isomorphism $A^1_{\rm ass} \cong A'$, and thus a subalgebra injection $i_A : A^1_{\rm ass} \to A : \Phi_A(a) \mapsto a \vdash \mathbf{1}$ which is a right inverse to the projection Φ_A, i.e. $\Phi_A \circ i_A = \mathrm{id}_{A^1_{\rm ass}}$.

In this work, we also have to take into account coalgebra structures and thus define the following:

Definition 2.4 *Let $(A, \Delta, \epsilon, \mathbf{1})$ be cocommutative C^3-coalgebra (a C^4-coalgebra)*

and two K-linear maps $\vdash, \dashv\colon A \otimes A \to A$. Then $(A, \Delta, \epsilon, \mathbf{1}, \vdash, \dashv)$ will be called a ***cocommutative bar-unital di-coalgebra*** *if and only if*

1. *$(A, \mathbf{1}, \vdash, \dashv)$ is a bar-unital balanced dialgebra.*

2. *Both $\vdash$ and $\dashv$ are morphisms of C^3-coalgebras.*

If in addition there is a morphism of C^3-coalgebras $S : A \to A$ such that $(A, \Delta, \epsilon, \mathbf{1}, \vdash, S)$ is a cocommutative right Hopf algebra and $(A, \Delta, \epsilon, \mathbf{1}, \dashv, S)$ is a cocommutative left Hopf algebra, then $(A, \Delta, \epsilon, \mathbf{1}, \vdash, \dashv, S)$ is called a ***cocommutative Hopf dialgebra****.*

We have used a relatively simple notion of one single compatible coalgebra structure motivated from differential geometry, see Section 3. In contrast to that, F. Goichot uses two a priori different coalgebra structures, see [12]. Moreover, a slightly more general context would have been to demand the existence of two different antipodes, a right antipode S for $\vdash$, and a left antipode S' for $\dashv$. The theory –including the classification in terms of ordinary Hopf algebras– could have been done as well, but we have refrained from doing so since it is not hard to see that such a more general Hopf dialgebra is **balanced iff** $S = S'$. This fact is crucial in the following refinement of Proposition 2.4:

Proposition 2.5 *Let $(A, \Delta, \epsilon, \mathbf{1}, \vdash, \dashv, S)$ be cocommutative Hopf dialgebra. Then the submodule of all primitive elements of A, $\mathsf{Prim}(A)$, is a Leibniz subalgebra of A equipped with the bracket (2.48).*

Proof: Let $x, y \in A$ be primitive. Then, using that $\vdash$ and $\dashv$ are morphisms of coalgebras, we get

$$\begin{aligned}\Delta(x \vdash y - y \dashv x) &= \mathbf{1} \otimes (x \vdash y - y \dashv x) + (x \vdash y - y \dashv x) \otimes \mathbf{1} \\ &\quad + (x \vdash \mathbf{1}) \otimes y + y \otimes (x \vdash \mathbf{1}) - y \otimes (\mathbf{1} \dashv x) - (\mathbf{1} \dashv x) \otimes y \\ &= \mathbf{1} \otimes (x \vdash y - y \dashv x) + (x \vdash y - y \dashv x) \otimes \mathbf{1}\end{aligned}$$

because A is balanced, and therefore $x \vdash y - y \dashv x$ is primitive. □

The first relationship with rack bialgebras is the following simple generalization of a cocommutative Hopf algebra equipped with the adjoint representation:

Proposition 2.6 *Let $(A, \Delta, \epsilon, \mathbf{1}, \vdash, \dashv, S)$ be cocommutative Hopf dialgebra. Define the following multiplication $\mu : A \otimes A \to A$ by*

$$\mu(a \otimes b) := a \triangleright b := \sum_{(a)} (a^{(1)} \vdash b) \dashv \big(S(a^{(2)})\big). \tag{2.50}$$

Then we have the following:

1. *The map $\triangleright$ defines on the K-module A two left module structures, one with respect to the algebra $(A, \mathbf{1}, \vdash)$, and one with respect to the algebra $(A, \mathbf{1}, \dashv)$, making the Hopf-dialgebra $(A, \Delta, \epsilon, \mathbf{1}, \vdash, \dashv, S)$ a module-Hopf dialgebra, i.e.*

$$\begin{aligned} a \triangleright (b \triangleright c) &= (a \vdash b) \triangleright c = (a \dashv b) \triangleright c && (2.51)\\ \Delta(a \triangleright b) &= \sum_{(a),(b)} (a^{(1)} \triangleright b^{(1)}) \otimes (a^{(2)} \triangleright b^{(2)}) && (2.52)\\ a \triangleright (b \vdash c) &= \sum_{(a)} (a^{(1)} \triangleright b) \vdash (a^{(2)} \triangleright c) && (2.53)\\ a \triangleright (b \dashv c) &= \sum_{(a)} (a^{(1)} \triangleright b) \dashv (a^{(2)} \triangleright c) && (2.54)\\ & && (2.55) \end{aligned}$$

2. *$(A, \Delta, \epsilon, \mathbf{1}, \mu)$ is a cocommutative rack bialgebra.*

Proof: *1.* First of all we have

$$\mu = \mu_{\dashv} \circ (\mu_{\vdash} \otimes S) \circ (\mathrm{id}_A \otimes \tau_{A,A}) \circ (\Delta \otimes \mathrm{id}_A)$$

where $\mu_{\vdash}$ and $\mu_{\dashv}$ stand for the multiplication maps $\vdash$ and $\dashv$, and this is clearly a composition of morphisms of C^3-coalgebras whence μ is a morphism of C^3-coalgebras proving eqn (2.52). Next, there is clearly $\mathbf{1} \triangleright b = b$ for all $b \in A$, and, **since the dialgebra is balanced**, we get for all $a \in A$

$$\begin{aligned} a \triangleright \mathbf{1} &= \sum_{(a)} (a^{(1)} \vdash \mathbf{1}) \dashv \big(S(a^{(2)})\big) = \sum_{(a)} \mathbf{1} \dashv a^{(1)} \dashv \big(S(a^{(2)})\big)\\ &= \sum_{(a)} \mathbf{1} \dashv \Big(a^{(1)} \vdash \big(S(a^{(2)})\big)\Big) = \epsilon(a)\mathbf{1} \dashv \mathbf{1} = \epsilon(a)\mathbf{1}. \end{aligned}$$

Next, let $a, b, c \in A$. Then

$$\begin{aligned} a \triangleright (b \triangleright c) &= \sum_{(a),(b)} \Big(a^{(1)} \vdash \big((b^{(1)} \vdash c) \dashv S(b^{(2)})\big)\Big) \dashv S(a^{(2)})\\ &= \sum_{(a),(b)} \Big(\big((a^{(1)} \vdash b^{(1)}) \vdash c\big) \dashv \big(S(b^{(2)}) \dashv S(a^{(2)})\big)\Big)\\ &= \sum_{(a),(b)} \Big(\big((a^{(1)} \vdash b^{(1)}) \vdash c\big) \dashv \big(S(b^{(2)}) \vdash S(a^{(2)})\big)\Big)\\ &= \sum_{(a),(b)} \Big(\big((a^{(1)} \overset{\dashv}{\vdash} b^{(1)}) \vdash c\big) \dashv \big(S(a^{(2)} \overset{\dashv}{\vdash} b^{(2)})\big)\Big)\\ &= (a \overset{\dashv}{\vdash} b) \triangleright c, && (2.56) \end{aligned}$$

proving eqs (2.51). Next, for all $a, a', a'' \in A$, we get

$$\begin{aligned}
\sum_{(a)} (a^{(1)} \triangleright a') \overset{\dashv}{\vdash} (a^{(2)} \triangleright a'') &= \sum_{(a)} \Big((a^{(1)} \vdash a') \dashv S(a^{(2)}) \Big) \overset{\dashv}{\vdash} \Big((a^{(3)} \vdash a'') \dashv S(a^{(4)}) \Big) \\
&= \sum_{(a)} (a^{(1)} \vdash a') \overset{\dashv}{\vdash} \Big(\big(S(a^{(2)}) \overset{\dashv}{\vdash} a^{(3)} \big) \overset{\dashv}{\vdash} \big(a'' \dashv S(a^{(4)}) \big) \Big) \\
&= \sum_{(a)} (a^{(1)} \vdash a') \overset{\dashv}{\vdash} \big(\big(\epsilon(a^{(2)}) \mathbf{1} \big) \overset{\dashv}{\vdash} \big(a'' \dashv S(a^{(3)}) \big) \big) \\
&= \sum_{(a)} \big(a^{(1)} \vdash (a' \overset{\dashv}{\vdash} a'') \big) \dashv S(a^{(2)}) = a \triangleright (a' \overset{\dashv}{\vdash} a''),
\end{aligned}$$

where –in the second to last equation– we have used the left antipode identity for the case $\dashv$ and the fact that $\sum_{(a)} S(a^{(1)}) \vdash a^{(2)}$ is a generalized left unit element for the case $\vdash$. It follows that $(A, \overset{\dashv}{\vdash})$ is an A-module-algebra proving eqs (2.53) and (2.54).

2. It remains to prove self-distributivity: For all $a, b, c \in A$, we get

$$\sum_{(a)} (a^{(1)} \triangleright b) \triangleright (a^{(2)} \triangleright c) \overset{(2.51)}{=} \sum_{(a)} \big((a^{(1)} \triangleright b) \dashv a^{(2)} \big) \triangleright c,$$

and in the end

$$\begin{aligned}
\sum_{(a)} (a^{(1)} \triangleright b) \dashv a^{(2)} &= \sum_{(a)} \big((a^{(1)} \vdash b) \dashv S(a^{(2)}) \big) \dashv a^{(3)} \\
&= \sum_{(a)} (a^{(1)} \vdash b) \dashv \big(S(a^{(2)}) \dashv a^{(3)} \big) \\
&= \sum_{(a)} (a^{(1)} \vdash b) \dashv \big(\epsilon(a^{(2)}) \mathbf{1} \big) = a \vdash b
\end{aligned}$$

proving the self-distributivity identity. □

The next theorem relates augmented cocommutative rack bialgebras with cocommutative Hopf dialgebras:

Theorem 2.6 *Let (B, Φ_B, H, ℓ) be a cocommutative augmented rack bialgebra. Then the K-module $(B \otimes H, \Delta_{B\otimes H}, \epsilon_B \otimes \epsilon_H, \mathbf{1}_B \otimes \mathbf{1}_H, \Phi, H)$ will be an augmented cocommutative Hopf dialgebra by means of the following definitions. Here we use Example 2.5 and take $h, h' \in H$ and $b \in B$:*

1. $\Phi : B \otimes H \to H : (b \otimes h) \mapsto \Phi(b \otimes h) := \Phi_B(b)h$.
2. $h'.(b \otimes h) := \sum_{(h')} ((h')^{(1)}.b) \otimes ((h')^{(2)}h)$ *and* $(b \otimes h).h' := b \otimes (hh')$.

3. $S(b\otimes h) := \mathbf{1}_B \otimes S_H\big(\Phi_B(b)h\big)$.

Moreover, the Leibniz bracket on the K-module of all primitive elements of $B\otimes H$, $\mathfrak{a} := \mathsf{Prim}(B)\oplus\mathsf{Prim}(H)$, is computed as follows for all $x, y \in \mathsf{Prim}(B)$ and all ξ, η in the Lie algebra $\mathsf{Prim}(H)$ (writing x and ξ for the more precise $x\otimes\mathbf{1}_H$ and $\mathbf{1}_B\otimes\xi$)

$$[x+\xi, y+\eta] = \Big([x,y] + \xi.y\Big) + \Big([\Phi_B(x),\eta] + [\xi,\eta]\Big) \tag{2.57}$$

where each bracket is of the form (2.48). Note that this Leibniz algebra is split over the Lie subalgebra $\mathsf{Prim}(H)$, the complementary two-sided ideal $\{x - \Phi_B(x) \mid x \in \mathsf{Prim}(B)\}$ being in the left center of $\mathfrak{a}$.

Proof: It is clear from the definitions that condition *2* defines a H-bimodule structure on $C\otimes H$ making it into a module C^3-coalgebra. Moreover, we compute for all $h, h', h'' \in H$ and $b \in B$

$$\begin{aligned}
\Phi\big(h'.(b\otimes h).h''\big) &= \sum_{(h')} \Phi\big(((h')^{(1)}.b)\otimes((h')^{(2)}hh''\big) = \sum_{(h')} \Phi_B((h')^{(1)}.b)(h')^{(2)}hh'' \\
&= \sum_{(h')} \mathrm{ad}_{(h')^{(1)}}\big(\Phi_B(b)\big)(h')^{(2)}hh'' = \sum_{(h')} (h')^{(1)}\Phi_B(b)S_H((h')^{(2)})(h')^{(3)}hh'' \\
&= \sum_{(h')} (h')^{(1)}\Phi_B(b)\epsilon_H((h')^{(2)})hh'' = h'\Phi_B(b)hh'' = h'\Phi(b\otimes h)h'',
\end{aligned}$$

whence Φ is a morphism of H-bimodules. Next, we get for all $b \in B$ and $h \in H$:

$$\begin{aligned}
\big(\mathrm{id}_{B\otimes H} *_{\vdash} S\big)(b\otimes h) &= \sum_{(b)(h)} (b^{(1)}\otimes h^{(1)}) \vdash S(b^{(2)}\otimes h^{(2)}) \\
&= \sum_{(b)(h)} \Phi(b^{(1)}\otimes h^{(1)}).\Big(\mathbf{1}_B \otimes S_H\big(\Phi_B(b^{(2)})h^{(2)}\big)\Big) \\
&= \sum_{(b)(h)} \big(\Phi_B(b^{(1)})h^{(1)}\big).\Big(\mathbf{1}_B \otimes S_H\big(\Phi_B(b^{(2)})h^{(2)}\big)\Big) \\
&= \sum_{(b)(h)} \epsilon_H\Big(\Phi_B(b^{(1)})h^{(1)}\Big)\mathbf{1}_B \otimes \Big(\Phi_B(b^{(2)})h^{(2)}S_H\big(\Phi_B(b^{(3)})h^{(3)}\big)\Big) \\
&= \mathbf{1}_B \otimes \big(\epsilon_B(b)\epsilon_H(h)\big)\mathbf{1}_H = \big(\epsilon_B\otimes\epsilon_H\big)(b\otimes h)\big(\mathbf{1}_B\otimes\mathbf{1}_H\big),
\end{aligned}$$

proving the right antipode identity, and

$$\begin{aligned}
\big(S *_{\dashv} \mathrm{id}_{B\otimes H}\big)(b\otimes h) &= \sum_{(b)(h)} S(b^{(1)}\otimes h^{(1)}) \dashv (b^{(2)}\otimes h^{(2)}) \\
&= \sum_{(b)(h)} \Big(\mathbf{1}_B \otimes S_H\big(\Phi_B(b^{(1)})h^{(1)}\big)\Big).\Phi(b^{(2)}\otimes h^{(2)}) \\
&= \sum_{(b)(h)} \Big(\mathbf{1}_B \otimes S_H\big(\Phi_B(b^{(1)})h^{(1)}\big)\Big).\big(\Phi_B(b^{(2)})h^{(2)}\big) \\
&= \sum_{(b)(h)} \mathbf{1}_B \otimes \Big(S_H\big(\Phi_B(b^{(1)})h^{(1)}\big)\Phi_B(b^{(2)})h^{(2)}\Big) \\
&= \mathbf{1}_B \otimes \big(\epsilon_B(b)\epsilon_H(h)\big)\mathbf{1}_H = \big(\epsilon_B\otimes\epsilon_H\big)(b\otimes h)\big(\mathbf{1}_B\otimes\mathbf{1}_H\big),
\end{aligned}$$

proving the left antipode identity. Finally for all $h\in H$ we get

$$h.(\mathbf{1}_B\otimes\mathbf{1}_H) = \sum_{(h)}(\epsilon_H(h^{(1)})\mathbf{1}_B)\otimes h^{(2)} = \mathbf{1}_B\otimes h = (\mathbf{1}_B\otimes\mathbf{1}_H).h$$

implying that the bar-unital dialgebra is balanced.
Formula (2.57) is straight-forward:

$$\begin{aligned}
&[x\otimes\mathbf{1}_H + \mathbf{1}_B\otimes\xi, y\otimes\mathbf{1}_H + \mathbf{1}_B\otimes\eta] \\
&\quad= (x\otimes\mathbf{1}_H)\vdash(y\otimes\mathbf{1}_H) - (y\otimes\mathbf{1}_H)\dashv(x\otimes\mathbf{1}_H) \\
&\qquad +(x\otimes\mathbf{1}_H)\vdash(\mathbf{1}_B\otimes\eta]) - (\mathbf{1}_B\otimes\eta])\dashv(x\otimes\mathbf{1}_H) \\
&\qquad +(\mathbf{1}_B\otimes\xi)\vdash(y\otimes\mathbf{1}_H) - (y\otimes\mathbf{1}_H)\dashv(\mathbf{1}_B\otimes\xi) \\
&\qquad +(\mathbf{1}_B\otimes\xi)\vdash(\mathbf{1}_B\otimes\eta) - (\mathbf{1}_B\otimes\eta)\dashv(\mathbf{1}_B\otimes\xi) \\
&\quad= (\Phi_B(x).y)\otimes\mathbf{1}_H + y\otimes\Phi_B(x) - y\otimes\Phi_B(x) \\
&\qquad +\epsilon_B(x)\mathbf{1}_B\otimes\eta + \mathbf{1}_B\otimes\big(\Phi_B(x)\eta\big) - \mathbf{1}_B\otimes\big(\eta\Phi_B(x)\big) \\
&\qquad +(\xi.y)\otimes\mathbf{1}_H + y\otimes\xi - y\otimes\xi \\
&\qquad \epsilon_H(\xi)\mathbf{1}_B\otimes\eta + \mathbf{1}_B\otimes(\xi\eta) - \mathbf{1}_B\otimes(\eta\xi) \\
&\quad= [x,y]\otimes\mathbf{1}_H + \mathbf{1}_B\otimes\big[\Phi_B(x),\eta\big] + (\xi.y)\otimes\mathbf{1}_H + \mathbf{1}_B\otimes[\xi,\eta],
\end{aligned}$$

because primitives are killed by counits, and the formula is proved.

