

# Relaxation approximation of Friedrich's systems under convex constraints

Jean-François Babadjian, Clément Mifsud, Nicolas Seguin

## ▶ To cite this version:

Jean-François Babadjian, Clément Mifsud, Nicolas Seguin. Relaxation approximation of Friedrich's systems under convex constraints. Networks and Heterogeneous Media, 2016, 11 (2), pp. 223 - 237. 10.3934/nhm.2016.11.223 . hal-01157484

## HAL Id: hal-01157484 https://hal.science/hal-01157484

Submitted on 29 May 2015

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers. L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés. Manuscript submitted to AIMS' Journals Volume  $\mathbf{X}$ , Number  $\mathbf{0X}$ , XX  $\mathbf{200X}$ 

pp. **X–XX** 

## RELAXATION APPROXIMATION OF FRIEDRICHS' SYSTEMS UNDER CONVEX CONSTRAINTS

### JEAN-FRANÇOIS BABADJIAN, CLÉMENT MIFSUD

Sorbonne Universités, UPMC Univ Paris 06, UMR 7598 Laboratoire Jacques-Louis Lions F-75005, Paris, France

### NICOLAS SEGUIN

Sorbonne Universités, UPMC Univ Paris 06, UMR 7598 Laboratoire Jacques-Louis Lions F-75005, Paris, France CNRS, UMR 7598, Laboratoire Jacques-Louis Lions F-75005, Paris, France INRIA-Paris-Rocquencourt, EPC Ange Domaine de Voluceau, BP105, 78153 Le Chesnay Cedex

(Communicated by the associate editor name)

ABSTRACT. This paper is devoted to present an approximation of a Cauchy problem for Friedrichs' systems under convex constraints. It is proved the strong convergence in  $L^2_{loc}$  of a parabolic-relaxed approximation towards the unique constrained solution.

1. Introduction. The aim of this paper is to prove the convergence of a relaxation approximation of weak solutions to Friedrichs' systems under convex constraints. The well-posedness has been established in [3] by means of a numerical scheme. We present here another way to construct such weak solutions thanks to a model that relaxes the constraints. We consider the following Cauchy problem: find  $W : [0,T] \times \mathbb{R}^n \to \mathbb{R}^m$  such that

$$\begin{cases} \partial_t W + \sum_{j=1}^n B_j \partial_j W = 0 & \text{in } ]0, T] \times \mathbb{R}^n, \\ W(t, x) \in K & \text{if } (t, x) \in [0, T] \times \mathbb{R}^n, \\ W(0, x) = W^0(x) & \text{if } x \in \mathbb{R}^n, \end{cases}$$
(1)

where K is a fixed (*i.e.* independent of the time and space variables) non empty closed and convex subset of  $\mathbb{R}^m$  containing 0 in its interior, the matrices  $B_j$  are  $m \times m$  symmetric matrices independent of time and space, and T > 0. The main difficulty is due to the constraints which introduce nonlinear effects to the linear Friedrichs' system [5]. This type of hyperbolic problems has been introduced in [3] where a notion of weak solutions to problem 1 has been defined.

 $<sup>2010\</sup> Mathematics\ Subject\ Classification.\ Primary:\ 35L45,\ 35L60;\ Secondary:\ 35A35.$ 

Key words and phrases. Friedrichs' systems, Convex constraints.

**Definition 1.1.** Let  $W^0 \in L^2(\mathbb{R}^n, K)$ , and T > 0. A function W is a weak constrained solution of 1 if  $W \in L^2([0,T] \times \mathbb{R}^n, K)$  satisfies

$$\int_{0}^{T} \int_{\mathbb{R}^{n}} \left( |W - \kappa|^{2} \partial_{t} \phi + \sum_{j=1}^{n} \langle W - \kappa; B_{j}(W - \kappa) \rangle \partial_{j} \phi \right) dt dx + \int_{\mathbb{R}^{n}} |W^{0}(x) - \kappa|^{2} \phi(0, x) dx \ge 0, \quad (2)$$

for all  $\kappa \in K$  and  $\phi \in \mathcal{C}^{\infty}_{c}([0,T[\times \mathbb{R}^{n}) \text{ with } \phi(t,x) \geq 0 \text{ for all } (t,x) \in [0,T[\times \mathbb{R}^{n}.$ 

We recall here the main result of [3].