□

A lengthy, but straight-forward reasoning shows that the above construction assigning $(B,\Phi_B,H,\ell)\to(B\otimes H,\Phi,H)$ defines a covariant functor from the category of all cocommutative rack bialgebras to the category of all cocommutative Hopf dialgebras.
A particular case of the preceding theorem is obtained by picking any C^4-coalgebra $(B,\Delta_B,\epsilon_B,\mathbf{1}_B)$ such that there is any H-module coalgebra structure ℓ on B (such that $h.\mathbf{1}_B = \epsilon_H(h)\mathbf{1}_B$) and by choosing the trivial map

$\Phi_B(b) = \epsilon_B(b)\mathbf{1}_H$: It follows that (B, Φ_B, H, ℓ) is an augmented rack bialgebra with left-trivial multiplication. It turns out that the Hopf dialgebra $B \otimes H$ formed out of this is already isomorphic to the general cocommutative Hopf dialgebra:

Theorem 2.7 *Let $(A, \Delta, \epsilon, \mathbf{1}, \vdash, \dashv, S)$ be a cocommutative Hopf dialgebra. Let E_A be the C^3-subcoalgebra of all generalized idempotent elements with respect to $\dashv$, and let $H_A = \mathbf{1} \dashv A$ be the Hopf subalgebra according to the Suschkewitsch decomposition of the left Hopf algebra $(A, \Delta, \epsilon, \mathbf{1}, \dashv, S)$, see Theorem 2.5. Then we have:*

1. *By means of the Suschkewitsch isomorphism $A \to E_A \otimes H_A$ for the left Hopf algebra $(A, \Delta, \epsilon, \mathbf{1}, \dashv, S)$, we can transfer the cocommutative dialgebra structure of A to $E_A \otimes H_A$: There is a well-defined left module-coalgebra action ℓ of H_A on E_A defined by (for all $c \in E_H$, $h \in H_A$, $a \in A$ such that $h = \mathbf{1} \dashv a$)*

$$\ell_h(c) = h.c = \sum_{(a)} (a^{(1)} \vdash c) \dashv S(a^{(2)}), \tag{2.58}$$

and the transferred multiplications $\vdash'$ and $\dashv'$ and the antipode S' on $E_A \otimes H_A$ read (for all $c, c' \in E_H$, $h, h' \in H_A$)

$$\begin{aligned} (c \otimes h) \vdash' (c' \otimes h') &= \epsilon(c) \sum_{(h)} (h^{(1)}.c') \otimes (h^{(2)} h'), && (2.59)\\ (c \otimes h) \dashv' (c' \otimes h') &= \epsilon(c')c \otimes (hh'), && (2.60)\\ S'(c \otimes h) &= \epsilon(c)\big(\mathbf{1}_{E_A} \otimes \big(S_{H_A}(h)\big)\big). && (2.61) \end{aligned}$$

2. *The K-linear map $A \to \mathbf{1} \dashv A : a \mapsto \mathbf{1} \dashv a$ descends to an isomorphism of associative algebras $A^1_{\mathrm{ass}} \cong H_A$.*

3. ***S. Covez, 2006**: The Leibniz subalgebra $\mathsf{Prim}(A)$ of A (equipped with the Leibniz bracket (2.48) is a **split semidirect sum** out of the two-sided ideal $\mathsf{Prim}(E_A) \subset \mathfrak{z}\big(\mathsf{Prim}(A)\big)$ and the Lie subalgebra $\mathsf{Prim}(H_A)$, i.e. for all $z, z' \in \mathsf{Prim}(E_A)$ and $\xi, \xi' \in \mathsf{Prim}(H_A)$, we have*

$$[z + \xi, z' + \xi'] = \xi.z + [\xi, \xi']. \tag{2.62}$$

4. *Let (B, Φ_B, H, ℓ) be a cocommutative augmented rack bialgebra. Then for the Hopf dialgebra $B \otimes H$ of Theorem 2.6, we get that the Hopf subalgebra $H_{B \otimes H}$ equals $\mathbf{1}_B \otimes H \cong H$, and*

$$E_{B \otimes H} = \left\{ \sum_{(b)} b^{(1)} \otimes \Big(S_H\big(\Phi_B(b^{(2)})\big)\Big) \in B \otimes H \;\middle|\; b \in B \right\}$$

which is isomorphic to B as a C^4-coalgebra of $B \otimes H$.

Proof: *1.* Note first that the right hand side of eqn (2.58) is just $a \triangleright c$ of Proposition 2.6 which had been shown to be a left module-Hopf dialgebra action of $(A, \dashv)$ and of $(A, \vdash)$ on A. Observe that for all $a, a' \in A$

$$(\mathbf{1} \dashv a) \triangleright a' \overset{(2.51)}{=} \mathbf{1} \triangleright (a \triangleright a') = a \triangleright a'$$

whence the H_A-action ℓ is well-defined on A. Moreover we compute for all $h \in H_A$ and all $a, a', a'' \in A$ such that $h = \mathbf{1} \dashv a$:

$$\begin{aligned} \sum_{(h)} (h^{(1)}.a') \dashv (h^{(2)}.a'') &= \sum_{(a)} (a^{(1)} \triangleright a') \dashv (a^{(2)} \triangleright a'') \\ &\overset{(2.54)}{=} a \triangleright (a' \dashv a'') = h.(a' \dashv a'') \end{aligned}$$

whence $(A, \dashv)$ is also a H_A-module-algebra. Now let $c \in E_H$. By definition, c is a generalized idempotent (w.r.t. $\dashv$), hence $c = \sum_{(c)} c^{(1)} \dashv c^{(2)}$, and thus for all $h \in H$

$$h.c = h. \sum_{(c)} c^{(1)} \dashv c^{(2)} = \sum_{(h),(c)} (h^{(1)}.c^{(1)}) \dashv (h^{(2)}.c^{(2)}) = \sum_{(h.c)} (h.c)^{(1)} \dashv (h.c)^{(2)}$$

whence $h.c$ is also in E_A, and E_A is a H_A-submodule of A.
Recall the Suschkewitsch decomposition of the left Hopf algebra $(A, \Delta, \epsilon, \mathbf{1}, \vdash, S)$ where one can use Theorem 2.5 and dualize all the formulas:

$$\begin{aligned} \Psi : A \to E_A \otimes H_A \to A \quad &: \quad a \mapsto \sum_{(a)} \big(a^{(1)} \dashv S(a^{(2)})\big) \otimes (\mathbf{1} \dashv a^{(3)}). \\ \Psi^{-1} : E_A \otimes H_A \to A \quad &: \quad \big(c \otimes (\mathbf{1} \dashv a)\big) \mapsto x \dashv a. \end{aligned}$$

Formulas (2.60) and (2.61) consequences of Theorem 2.5. The only formula which remains to be shown is eqn (2.59). Note first that every generalized idempotent $c \in E_{\mathcal{H}}$ (w.r.t. $\dashv$) is also a generalized idempotent with respect to $\vdash$. Indeed, since all the components $c^{(1)}$ and $c^{(2)}$ in $\Delta(c) = \sum_{(c)} c^{(1)} \otimes c^{(2)}$ can be chosen in $E_{\mathcal{H}}$, we get

$$\begin{aligned} \sum_{(c)} c^{(1)} \vdash c^{(2)} &= \sum_{(c)} \big(c^{(1)} \dashv S(c^{(2)})\big) \vdash c^{(3)} = \sum_{(c)} \big(c^{(1)} \vdash S(c^{(2)})\big) \vdash c^{(3)} \\ &= \sum_{(c)} \epsilon(c^{(1)}) \mathbf{1} \vdash c^{(2)} = c. \end{aligned}$$

Next for all $c, c' \in E_{\mathcal{H}}$, $h, h' \in H_{\mathcal{H}}$, and $a, a' \in a$ such that $\mathbf{1} \dashv a = h$ and $\mathbf{1} \dashv a' = h'$, we get –since c is a generalized left unit element (w.r.t. $\vdash$)–

$$\begin{aligned} (c \otimes h) \vdash' (c' \otimes h') &= \Psi\big(\Psi^{-1}(c \otimes h) \vdash \Psi^{-1}(c' \otimes h')\big) = \Psi\big((c \dashv a) \vdash (c' \dashv a')\big) \\ &= \Psi\big((c \vdash a) \vdash (c' \dashv a')\big) = \epsilon(c) \Psi\big(a \vdash (c' \dashv a')\big) \end{aligned}$$

and this is equal to

$$\begin{aligned}
\epsilon(c) \sum_{(a),(c'),(a')} & \Big(\big(a^{(1)} \vdash ((c')^{(1)} \dashv (a')^{(1)})\big) \dashv \big(S\big(a^{(2)} \vdash ((c')^{(2)} \dashv (a')^{(2)})\big)\big) \Big) \\
& \otimes \big(\mathbf{1} \dashv \big(a^{(3)} \vdash ((c')^{(3)} \dashv (a')^{(3)})\big)\big) \\
= \; \epsilon(c) \sum_{(a),(c'),(a')} & \Big(a^{(1)} \vdash \big((c')^{(1)} \dashv ((a')^{(1)} \dashv S((a')^{(2)})) \dashv S((c')^{(2)}) \dashv S(a^{(2)})\big) \Big) \\
& \otimes \big(\mathbf{1} \dashv \big((a^{(3)} \dashv (c')^{(3)}) \dashv (a')^{(3)}\big)\big) \\
= \; \epsilon(c) \sum_{(a),(c')} & \Big(a^{(1)} \vdash \big(((c')^{(1)} \dashv S((c')^{(2)})) \dashv S(a^{(2)})\big) \Big) \\
& \otimes \big(\mathbf{1} \dashv \big(a^{(3)} \dashv a'\big)\big) \\
= \; \epsilon(c) \sum_{(a)} & \Big(a^{(1)} \vdash \big(c' \dashv S(a^{(2)})\big) \Big) \otimes \big(\mathbf{1} \dashv \big(a^{(3)} \dashv a'\big)\big) \\
= \; \epsilon(c) \sum_{(h)} & \big(h^{(1)}.c'\big) \otimes \big(h^{(2)} h'\big)
\end{aligned}$$

proving eqn (2.59).
2. Clear for any bar-unital balanced dialgebra.
3. Straight-forward computation using $\mathsf{Prim}(\mathcal{H}) = \mathsf{Prim}(E_{\mathcal{H}}) \oplus \mathsf{Prim}(H_{\mathcal{H}})$ where the latter is well-known to be a Lie algebra and the former is abelian.
4. For each $b \in b$ and $h \in H$, we get

$$(\mathbf{1}_B \otimes \mathbf{1}_H) \dashv (b \otimes h) = (\mathbf{1}_B \otimes \big(\Phi_B(b)h\big),$$

proving the first statement. Moreover

$$\begin{aligned}
\sum_{(b),(h)} (b^{(1)} \otimes h^{(1)}) \dashv \big(S(b^{(2)} \otimes h^{(2)})\big)\big) \; &= \; \sum_{(b),(h)} b^{(1)} \otimes h^{(1)} S_H(h^{(2)}) S_H\big(\Phi_B(b^{(2)})\big) \\
&= \; \sum_{(b)} b^{(1)} \otimes S_H\big(\Phi_B(b^{(2)})\big),
\end{aligned}$$

proving the form of the generalized idempotents, and since the K-linear map $B \to B \otimes H$ given by $\big(\mathrm{id}_B \otimes (S_H \circ \Phi_B)\big) \circ \Delta$ is an injective morphism of C^3-coalgebra, the statement is proved. □

The third statement had been proved by Simon Covez in his Master thesis [6] in the differential geometric context of digroups, compare with Section 3.

Example 2.6
As an example, let us compute the Suschkewitsch decomposition for the augmented rack bialgebra $K[X]$ where $p : X \to G$ is an augmented pointed rack,

see Example 2.4. By the above theorem, part *4.*, its associated cocommutative augmented Hopf dialgebra decomposes as $B \otimes H$, where the Hopf algebra $H = K[G]$ is the standard group algebra and $B = K[X]$. The generalized idempotents are in this case

$$E_{B\otimes H} \;=\; \{b \otimes p(b)^{-1} \,|\, b \in B\}.$$

$\diamond$

We finish this section with a formula relating *universal algebras*: The functor associating to any dialgebra A its Leibniz algebra A^- via eqn (2.48) is well-known to have a left adjoint (see e.g. [20]) associating to any Leibniz algebra $(\mathfrak{h}, [\ ,\ ])$ its (in general non bar-unital) *universal enveloping dialgebra* $\mathsf{Ud}(\mathfrak{h})$ *associated to* $\mathfrak{h}$ defined by

$$\mathsf{Ud}(\mathfrak{h}) = \mathfrak{h} \otimes \mathsf{U}(\overline{\mathfrak{h}}). \tag{2.63}$$

But also in the category of bar-unital balanced dialgebras, there is such a left adjoint: To any Leibniz algebra $(\mathfrak{h}, [\ ,\ ])$, we associate its *universal balanced bar-unital enveloping dialgebra* $\widetilde{\mathsf{Ud}}(\mathfrak{h})$

$$\widetilde{\mathsf{Ud}}(\mathfrak{h}) = \mathsf{UAR}(\mathfrak{h}) \otimes \mathsf{U}(\overline{\mathfrak{h}}). \tag{2.64}$$

Before proving the theorem, we note that $\widetilde{\mathsf{Ud}}(\mathfrak{h}) = \mathsf{Ud}(\mathfrak{h}) \oplus \mathsf{U}(\overline{\mathfrak{h}})$ is obtained by adjoining a balanced bar-unit to $\mathsf{Ud}(\mathfrak{h})$.

Theorem 2.8 *For any Leibniz algebra $(\mathfrak{h}, [\ ,\ ])$, the assignment $\mathfrak{h} \to \widetilde{\mathsf{Ud}}(\mathfrak{h})$ defines a left adjoint functor to the functor associating to any bar-unital balanced dialgebra its commutator Leibniz algebra.*

Proof: Clearly, $\widetilde{\mathsf{Ud}}(\mathfrak{h})$ is the cocommutative Hopf dialgebra associated to the universal augmented rack bialgebra $(\mathsf{UAR}(\mathfrak{h}), \Phi_{\mathfrak{h}}, \mathsf{U}(\overline{\mathfrak{h}}), \ell)$ (cf. Theorem 2.6) which in turn is associated to the Leibniz algebra $(\mathfrak{h}, [\ ,\ ])$ (cf. Theorem 2.4). Since both assignments are functorial, it follows that the assignment $\mathfrak{h} \to \widetilde{\mathsf{Ud}}(\mathfrak{h})$ is a functor. It remains to prove the universal property: Let $(\mathfrak{h}, [\ ,\ ])$ be a Leibniz algebra, let $(A, \mathbf{1}, \vdash, \dashv)$ a bar-unital balanced dialgebra, and let $\varphi : \mathfrak{h} \to A^-$ be a morphism of Leibniz algebras. It follows that the K-linear map $\Phi_A \circ \varphi : \mathfrak{h} \to A^1_{\text{ass}}$ vanishes on the two-sided ideal $Q(\mathfrak{h})$ and descends to a morphism $\overline{\varphi}$ of the quotient Lie algebra $\overline{\mathfrak{h}}$ to A^1_{ass} with its commutator Lie bracket such that $\overline{\varphi} \circ p = \Phi_A \circ \varphi$. Hence there is a unique morphism $\mathsf{U}(\overline{\varphi}) : \mathsf{U}(\overline{\mathfrak{h}}) \to A^1_{\text{ass}}$ of unital associative algebras extending $\overline{\varphi}$. Define the K-linear map $\hat{\varphi} : \widetilde{\mathsf{Ud}}(\mathfrak{h}) = \mathsf{U}(\overline{\mathfrak{h}}) \oplus \mathsf{Ud}(\mathfrak{h}) \to A$ by (for all $u, v \in \mathsf{U}(\overline{\mathfrak{h}})$ and $x \in \mathfrak{h}$):

$$\hat{\varphi}(u) = i_A\big(\mathsf{U}(\overline{\varphi})(u)\big) \quad \text{and} \quad \hat{\varphi}(x \otimes v) = \varphi(x) \dashv i_A\big(\mathsf{U}(\overline{\varphi})(v)\big)$$

where we recall the natural injection of unital algebras $i_A : A^1_{\mathrm{ass}} \to A$ given by $i_A\big(\Phi_A(a)\big) = \mathbf{1} \dashv a$ for all $a \in A$. We shall show that $\hat{\varphi}$ is a morphism of augmented dialgebras: We compute for all $u, u', u'' \in \mathsf{U}(\overline{\mathfrak{h}})$, using that i_A and $\mathsf{U}(\overline{\varphi})$ are morphisms of unital associative algebras and that in the image of i_A, we can use the multiplication symbols $\vdash$ and $\dashv$ arbitrarily:

$$\hat{\varphi}(u'uu'') = i_A\big(\mathsf{U}(\overline{\varphi})(u')\big) \vdash i_A\big(\mathsf{U}(\overline{\varphi})(u)\big) \dashv i_A\big(\mathsf{U}(\overline{\varphi})(u'')\big) = \hat{\varphi}(u') \vdash \hat{\varphi}(u) \dashv \hat{\varphi}(u'')$$

showing the fact that $\hat{\varphi}$ preserves the bimodule structures on the first component of $\widetilde{\mathsf{Ud}}(\mathfrak{h})$. Next we have for all $x_1, x \in \mathfrak{h}$

$$\begin{aligned} \varphi\big(p(x_1).x\big) &= \varphi([x_1, x]) = \varphi(x_1) \vdash \varphi(x) - \varphi(x) \dashv \varphi(x_1) \\ &= i_A\big(\overline{\varphi}\big(p(x_1)\big)\big) \vdash \varphi(x) - \varphi(x) \dashv i_A\big(\overline{\varphi}\big(p(x_1)\big)\big) \end{aligned}$$

and by induction on k in $u' = p(x_1) \cdots p(x_k) \in \mathsf{U}(\overline{\mathfrak{h}})$ and $x_1, \ldots, x_k \in \mathfrak{h}$, we prove

$$\varphi(u'.x) = \sum_{(u')} i_A\big(\overline{\varphi}(u'^{(1)})\big) \vdash \varphi(x) \dashv \big(\overline{\varphi}\big(S(u'^{(2)})\big)\big).$$

Now, for all $u', u'', v \in \mathsf{U}(\overline{\mathfrak{h}})$ and $x \in \mathfrak{h}$, we get:

$$\begin{aligned} \hat{\varphi}(u'.(x \otimes v).u'') &= \sum_{(u')} \hat{\varphi}\big(((u')^{(1)}.x) \otimes ((u')^{(2)} v u'')\big) \\ &= \sum_{(u')} \varphi((u')^{(1)}.x) \dashv i_A\big(\mathsf{U}(\overline{\varphi})((u')^{(2)} v u'')\big) \\ &= \sum_{(u')} i_A\big(\mathsf{U}(\overline{\varphi})((u')^{(1)})\big) \vdash \varphi(x) \dashv i_A\big(\mathsf{U}(\overline{\varphi})\big(S((u')^{(2)})(u')^{(3)} v u''\big)\big) \\ &= i_A\big(\mathsf{U}(\overline{\varphi})(u')\big) \vdash \varphi(x) \dashv i_A\big(\mathsf{U}(\overline{\varphi})(v)\big) \dashv i_A\big(\mathsf{U}(\overline{\varphi})(u'')\big) \\ &= i_A\big(\mathsf{U}(\overline{\varphi})(u')\big) \vdash \hat{\varphi}(x \otimes v) \dashv i_A\big(\mathsf{U}(\overline{\varphi})(u'')\big) \end{aligned}$$

showing the fact that $\hat{\varphi}$ preserves the bimodule structures on the second component of $\widetilde{\mathsf{Ud}}(\mathfrak{h})$. Hence $\hat{\varphi}$ is a morphism of bar-unital (augmented) dialgebras. The uniqueness of $\hat{\varphi}$ follows from the universal property of $\mathsf{U}(\overline{\mathfrak{h}})$ □

3 Coalgebra Structures for pointed manifolds with multiplication

In this section, the symbol $\mathbb{K}$ denotes either the field of all real numbers, $\mathbb{R}$, or the field of all complex numbers, $\mathbb{C}$. We define here the monoidal category of pointed manifolds, and exhibit the *Serre functor* sending a pointed manifold to the coalgebra of point-distributions supported in the distiguished point.