**Theorem 1.2.** Assume that  $W^0 \in L^2(\mathbb{R}^n, K)$ . There exists a unique weak constrained solution  $W \in L^2([0, T] \times \mathbb{R}^n, K)$  to 1 in the sense of Definition 1.1. In addition, this solution belongs to  $C([0, T], L^2(\mathbb{R}^n, K))$ , and if further  $W^0 \in H^1(\mathbb{R}^n, K)$ , then  $W \in L^{\infty}([0, T], H^1(\mathbb{R}^n, K))$ .

As already mentioned, the well-posedness of problem 1 has been established in [3] thanks to a numerical method. We relax here the constraints  $W(t, x) \in K$  for a.e.  $(t, x) \in ]0, T[\times \mathbb{R}^n$  as

$$\partial_t W_\epsilon + \sum_{j=1}^n B_j \partial_{x_j} W_\epsilon = \frac{P_K(W_\epsilon) - W_\epsilon}{\epsilon},\tag{3}$$

where  $P_K$  denotes the orthogonal projection onto the closed convex set K, and  $\epsilon > 0$  is a small parameter. Formally, if we multiply equation **3** by  $\epsilon$ , and let  $\epsilon$  tend to 0, we get that the "limit" of  $W_{\epsilon}$ , denoted by W, satisfies  $P_K(W) = W$ , which ensures that  $W \in K$ . In addition, Definition **1.1** has been motivated in [**3**] by a formal derivation from the relaxation system **3**. To see it, it suffices to take the scalar product of equation of (**3**) with  $W_{\epsilon} - \kappa$ , where  $\kappa \in K$  is arbitrary. We then use the first order characterization of the projection which ensures that the right hand side is non-positive, to get the inequality of Definition **1.1**. The purpose of this work is to rigorously justify these formal steps.

The relaxation model presented here is very similar to viscous approximation of constrained models found in mechanics, and especially in plasticity. We start from the system of dynamical linear elasticity in three space dimension which can be written as

$$\begin{cases} \partial_t F + \nabla_x v = 0, \\ \partial_t v + \operatorname{div} \sigma = 0, \end{cases}$$

$$\tag{4}$$

for all  $t \in [0,T]$  and  $x \in \mathbb{R}^3$ . In the previous system, F(t,x) is a  $3 \times 3$  matrix which stands for the displacement gradient,  $v(t,x) \in \mathbb{R}^3$  is the velocity, *i.e.* the displacement time derivative, and  $\sigma = \mu (F + F^T) + \lambda (\operatorname{tr} F) I_3$  is the symmetric Cauchy stress tensor (here  $\lambda$  and  $\mu$  are the Lamé coefficients, and  $I_3$  is the idendity matrix in  $\mathbb{R}^3$ ). This system can be rewritten (thanks to a change of variables see [6]) in the Friedrichs' framework as

$$\partial_t U + A_1 \partial_{x_1} U + A_2 \partial_{x_2} U + A_3 \partial_{x_3} U = 0,$$

where U is a vector in  $\mathbb{R}^9$  (containing the three components of the velocity v, and the six components of the stress  $\sigma$ ) and  $A_1, A_2$  and  $A_3$  are symmetric matrices.

We now introduce the convex contraint coming from plasticity, see [11]. Indeed, the theory of perfect plasticity is characterized by the fact that the stress tensor  $\sigma$  is constrained to stay inside a fixed closed convex set K of symmetric  $3 \times 3$  matrices.

The total strain is then additively decomposed as the sum of (i) the elastic strain, denoted by e, which is still related to the stress by the linear relation  $\sigma = \lambda tr e + 2\mu e$ ; (ii) and the plastic strain, denoted by p, whose rate is oriented in a normal direction to K at  $\sigma$ . Summarizing, one has

$$\frac{F+F^T}{2} = e+p, \ \sigma = \lambda \text{tr} e + 2\mu e \in K, \ \partial_t p \in \partial I_K(\sigma),$$
(5)

where  $\partial I_K(\sigma)$  denotes the subdifferential of  $I_K$ , the indicator function of K, at the point  $\sigma$ . Using Fenchel-Moreau regularization of  $I_K$  (see [8]), the last condition in 5 can be relaxed as

$$\partial_t p = \frac{1}{\epsilon} \left( P_K(\sigma) - \sigma \right)$$

where  $\epsilon > 0$  is a viscosity parameter. We can now reformulate, at least formally, the dynamical problem of visco-plasticity (see [9, 11]) as

$$\begin{cases} \partial_t U + A_1 \partial_{x_1} U + A_2 \partial_{x_2} U + A_3 \partial_{x_3} U = \frac{P_{\tilde{K}}(U) - U}{\epsilon} & \text{on } [0, T] \times \mathbb{R}^3, \\ U(t, x) \in \tilde{K} & \text{if } (t, x) \in [0, T] \times \mathbb{R}^3, \\ U(0, x) = U^0(x) & \text{if } x \in \mathbb{R}^3, \end{cases}$$
(6)

where, again,  $U \in \mathbb{R}^9$ , and  $\tilde{K} = \{u \in \mathbb{R}^9 : \sigma \in K\}$ , and  $A_1$ ,  $A_2$  and  $A_3$  are the same matrices than in the elasto-dynamic case. As  $\epsilon$  tends to zero, one expects the solution to 6 to converge to that of the model of perfect plasticity (see [12] in the quasistatic case).

**Notation.** In the sequel, we denote by  $\langle | \rangle$  the scalar product of  $L^2(\mathbb{R}^n, \mathbb{R}^m)$  and by  $\langle ; \rangle$  the canonical scalar product of  $\mathbb{R}^m$  (and |.| the associated norm). Also, to shorten notation, we write  $L^2_{t,x}$  (resp.  $H^1_{t,x}$ ) instead of  $L^2(0,T; L^2(\mathbb{R}^n, \mathbb{R}^m))$  (resp.  $H^1((0,T) \times \mathbb{R}^n, \mathbb{R}^m)$ ),  $L^2_x$  instead of  $L^2(\mathbb{R}^n, \mathbb{R}^m)$ .

This paper is organized as follows. In the first section, we establish the existence and uniqueness of the relaxation model thanks to a parabolic approximation. In the second section, we prove that the relaxed solution  $W_{\epsilon}$  to 3 satisfies the inequalities of Definition 1.1. Finally, to get the existence of a solution as a limit when  $\epsilon$  tends to zero of relaxed solutions  $W_{\epsilon}$ , we prove the strong convergence of the sequence  $(W_{\epsilon})_{\epsilon>0}$  in the space  $L^2((0,T) \times \omega)$ , where  $\omega$  is a open bounded subset of  $\mathbb{R}^n$ , to a weak solution to the contrained Friedrichs' systems.

2. **Parabolic approximation.** In order to find a solution to the relaxation problem 3, we use a parabolic type regularization. To this aim, we consider a classical sequence of mollifiers in  $\mathbb{R}^n$ , denoted by  $(\rho_\eta)_{\eta>0}$ .

**Theorem 2.1.** Let  $W^0 \in H^1(\mathbb{R}^n, K)$ . For every  $\epsilon > 0$  and  $\eta > 0$ , the system

$$\begin{cases} \partial_t W_{\epsilon,\eta} - \eta \Delta W_{\epsilon,\eta} + \sum_{j=1}^n B_j \partial_j W_{\epsilon,\eta} = \frac{P_K(W_{\epsilon,\eta}) - W_{\epsilon,\eta}}{\epsilon}, & \text{on } ]0, T] \times \mathbb{R}^n, \\ W_{\epsilon,\eta}(0, x) = W^0(x) * \rho_\eta, & \text{if } x \in \mathbb{R}^n, \end{cases}$$
(7)

admits a unique solution  $W_{\epsilon,\eta}$  with the following properties:

 $W_{\epsilon,\eta} \in L^2(0,T; H^2(\mathbb{R}^n,\mathbb{R}^m)), \quad \partial_t W_{\epsilon,\eta} \in L^2(0,T; H^1(\mathbb{R}^n,\mathbb{R}^m)),$ 

and

$$\partial_{tt} W_{\epsilon,\eta} \in L^2(0,T; H^{-1}(\mathbb{R}^n,\mathbb{R}^m)).$$

Furthermore, we have the following estimates

$$\sup_{0 \le t \le T} \|W_{\epsilon,\eta}(t)\|_{L^2_x}^2 \le \|W^0\|_{L^2_x}^2,$$
(8)

$$\sup_{0 \le t \le T} \left\| W_{\epsilon,\eta}(t) \right\|_{H^1_x} \le C_{\epsilon} \left\| W^0 \right\|_{H^1_x}, \tag{9}$$

$$\sup_{0 \le t \le T} \left\| \partial_t W_{\epsilon,\eta}(t) \right\|_{L^2_x} \le C_{\epsilon} \left\| W^0 \right\|_{H^1_x}, \tag{10}$$

for some constant  $C_{\epsilon} > 0$  independent of  $\eta$ .