We recall further that this is a strong monoidal functor. Further down, we will study Lie (semi) groups, Lie racks, and Lie digroups as examples of this construction, motivating geometrically the notions of a rack bialgebra and of a Hopf dialgebra.

3.1 Pointed manifolds with multiplication(s)

Recall first the *category of all pointed manifolds* $\mathcal{M}f*$ whose objects consist of pairs (M, e) where M is a non-empty differentiable manifold and e is an element of M and whose morphisms $(M, e) \to (M', e')$ are given by all smooth maps $\phi : M \to M'$ of the underlying manifolds such that $\phi(e) = e'$. Recall that the cartesian product $\times$ makes $\mathcal{M}f*$ into a *monoidal category* (see e.g. [21, p.161-170; 251-257] for definitions) by setting $(M, e_1) \times (N, e_2) := \big(M \times N, (e_1, e_2)\big)$ with the one-point set $\big(\{\mathrm{pt}\}, \mathrm{pt}\big)$ as unit object and the usual associators, left-unit and right-unit identifications borrowed from the category of sets. This monoidal category is *symmetric* by means of the usual (tensor) flip map $\tau_T = \tau_{T\ (M,N)} : M \times N \to N \times M : (x, y) \mapsto (y, x)$ where the pair of distinguished points is also interchanged.
By simply forgetting about the differentiable structure we get the category of *pointed sets*.

Recall that a *pointed manifold with multiplication* is a triple $(M, e, \mathbf{m})$ where (M, e) is a pointed manifold, and $\mathbf{m} : (M, e) \times (M, e) \to (M, e)$ is a smooth map of pointed manifolds, i.e. is a smooth map $M \times M \to M$ such that $\mathbf{m}(e, e) = e$. Moreover, a pointed manifold with multiplication will be called *left-regular* (resp. *right-regular*) if all the left (resp. right) multiplication maps $y \mapsto \mathbf{m}(x, y)$ (resp. $y \mapsto \mathbf{m}(y, x)$ are diffeomorphisms. Morphisms of pointed manifolds with multiplication $(M, e, \mathbf{m}) \to (M', e', \mathbf{m}')$ are smooth maps of pointed manifolds $\phi : (M, e) \to (M', e')$ such that

$$\phi \circ \mathbf{m} = \mathbf{m}' \circ (\phi \times \phi). \tag{3.1}$$

The obvious generalization are a finite number of maps $M^{\times n} \to M$ with $n \geq 1$) arguments.
Again by forgetting about differentiable structures, we get the category of *pointed sets with multiplication*.

3.2 Coalgebra Structure for distributions supported in one point

For any pointed manifold (M, e) recall the $\mathbb{K}$-vector space

$$\mathcal{E}'_e(M) := \\ \big\{T : \mathcal{C}^\infty(M, \mathbb{K}) \to \mathbb{K} \mid T \text{ is a continuous linear map and } \mathrm{supp}(T) = \{e\}\big\} \tag{3.2}$$

of all distributions supported in the singleton $\{e\}$, see e.g. [24, Ch.6,7] for definitions). Now let $\phi : M \to M'$ be a smooth map such that $\phi(e) = e'$. For any distribution $S \in \mathcal{E}'_e(M)$ and any smooth function $f' \in \mathcal{C}^\infty(M', \mathbb{K})$ the well-known prescription

$$(\phi_* S)(f') := S(f' \circ \phi) \tag{3.3}$$

gives a well-defined distribution $\phi_* S$ on the target manifold M' supported in $e' = \phi(e)$, and the map $\phi_* : \mathcal{E}'_e(M) \to \mathcal{E}'_{e'}(M')$ is a $\mathbb{K}$-linear map (which is continuous). Clearly for three pointed manifolds (M, e), (M', e'), and (M'', e'') with smooth maps $\phi : (M, e) \to (M', e')$ and $\psi : (M', e') \to (M'', e'')$ we get

$$(\psi \circ \phi)_* = \psi_* \circ \phi_* \quad \text{and} \quad \big(\mathrm{id}_M\big)_* = \mathrm{id}_{\mathcal{E}'_e(M)}. \tag{3.4}$$

This defines a covariant functor $F : \mathcal{M}f* \to \mathbb{K}\mathbf{Vect}$ to the category of all $\mathbb{K}$-vector spaces by associating to any pointed manifold (M, e) the $\mathbb{K}$-vector space $F(M, e) := \mathcal{E}'_e(M)$, and to any smooth map $(M, e) \to (M, e')$ the linear map $F(\phi) := \phi_* : \mathcal{E}'_e(M) \to \mathcal{E}'_{e'}(M')$. We call this functor *Serre functor* in tribute to the predominant role it plays in [26]. It is one of the main objects of this article.

There is, however, much more structure in this functor: First any distribution space $\mathcal{E}'_e(M)$ contains a canonical linear form $\epsilon = \epsilon_e : \mathcal{E}'_e(M) \to \mathbb{K}$ defined by

$$\epsilon_e(T) := T(1), \tag{3.5}$$

where 1 denotes the constant function $M \to \mathbb{K}$ whose only value is equal to $1 \in \mathbb{K}$. Moreover each space $\mathcal{E}'_e(M)$ contains a canonical element $\mathbf{1} = \mathbf{1}_e$ defined by the well-known *delta distribution*

$$\mathbf{1}_e = \delta_e : f \mapsto f(e), \tag{3.6}$$

and we clearly have

$$\epsilon_e(\mathbf{1}_e) = 1. \tag{3.7}$$

Moreover, both ϵ_e and $\mathbf{1}_e$ are natural in the following sense: let $\phi : (M, e) \to (M', e')$ be a smooth map of pointed differentiable manifolds. Then it is straight-forward to check that

$$\epsilon_{e'} \circ \phi_* = \epsilon_e \quad \text{and} \quad \phi_*(\mathbf{1}_e) = \mathbf{1}_{e'}. \tag{3.8}$$

Recall the well-known *tensor product* or *direct product* of two distributions (cf. [30, p. 403]): More generally, let M and N be two differentiable manifolds, and let $S \in \mathcal{D}'(M)$ and $T \in \mathcal{D}'(N)$ be two distributions (where the symbol $\mathcal{D}'(M)$ denotes the continuous dual space of the test function space $\mathcal{D}(M)$ of all smooth $\mathbb{K}$-valued functions with compact support). Let $f : M \times N \to \mathbb{K}$ be a smooth function with compact support $K \subset M \times N$, and let $K_1 := \mathrm{pr}_M(K) \subset M$, $K_2 = \mathrm{pr}_N(K) \subset N$, whence K is a subset of the compact set $K_1 \times K_2$. Let $T^{(2)} : \mathcal{D}(M \times N) \to \mathrm{Fun}(M, \mathbb{K})$ be the following map: For each $x \in M$, let $f_x \in \mathcal{D}(N)$ be the partial function $y \mapsto f(x, y)$. Then we set

$$\big(T^{(2)}(f)\big)(x) = T(f_x).$$

The superscript (2) means here that we see f as a function of its second variable only, when applying the distribution T.

Upon using the approximation theorem of any distribution by a sequence of regular distributions (see e.g. [24, p.157, Thm.6.32]), one can show that $T^{(2)}(f) : M \to \mathbb{K}$ is a smooth function having compact support in K_1. It is clear that $T^{(2)} : \mathcal{D}(M \times N) \to \mathcal{D}(M)$ is linear, and it can be shown by the same approximation theorem that $T^{(2)}$ is continuous, see [30, p.416]. It follows that the map

$$(S, T) \mapsto \Big(f \mapsto S\big(T^{(2)}(f)\big)\Big)$$

is a well-defined $\mathbb{K}$-bilinear map $\mathcal{D}'(M) \times \mathcal{D}'(N) \to \mathcal{D}'(M \times N)$, and there is thus a unique linear map (where $\otimes$ denotes the usual algebraic tensor product over $\mathbb{K}$)

$$F_{2\ M,N} : \mathcal{D}'(M) \otimes \mathcal{D}'(N) \to \mathcal{D}'(M \times N) \tag{3.9}$$

such that for all $f \in \mathcal{D}(M \times N)$ we have

$$\big(F_{2\ M,N}(S \otimes T)\big)(f) = S\big(T^{(2)}(f)\big).$$

Note also that it can be shown that the right hand side is equal to $T\big(S^{(1)}(f)\big)$ where the notation is self-explanatory. Furthermore, it is not hard to see that for two distributions supported in one point, i.e. $S \in \mathcal{E}'_{e_1}(M)$ and $T \in \mathcal{E}'_{e_2}(N)$ the distribution $F_{2\ M,N}(S \otimes T)$ is supported in (e_1, e_2), i.e. is an element of $\mathcal{E}'_{(e_1,e_2)}(M \times N)$. We shall denote the restriction of the map $F_{2\ M,N}$ to

$\mathcal{E}'_{e_1}(M) \otimes \mathcal{E}'_{e_2}(N)$ by the same symbol $F_{2\ M,N}$. For three pointed manifolds (M, e_1), (N, e_2), and (P, e_3), let $\alpha_{M,N,P} : M \times (N \times P) \to (M \times N) \times P$ be the usual associator for the monoidal category of all sets (see [21]), and for three vector spaces V, W, X over $\mathbb{K}$, let $\beta_{V,W,X} : V \otimes (W \otimes X) \to (V \otimes W) \otimes X$ be the well-known associator for the monoidal category of all vector spaces. By using the definitions, it is not hard to see that the following identity holds

$$\begin{aligned} & F_{2\ (M\times N),P} \circ \big(F_{2\ M,N} \otimes \mathrm{id}_{\mathcal{E}'_{e_3}(P)}\big) \circ \beta_{\mathcal{E}'_{e_1}(M),\mathcal{E}'_{e_2}(N),\mathcal{E}'_{e_3}(P)} \\ = \ & \big(\alpha_{M,N,P}\big)_* \circ F_{2\ M,(N\times P)} \circ \big(\mathrm{id}_{\mathcal{E}'_{e_1}(M)} \otimes F_{2\ N,P}\big) \end{aligned} \tag{3.10}$$

hence eqn (3) of [21, p.255]. In the same vein, the two diagrams in eqn (4) of [21, p.256] are satisfied upon setting $F_0 = \epsilon_{\mathrm{pt}}$ and $\lambda : \mathbb{K} \otimes V \to V$ and $\rho : V \otimes \mathbb{K} \to V$ the usual left-unit and right-unit identifications in the monoidal category of vector spaces.
Let (M', e'_1) and (N', e'_2) two other pointed differentiable manifolds, and let $\phi : (M, e_1) \to (M', e'_1)$ and $\psi : (N, e_2) \to (N', e'_2)$ two smooth maps of pointed differentiable manifolds. It is a straight-forward check that the map $F_{2\ M,N}$ to is natural in the following sense

$$F_{2\ M',N'} \circ \big(\phi_* \otimes \psi_*\big) = (\phi \times \psi)_* \circ F_{2\ M,N}. \tag{3.11}$$

Moreover, note that the map $F_0 = \epsilon_{\mathrm{pt}}$ (see eqn (3.5)) defines an isomorphism of $\mathcal{E}'_{\mathrm{pt}}(\{\mathrm{pt}\})$ to $\mathbb{K}$ which had already been seen to be natural.
As a result, the functor F is a *monoidal functor* in the sense of [21, p.255-257]. Moreover, since the category $\mathcal{M}f*$ is even a *symmetric monoidal category* by means of the canonical flip map $\tau_{M,N} : M \times N \to N \times M : (x, y) \to (y, x)$, see e.g. [21, p.252-253], and the monoidal category $\mathbb{K}$-**vect** is also symmetric, it is not hard to see that the monoidal functor is also symmetric, see e.g. [21, p.257] for definitions.

We shall now show that the monoidal functor F is *strong*, i.e. that $F_0 = \epsilon_{\mathrm{pt}}$ and $F_{2\ M,N}$ are isomorphisms. This is clear for ϵ_{pt}. Recall that for each distribution T in $\mathcal{E}'_e(V)$ (where V is a nonempty open set in $\mathbb{R}^m$ containing the point e), there is nonnegative integer l (called the *order* of the distribution) such that

$$T = \sum_{r=0}^{l} \sum_{\mathbf{k}\in\mathbb{N}^m, |\mathbf{k}|=r} c_{\mathbf{k}} \frac{\partial^{\mathbf{k}}}{\partial x_{\mathbf{k}}}(\delta_e)$$

where $c_{\mathbf{k}} \in \mathbb{K}$ for each multi-index $\mathbf{k}$, see e.g. [24, p.150, Thm. 6.25]. In a slightly more algebraic manner we can express this as follows: let E be a finite-dimensional real vector space, let $V \subset E$ be an open set containing $e \in E$. Then we have the following linear isomorphism $\Phi_{\mathsf{S}} : \mathsf{S}(E) \to \mathcal{E}'_e(V)$ given

by $\Phi_{\mathsf{S}}(\mathbf{1}) = \delta_e$ and for any positive integer k and vectors $w^{(1)}, \ldots, w_{(k)} \in E$ and $f \in \mathcal{C}^\infty(V, \mathbb{K})$

$$\begin{aligned} &\Phi_{\mathsf{S}}(w^{(1)} \bullet \ldots \bullet w_{(k)})(f) \\ &\quad = \quad \frac{\partial^k \Big(f\big(e + s_1 w^{(1)} + \cdots + s_k w_{(k)}\big)\Big)}{\partial s_1 \cdots \partial s_k}\Bigg|_{s_1=0,\ldots,s_k=0} \end{aligned} \tag{3.12}$$

where $\bullet$ denotes the commutative multiplication in the symmetric algebra, see Appendix A. Using the fact that the inclusion map $\iota_{U_\alpha} : U_\alpha \to M$ of any chart domain of M such that $e \in U_\alpha$ defines an isomorphism $\iota_{U_\alpha *} : \mathcal{E}'_e(U_\alpha) \to \mathcal{E}'_e(M)$, and that any chart $\varphi_\alpha : U_\alpha \to V_\alpha \subset \mathbb{R}^m$ defines an isomorphism $\varphi_{\alpha *} : \mathcal{E}'_e(U_\alpha) \to \mathcal{E}'_{\varphi_\alpha(e)}(V_\alpha)$, we can conclude that there is a linear isomorphism

$$\Phi_\alpha =: \iota_{U_\alpha *} \circ \varphi_{\alpha *}^{-1} \circ \iota_{V_\alpha *}^{-1} \circ \Phi_{\mathsf{S}} : \mathsf{S}(\mathbb{R}^m) \to \mathcal{E}'_e(M) \tag{3.13}$$

with the symmetric coalgebra $\mathsf{S}(\mathbb{R}^m)$ on $\mathbb{R}^m$ (see Appendix A) computed as follows

$$w^{(1)} \bullet \cdots \bullet w_{(k)} \quad \mapsto \quad \Big(f \mapsto \sum_{i_1,\ldots,i_k=1}^{m} \frac{\partial^k (f|_{U_\alpha} \circ \varphi_\alpha^{-1})}{\partial x_{i_1} \cdots \partial x_{i_k}}\big(\varphi_\alpha(e)\big) w_{(1)i_1} \cdots w_{(k)i_k} \Big). \tag{3.14}$$

where we write $\mathbb{R}^m \ni w_{(j)} = \sum_{i=1}^m w_{(j)i} e_i$ where all the $w_{(j)i}$ are real numbers and $e_1, \ldots, e_m$ is the canonical base of $\mathbb{R}^m$. Note that for the particular case of M being an open set V of $\mathbb{R}^m$ and the chart φ_α being the identity map the map Φ_α (see eqs (3.13) and (3.14)) coincides with the canonical map Φ_{S}, see eqn (3.12).