*Proof.* The proof essentially follows that of Theorem 1, Part II, Section 7.3.2 in [4]. Let  $X = L^{\infty}(0,T; H^1(\mathbb{R}^n,\mathbb{R}^m))$  and  $V \in X$ . We consider the problem

$$\begin{cases} \partial_t U - \eta \Delta U = \frac{P_K(V) - V}{\epsilon} - \sum_{j=1}^n B_j \partial_j V, & \text{on } ]0, T] \times \mathbb{R}^n, \\ U(0, x) = W^0(x) * \rho_\eta, & \text{if } x \in \mathbb{R}^n. \end{cases}$$
(11)

Since  $0 \in K$ , we have the following inequality

 $\forall k$ 

$$\in \mathbb{R}^m, \qquad |(P_K(k) - k)| \le |k|$$

which shows that  $\frac{P_K(V)-V}{\epsilon} - \sum_{j=1}^n B_j \partial_j V \in L^2(0,T;L^2(\mathbb{R}^n,\mathbb{R}^m))$ . Using the theory of parabolic equations, we get that equation 11 admits a unique solution U with  $U \in L^2(0,T;H^2(\mathbb{R}^n,\mathbb{R}^m))$  and  $\partial_t U \in L^2(0,T;L^2(\mathbb{R}^n,\mathbb{R}^m))$ . Let  $\tilde{V} \in X$  and  $\tilde{U}$  be the solution to 11 associated with  $\tilde{V}$ . The function  $\hat{U} = U - \tilde{U}$  is a solution to

$$\begin{cases} \partial_t \hat{U} - \eta \Delta \hat{U} = \frac{P_K(V) - P_K(\tilde{V}) - (V - \tilde{V})}{\epsilon} - \sum_{j=1}^n B_j \partial_j (V - \tilde{V}), & \text{on } ]0, T] \times \mathbb{R}^n, \\ \hat{U}(0, x) = 0, & \text{if } x \in \mathbb{R}^n. \end{cases}$$
(12)

Thanks to the theory of parabolic equations, we have the following estimate, we obtain the following estimate

$$\begin{aligned} \underset{0 \leq t \leq T}{\text{ess-sup}} \left\| \hat{U}(t) \right\|_{H^{1}_{x}} \\ &\leq C(\eta) \left\| \frac{P_{K}(V) - P_{K}(\tilde{V}) - (V - \tilde{V})}{\epsilon} - \sum_{j=1}^{n} B_{j} \partial_{j}(V - \tilde{V}) \right\|_{L^{2}_{t,x}} \\ &\leq C(\eta) \max\left(\frac{2}{\epsilon}, \|B_{j}\|\right) \left\| V - \tilde{V} \right\|_{L^{2}_{t}H^{1}_{x}} \\ &\leq C(\eta) \max\left(\frac{2}{\epsilon}, \|B_{j}\|\right) \sqrt{T} \underset{0 \leq t \leq T}{\text{ess-sup}} \left\| V(t) - \tilde{V}(t) \right\|_{H^{1}_{x}}. \end{aligned}$$

Therefore, the mapping

$$\psi: \begin{cases} X \to \left\{ U \in L_t^2 H_x^2 \text{ and } \partial_t U \in L_{t,x}^2 \right\} \subset X \\ \hat{V} \mapsto \hat{U} \end{cases}$$

is Lipschitz-continuous with Lipschitz constant bounded by  $C(\eta) \max\left(\frac{2}{\epsilon}, \|B_j\|\right) \sqrt{T}$ .

We now divide [0,T] into sub-intervals  $[0,T_1], [T_1, 2T_1], [2T_1, 3T_1], \dots, [NT_1, T]$  such that

$$C(\eta) \max\left(\frac{2}{\epsilon}, \|B_j\|\right) \sqrt{\max(T_1, T - NT_1)} < 1.$$

For every  $i \in \{0, \ldots N-1\}$ , the Banach fixed-point Theorem ensures the existence and uniqueness of a solution to 7 on the interval  $[iT_1, (i+1)T_1]$  (with initial condition  $W^0 * \rho_\eta$  if i = 0, and  $W_{\epsilon,\eta}(iT_1)$  if  $i \ge 1$  obtained at the previous step). We then obtain a solution  $W_{\epsilon,\eta}$  on the entire interval [0,T] by gluing the solutions on each sub-intervals, so that  $W_{\epsilon,\eta} \in L^2(0,T; H^2(\mathbb{R}^n,\mathbb{R}^m))$ . According to the initial condition on each sub-intervals, the function  $t \mapsto W_{\epsilon,\eta}(t)$  is continuous in  $L^2_x$  at every  $t = iT_1$ , so that  $\partial_t W_{\epsilon,\eta} \in L^2(0,T; L^2(\mathbb{R}^n,\mathbb{R}^m))$ , and, in particular  $W_{\epsilon,\eta} \in$  $H^1(]0,T[\times\mathbb{R}^n,\mathbb{R}^m)$ .