For two pointed manifolds (M, e_1) and (N, e_2) and given charts $(U_\alpha, \varphi_\alpha)$ of M such that $e_1 \in U_\alpha$ and $(\tilde{U}_\beta, \tilde{\varphi}_\beta)$ of N such that $e_2 \in \tilde{U}_\beta$, we thus have linear isomorphisms $\Phi_\alpha : \mathsf{S}(\mathbb{R}^m) \to \mathcal{E}'_{e_1}(M)$, $\tilde{\Phi}_\beta : \mathsf{S}(\mathbb{R}^n) \to \mathcal{E}'_{e_2}(N)$, and $\Phi_{\alpha,\beta} : \mathsf{S}(\mathbb{R}^{m+n}) \to \mathcal{E}'_{(e_1,e_2)}(M \times N)$ (upon using the product chart $(U_\alpha \times \tilde{U}_\beta, \varphi_\alpha \times \tilde{\varphi}_\beta)$). Using the above definitions, one can compute that

$$F_{2\ M,N} \circ \Big(\Phi_\alpha \otimes \tilde{\Phi}_\beta \Big) = \Phi_{\alpha,\beta} \circ \Theta_{m,n}$$

where $\Theta_{m,n} : \mathsf{S}(\mathbb{R}^m) \otimes \mathsf{S}(\mathbb{R}^n) \to \mathsf{S}(\mathbb{R}^{m+n})$ denotes the natural isomorphism of commutative associative unital algebras induced by the obvious inclusions $\mathbb{R}^m \hookrightarrow \mathbb{R}^{m+n}$ (first m coordinates) and $\mathbb{R}^n \hookrightarrow \mathbb{R}^{m+n}$ (last n coordinates). It follows that the natural map $F_{2\ M,N}$ is equal to $\Phi_{\alpha,\beta} \circ \Theta_{m,n} \circ \big(\Phi_\alpha^{-1} \otimes \tilde{\Phi}_\beta^{-1}\big)$ and is thus a linear isomorphism, whence the functor F is a strong monoidal functor.

In order to define more structure, let us consider the well-known *diagonal map* $\mathrm{diag}_M : M \to M \times M$ defined by

$$\mathrm{diag}_M(x) = (x, x)$$

for all $x \in M$. Clearly, diag_M is a smooth map of pointed manifolds $(M, e) \to \big(M \times M, (e, e)\big)$. Moreover, the diagonal map is clearly natural in the sense that

$$\mathrm{diag}_{M'} \circ \phi = \big(\phi \times \phi\big) \circ \mathrm{diag}_M$$

for any smooth map $\phi : (M, e) \to (M', e')$ of pointed differentiable manifolds. In other words, the class of diagonal maps diag_M constitutes a natural transformation from the identity functor to the *diagonal functor* $(M, e) \to \big(M \times M, (e, e)\big)$ and $\phi \mapsto \phi \times \phi$. Define the following linear map $\Delta = \Delta_e = \Delta_{(M,e)} : \mathcal{E}'_e(M) \to \mathcal{E}'_e(M) \otimes \mathcal{E}'_e(M)$ by

$$\Delta_e := F_2^{-1}{}_{M,M} \circ \mathrm{diag}_{M\ *}. \tag{3.15}$$

This definition has avatars with more than two tensor factors. Indeed, observe that the naturality relation (3.11) implies for $\phi = \mathrm{id}_M$ and $\psi = \mathrm{diag}_M$ that

$$\begin{aligned}&(\mathrm{id}_M \times \mathrm{diag}_M)_* \\ &\quad = \ F_{2\ M,(M\times M)} \circ (\mathrm{id}_{\mathcal{E}'_e(M)} \times F_{2\ M,M}) \circ (\mathrm{id}_{\mathcal{E}'_e(M)} \times \Delta_e) \circ F_2^{-1}{}_{M,M}.\end{aligned}$$

Similarly, we have relations of this type for any number of tensor factors.

In the following, we invite the reader to look again at Appendix A for definitions and notations about coalgebras.
We have the following

Theorem 3.1 *With the above notations:*

1. *The $\mathbb{K}$-vector space $\mathcal{E}'_e(M)$ equipped with the linear maps Δ_e (cf. eqn (3.15)), ϵ_e (cf. eqn (3.5), and $\mathbf{1}_e$ (cf. eqn (3.6) is a C^5-coalgebra which is (non canonically) isomorphic to the standard symmetric coalgebra*

$$\big(\mathsf{S}(\mathbb{R}^m), \epsilon, \Delta, \mathbf{1}\big).$$

2. *The above strong symmetric monoidal functor F extends to a functor –also denoted by F– from $\mathcal{M}f*$ to the symmetric monoidal category of C^5-coalgebras over $\mathbb{K}$.*

3. *The subspace of all primitive elements of the coaugmented coalgebra $\mathcal{E}'_e(M)$ is natural isomorphic to the tangent space T_eM. Moreover for each smooth map $\phi : (M, e) \to (M', e')$ of pointed manifolds the coalgebra morphism $\phi_* : \mathcal{E}'_e(M) \to \mathcal{E}'_{e'}(M')$ induces the tangent map $T_e\phi : T_eM \to T_{e'}M'$.*

Proof:

(a) **coassociativity of Δ_e:** This follows from the coassociativity-diagram of $\mathrm{diag}_{M\ *}$ by first taking the induced diagram between distribution spaces which reads then

$$(\alpha_{M,M,M})_* \circ (\mathrm{id}_M \times \mathrm{diag}_M)_* \circ \mathrm{diag}_{M\ *} = (\mathrm{diag}_M \times \mathrm{id}_M)_* \circ \mathrm{diag}_{M\ *}.$$

Starting for example on the left hand side, one replaces the map $\mathrm{diag}_{M\ *}$ by $F_{2\ M,M} \circ \Delta_e$, and also the map $(\mathrm{id}_M \times \mathrm{diag}_M)_*$ by

$$F_{2\ M,(M\times M)} \circ (\mathrm{id}_{\mathcal{E}'_e(M)} \times F_{2\ M,M}) \circ (\mathrm{id}_{\mathcal{E}'_e(M)} \times \Delta_e) \circ F^{-1}_{2\ M,M}.$$

Now one observes that one may apply the relation (3.10) on the left hand side. One obtains

$$\begin{aligned} & F_{2\ (M\times M),M} \circ (F_{2\ M,M} \times \mathrm{id}_{\mathcal{E}'_e(M)}) \circ \beta \circ (\mathrm{id}_{\mathcal{E}'_e(M)} \times \Delta_e) \circ \Delta_e \\ = \ & F_{2\ (M\times M),M} \circ (F_{2\ M,M} \times \mathrm{id}_{\mathcal{E}'_e(M)}) \circ (\Delta_e \times \mathrm{id}_{\mathcal{E}'_e(M)}) \circ \Delta_e. \end{aligned}$$

One deduces coassociativity.

(b) **cocommutativity of Δ_e:** This follows from the symmetry of F (already noted before) and the cocommutativity of $\mathrm{diag}_{M\ *}$.

(c) **counitality of Δ_e:** This follows from the counitality of $\mathrm{diag}_{M\ *}$, i.e.

$$(\mathrm{proj}_e \times \mathrm{id}) \circ \mathrm{diag}_M = \mathrm{incl}^1_e, \quad \text{and} \quad (\mathrm{id} \times \mathrm{proj}_e) \circ \mathrm{diag}_M = \mathrm{incl}^2_e,$$

where $\mathrm{proj}_e : M \to \{e\}$, $\mathrm{incl}^1_e : M \to \{e\} \times M$ and $\mathrm{incl}^2_e : M \to M \times \{e\}$ are the canonical maps. Indeed, these equations induce the corresponding equations between distribution spaces, and translating direct products into tensor products (and thus $(\mathrm{proj}_e)_*$ into ϵ and $(\mathrm{diag}_M)_*$ into Δ_e), one obtains counitality.

(d) **connectedness of Δ_e:** The coalgebra $\mathcal{E}'_e(M)$ is isomorphic to the symmetric algebra $\mathsf{S}(\mathbb{R}^n)$, and the latter is connected.

This shows part (1) of the statement, as the isomorphy to the standard symmetric coalgebra has been shown above.

The only thing which has to be shown for the second statement is the preservation of the coalgebra structure on the level of morphisms, which is clear.

For the third part, consider the linear map $T_eM \to \mathsf{Prim}(\mathcal{E}'_e(M))$ (see Appendix A for the definition of the primitives $\mathsf{Prim}(C)$ of a coalgebra C) defined by

$$v \mapsto \big(f \mapsto df_e(v)\big) :$$

Indeed the right hand side is clearly in $\mathcal{E}'_e(M)$, and the Leibniz rule for the derivative shows that this is in $\mathsf{Prim}(\mathcal{E}'_e(M))$. Moreover the above map is clearly injective, and since $\mathsf{Prim}(\mathsf{S}(\mathbb{R}^n)) = \mathbb{R}^n$ and $\dim(T_eM) = n$ it follows that the above map is an isomorphism of real vector spaces. The naturality is a simple computation. □

The last statement means that the composed functor $\mathsf{Prim} \circ F$ of the Serre functor F and the functor associating to any coalgebra C its space of primitive elements, $\mathsf{Prim}(C)$, is naturally isomorphic to the tangent functor T_* associating to any pointed differentiable manifold (M, e) its tangent space T_eM.

Remark 3.1
There is neither a canonically defined (i.e. not depending on the choice of a chart) projection from the coalgebra to its primitives, so the coalgebras $\mathcal{E}'_e(M)$ are isomorphic to the cofree $\mathsf{S}(\mathbb{R}^m)$, but in general not naturally, nor a canonically defined commutative multiplication (the classical convolution of distributions of compact support which needs the additive vector space structure). ◇

Remark 3.2
Note also the disjoint union $\bigcup_{x\in M} \mathcal{E}'_x(M)_{(k)}$ carries the structure of a smooth vector bundle over M: Its smooth sections coincide with the space of all differential operators of order k. ◇

Remark 3.3
In case $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ are pointed open sets, the coalgebra morphism ϕ_* of a smooth map $\phi : U \to V$ of pointed manifolds is isomorphic to the coalgebra morphism $\mathsf{S}(\mathbb{R}^m) \to \mathsf{S}(\mathbb{R}^n)$ induced by the jet of infinite order of ϕ at the distinguished point e of U, $j^\infty(\phi)_e$, see e.g. [15] for further information. The functorial equation $(\phi \circ \psi)_* = \phi_* \circ \psi_*$ can be computed out of the chain rule for higher derivatives. ◇

3.3 Pointed manifolds with multiplication and their associated bialgebras

We can now apply the Serre functor defined in the preceding Section 3.2 to pointed manifolds with multiplication:

Theorem 3.2 *Let $(M, e, \mathbf{m})$ be a pointed manifold with multiplication. Then the C^5-coalgebra $\mathcal{E}'_e(M)$ carries a multiplication, i.e. a linear map $\mu = \mathbf{m}_* \circ F_{2\ M,M} : \mathcal{E}'_e(M) \otimes \mathcal{E}'_e(M) \to \mathcal{E}'_e(M)$ which is a morphism of C^5-coalgebras. In case $\mathbf{m}$ is left-unital (resp. right unital), the nonassociative C^3I-algebra $\mathcal{E}'_e(M)$ is left regular (resp. right regular)*

Proof: The map μ exists and is linear by functoriality. We have trivially

$$\mathrm{diag}_M \circ \mathbf{m} = (\mathbf{m} \times \mathbf{m}) \circ (\mathrm{id}_M \times \tau_{M,M} \times \mathrm{id}_M) \circ (\mathrm{diag}_M \times \mathrm{diag}_M),$$

and this shows that μ is a morphism of coalgebras by translating diag_M into Δ_e using as before the maps of type F_2. The regularity statements are a consequence of the connectedness of the C^3-coalgebra $\mathcal{E}'_e(M)$, see Lemma 2.1. □

In the following, we shall enumerate some important (sub)categories of pointed differentiable manifolds with multiplications.

3.3.1 Lie groups and universal enveloping algebras

Let $\big(G, \mathbf{m}, e, (\ )^{-1}\big)$ a Lie group. The following theorem is well-known (see [26]):

Theorem 3.3 *The associated coalgebra with multiplication μ of the Lie group $\big(G, \mathbf{m}, e, (\ )^{-1}\big)$ is an associative unital bialgebra (in fact, a Hopf algebra) isomorphic to the* ***universal enveloping algebra of the Lie algebra*** *$\mathfrak{g} = T_eG$ of G.*

We just indicate the isomorphism: For any $\xi \in \mathfrak{g}$, let ξ^+ denote the *left invariant vector field* $\xi^+(g) := T_eL_g(\xi)$ *generated by its value* $\xi \in \mathfrak{g} = T_eG$. Then the map $\Phi_{\mathsf{U}} : \mathsf{U}(\mathfrak{g}) \to F(G)$ is given by (for all $k \in \mathbb{N}$, $\xi_1, \ldots, \xi_k \in \mathfrak{g}$ and $f' \in \mathcal{C}^\infty(G, \mathbb{K})$)

$$\Phi_{\mathsf{U}}(\xi_1 \cdots \xi_k)\big(f'\big) = \Big(\big(\mathbb{L}_{\xi_1^+} \circ \cdots \circ \mathbb{L}_{\xi_k^+}\big)(f')\Big)(e) \tag{3.16}$$

where $\mathbb{L}_X$ denotes the Lie derivative in the direction of the vector field X. Note that the identities for the inverse map $g \mapsto g^{-1}$ can be written as

$$\mathbf{m} \circ \big((\ )^{-1} \times \mathrm{id}_G\big) \circ \mathrm{diag}_G = (g \mapsto e) = \mathbf{m} \circ \big(\mathrm{id}_G \times (\ )^{-1}\big) \circ \mathrm{diag}_G, \tag{3.17}$$

and an application of the functor F gives the convolution identities for the antipode, defined by $S = \big((\ )^{-1}\big)_*$.

3.3.2 Lie semigroups and Lie monoids

It is easy to see but presumably less known that the result of the preceding subsection remains true for a *Lie monoid* $\big(G, \mathbf{m}, e\big)$:

Theorem 3.4 *1. The associated coalgebra with multiplication μ of the Lie monoid $\big(G, \mathbf{m}, e\big)$ is an associative unital bialgebra (in fact a Hopf algebra) isomorphic to the* ***universal enveloping algebra of a Lie algebra*** $\mathfrak{g} \cong T_eG$.

2. The associated coalgebra with multiplication μ of the right Lie group $(G, \mathbf{m}, e, (\ )^{-1})$ is a right Hopf algebra.

In order to see the first statement note that it is clear that the associated coalgebra $C := F(G)$ carries an associative unital multiplication $\mu = \mathbf{m}_* \circ F_{2\ G,G}$. The fact that the coalgebra is always connected implies by the Takeuchi-Sweedler argument (see Appendix A) that the identity map id_C has a convolution inverse, and is thus a Hopf algebra. Since the coalgebra C is connected and cocommutative, it follows from the *Cartier-Milnor-Moore Theorem* (see e.g. [22]) that the Hopf algebra $F(G)$ is isomorphic to the universal enveloping algebra over the Lie subalgebra $\mathfrak{g}$ of its primitive elements which is equal to T_eG.
The second statement is an immediate consequence of the functorial properties of F.

3.3.3 (Lie) dimonoids and digroups

Recall that a *Lie dimonoid* (see e.g. [20]) is a pointed differentiable manifold (D, e) equipped with two smooth *associative multiplications* $D \times D \to D$, written $(x, y) \mapsto x \vdash y$ and $(x, y) \mapsto x \dashv y$ (and preserving points, i.e. $e \vdash e = e = e \dashv e$), such that the dialgebra conditions eqs (2.42), (2.43), (2.44), (2.45), and (2.46) hold for all $x, y, z \in D$ and e (replacing $\mathbf{1}$): Hence $(D, \vdash, e)$ is a left unital Lie semigroup and $(D, \dashv, e)$ is a right unital Lie semigroup, and as for dialgebras, we shall say *bar-unital dimonoid* to stress the fact that the bar-unit e is among the data for the dimonoid.
Let us call a Lie dimonoid $(D, e, \vdash, \dashv)$ *balanced* iff in addition for all $x \in D$ the analogue of eqn (2.47) holds, i.e. $x \vdash e = e \dashv x$. Any Lie monoid $(G, e, \mathbf{m})$ is a Lie dimonoid by setting $\vdash = \dashv = \mathbf{m}$.
Another class of examples is obtained by the following important *augmented dimonoid* construction (cf Example 2.5): Let G be a Lie group, let (D, e_D) be a pointed differentiable manifold, let G smoothly act on the left and on the right of M (written $(g, x) \mapsto gx$ and $(x, g) \mapsto xg$) such that $(gx)g' = g(xg')$

for all $g, g' \in g$ and $x \in D$, and let $f : (D, e_D) \to (G, e)$ be a smooth map of pointed manifolds such that for all $g, g' \in G$ and $x \in D$

$$f(gxg') = gf(x)g'. \tag{3.18}$$

Then the pointed manifold $\big(D, e_D, \vdash, \dashv\big)$ will be a (bar-unital) dimonoid by setting

$$x \vdash y := f(x)y \quad \text{and} \quad x \dashv y := xf(y). \tag{3.19}$$

A *general Lie digroup* is defined (according to Liu, [18, Definition 1.1]) to be a (bar-unital) dimonoid $(D, e, \vdash, \dashv)$ such that the left unital Lie semigroup $(D, e, \vdash)$ is a right group and the right unital Lie semigroup $(D, e, \dashv)$ is a left group (see Appendix B for definitions): Here the right inverse of x with respect to $\vdash$ does in general not coincide with the left inverse of x with respect to $\dashv$. For an example, take any Lie group G, set $(D, e_D) = \big((G \times G, (e, e)\big)$, define the two canonical G-actions $g(g_1, g_2) := (gg_1, g_2)$ and $(g_1, g_2)g := (g_1, g_2 g)$ (for all $g, g_1, g_2 \in g$), and let $f : G \times G \to G$ be the group multiplication. Then $(D, e_D, \vdash, \dashv)$ will be a general digroup with $(g_1, g_2)_{\vdash}^{-1} = (g_2^{-1} g_1^{-1}, e)$ and $(g_1, g_2)_{\dashv}^{-1} = (e, g_2^{-1} g_1^{-1})$.
In [14, Definition 4.1] Kinyon defines a *Lie digroup* as a general Lie digroup such that in addition for each x its right inverse (w.r.t. to $\vdash$) is equal to its left inverse (w.r.t. $\dashv$). This can be shown to be equivalent to demanding that the general Lie digroup $(D, e, \vdash, \dashv)$ be balanced.
Again using the Suschkewitsch decomposition Theorem (which applies in case the underlying manifold is connected), it is not hard to see that the category of all connected Lie digroups (in the sense of Kinyon) is equivalent to the category of all left $\mathcal{G}$-spaces, i.e. whose objects are pairs (G, X) where G is a connected Lie group and X is a pointed connected left G-space (i.e. the distinguished point of X is a fixed point of the G-action) with obvious morphisms. Recall that the Lie digroup is given by $X \times G$ equipped with the point (e_X, e) and the two multiplications $(x_1, g_1) \vdash (x_2, g_2) = (g_1 x_2, g_1 g_2)$ and $(x_1, g_1) \dashv (x_2, g_2) = (x_1, g_1 g_2)$ for all $x_1, x_1 \in X$ and $g_1, g_2 \in G$.