To obtain the announced regularity, we use the following result (whose proof relies on the chain rule in Sobolev spaces).

**Lemma 2.2.** Let  $U \in H^1(]0, T[\times \mathbb{R}^n, \mathbb{R}^m)$ . Then the function  $P_K \circ U - U$  also belongs to  $H^1(]0, T[\times \mathbb{R}^n, \mathbb{R}^m)$ , and there exists a constant M > 0, independent of U, such that

$$||P_K \circ U - U||_{H^1_{t,x}} \le M ||U||_{H^1_{t,x}}$$

This result ensures that  $\frac{P_K(W_{\epsilon,\eta})-W_{\epsilon,\eta}}{\epsilon} \in H^1(]0, T[\times \mathbb{R}^n, \mathbb{R}^m)$  and then, using again to the regularity theory of parabolic equations, we obtain that

$$W_{\epsilon,\eta} \in L^2(0,T; H^2(\mathbb{R}^n, \mathbb{R}^m)), \qquad \partial_t W_{\epsilon,\eta} \in L^2(0,T; H^1(\mathbb{R}^n, \mathbb{R}^m))$$

and

$$\partial_{tt} W_{\epsilon,\eta} \in L^2(0,T; H^{-1}(\mathbb{R}^n,\mathbb{R}^m))$$

Now we derive the estimates. We are going to use the following result (see [4]).

**Lemma 2.3.** Let  $U \in L^2(0, T, H^1(\mathbb{R}^n, \mathbb{R}^m))$  with  $\partial_t U \in L^2(0, T; H^{-1}(\mathbb{R}^n, \mathbb{R}^m))$ . Then, the function

$$t \mapsto \left\| U(t) \right\|_{L^2}^2$$

is absolutely continuous, and for a.e.  $t \in [0, T]$ ,

$$\frac{d}{dt} \left( \frac{1}{2} \| U(t) \|_{L^2_x}^2 \right) = \langle U(t), \partial_t U(t) \rangle_{H^1_x, H^{-1}_x}.$$

Applying this result to  $W_{\epsilon,\eta}$ , we get that for a.e.  $t \in [0,T]$ ,

$$\frac{d}{dt} \left( \frac{1}{2} \left\| W_{\epsilon,\eta}(t) \right\|_{L^2_x}^2 \right) = \left\langle W_{\epsilon,\eta}(t) \left| \partial_t W_{\epsilon,\eta}(t) \right\rangle_{L^2_x} \\
= \left\langle W_{\epsilon,\eta}(t) \left| \frac{P_K(W_{\epsilon,\eta})(t) - W_{\epsilon,\eta}(t)}{\epsilon} - \sum_{j=1}^n B_j \partial_j W_{\epsilon,\eta}(t) + \eta \Delta W_{\epsilon,\eta}(t) \right\rangle, \quad (13)$$

where we used the fact that  $W_{\epsilon,\eta}$  is a solution to the partial differential equation 7. Since  $0 \in K$ , we have

$$\langle W_{\epsilon,\eta}(t) | P_K(W_{\epsilon,\eta})(t) - W_{\epsilon,\eta}(t) \rangle_{L^2_x}$$
  
=  $\langle P_K(W_{\epsilon,\eta})(t) | P_K(W_{\epsilon,\eta})(t) - W_{\epsilon,\eta}(t) \rangle - \| P_K(W_{\epsilon,\eta})(t) - W_{\epsilon,\eta}(t) \|_{L^2_x}^2 \le 0.$  (14)

On the other hand, if  $v \in \mathcal{C}_c^{\infty}(\mathbb{R}^n, \mathbb{R}^m)$ , an integration by parts shows that

$$\langle v \, | \, \eta \Delta v \rangle_{L^2} = -\eta \, \| Dv \|_{L^2_x}^2 \le 0,$$

and

$$\left\langle v \left| \sum_{j=1}^{n} B_j \partial_j v \right\rangle = 0, \right.$$

since the matrices  $B_j$  are symmetric and independent of the space variables. By approximation, these formulas are true for  $v \in H^1(\mathbb{R}^n, \mathbb{R}^m)$  as well, and in particular,

$$\left\langle W_{\epsilon,\eta}(t) \left| \sum_{j=1}^{n} B_j \partial_j W_{\epsilon,\eta}(t) + \eta \Delta W_{\epsilon,\eta}(t) \right\rangle \le 0.$$
 (15)