The following theorem is a direct consequence of the functorial properties of the functor F:

Theorem 3.5 *Let $(D, e, \vdash, \dashv)$ be a (bar-unital) Lie dimonoid $(D, e, \vdash, \dashv)$.*

1. *The underlying vector space of the associated coalgebra $F(D)$ to the (bar-unital) Lie dimonoid $(D, e, \vdash, \dashv)$ equipped with $\mathbf{1}$ and the multiplications $\mu_{\vdash}, \mu_{\dashv}$ is an associative bar-unital dialgebra.*
   In case D is balanced, $F(D)$ is a cocommutative Hopf dialgebra.

2. *In case $(D, e, \vdash, \dashv)$ is a Lie digroup (in the sense of Kinyon), $F(D)$ is a cocommutative Hopf dialgebra.*

3.3.4 (Lie) racks

Recall that a *Lie rack* is a pointed manifold with multiplication $(M, e, \mathbf{m})$ satisfying the following identities for all $x, y, z \in M$ where the standard notation is $\mathbf{m}(x, y) = x \triangleright y$

$$\begin{aligned} e \triangleright x &= x, && (3.20)\\ x \triangleright e &= e, && (3.21)\\ x \triangleright (y \triangleright z) &= (x \triangleright y) \triangleright (x \triangleright z) && (3.22) \end{aligned}$$

In addition, one demands that $(M, e, \mathbf{m})$ be left-regular, i.e. for all $x \in M$ the left multiplication maps $L_x : y \mapsto x \triangleright y$ should be a diffeomorphism.
Note the following version of the self-distributivity identity (3.22) in terms of maps:

$$\begin{aligned} &\mathbf{m} \circ (\mathrm{id}_M \times \mathbf{m})\\ &\quad = \mathbf{m} \circ (\mathbf{m} \times \mathbf{m}) \circ (\mathrm{id}_M \times \tau_{M,M} \times \mathrm{id}_M) \circ (\mathrm{diag}_M \times \mathrm{id}_M \times \mathrm{id}_M) && (3.23) \end{aligned}$$

Example 3.1
Note that every pointed differentiable manifold (M, e) carries a *trivial Lie rack structure* defined for all $x, y \in M$ by

$$x \triangleright_0 y := y, \tag{3.24}$$

and this assignment is functorial. ◇

Example 3.2
Any Lie group G becomes a Lie rack upon setting for all $g, g' \in G$

$$g \triangleright g' := gg'g^{-1}, \tag{3.25}$$

again defining a functor from the category of Lie groups to the category of all Lie racks. Examples of racks which are not the conjugation rack underlying a group abound, for example, every conjugation class and every union of conjugation classes in a group (defining an immersed submanifold) in a Lie group is a Lie rack. ◇

Example 3.3
Let $(D, e, \vdash, \dashv)$ be a (balanced) *digroup*. Then formula (13) of [14],

$$x \triangleright y := x \vdash y \dashv x^{-1}$$

equips the pointed manifold $(D, e, \triangleright)$ with the structure of a Lie rack. ◇

Any Lie rack $(M, e, \triangleright)$ can be *gauged* by any smooth map $f : (M, e) \to (M, e)$ of pointed manifolds satisfying for all $x, y \in M$

$$f(x \triangleright y) = x \triangleright f(y).$$

A straight-forward computation shows that the pointed manifold (M, e) equipped with the *gauged* multiplication $\triangleright_f$ defined by

$$x \triangleright_f y := f(x) \triangleright y$$

is a Lie rack $(M, e, \triangleright_f)$.

Furthermore, recall that an *augmented Lie rack* (see [11]) (M, ϕ, G, ℓ) consists of a pointed differentiable manifold (M, e_M), of a Lie group G, of a smooth map $\phi : M \to G$ (of pointed manifolds), and of a smooth left G-action $\ell : G \times M \to M$ (written $(g, x) \mapsto \ell(g, x) = \ell_g(x) = gx$) such that for all $g \in G$, $x \in M$

$$ge_M = e_M, \tag{3.26}$$
$$\phi(gx) = g\phi(x)g^{-1}. \tag{3.27}$$

It is a routine check that the multiplication $\triangleright$ on M defined for all $x, y \in M$ by

$$x \triangleright y := \ell_{\phi(x)}(y) \tag{3.28}$$

satisfies all the axioms (3.20), (3.21), and (3.22) of a Lie rack, thus making $(M, e_M, \triangleright)$ into a Lie rack such that the map ϕ is a morphism of Lie racks, i.e. for all $x, y \in M$

$$\phi(x \triangleright y) = \phi(x)\phi(y)\phi(x)^{-1}. \tag{3.29}$$

A morphism $(\Psi, \psi) : (M, \phi, G, \ell) \to (M', \phi', G', \ell')$ of augmented Lie racks is a pair of maps of pointed differentiable manifolds $\Psi : M \to M'$ and $\psi : G \to G'$ such that ψ is homomorphism of Lie groups and such that all reasonable diagrams commute, viz: for all $g \in G$

$$\phi' \circ \Psi = \psi \circ \phi \tag{3.30}$$
$$\Psi \circ \ell_g = \ell'_{\psi(g)} \circ \Psi \tag{3.31}$$

Note that the trivial Lie rack structure of a pointed manifold (M, e) comes from an augmented Lie rack over the trivial Lie group $G = \{e\}$.

Let $(M, e, \triangleright)$ be a Lie rack. Applying the functor F we get the following

Theorem 3.6 *The associated coalgebra $F(M)$ with multiplication μ of the Lie rack $\big(M, e, \mathbf{m}\big)$ is a rack bialgebra, i.e. satisfying for all $a, b, c \in C$, using the same notation $a \triangleright b$ for $\mu(a \otimes b)$:*

$$\begin{aligned} \mathbf{1} \triangleright a &= a, && (3.32)\\ a \triangleright \mathbf{1} &= \epsilon(a)\mathbf{1}, && (3.33)\\ a \triangleright (b \triangleright c) &= \sum_{(a)} (a^{(1)} \triangleright b) \triangleright (a^{(2)} \triangleright c). && (3.34) \end{aligned}$$

Proof:

(2.4) By definition, $S \otimes T \in \mathcal{E}'_e(M) \otimes \mathcal{E}'_e(M)$ are sent by $\mu = \triangleright_*$ to the distribution $f \mapsto F_{2\ M,M}(S \otimes T)(f \circ \triangleright)$. We evaluate this formula for $S = \mathbf{1}$. This gives the distribution $f \mapsto \mathbf{1}(T^{(2)}(f \circ \triangleright))$. But $T^{(2)}$ means that the function is seen as function of its second variable, i.e. $T^{(2)}(f \circ \triangleright)(y) = T(f(y \triangleright -))$. On the other hand, the delta distribution $\mathbf{1}$ evaluates a function in e, thus

$$\mathbf{1}(T^{(2)}(f \circ \triangleright)) \,=\, T(f(e \triangleright -)) \,=\, T(f),$$

because $e \triangleright y = y$ for all $y \in M$. This shows $\mathbf{1} \triangleright T = T$.

(2.5) Exchanging the roles of the two variables in the above computation, we obtain for $T \triangleright \mathbf{1}$ the distribution $T(\mathbf{1}^{(2)}(f \circ \triangleright))$ or in other words $\mathbf{1}(T^{(1)}(f \circ \triangleright))$, i.e. the above element y is now in the second place. We obtain

$$\mathbf{1}(T^{(1)}(f \circ \triangleright)) \,=\, T(f(- \triangleright e)) \,=\, T(f(e)) \,=\, T(1) \,=\, \epsilon(T).$$

This shows $T \triangleright \mathbf{1} = \epsilon(T)\mathbf{1}$.

(2.6) As remarked before, the definition of Δ_e, namely $\Delta_e = F_{2\ M,M}^{-1} \circ \mathrm{diag}_{M\ *}$, induces thanks to the naturality relation (3.11) relations like

$$(\mathrm{id}_M \times \mathrm{diag}_M)_* = F_{2\ M,(M\times M)} \circ (\mathrm{id}_{\mathcal{E}'_e(M)} \times F_{2\ M,M}) \circ (\mathrm{id}_{\mathcal{E}'_e(M)} \times \Delta_e) \circ F_{2\ M,M}^{-1},$$

and

$$\begin{aligned} &\Delta_e \times \mathrm{id}_{\mathcal{E}'_e(M)} \times \mathrm{id}_{\mathcal{E}'_e(M)}\\ &\quad= \big(F_{2\ M,M}^{-1} \times \mathrm{id}_{\mathcal{E}'_e(M)} \times \mathrm{id}_{\mathcal{E}'_e(M)}\big) \circ \big(F_{2\ (M\times M),M}^{-1} \times \mathrm{id}_{\mathcal{E}'_e(M)}\big)\\ &\qquad\circ F_{2\ (M\times M)\times M,M}^{-1} \circ \big(\mathrm{diag}_M \times \mathrm{id}_M \times \mathrm{id}_M\big)_*\\ &\qquad\circ F_{2\ (M\times M),M} \circ \big(F_{2\ M,M} \times \mathrm{id}_{\mathcal{E}'_e(M)}\big). \end{aligned}$$

Therefore, starting from the relation induced on $\mathcal{E}'_e(M)$ by relation (3.23), one replaces $(\mathrm{diag}_M \times \mathrm{id}_M \times \mathrm{id}_M)_*$ by the above and obtains finally an equation equivalent to equation (2.6).

□

Remark 3.4
This theorem should be compared to Proposition 3.1 in [4]. In [4], the authors work with the vector space $\mathbb{K}[M]$ generated by the rack M, while we work with point-distributions on a Lie rack M. In some sense, we extend their Proposition 3.1 "to all orders". Observe however that their structure is slightly different (motivated in their Remark 7.2). ◇

We get a similar theorem for an augmented Lie rack: Let $\mathfrak{g}$ denote the Lie algebra of the Lie group G, then we have the

Theorem 3.7 *The associated coalgebra C with multiplication μ of an augmented Lie rack (M, ϕ, G, ℓ) is a cocommutative augmented rack bialgebra $(C, \phi_*, \mathsf{U}(\mathfrak{g}), \ell)$*

We shall close the subsection with a geometric explanation of some of the structures appearing in Subsection 2.1: Let $\big(\mathfrak{h}, [\ ,\ ]\big)$ be a real finite-dimensional Leibniz algebra. Then for any real number $\hbar$, there is the following Lie rack structure on the manifold $\mathfrak{h}$ defined by

$$x \blacktriangleright_\hbar y := e^{\hbar \mathrm{ad}_x}(y) \tag{3.35}$$

For later use we note that on the space $\mathfrak{h}[[\hbar]]$ of all formal power series the above formula makes sense if x, y are also formal power series.
Moreover, pick a two-sided ideal $\mathfrak{z} \subset \mathfrak{h}$ with $Q(\mathfrak{h}) \subset \mathfrak{z} \subset \mathfrak{z}(\mathfrak{h})$ so that the quotient algebra $\mathfrak{g} := \mathfrak{h}/\mathfrak{z}$ is a Lie algebra. Let $p : \mathfrak{h} \to \mathfrak{g}$ be the canoncial projection. Let G be the connected simply connnected Lie group having Lie algebra $\mathfrak{g}$. Since $\mathfrak{g}$ acts on $\mathfrak{h}$ as derivations, there is a unique Lie group action ℓ of G on $\mathfrak{h}$ by automorphisms of Leibniz algebras. Consider the smooth map

$$\phi : \mathfrak{h} \to G : x \mapsto \exp\big(p(x)\big). \tag{3.36}$$

Clearly $\phi(g.x) = g\phi(x)g^{-1}$ for all $x \in \mathfrak{h}$ and $g \in G$ whence $(\mathfrak{h}, \phi, G, \ell)$ is an augmented Lie rack, and it is not hard to see that the Lie rack structure coincides with (3.35) for $\hbar = 1$.

Theorem 3.8 *The C^5-rack bialgebra associated to the augmented Lie rack $(\mathfrak{h}, \phi, G, \ell)$ by means of the Serre functor is isomorphic to the universal envelopping algebra of infinite order, $\mathsf{UAR}^\infty(\mathfrak{h})$, see Definition 2.3 and Theorem 2.1.*

Proof: First we compute $\phi_* = \exp_* \circ p_*$. Since $p : \mathfrak{h} \to \mathfrak{g}$ is linear, it is easy to see using formula (3.12) that for all $k \in \mathbb{N}$ and $x_1, \ldots, x_k \in \mathfrak{h}$

$$p_*\big(\Phi_{\mathsf{S}}(x_1 \bullet \cdots \bullet x_k)\big) = \Phi_{\mathsf{S}}\big(p(x_1) \bullet \cdots \bullet p(x_k)\big) = \Phi_{\mathsf{S}}\Big(\mathsf{S}(p)\big(x_1 \bullet \cdots \bullet x_k\big)\Big),$$

see (3.12) for a definition of Φ_{S}. Next, for all $k \in \mathbb{N}$ and $\xi_1, \ldots, \xi_k \in \mathfrak{g}$, we shall show the formula (for all $f' \in \mathcal{C}^\infty(G, \mathbb{K})$)

$$\begin{aligned}\Big(\exp_*\big(\Phi_{\mathsf{S}}(\xi_1 \bullet \cdots \bullet \xi_k)\big)\Big)(f') &= \sum_{\sigma \in S_k} \frac{1}{k!}\big(\Phi_{\mathsf{U}}(\xi_{\sigma(1)} \cdots \xi_{\sigma(k)})\big)(f') \\ &= \Big(\Phi_{\mathsf{U}}\big(\omega(\xi_1 \bullet \cdots \bullet \xi_k)\big)\Big)(f')\end{aligned}$$

(see eqn (3.16 for a definition of Φ_{U}). Both sides of this equation are symmetric k-linear maps in the arguments $\xi_1, \ldots, \xi_k$, hence by the polarization Lemma (see e.g. [29]), it suffices to check equality in case $\xi_1 = \cdots = \xi_k = \xi$. Since for each real number t the map $g \mapsto F_t^\xi(g) := g\exp(t\xi)$ is the flow of the left invariant vector field ξ^+, we get

$$\begin{aligned}\Big(\exp_*\big(\Phi_{\mathsf{S}}(\xi \bullet \cdots \bullet \xi)\big)\Big)(f') &= \frac{\partial^k}{\partial t^k}\Big(f'\big(\exp(t\xi)\big)\Big)\Big|_{t=0} = \frac{\partial^k}{\partial t^k}\Big(f'\big(F_t^\xi(e)\big)\Big)\Big|_{t=0} \\ &= \frac{\partial^k}{\partial t^k}\Big(\big((F_t^\xi)^* f'\big)(e)\big)\Big)\Big|_{t=0} \\ &= \Big((F_t^\xi)^*\big(\big(\mathbb{L}_{\xi^+} \circ \cdots \circ \mathbb{L}_{\xi^+}\big)(f')\big)\Big)(e)\Big|_{t=0} \\ &= \big(\big(\mathbb{L}_{\xi^+} \circ \cdots \circ \mathbb{L}_{\xi^+}\big)(f')\big)(e) \\ &= \Big(\Phi_{\mathsf{U}}\big(\omega(\xi \bullet \cdots \bullet \xi)\big)\Big)(f')\end{aligned}$$

proving the above formula. It follows that

$$\phi_* = \exp_* \circ p_* = \Phi_{\mathsf{U}} \circ \omega \circ \mathsf{S}(p) \circ \Phi_{\mathsf{S}}^{-1}. \tag{3.37}$$

Next, we compute ℓ_*. We get for positive integers k, l, $\xi_1, \ldots, \xi_k \in \mathfrak{g}$, $x \in \mathfrak{h}$, and $f \in \mathcal{C}^\infty(\mathfrak{h}, \mathbb{K})$:

$$\begin{aligned}&\Big(\ell_*\big(\Phi_{\mathsf{U}}(\xi_1 \cdots \xi_k) \otimes \Phi_{\mathsf{S}}(x^{\bullet l})\big)\Big)(f) \\ &= \frac{\partial^{k+l}}{\partial s_1 \cdots \partial s_k \partial t^l}\Big(f\big(\big(\ell_{\exp(s_1\xi_1)} \circ \cdots \circ \ell_{\exp(s_k\xi_k)}\big)(tx)\big)\Big)\Big|_{s_1 = \cdots = s_k = 0 = t} \\ &= \frac{\partial^k}{\partial s_1 \cdots \partial s_k} \sum_{j_1 \cdots j_l = 1}^{\dim(\mathfrak{h})} \left(\frac{\partial^l f}{\partial x_{j_1} \cdots \partial x_{j_l}}(0)\ell_s(x)_{j_1} \cdots \ell_s(x)_{j_l}\right)\Bigg|_{s_1 = \cdots = s_k = 0}\end{aligned}$$

where in the last line we have used a basis of $\mathfrak{h}$, have written $y_1, \ldots, y_n$ ($n = \dim(\mathfrak{h})$) for the components of each vector $y \in \mathfrak{h}$, and used the notation ℓ_s for the linear

map $\ell_{\exp(s_1\xi_1)} \circ \cdots \circ \ell_{\exp(s_k\xi_k)}$. By induction on k it is easy to prove that

$$\frac{\partial^k}{\partial s_1 \cdots \partial s_k}\Big(\ell_s(x) \bullet \cdots \bullet \ell_s(x)\Big)\Big|_{s_1=\cdots=s_k=0} = \mathrm{ad}^s_{\xi_1\cdots\xi_k}\big(x^{\bullet l}\big),$$

and using again the Polarisation Lemma, we finally get for all $u \in \mathsf{U}(\mathfrak{g})$ and $\alpha \in \mathsf{S}(\mathfrak{h})$

$$\ell_*\big(\Phi_{\mathsf{U}}(u) \otimes \Phi_{\mathsf{S}}(\alpha)\big) = \Phi_{\mathsf{S}}\big(\mathrm{ad}^s_u(\alpha)\big) = \Phi_{\mathsf{S}}\big(u.\alpha\big), \tag{3.38}$$

and the isomorphism with the augmented rack bialgebra $\mathsf{UAR}^\infty(\mathfrak{h}) = \mathsf{S}(\mathfrak{h})$ is established. □

Remark 3.5
Observe that the Serre functor can be rendered completely algebraic, i.e. for example for an algebraic Lie rack R (meaning that the underlying pointed manifold is a smooth algebraic variety and the rack product is algebraic), one can take as its Serre functor image $F(R)$ the space of derivations along the evaluation map in the distinguished point. This gives a new and completely algebraic way to associate to a Lie rack its tangent Leibniz algebra. ◇

4 Deformation quantization of rack bialgebras

4.1 Deformation quantization via an explicit formula

In this subsection, K denotes the field of real numbers $\mathbb{R}$ or the field of complex numbers $\mathbb{C}$.

Let $(\mathfrak{h}, [\,,\,])$ be a finite dimensional Leibniz algebra of dimension n, and denote by $\mathfrak{h}^*$ its linear dual. In order to make computations more elementary we shall use a fixed basis $e_1, \ldots, e_n$ of $\mathfrak{h}$, but it is a routine check that all the relevant formulas are invariant under a change of basis. Let $e^1, \ldots, e^n$ be the corresponding dual basis of $\mathfrak{h}^*$, i.e. by definition

$$e^i(e_j) \,=\, \delta_{ij},$$

for all $i, j = 1, \ldots, n$. Furthermore, let c^k_{ij} for $i, j, k = 1, \ldots, n$ be the structure constants of the Leibniz algebra $\mathfrak{h}$ with respect to the basis $e_1, \ldots, e_n$, i.e.

$$c^k_{ij} \,=\, e^i([e_j, e_k])$$

for all $i, j, k = 1, \ldots, n$. We will denote by $x, y, z, \ldots$ elements of $\mathfrak{h}$, while $\alpha, \beta, \gamma, \ldots$ will denote elements of $\mathfrak{h}^*$. Denote by $\alpha_1, \ldots, \alpha_n$ the coordinates

of $\alpha \in \mathfrak{h}^*$ with respect to the basis $e^1, \ldots, e^n$. For all $x \in \mathfrak{h}$, denote by $\hat{x} \in \mathcal{C}^\infty(\mathfrak{h}^*, K)$ the linear function given by

$$\hat{x}(\alpha) := \alpha(x),$$

for all $\alpha \in \mathfrak{h}^*$. In the same vein, let $e^{\hat{x}}$ be the exponential function given by

$$e^{\hat{x}}(\alpha) := e^{\alpha(x)} = e^{\hat{x}(\alpha)},$$

for all $\alpha \in \mathfrak{h}^*$. For all integers $i = 1, \ldots, n$, define a first order differential operator $\widetilde{\mathrm{ad}_i}$ on smooth functions $f : \mathfrak{h}^* \to K$ by

$$(\widetilde{\mathrm{ad}_i}(f))(\alpha) := \sum_{j,k=1}^{n} \alpha_k \, c_{ij}^k \, \frac{\partial f}{\partial \alpha_j}(\alpha).$$

The following star-product formula, where $\hbar$ is a formal parameter (which may be replaced by a real number in situations where the formula is convergent), will render $\mathfrak{h}^*$ a *quantum rack* in the sense of [9].

Let $f, g \in \mathcal{C}^\infty(\mathfrak{h}^*, K)$.

$$(f \triangleright_\hbar g)(\alpha) := \sum_{r=0}^{\infty} \frac{\hbar^r}{r!} \sum_{i_1,\ldots,i_r=1}^{n} \frac{\partial^r f}{\partial \alpha_{i_1} \ldots \partial \alpha_{i_r}}(0) \Big((\widetilde{\mathrm{ad}_{i_1}} \circ \ldots \circ \widetilde{\mathrm{ad}_{i_r}})(g) \Big)(\alpha). \quad (4.1)$$

Theorem 4.1 *For all $x, y \in \mathfrak{h}$, we have*

$$e^{\hat{x}} \triangleright_\hbar e^{\hat{y}} = e^{\widehat{x \blacktriangleright_\hbar y}},$$

where $\blacktriangleright_\hbar \colon \mathfrak{h} \times \mathfrak{h} \to \mathfrak{h}$ is the Lie rack structure (3.35).

Proof: The proof of the theorem relies on the following Lemmas:

Lemma 4.1 *The map "hat" $\hat{} \colon \mathfrak{h} \to \mathcal{C}^\infty(\mathfrak{h}^*, K)$ which sends $x \in \mathfrak{h}$ to the linear function $\hat{x}$ extends to an injective morphism of commutative associative unital algebras $\Psi : \mathsf{S}(\mathfrak{h}) \to \mathcal{C}^\infty(\mathfrak{h}^*, K)$ such that*

$$\Psi(x_1 \bullet \ldots \bullet x_k) = \hat{x_1} \ldots \hat{x_k}$$

for all integers k and all $x_1, \ldots, x_k \in \mathfrak{h}$.

Proof: This follows immediately from the freeness property of the algebra $\mathsf{S}(\mathfrak{h})$. □

Lemma 4.2 *The morphism Ψ intertwines the adjoint actions $\widetilde{\mathrm{ad}}_i$ and $\mathrm{ad}^s_{e_i}$ (see eqn (2.30)), i.e. for all $i = 1, \ldots, n$, we have*

$$\widetilde{\mathrm{ad}}_i(\Psi(a)) \;=\; \Psi(\mathrm{ad}^s_{e_i}(a))$$

for all $a \in \mathsf{S}(\mathfrak{h})$.

Proof: Indeed, it is enough to show this for $x \in \mathfrak{h} \subset \mathsf{S}(\mathfrak{h})$ as both adjoint actions are derivations. Now we have for $\alpha \in \mathfrak{h}^*$:

$$\begin{aligned}\Psi(\mathrm{ad}^s_{e_i}(x))(\alpha) &= \widehat{[e_i, x]}(\alpha) = \alpha([e_i, x]) = \sum_{j,k=1}^{n} \alpha_k \, e^k([e_i, e_j]) \, x_j \\ &= \sum_{j,k=1}^{n} c^k_{ij} \, \alpha_k \, \frac{\partial \hat{x}}{\partial \alpha_j}(\alpha) = \widetilde{\mathrm{ad}}_i(\Psi(a))(\alpha)\end{aligned}$$

□

Lemma 4.3 *For all $b \in \mathsf{S}(\mathfrak{h})$ and all $x_1, \ldots, x_r \in \mathfrak{h}$, we have*

$$\Psi(x_1 \bullet \ldots \bullet x_r) \rhd_\hbar \Psi(b) \;=\; \hbar^r \, \Psi((x_1 \bullet \ldots \bullet x_r) \rhd b),$$

where the left-hand $\rhd$ is the rack multiplication in the rack bialgebra $\mathsf{S}(\mathfrak{h})$.

Proof: First of all, note that by linearity it is enough to show this for $x_1, \ldots, x_r = e_{i_1}, \ldots, e_{i_r}$ with $i_1, \ldots, i_r \in \{1, \ldots, n\}$. By eqn (2.30), we have

$$(e_{i_1} \bullet \ldots \bullet e_{i_r}) \rhd b \;=\; \frac{1}{k!} \sum_{\sigma \in S_k} (\mathrm{ad}^s_{i_{\sigma(1)}} \circ \ldots \circ \mathrm{ad}^s_{i_{\sigma(r)}})(b).$$

Applying Ψ gives then

$$\begin{aligned}\Psi((e_{i_1} \bullet \ldots \bullet e_{i_r}) \rhd b) &= \frac{1}{k!} \sum_{\sigma \in S_k} \Psi((\mathrm{ad}^s_{i_{\sigma(1)}} \circ \ldots \circ \mathrm{ad}^s_{i_{\sigma(r)}})(b)) \\ &= \frac{1}{k!} \sum_{\sigma \in S_k} \widetilde{\mathrm{ad}}_{i_{\sigma(1)}} \circ \ldots \circ \widetilde{\mathrm{ad}}_{i_{\sigma(r)}}(\Psi(b)),\end{aligned}$$

by the previous lemma. Now compute

$$\frac{\partial^k \Big(\Psi(e_{i_1} \bullet \ldots \bullet e_{i_r}) \Big)}{\partial \alpha_{j_1} \ldots \partial \alpha_{j_k}}(0).$$

This expression is non zero only if $k = r$ and $\{i_1, \ldots, i_k\} = \{j_1, \ldots, j_k\}$. In this case, the result is 1. One deduces the asserted formula. □

Now we come back to the proof of the theorem. The assertion of the theorem is the equality:

$$e^{\hat{x}} \triangleright_\hbar e^{\hat{y}} \;=\; e^{\widehat{x \blacktriangleright_\hbar y}}.$$

Summing up the assertion of the previous lemma (taking $x_1 = \ldots = x_r = x$), we obtain:

$$\sum_{r=0}^{\infty} \frac{1}{r!} \Psi(\underbrace{x \bullet \ldots \bullet x}_{r \text{ times}}) \triangleright_\hbar \Psi(b) \;=\; \Psi\left(\Big(\sum_{r=0}^{\infty} \frac{\hbar^r}{r!} (\underbrace{x \bullet \ldots \bullet x}_{r \text{ times}}) \Big) \triangleright b \right),$$

and thus (as the rack product in $\mathsf{S}(\mathfrak{h})$ is given by the adjoint action, using also that Ψ is multiplicative)

$$e^{\hat{x}} \triangleright_\hbar \Psi(b) \;=\; \Psi(e^{\hbar \operatorname{ad}_x}(b)). \tag{4.2}$$

This extends then to the asserted formula using that $e^{\hbar \operatorname{ad}_x}$ is an automorphism of $\mathsf{S}(\mathfrak{h})$ (because it is the exponential of a derivation). □

Corollary 4.1 *The above defined star-product induces the structure of a rack with respect to the product $\triangleright_\hbar$ on the set of exponential functions on $\mathfrak{h}^*$, and this star-product is opposite to the star-product found in [9].*

Proof: Via the formula of the theorem, the self-distributivity property of the rack product $\blacktriangleright$ in the rack bialgebra $\mathsf{S}(\mathfrak{h})$ translates into the self-distributivity property of $\triangleright_\hbar$ on the set of exponential functions. Since the star-product defined in [9] is a series of bidifferential operators, and since such a series is uniquely determined by its values on exponential functions, the present star-product coincides with the one found in [9] thanks to the statement of the preceding theorem. □

4.2 General deformation theory for rack bialgebras

In this section, $(R, \Delta, \epsilon, \mu, \mathbf{1})$ is a **cocommutative** rack-bialgebra over a general commutative ring K, and we use the notation $r \triangleright s$ to denote the rack product $\mu(r \otimes s)$ of two elements r and s of R. In this subsection, we will often drop the symbol Σ in Sweedler's notation of (iterated) comultiplications, so that the n-iterated comultiplication of r in R reads

$$r^{(1)} \otimes \cdots \otimes r^{(n)} := (\Delta \otimes \mathrm{Id}^{\otimes n-1}) \circ \cdots \circ \Delta(r)$$

Let $K_\hbar = K[[\hbar]]$ denote the K-algebra of formal power series in the indeterminate $\hbar$ with coefficients in K. If V is a vector space over K, $V_\hbar$ stands

for V[[$\hbar$]]. Recall that if W is a K-module, a $K_\hbar$-linear morphism from $V_\hbar$ to $W_\hbar$ is the same as a power series in $\hbar$ with coefficients in $\mathrm{Hom}_K(V,W)$ via the canonical map

$$\mathrm{Hom}_{K_\hbar}(V_\hbar, W_\hbar) \cong \mathrm{Hom}_K(V,W)_\hbar.$$

This identification will be used without extra mention in the following.

Definition 4.1 *A **formal deformation of the rack product** μ is a formal power series $\mu_\hbar$*

$$\mu_\hbar := \sum_{n\geq 0} \hbar^n \mu_n$$

in $\mathrm{Hom}_K(R\otimes R, R)_\hbar$*, such that*

1. $\mu_0 = \mu$,
2. $(R_\hbar, \Delta, \epsilon, \mu_\hbar, \mathbf{1})$ *is a rack bialgebra over* $K_\hbar$.

Example 4.1
For a Leibniz algebra $\mathfrak{h}$, we have introduced in Definition 2.3 a cocommutative augmented rack bialgebra $\mathsf{UAR}^\infty(\mathfrak{h})$. Furthermore, the rack star product defined in eqn (4.1), restricted to $S(\mathfrak{h})_\hbar$, is a deformation of the trivial rack product of $S(\mathfrak{h})$ given for all $r, s \in S(\mathfrak{h})$ by

$$r \rhd s := \epsilon(r)s$$

The self-distributivity relation is shown in a way very similar to the proof of Theorem 4.1, see eqn (4.2). $\diamond$

As in the classical setting of deformation theory of associatice products, we will relate our deformation theory of rack products to cohomology. For this, let us first examine an introductory example:

Example 4.2
Let $(R, \rhd)$ be a rack bialgebra, and suppose there exists a deformation $\rhd_\hbar = \rhd + \hbar\omega$ of $\rhd$. The new rack product $\rhd_\hbar$ should satisfy the self-distributivity identity, i.e. for all $a, b, c \in R$

$$a \rhd_\hbar (b \rhd_\hbar c) \;=\; (a^{(1)} \rhd_\hbar b) \rhd_\hbar (a^{(2)} \rhd_\hbar c)$$

To the order $\hbar^0$, this is only the self-distributivity relation for $\rhd$. But to order $\hbar^1$ (neglecting order $\hbar^2$ and higher), we obtain:

$$\omega(a, b\rhd c) + a\rhd\omega(b,c) \;=\; \omega(a^{(1)}\rhd b, a^{(2)}\rhd c) + \omega(a^{(1)}, b)\rhd(a^{(2)}\rhd c) + (a^{(1)}\rhd b)\rhd\omega(a^{(2)}, c).$$

It will turn out that this is the cocycle condition for ω in the deformation complex which we are going to define. More precisely, we will have

1. $d_{2,0}\omega(a,b,c) \;=\; \omega(a, b \rhd c)$,

2. $d_{1,1}\omega(a,b,c) \;=\; a \rhd \omega(b,c)$,

3. $d_{1,0}\omega(a,b,c) \;=\; \omega(a^{(1)} \rhd b, a^{(2)} \rhd c)$,

4. $d^2_3\omega(a,b,c) \;=\; \omega(a^{(1)}, b) \rhd (a^{(2)} \rhd c)$,

5. $d_{2,1}\omega(a,b,c) \;=\; (a^{(1)} \rhd b) \rhd \omega((a^{(2)}, c)$.

This may perhaps help to understand the general definition of the operators $d^n_{i,\mu}$ for $i = 1, \ldots, n$ and $\mu \in \{0, 1\}$ further down.

On the other hand, the requirement that $\rhd_\hbar$ should be a morphism of coalgebras (with respect to the undeformed coproduct $\triangle$ of R) means

$$\triangle \circ \rhd_\hbar \;=\; (\rhd_\hbar \otimes \rhd_\hbar) \circ \triangle^{[2]}.$$

This reads for $a, b \in R$ to the order $\hbar$ (neglecting higher powers of $\hbar$) as

$$\omega(a,b)^{(1)} \otimes \omega(a,b)^{(2)} \;=\; \omega(a^{(1)}, b^{(1)}) \otimes (a^{(2)} \rhd b^{(2)}) + (a^{(1)} \rhd b^{(1)}) \otimes \omega(a^{(2)}, b^{(2)}).$$

This is exactly the requirement that ω is a coderivation along $\rhd = \mu$, to be defined below. $\diamond$

Recall that R being a rack bialgebra means in particular that $\mu : R^{\otimes 2} \to R$ is a morphism of coassociative coalgebras. For all positive integer n, let $\mu^n : R^{\otimes n} \to R$ be the linear map defined inductively by setting

- $\mu^1 := \mathrm{Id} : R \to R$,
- $\mu^2 := \mu : R^{\otimes 2} \to R$,
- $\mu^n := \mu \circ (\mu^1 \otimes \mu^{n-1})$, $n \geq 3$,

so that

$$\mu^n(r_1, \cdots, r_n) \;=\; r_1 \rhd (r_2 \rhd (\cdots \rhd (r_{n-1} \rhd r_n) \cdots))$$

for all $r_1, \ldots, r_n$ in R.

Proposition 4.1 *For all $n \geq 1$, the map μ^n is a morphism of coalgebras satisfying*

$$\mu^i(r_1^{(1)}, \cdots, r_{i-1}^{(1)}, r_i) \rhd \mu^{n-1}(r_1^{(2)}, \cdots, r_{i-1}^{(2)}, r_{i+1}, \cdots, r_n) \;=\; \mu^n(r_1, \cdots, r_n), \quad (4.3)$$

$$\mu^n(r_1, \cdots, r_{i-1}, r_i^{(1)} \rhd r_{i+1}, \cdots, r_i^{(n+1-i)} \rhd r_{n+1}) \;=\; \mu^{n+1}(r_1, \cdots, r_{n+1}) \quad (4.4)$$

for all positive integers i and n such that $1 \leq i < n$ and for all r_1, ..., r_n in R.

Proof:

- eqn (4.3): Let us show that the assertion of eqn (4.3) is true for all n and i with $1 \leq i < n$ by induction over i. Suppose that the induction hypothesis is true and compute

$$\mu^i(r_1^{(1)},\cdots,r_i^{(1)},r_{i+1}) \rhd \mu^{n-1}(r_1^{(2)},\cdots,r_i^{(2)},r_{i+2},\cdots,r_n)$$
$$\left(r_1^{(1)} \rhd \mu^i(r_2^{(1)},\cdots,r_i^{(1)},r_{i+1})\right) \rhd \left(r_1^{(2)} \rhd \mu^{n-2}(r_2^{(2)},\cdots,r_i^{(2)},r_{i+2},\cdots,r_n)\right),$$

  which gives, thanks to the self-distributivity relation in the rack algebra R,

$$\begin{aligned} r_1 \rhd &\left(\mu^i(r_2^{(1)},\cdots,r_i^{(1)},r_{i+1}) \rhd \mu^{n-2}(r_2^{(2)},\cdots,r_i^{(2)},r_{i+2},\cdots,r_n)\right) \\ &= r_1 \rhd \mu^{n-1}(r_2,\cdots,r_n) \;=\; \mu^n(r_1,\cdots,r_n), \end{aligned}$$

  where we have used the induction hypothesis. This proves the assertion.