Gathering 13, 14 and 15, we obtain that

$$\frac{d}{dt}\left(\frac{1}{2}\left\|W_{\epsilon,\eta}\right\|_{L^2_x}^2\right) \le 0,$$

and using Gronwall's Lemma, we derive the first estimate 8.

$$\sup_{0 \le t \le T} \|W_{\epsilon,\eta}(t)\|_{L^2_x}^2 = \|W^0 * \rho_\eta\|_{L^2_x}^2 \le \|W^0\|_{L^2_x}^2.$$
(16)

We apply the same argument to the spatial weak derivates  $\partial_k W_{\epsilon,\eta}$  of  $W_{\epsilon,\eta}$ . Deriving the partial differential equation 7 in the sense of distribution, we infer that

$$\partial_t \partial_k W_{\epsilon,\eta} - \eta \Delta \partial_k W_{\epsilon,\eta} + \sum_{j=1}^n B_j \partial_j \partial_k W_{\epsilon,\eta} = \partial_k \frac{P_K(W_{\epsilon,\eta}) - W_{\epsilon,\eta}}{\epsilon}$$

The previous equality actually holds in  $L^2(0, T, L^2(\mathbb{R}^n, \mathbb{R}^m))$  thanks to the regularity of  $W_{\epsilon,\eta}$ , and we can apply Lemma 2.3 to obtain that Êfor a.e.  $t \in [0, T]$ ,

$$\frac{d}{dt} \left( \frac{1}{2} \left\| \partial_k W_{\epsilon,\eta}(t) \right\|_{L^2_x}^2 \right) = \left\langle \partial_k W_{\epsilon,\eta}(t), \partial_t \partial_k W_{\epsilon,\eta}(t) \right\rangle_{L^2_x} \\ = \left\langle \left. \partial_k W_{\epsilon,\eta}(t) \right| \left. \partial_k \frac{P_K(W_{\epsilon,\eta})(t) - W_{\epsilon,\eta}(t)}{\epsilon} - \sum_{j=1}^n B_j \partial_j \partial_k W_{\epsilon,\eta}(t) + \eta \Delta \partial_k W_{\epsilon,\eta}(t) \right\rangle.$$

Arguing as in 15, we get

$$\frac{d}{dt} \left( \frac{1}{2} \left\| \partial_k W_{\epsilon,\eta}(t) \right\|_{L^2_x}^2 \right) \le \left\langle \partial_k W_{\epsilon,\eta}(t) \left| \partial_k \frac{P_K(W_{\epsilon,\eta}(t)) - W_{\epsilon,\eta}(t)}{\epsilon} \right\rangle \\ \le \frac{M}{\epsilon} \left\| \partial_k W_{\epsilon,\eta}(t) \right\|_{L^2_x}^2,$$

where we used Lemma 2.2. Using again Gronwall's Lemma, it yields

$$\sup_{0 \le t \le T} \left\| \partial_k W_{\epsilon,\eta}(t) \right\|_{L^2_x}^2 \le \exp\left(\frac{2TM}{\epsilon}\right) \left\| \partial_k W^0 * \rho_\eta \right\|_{L^2_x}^2 \\ \le \exp\left(\frac{2TM}{\epsilon}\right) \left\| \partial_k W^0 \right\|_{L^2_x}^2, \tag{17}$$

which completes the proof of estimate 9.

We finally derive the last estimate for  $\partial_t W_{\epsilon,\eta}$ . Again, we derive the partial differential equation 7 with respect to t in the distributional sense to get

$$\partial_{tt} W_{\epsilon,\eta} - \eta \Delta \partial_t W_{\epsilon,\eta} + \sum_{j=1}^n B_j \partial_j \partial_t W_{\epsilon,\eta} = \partial_t \frac{P_K(W_{\epsilon,\eta}) - W_{\epsilon,\eta}}{\epsilon}$$

in  $L^2(0,T,H^{-1}(\mathbb{R}^n,\mathbb{R}^m))$ . Using again Lemma 2.3, we get that for a.e.  $t \in [0,T]$ ,

$$\frac{d}{dt} \left( \frac{1}{2} \left\| \partial_t W_{\epsilon,\eta}(t) \right\|_{L^2_x}^2 \right) = \langle \partial_t W_{\epsilon,\eta}(t) \left| \partial_{tt} W_{\epsilon,\eta}(t) \rangle_{H^1_x, H^{-1}_x} \right. \\ \left. = \left\langle \left. \partial_t W_{\epsilon,\eta}(t) \right| \eta \Delta \partial_t W_{\epsilon,\eta}(t) - \sum_{j=1}^n B_j \partial_j \partial_t W_{\epsilon,\eta}(t) + \partial_t \frac{P_K(W_{\epsilon,\eta})(t) - W_{\epsilon,\eta}(t)}{\epsilon} \right\rangle_{H^1_x, H^{-1}_x} \right.$$