- eqn (4.4): The assertion follows here again from an easy induction using the self-distributivity relation.

□

If (C,Δ_C) and (D,Δ_D) are two coassociative coalgebras and $\phi : C \to D$ is a morphism of coalgebras, we denote by $\mathrm{Coder}(C,V,\phi)$ the vector space of *coderivations from C to V along ϕ*, i.e. the vector space of linear maps $f : C \to D$ such that

$$\Delta_D \circ f \;=\; (f \otimes \phi + \phi \otimes f) \circ \Delta_C$$

Let us note the following permanence property of coderivations along a map under partial convolution which will be useful in the proof of the following theorem. For a coalgebra A, maps $f : A \otimes B \to V$ and $g : A \otimes C \to V$ and some product $\rhd : V \otimes V \to V$, the *partial convolution* of f and g is the map $f \star_{\mathrm{part}} g : A \otimes B \otimes C \to V$ defined for all $a \in A$, $b \in B$ and $c \in C$ by

$$(f \star_{\mathrm{part}} g)(a \otimes b \otimes c) \;:=\; f(a^{(1)} \otimes b) \rhd g(a^{(2)} \otimes c).$$

Lemma 4.4 *Let A, B, C and V be coalgebras, V carrying a product $\rhd$ which is supposed to be a coalgebra morphism. Let $f : A \otimes B \to V$ be a coderivation along ϕ and $g : A \otimes C \to V$ be a coalgebra morphism. Then the partial convolution $f \star_{\mathrm{part}} g$ is a coderivation along $\phi \star_{\mathrm{part}} g$.*

Proof: We compute for all $a \in A$, $b \in B$ and $c \in C$

$$\begin{aligned}
\triangle_V \circ (f \star_{\text{part}} g)(a \otimes b \otimes c) &= \triangle_V(f(a^{(1)} \otimes b) \rhd g(a^{(2)} \otimes c)) \\
&= (f(a^{(1)} \otimes b))^{(1)} \rhd (g(a^{(2)} \otimes c))^{(1)} \otimes (f(a^{(1)} \otimes b))^{(2)} \rhd (g(a^{(2)} \otimes c))^{(2)} \\
&= (f(a^{(1)} \otimes b))^{(1)} \rhd g(a^{(2)} \otimes c^{(1)}) \otimes (f(a^{(1)} \otimes b))^{(2)} \rhd g(a^{(3)} \otimes c^{(2)}) \\
&= \phi(a^{(1)} \otimes b^{(1)}) \rhd g(a^{(2)} \otimes c^{(1)}) \otimes f(a^{(3)} \otimes b^{(2)}) \rhd g(a^{(4)} \otimes c^{(2)}) + \\
&+ f(a^{(1)} \otimes b^{(1)}) \rhd g(a^{(2)} \otimes c^{(1)}) \otimes \phi(a^{(3)} \otimes b^{(2)}) \rhd g(a^{(4)} \otimes c^{(2)}) \\
&= (\phi \star_{\text{part}} g)(a^{(1)} \otimes b^{(1)} \otimes c^{(1)}) \otimes (f \star_{\text{part}} g)(a^{(2)} \otimes b^{(2)} \otimes c^{(2)}) + \\
&+ (f \star_{\text{part}} g)(a^{(1)} \otimes b^{(1)} \otimes c^{(1)}) \otimes (\phi \star_{\text{part}} g)(a^{(2)} \otimes b^{(2)} \otimes c^{(2)}) \\
&= \big((\phi \star_{\text{part}} g) \otimes (f \star_{\text{part}} g) + (f \star_{\text{part}} g) \otimes (\phi \star_{\text{part}} g)\big) \circ \triangle_{A \otimes B \otimes C}(a \otimes b \otimes c).
\end{aligned}$$

□

Definition 4.2 *The* ***deformation complex of*** R *is the graded vector space* $C^*(R;R)$ *defined in degree* n *by*

$$C^n(R;R) := \text{Coder}(R^{\otimes n}, R, \mu^n)$$

endowed with the differential $d_R : C^*(R;R) \to C^{*+1}(R;R)$ *defined in degree* n *by*

$$d_R^n := \sum_{i=1}^{n} (-1)^{i+1}(d_{i,1}^n - d_{i,0}^n) \; + \; (-1)^{n+1} d_{n+1}^n$$

where the maps $d_{i,1}^n$ *and* $d_{i,0}^n$ *are defined respectively by*

$$d_{i,1}^n \omega(r_1, \cdots, r_{n+1}) := \sum_{(r_1),\cdots,(r_i)} \mu^i(r_1^{(1)}, \cdots, r_{i-1}^{(1)}, r_i) \rhd \omega(r_1^{(2)}, \cdots, r_{i-1}^{(2)}, r_{i+1}, \cdots, r_{n+1})$$

and

$$d_{i,0}^n \omega(r_1, \cdots, r_{n+1}) := \sum_{(r_i)} \omega(r_1, \cdots, r_{i-1}, r_i^{(1)} \rhd r_{i+1}, \cdots, r_i^{(n+1-i)} \rhd r_{n+1})$$

and d_{n+1}^n *by*

$$\begin{aligned}
&d_{n+1}^n \omega(r_1, \cdots, r_{n+1}) \\
&\quad := \sum_{(r_1),\cdots,(r_{n-1})} \omega(r_1^{(1)}, \cdots, r_{n-1}^{(1)}, r_n) \rhd \mu^n(r_1^{(2)}, \cdots, r_{n-1}^{(2)}, r_{n+1})
\end{aligned}$$

for all ω *in* $C^n(R;R)$ *and* $r_1, \ldots, r_{n+1}$ *in* R.

Theorem 4.2 $d_{\mathbb{R}}$ *is a well defined differential.*

Proof: That d_R is well defined means that it sends coderivations to coderivations. It suffices to show that this is already true for all maps $d^n_{i,1}$, $d^n_{i,0}$ and d^n_{n+1}, which is the case. For this, we use Lemma 4.4. Indeed, a cochain $\omega \in C^n(R;R)$ is a coderivation along μ^n. By Proposition 4.1, μ^n is a coalgebra morphism. On the other hand, it is clear from the formula for $d^n_{i,1}$ that $d^n_{i,1}$ is a partial convolution with respect to the first $i-1$ tensor labels of μ^i and ω. Therefore the Lemma applies to give that the result is a coderivation along the partial convolution of μ^i and μ^n, which is just μ^{n+1} again by Proposition 4.1. This shows that $d^n_{i,1}\omega$ belongs to $\mathrm{Coder}(R^{\otimes n}, R, \mu^{n+1})$ as expected. The maps $d^n_{i,0}$ and d^n_{n+1} can be treated in a similar way.

The fact that d_R squares to zero is related to the so-called *cubical identities* satisfied by the maps $d_{i,1}$ and the maps $d_{i,0}$, namely

$$d^{n+1}_{j,\mu} \circ d^n_{i,\nu} \;=\; d^{n+1}_{i+1,\nu} \circ d^n_{j,\mu} \quad \text{for} \quad j \leq i \quad \text{and} \quad \mu, \nu \in \{0,1\},$$

and auxiliary identities which express the compatibility of the maps $d_{i,1}$ and $d_{i,0}$ with d^n_{n+1}, and an identity involving d^n_{n+1} and d^{n+1}_{n+2}. One could call this kind of object an *augmented cubical vector space*.

We will not show the usual cubical relations, i.e. those which do not refer to the auxiliary coboundary map d^n_{n+1}, because these are well-known to hold for rack cohomology, see [8], Corollary 3.12, and our case is easily adapted from there. One possibility of adaptation (in case one works over the real or complex numbers) is to take a Lie rack, write its rack homology complex (with trivial coefficients in the real or complex numbers), and to apply the Serre functor.

Let us show that the two following extra relations involving the extra face d^n_{n+1} hold:

$$d^{n+1}_{i,\mu} \circ d^n_{n+1} = d^{n+1}_{n+2} \circ d^n_{i,\mu} \tag{4.5}$$

for all $1 \leq i \leq n$ and μ in $\{0,1\}$ and

$$d^{n+1}_{n+1,0} \circ d^n_{n+1} = d^{n+1}_{n+2} \circ d^n_{n+1} + d^{n+1}_{n+1,1} \circ d^n_{n+1} \tag{4.6}$$

Indeed, if ω is a n-cochain and r_1, ..., r_{n+2} are elements in R, then

$$\begin{aligned}
&(d^{n+1}_{i,1} \circ d^n_{n+1}\omega)(r_1,\cdots,r_{n+2}) \\
&\quad = \mu^i(r^{(1)}_1,\cdots,r^{(1)}_{i-1},r_i) \rhd d^n_{n+1}\omega(r^{(2)}_1,\cdots,r^{(2)}_{i-1},r_{i+1},,\cdots,r_{n+2}) \\
&\quad = \mu^i(r^{(1)}_1,\cdots,r^{(1)}_{i-1},r_i) \rhd \big(\omega(r^{(2)}_1,\cdots,r^{(2)}_{i-1},r^{(1)}_{i+1},,\cdots,r^{(1)}_n,r_{n+1}) \rhd \\
&\qquad \rhd \mu^n(r^{(3)}_1,\cdots,r^{(3)}_{i-1},r^{(2)}_{i+1},,\cdots,r^{(2)}_n,r_{n+2})\big)
\end{aligned}$$

By Proposition 4.1 and thanks to the self-distributivity of the rack product, this

equality can be rewritten as

$$\begin{aligned}
&(d^{n+1}_{i,1} \circ d^n_{n+1}\omega)(r_1,\cdots,r_{n+2}) \\
&\quad = \big(\mu^i(r_1^{(1)},\cdots,r_{i-1}^{(1)},r_i^{(1)}) \rhd \omega(r_1^{(2)},\cdots,r_{i-1}^{(2)},r_{i+1}^{(1)},,\cdots,r_n^{(1)},r_{n+1})\big)\rhd \\
&\qquad \rhd \big(\mu^i(r_1^{(3)},\cdots,r_{i-1}^{(3)},r_i^{(2)}) \rhd \mu^n(r_1^{(4)},\cdots,r_{i-1}^{(4)},r_{i+1}^{(2)},,\cdots,r_n^{(2)},r_{n+2})\big) \\
&\quad = d^n_{i,1}\omega(r_1^{(1)},\cdots,r_n^{(1)},r_{n+1}) \rhd \mu^n(r_1^{(2)},\cdots,r_n^{(2)},r_{n+2}) \\
&\quad = (d^{n+1}_{n+2} \circ d^n_{i,1}\omega)(r_1,\cdots,r_{n+2}),
\end{aligned}$$

which proves that Relation (4.5) holds when $\mu = 1$. The case $\mu = 0$ goes as follows:

$$\begin{aligned}
(d^{n+1}_{i,0}\circ d^n_{n+1}\omega)(r_1,\cdots,r_{n+2}) &= d^n_{n+1}\omega(r_1,\cdots,r_{i-1},r_i^{(1)} \rhd r_{i+1},\cdots,r_i^{(n+2-i)} \rhd r_{n+2}) \\
&= \omega(r_1^{(1)},\cdots,r_{i-1}^{(1)},r_i^{(1)} \rhd r_{i+1}^{(1)},\cdots,r_i^{(n-i)} \rhd r_n^{(1)}, r_i^{(n+1-i)} \rhd r_{n+1}) \rhd \\
&\quad \rhd \mu^n(r_1^{(2)},\cdots,r_{i-1}^{(2)},r_i^{(n+2-i)} \rhd r_{i+1}^{(2)},\cdots,r_i^{(2n-2i+1)} \rhd r_n^{(2)}, r_i^{(2n-2i+2)} \rhd r_{n+2})
\end{aligned}$$

where we have used that the rack product is a morphism of coalgebras. Recall the following equation from Proposition 4.1:

$$\mu^n(s_1,\cdots,s_{i-1},s_i^{(1)} \rhd s_{i+1},\cdots,s_i^{(n+1-i)} \rhd s_{n+1}) = \mu^{n+1}(s_1,\cdots,s_{n+1})$$

for all s_1, ..., s_{n+1} in R and $1 \leq i \leq n$. This allows to rewrite the preceeding equality as

$$\begin{aligned}
&(d^{n+1}_{i,0}\circ d^n_{n+1}\omega)(r_1,\cdots,r_{n+2}) \\
&\quad = \omega(r_1^{(1)},\cdots,r_{i-1}^{(1)},r_i^{(1)} \rhd r_{i+1}^{(1)},\cdots,r_i^{(n-i)} \rhd r_n^{(1)}, r_i^{(n+1-i)} \rhd r_{n+1}) \rhd \\
&\qquad \rhd \mu^{n+1}(r_1^{(2)},\cdots,r_{i-1}^{(2)},r_i^{(n+2-i)},r_{i+1}^{(2)},\cdots,r_n^{(2)},r_{n+2}) \\
&\quad = d^n_{i,0}\,\omega(r_1^{(1)},\cdots,r_n^{(1)},r_{n+1}) \;\rhd\; \mu^{n+1}(r_1^{(2)},\cdots,r_n^{(2)},r_{n+2}) \\
&\quad = (d^{n+1}_{n+2} \circ d^n_{i,0})(r_1,\cdots,r_{n+2})
\end{aligned}$$

which proves that (4.5) holds when $\mu = 0$. Relation (4.6) relies on the fact that cochains are coderivations. Indeed,

$$\begin{aligned}
(d^{n+1}_{n+1,0}\circ d^n_{n+1}\omega)(r_1,\cdots,r_{n+2}) &= d^n_{n+1}\omega(r_1,\cdots,r_n,r_{n+1} \rhd r_{n+2}) \\
&= \omega(r_1^{(1)},\cdots,r_{n-1}^{(1)},r_n) \rhd \mu^n(r_1^{(2)},\cdots,r_{n-1}^{(2)},r_{n+1} \rhd r_{n+2}) \\
&= \omega(r_1^{(1)},\cdots,r_{n-1}^{(1)},r_n) \rhd \mu^{n+1}(r_1^{(2)},\cdots,r_{n-1}^{(2)},r_{n+1},r_{n+2}) \\
&= \omega(r_1^{(1)},\cdots,r_{n-1}^{(1)},r_n) \rhd \Big(\mu^n(r_1^{(2)},\cdots,r_{n-1}^{(2)},r_{n+1}) \rhd \mu^n(r_1^{(3)},\cdots,r_{n-1}^{(3)},r_{n+2})\Big)
\end{aligned}$$

where we have used Proposition 4.1 in the last equality. By self-distributivity of

$\triangleright$ and because ω is a coderivation, this gives

$$\begin{aligned}(d^{n+1}_{n+1,0}\circ d^n_{n+1}\omega)(r_1;\cdot\cdot,r_{n+2}) = {} & \big(\omega(r_1^{(1)};\cdot\cdot,r_{n-1}^{(1)},r_n)^{(1)}\triangleright\mu^n(r_1^{(2)};\cdot\cdot,r_{n-1}^{(2)},r_{n+1})\big)\triangleright\\ & \big(\omega(r_1^{(1)};\cdot\cdot,r_{n-1}^{(1)},r_n)^{(2)}\triangleright\mu^n(r_1^{(3)};\cdot\cdot,r_{n-1}^{(3)},r_{n+2})\big)\\ = {} & \big(\omega(r_1^{(1)};\cdot\cdot,r_{n-1}^{(1)},r_n^{(1)})\triangleright\mu^n(r_1^{(2)};\cdot\cdot,r_{n-1}^{(2)},r_{n+1})\big)\triangleright\\ & \big(\mu^n(r_1^{(3)};\cdot\cdot,r_{n-1}^{(3)},r_n^{(2)})\triangleright\mu^n(r_1^{(4)};\cdot\cdot,r_{n-1}^{(4)},r_{n+2})\big)\\ & +\big(\mu^n(r_1^{(1)};\cdot\cdot,r_{n-1}^{(1)},r_n^{(1)})\triangleright\mu^n(r_1^{(2)};\cdot\cdot,r_{n-1}^{(2)},r_{n+1})\big)\triangleright\\ & \big(\omega(r_1^{(3)};\cdot\cdot,r_{n-1}^{(3)},r_n^{(2)})\triangleright\mu^n(r_1^{(4)};\cdot\cdot,r_{n-1}^{(4)},r_{n+2})\big)\end{aligned}$$

Applying Proposition 4.1 again enables us to rewrite this last equality as

$$\begin{aligned}&(d^{n+1}_{n+1,0}\circ d^n_{n+1}\omega)(r_1;\cdot\cdot,r_{n+2})\\ &\quad=\big(\omega(r_1^{(1)};\cdot\cdot,r_n^{(1)})\triangleright\mu^n(r_1^{(2)};\cdot\cdot,r_{n-1}^{(2)},r_{n+1})\big)\triangleright\mu^{n+1}(r_1^{(3)};\cdot\cdot,r_{n-1}^{(3)},r_n^{(2)},r_{n+2})\\ &\qquad+\mu^{n+1}(r_1^{(1)};\cdot\cdot,r_n^{(1)},r_{n+1})\triangleright\big(\omega(r_1^{(2)};\cdot\cdot,r_n^{(2)})\triangleright\mu^n(r_1^{(3)};\cdot\cdot,r_{n-1}^{(3)},r_{n+2})\big)\\ &\quad=d^n_{n+1}\omega(r_1^{(1)},\cdots,r_n^{(1)},r_{n+1})\triangleright\mu^{n+1}(r_1^{(2)};\cdot\cdot,r_n^{(2)},r_{n+2})\\ &\qquad+\mu^{n+1}(r_1^{(1)};\cdot\cdot,r_n^{(1)},r_{n+1})\triangleright d^n_{n+1}\omega(r_1^{(2)};\cdot\cdot,r_n^{(2)},r_{n+2})\\ &\quad=\big((d^{n+1}_{n+2}\circ d^n_{n+1}+d^{n+1}_{n+1,1}\circ d^n_{n+1})\,\omega\big)(r_1,\cdots,r_{n+2})\end{aligned}$$

which proves (4.6).