As before, we infer that

$$\left\langle \partial_t W_{\epsilon,\eta}(t) \left| \sum_{j=1}^n B_j \partial_j \partial_t W_{\epsilon,\eta}(t) \right\rangle_{L^2_x} = 0, \quad \left\langle \partial_t W_{\epsilon,\eta}(t) \left| \eta \Delta \partial_t W_{\epsilon,\eta}(t) \right\rangle_{H^1_x, H^{-1}_x} \le 0, \right. \right.$$

which shows, thanks to Lemma 2.2, that

$$\frac{d}{dt} \left( \frac{1}{2} \left\| \partial_t W_{\epsilon,\eta}(t) \right\|_{L^2_x}^2 \right) \leq \frac{1}{\epsilon} \left\| \partial_t W_{\epsilon,\eta}(t) \right\|_{L^2_x} \left\| \partial_t (P_K(W_{\epsilon,\eta})(t) - W_{\epsilon,\eta}(t)) \right\|_{L^2_x} \leq \frac{M}{\epsilon} \left\| \partial_t W_{\epsilon,\eta}(t) \right\|_{L^2_x}^2.$$

At this point, we would like to use Gronwall's Lemma. To do that, we need to know the value of  $\partial_t W_{\epsilon,\eta}$  at t = 0. To this aim, let us take a test function  $\phi \in C_c^{\infty}(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^m)$  with  $\phi(T, \cdot) = 0$ . On the one hand, according to Fubini's Theorem and Green's formula on [0, T], Êwe have

$$\int_0^T \int_{\mathbb{R}^n} \left\langle \partial_t W_{\epsilon,\eta}; \partial_t \phi \right\rangle = \int_{\mathbb{R}^n} \left[ -\left\langle \partial_t W_{\epsilon,\eta}(0,\cdot); \phi(0,\cdot) \right\rangle - \int_0^T \left\langle \partial_{tt} W_{\epsilon,\eta}; \phi \right\rangle \right],$$

since  $\partial_t W_{\epsilon,\eta} \in H^1(0,T; H^{-1}(\mathbb{R}^n, \mathbb{R}^m))$  and  $\phi$  is smooth. On the other hand, according to equation 7

$$\begin{split} \int_{0}^{T} \int_{\mathbb{R}^{n}} \left\langle \partial_{t} W_{\epsilon,\eta}; \partial_{t} \phi \right\rangle \\ &= \int_{0}^{T} \int_{\mathbb{R}^{n}} \left\langle \eta \Delta W_{\epsilon,\eta} - \sum_{j=1}^{n} B_{j} \partial_{j} W_{\epsilon,\eta} + \frac{P_{K}(W_{\epsilon,\eta}) - W_{\epsilon,\eta}}{\epsilon}; \partial_{t} \phi \right\rangle \\ &= \int_{\mathbb{R}^{n}} \left\{ -\int_{0}^{T} \left\langle \partial_{t} \left[ \eta \Delta W_{\epsilon,\eta} - \sum_{j=1}^{n} B_{j} \partial_{j} W_{\epsilon,\eta} + \frac{P_{K}(W_{\epsilon,\eta}) - W_{\epsilon,\eta}}{\epsilon} \right]; \phi \right\rangle \\ &- \left\langle \eta \Delta W^{0} * \rho_{\eta} - \sum_{j=1}^{n} B_{j} \partial_{j} W^{0} * \rho_{\eta} + \frac{P_{K}(W^{0} * \rho_{\eta}) - W^{0} * \rho_{\eta}}{\epsilon}; \phi(0, \cdot) \right\rangle \right\}. \end{split}$$

Since  $\phi(0, \cdot)$  is arbitrary, we obtain that

$$\partial_t W_{\epsilon,\eta}(0,\cdot) = \eta \Delta W^0 * \rho_\eta - \sum_{j=1}^n B_j \partial_j W^0 * \rho_\eta + \frac{P_K(W^0 * \rho_\eta) - W^0 * \rho_\eta}{\epsilon}.$$

We are now in position to apply Gronwall's Lemma which implies that

$$\sup_{0 \le t \le T} \left\| \partial_t W_{\epsilon,\eta}(t) \right\|_{L^2_x}^2 \le \exp\left(\frac{2TM}{\epsilon}\right) \left\| \partial_t W_{\epsilon,\eta}(0,\cdot) \right\|_{L^2_x}^2.$$

Since,

$$\left\|\Delta W^0 * \rho_\eta\right\|_{L^2_x} \le \frac{C}{\eta} \left\|\nabla W^0\right\|_{L^2_x},$$

for some constant C > 0 independent of  $\eta$ , we deduce that

$$\sup_{0 \le t \le T} \left\| \partial_t W_{\epsilon,\eta}(t) \right\|_{L^2_x} \le C_{\epsilon} \left\| W^0 \right\|_{H^1_x}$$

where  $C_{\epsilon} > 0$  is another constant independent of  $\eta$ , which completes the proof of the last estimate 10.