Let us show now how $d_R\circ d_R=0$ can be deduced from (4.5), (4.6) and from the cubical relations. In degree n, we have

$$\begin{aligned}d_R\circ d_R = {} & \big(\sum_{i=1}^{n+1}(-1)^{i+1}(d^{n+1}_{i,1}-d^{n+1}_{i,0})+(-1)^{n+2}d^{n+1}_{n+2}\big)\circ\big(\sum_{i=1}^{n}(-1)^{i+1}(d^n_{i,1}-d^n_{i,0})\\ & +(-1)^{n+1}d^n_{n+1}\big)\\ = {} & \sum_{i=1}^{n+1}\sum_{j=1}^{n}(-1)^{i+j}(d^{n+1}_{i,1}\circ d^n_{j,1}-d^{n+1}_{i,1}\circ d^n_{j,0}-d^{n+1}_{i,0}\circ d^n_{j,1}+d^{n+1}_{i,0}\circ d^n_{j,0})\\ & +\sum_{i=1}^{n}(-1)^{n+i+1}(d^{n+1}_{n+2}\circ d^n_{i,1}-d^{n+1}_{n+2}\circ d^n_{i,0}-d^{n+1}_{i,1}\circ d^n_{n+1}+d^{n+1}_{i,0}\circ d^n_{n+1})\\ & -d^{n+1}_{n+2}\circ d^n_{n+1}-d^{n+1}_{n+1,1}\circ d^n_{n+1}+d^{n+1}_{n+1,0}\circ d^n_{n+1}\end{aligned}$$

The first double sum is equal to zero thanks to the cubical relations, the second sum is zero thanks to relation (4.5). Relation (4.6) implies that the last one vanishes. This shows that d_R is indeed a differential and concludes the proof of the proposition. $\square$

Definition 4.3 *The cohomology of the deformation complex $(C^*(R;R), d_R)$ is called the adjoint cohomology of the rack bialgebra R and is denoted by $H^*(R;R)$.*

Definition 4.4 *An **infinitesimal deformation** of the rack product is a deformation of the rack product over the K-algebra of dual numbers $\bar{K}_\hbar := K_\hbar/(\hbar^2)$, i.e. a linear map $\mu_1 : R^{\otimes 2} \to R$ such that $\bar{R}_\hbar := R \otimes \bar{K}_\hbar$ is a rack bialgebra over $\bar{K}_\hbar$ when equipped with $\mu_0 + \hbar\mu_1$.*
*Two infinitesimal deformations $\mu_0 + \hbar\mu_1$ and $\mu_0 + \hbar\mu_1'$ are said to be **equivalent** if there exists an automorphism $\phi : \bar{R}_\hbar \to \bar{R}_\hbar$ of the coalgebra of $(\bar{R}_\hbar, \Delta, \epsilon)$ of the form $\phi := \mathrm{id}_R + \hbar\alpha$ such that*

$$\phi \circ (\mu_0 + \hbar\mu_1) = (\mu_0 + \hbar\mu_1') \circ \phi.$$

As usual, being equivalent is an equivalence relation and one has the following cohomological interpretation of the set of equivalence classes of infinitesimal deformations, denoted $Def(\mu_0, \bar{K}_\hbar)$:

Proposition 4.2

$$Def(\mu_0, \bar{\mathbb{K}}_\hbar) \;=\; H^2(R;R)$$

The identificaton is obtained by sending each equivalence class $[\mu_0 + \hbar\mu_1]$ in $Def(\mu_0, \bar{K}_\hbar)$ to the cohomology class $[\mu_1]$ in $H^2(R;R)$.

Proof: One checks easily that the correspondence is well defined (if $\mu_0 + \hbar\mu_1$ is an infinitesimal deformation, then μ_1 is a 2-cocycle, see Example 4.2) and that it is bijective when restricted to equivalence classes. □

Remark 4.1

(a) The choice of taking coderivations in the deformation complex is explained as follows: The rack product μ is a morphism of coalgebras, and we want to deform it as a morphism of coalgebras with respect to the fixed coalgebra structure we started with. Tangent vectors to μ in $\mathrm{Hom}_{\mathrm{coalg}}(C \otimes C, C)$ are exactly coderivations along μ. This is the first step: Deformations as morphisms of coalgebras. Then as a second step, we look for 1-cocycles, meaning that we determine those morphisms of coalgebras which give rise to rack bialgebra structures. The deformation complex in [4] takes into account also the possibility of deforming the coalgebra structure, and we recover our complex by restriction.

(b) Given a Leibniz algebra $\mathfrak{h}$, there is a natural restriction map from the cohomology complex with adjoint coefficients of $\mathfrak{h}$ to the deformation complex of its augmented enveloping rack bialgebra $\mathsf{UAR}(\mathfrak{h})$. The induced map in cohomology is not necessarily an isomorphism, as the abelian case shows. Observe that the deformation complex of the rack bialgebra $K[R]$ for a rack R does not contain the complex of rack cohomology for two reasons: First, this latter complex is ill-defined for adjoint coefficients, and second, there are not enough coderivations as all elements are set-like. A way out for this last problem would be to pass to completions.

$\diamond$

A Some definitions around coalgebras

Let C be a module over a commutative associative unital ring K (which we shall assume to contain $\mathbb{Q}$). Recall that a linear map $\Delta : C \to C \otimes_K C = C \otimes C$ is called a *coassociative comultiplication* iff $\big(\Delta \otimes \mathrm{id}_C\big) \circ \Delta = \big(\mathrm{id}_C \otimes \Delta\big) \circ \Delta$, and the pair (C, Δ) is called a (coassociative) *coalgebra over* K. Let (C', Δ') be another coalgebra. Recall that a K-linear map $\Phi : C \to C'$ is called a *homomorphism of coalgebras* iff $\Delta' \circ \phi = (\phi \otimes \phi) \circ \Delta$. The coalgebra (C, Δ) is called *cocommutative* iff $\tau \circ \Delta = \Delta$ where $\tau : C \otimes C \to C \otimes C$ denotes the canonical flip map. Recall furthermore that a linear map $\epsilon : C \to K$ is called a *counit* for the coalgebra (C, Δ) iff $\big(\epsilon \otimes \mathrm{id}_C\big) \circ \Delta = \big(\mathrm{id}_C \otimes \epsilon\big) \circ \Delta = \mathrm{id}_C$. The triple (C, Δ, ϵ) is called a *counital coalgebra*. Moreover, a counital coalgebra (C, Δ, ϵ) equipped with an element $\mathbf{1}$ is called *coaugmented* iff $\Delta(\mathbf{1}) = \mathbf{1} \otimes \mathbf{1}$ and $\epsilon(\mathbf{1}) = 1 \in K$. Let $C^+ \subset C$ denote the kernel of ϵ. Recall that a morphism $\phi : (C, \Delta, \epsilon, \mathbf{1}) \to (C', \Delta', \epsilon', \mathbf{1}')$ of counital coaugmented coalgebras over K is a K-linear map satifying $(\phi \otimes \phi) \circ \Delta = \Delta' \circ \phi$, $\epsilon' \circ \phi = \epsilon$, and $\phi(\mathbf{1}) = \mathbf{1}'$. Moreover, for any counital coaugmented coalgebra the K-submodule of all *primitive elements* is defined by

$$\mathsf{Prim}(C) := \{x \in C \mid \Delta(x) = x \otimes \mathbf{1} + \mathbf{1} \otimes x\}. \tag{A.1}$$

Every morphism of counital coaugmented coalgebra clearly maps primitive elements to primitive elements, thus defining a functor Prim from the category of counital coaugmented coalgebras to the category of K-modules. Finally, following Quillen [22], we shall call a counital coaugmented coalgebra *connected* iff the following holds: The sequence of submodules $(C_{(r)})_{r \in \mathbb{N}}$ defined by $C_{(0)} = K\mathbf{1}$ and recursively by

$$C_{(k+1)} := \{x \in C \mid \Delta(x) - x \otimes \mathbf{1} - \mathbf{1} \otimes x \ \in \ C_{(k)}\} \tag{A.2}$$

is easily seen to be an ascending sequence of coaugmented counital subcoalgebras of $(C, \Delta, \epsilon, \mathbf{1})$, and if the union of all the $C_{(k)}$ is equal to C, then $(C, \Delta, \epsilon, \mathbf{1})$ is called connected. We refer to each $C_{(k)}$ as the *subcoalgebra of order k*. Clearly, each $C_{(k)}$ is connected, and $C_{(1)} = K\mathbf{1} \oplus \mathsf{Prim}(C)$. Moreover, each morphism of counital coaugmented coalgebras maps each subcoalgebra of order k to the subcoalgebra of order k of the target coalgebra thus defining a functor $C \to C_{(k)}$ from the category of coaugmented counital coalgebras to itself. We shall use the following acronyms:

Definition A.1 *We call a coassociative, counital, coaugmented coalgebra a C^3-coalgebra. In case the C^3-coalgebra is in addition cocommutative, we shall speak of a C^4-coalgebra. Finally, a connected C^4-coalgebra will be coined a C^5-coalgebra.*

Recall also that the tensor product of two counital coaugmented coalgebras $(C, \Delta, \epsilon, \mathbf{1})$ and $(C', \Delta', \epsilon', \mathbf{1}')$ is given by $(C \otimes C', (\mathrm{id}_C \otimes \tau \otimes \mathrm{id}_{C'}) \circ (\Delta \otimes \Delta'), \epsilon \otimes \epsilon', \mathbf{1} \otimes \mathbf{1}')$. Tensor products of connected coalgebras are connected.

Recall the standard example: Let V be a K-module and $\mathsf{S}(V) = \oplus_{r=0}^{\infty} \mathsf{S}^r(V)$ be the *symmetric algebra generated by* V, i.e. the free algebra $\mathsf{T}(V)$ (for which we denote the tensor multiplication by suppressing the symbol) modulo the two-sided ideal I generated by $xy - yx$ for all $x, y \in V$. Denoting the commutative associative multiplication in $\mathsf{S}(V)$ (which is induced by the free multiplication) by $\bullet$, i.e.

$$x_1 \bullet \cdots \bullet x_k := x_1 \cdots x_k \bmod I,$$

we have $\Delta(x) = x \otimes \mathbf{1} + \mathbf{1} \otimes x$ for all $x \in V$ and $\Delta(x_1 \bullet \cdots \bullet x_k) = (x_1 \otimes \mathbf{1} + \mathbf{1} \otimes x_1) \bullet \cdots \bullet (x_k \otimes \mathbf{1} + \mathbf{1} \otimes x_k)$ for all positive integers k and $x_1, \ldots, x_k \in V$. Recall that $S^0(V)$ is the free K-module $K\mathbf{1}$ and the counit is defined by $\epsilon(\lambda\mathbf{1}) = \lambda$ for all $\lambda \in K$ and by declaring that ϵ vanishes on $\oplus_{r=1}^{\infty} \mathsf{S}^r(V)$. Moreover, the submodules $\big(\mathsf{S}(V)\big)_{(n)}$ are given by $\oplus_{r=0}^{n} \mathsf{S}^r(V)$, whence $\mathsf{S}(V)$ is clearly connected, so it is a C^5-coalgebra whose submodule of primitive elements equals V.

Moreover, for a given coalgebra (C, Δ) and a given nonassociative algebra (A, μ) where $\mu : A \otimes A \to A$ is a given K-linear map, recall the *convolution multiplication* in the K-module $\mathrm{Hom}_K(C, A)$ defined in the usual way for any two K-linear maps $\phi, \psi : c \to A$ by

$$\phi * \psi := \mu \circ (\phi \otimes \psi) \circ \Delta. \tag{A.3}$$

In case Δ is coassociative and μ associative, $*$ will be associative. The following fact is rather important: If C is connected and if the K-linear map

$\varphi : C \to A$ vanishes on $\mathbf{1}_C$, then *any convolution power series of* φ *converges*, i.e. the evaluation of some formal series $\sum_{r=0}^{\infty} \lambda_r \varphi^{*r}$ (with $\lambda_r \in K$ and $\varphi^{*0} := \mathbf{1}_A \epsilon_C$) on $c \in C$ always reduces to a finite number of terms. In particular, let $\psi : C \to A$ be a K-linear map such that $\psi(\mathbf{1}_C) = \mathbf{1}_A$. Then –as has been observed by Takeuchi and Sweedler (see [28, Lemma 14] or [27, Lemma 9.2.3]– ψ has always a *convolution inverse*, i.e. there is a unique K-linear map $\psi' : C \to A$ such that $\psi * \psi' = \mathbf{1}_A \epsilon_C = \psi' * \psi$, where ψ' is defined by the geometric series $\psi' = \sum_{r=0}^{\infty} (\mathbf{1}_A \epsilon_C - \psi)^{*r}$.

B Semigroups

We collect some properties of semigroups which are very old, but a bit less well-known than properties of groups. The standard reference to these topics is the book [5] by A. H. Clifford and G. B. Preston.

Recall that a *semigroup* Γ is a set equipped with an associative multiplication $\Gamma \times \Gamma \to \Gamma$, written $(x, y) \mapsto xy$. An element e of Γ is called a *left unit element* (resp. a *right unit element* resp. a *unit element*) iff for all $x \in \Gamma$ we have $ex = x$ (resp. $xe = x$ resp. iff e is both left and right unit element). A pair (Γ, e) of a semigroup Γ and an element e is called *left unital* (resp. *right unital* resp. *unital*) iff e is a left unit element (resp. a right unit element resp. a unit element). A unital semigroup is also called a *monoid*. It is well-known that the unit element of a monoid is the unique unit element (unlike left or right unit elements in general). Let (Γ, e) be a right unital or a left unital semigroup. Recall that for a given element $x \in \Gamma$ an element $y \in \Gamma$ is called a *left inverse of* x (resp. a *right inverse of* x resp. an *inverse of* x) iff $yx = e$ (resp. $xy = e$ resp. iff y is both a left and a right inverse of x). Clearly, a unital semigroup (Γ, e) such that every element has an inverse is a *group*. In that case it is well-known that for each x there is exactly one inverse element, called x^{-1}.

Note that by a Lemma by *L. E. Dickson* (1905, see [5, p.4] for the reference) every *left* unital semigroup such that each element has at least one *left* inverse is already a group which can be shown by just using the definitions. Dually, every *right* unital semigroup such that each element has at least one *right* inverse is also a group.

More interesting is the case of a *left* (resp. right) unital semigroup (Γ, e) such that every element x has at least one *right* (resp. left) inverse element. In that case (which is an equivalent formulation of a so-called *right group* (resp. left group), see [5, p.37]), the conclusion of Dickson's Lemma does no longer hold. In order to see what is going on, there is first the following useful

Lemma B.1 *Let (Γ, e) be a left-unital semigroup, let a, b, c three elements of Γ such that*

$$ab = e \quad \text{and} \quad bc = e.$$

Then

$$c = ae, \ be = b,$$

and the left multiplications $L_a : x \mapsto ax$ and $L_b : x \mapsto bx$ are invertible. In particular, given the element a, its right inverse b is unique under the above hypotheses.

The proof is straight-forward.
The structure of right (resp.left) groups is completely settled in the *Suschkewitsch Decomposition Theorem*, 1928: Given a right group (Γ, e), it can be shown –using the above Lemma and elementary manipulations, see also [5, p.38, Thm 1.27]– that all the left multiplications $L_x : y \mapsto xy$ (resp. right multiplications $R_x : y \mapsto yx$) are invertible, that for each element there is exactly one right (resp. left) inverse (whence there is a map $\Gamma \to \Gamma$ assigning to each element x its right (resp. left) inverse x^{-1}), that the image of this right (resp. left) inverse map is equal to Γe (resp. $e\Gamma$) (which turns out to be a *subgroup* of (Γ, e)), and that (Γ, e) is isomorphic to the cartesian product $\big(\Gamma e \times E, (e, e)\big)$ (resp. $\big(E \times e\Gamma, (e, e)\big)$ where E is the set of all left (resp. right) unit elements in (Γ, e) (coinciding with the set of all idempotent elements). For right groups, the aforementioned isomorphism is given as follows:

$$\phi : \Gamma e \times E \to \Gamma \quad : \quad (a, f) \mapsto af, \tag{B.1}$$

$$\phi^{-1} : \Gamma \to \Gamma e \times E \quad : \quad x \mapsto (xe, x^{-1}x). \tag{B.2}$$

Note that both components of ϕ^{-1} are idempotent maps. There is a completely analogous statement for left groups.

Recall that a *Lie semigroup* is a differentiable manifold Γ equipped with a smooth associative multiplication $\mathbf{m} : \Gamma \times \Gamma \to \Gamma$. All the other definitions of semigroups mentioned above (such as left unital, right unital semigroups, monoids, groups, right groups, left groups etc.) carry over to the Lie, i.e. differentiable, case.
Moreover for *right Lie groups*, it is easy to see that all the left multiplications are diffeomorphisms (since their inverse maps are left multiplications with the inverse elements and therefore smooth). This fact and the regular value theorem applied to the equation $xy = e$ imply that the right inverse map is smooth since its graph is a closed submanifold of $\Gamma \times \Gamma$ and the restriction of the projection on the first factor of the graph is a diffeomorphism. As the maps $x \mapsto xe = (x^{-1})^{-1}$ and $x \mapsto x^{-1}x$ are smooth and idempotent, it follows that their images, the subgroup Γe, and the semigroup of all left unit

elements, E, are both smooth submanifolds of Γ and closed sets provided Γ is connected, see e.g. [3, p.54, Satz 5.13] for a proof. Hence Γe is a connected Lie group, and the Suschkewitsch decomposition $\Gamma \cong \Gamma e \times E$, see Appendix B, is a diffeomorphism. Conversely, any cartesian product of a Lie group G and a differentiable manifold E equipped with the multiplication $(g, x)(h, y) := (gh, y)$ is easily seen to be a right Lie group. An analogous statement holds for left Lie groups.
It is not hard to see that the category of all connected right Lie groups is equivalent to the product of category of all connected Lie groups and the category of all pointed connected manifolds.

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