3. Approximation of the convex constraints. We now consider the relaxation problem

$$\begin{cases} \partial_t W_{\epsilon} + \sum_{j=1}^n B_j \partial_j W_{\epsilon} = \frac{P_K(W_{\epsilon}) - W_{\epsilon}}{\epsilon}, & \text{on } ]0, T] \times \mathbb{R}^n, \\ W_{\epsilon}(0, x) = W^0(x), & \text{if } x \in \mathbb{R}^n. \end{cases}$$
(18)

Thanks to Theorem 2.1, we will construct the solution to the previous problem as the limit of the solution of the parabolic problem 7 when  $\eta$  tends to zero.

**Theorem 3.1.** There exists a unique solution  $W_{\epsilon} \in H^1(]0, T[\times \mathbb{R}^n, \mathbb{R}^m))$  to 18 satisfying, for all  $\phi \in C_c^{\infty}(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^m)$ ,

$$\int_{0}^{T} \int_{\mathbb{R}^{n}} \left\langle \partial_{t} W_{\epsilon} + \sum_{j=1}^{n} B_{j} \partial_{j} W_{\epsilon}; \phi \right\rangle dx \, dt = \int_{0}^{T} \int_{\mathbb{R}^{n}} \left\langle \frac{P_{K}(W_{\epsilon}) - W_{\epsilon}}{\epsilon}; \phi \right\rangle dx \, dt, \quad (19)$$

and  $W_{\epsilon}(0, \cdot) = W_0$  in  $L^2(\mathbb{R}^n, \mathbb{R}^m)$ . In addition,

$$\sup_{0 \le t \le T} \|W_{\epsilon}(t)\|_{L^{2}_{x}}^{2} \le \|W^{0}\|_{L^{2}_{x}}^{2}.$$
(20)

*Proof.* Thanks to Theorem 2.1, the sequence  $(W_{\epsilon,\eta})_{\eta>0}$  is bounded in the space  $H^1(]0, T[\times \mathbb{R}^n, \mathbb{R}^m)$ . We can thus extract a subsequence (not relabeled) such that

 $W_{\epsilon,\eta} \rightharpoonup W_{\epsilon}$  weakly in  $H^1(]0, T[\times \mathbb{R}^n, \mathbb{R}^m)$ .

In particular, 20 is a consequence of 8 by the lower semicontinuity of the norm with respect to weak convergence. Since the embedding of  $H^1(]0, T[\times \mathbb{R}^n, \mathbb{R}^m)$  into  $L^2_{\text{loc}}([0,T] \times \mathbb{R}^n, \mathbb{R}^m)$  is compact (cf [1]), we deduce that

$$W_{\epsilon,\eta} \to W_{\epsilon}$$
 strongly in  $L^2(]0, T[\times] - R, R[^n, \mathbb{R}^m),$ 

for each R > 0. Let  $\phi \in \mathcal{C}_c^{\infty}(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^m)$  and R such that the support of  $\phi$  is contained in  $[-R, R]^{n+1}$ . Since  $W_{\epsilon,\eta}$  is a weak solution of 11, we have

$$\int_0^T \int_{\mathbb{R}^n} \left( \left\langle \partial_t W_{\epsilon,\eta}; \phi \right\rangle + \eta \sum_{j=1}^n \left\langle \partial_j W_{\epsilon,\eta}; \partial_j \phi \right\rangle + \sum_{j=1}^n \left\langle B_j \partial_j W_{\epsilon,\eta}; \phi \right\rangle \right) dx \, dt$$
$$= \int_0^T \int_{\mathbb{R}^n} \left\langle \frac{P_K(W_{\epsilon,\eta}) - W_{\epsilon,\eta}}{\epsilon}; \phi \right\rangle \, dx \, dt.$$

Using the weak convergence, we infer that

$$\begin{split} \int_0^T \int_{\mathbb{R}^n} \left( \left\langle \partial_t W_{\epsilon,\eta}; \phi \right\rangle + \eta \sum_{j=1}^n \left\langle \partial_j W_{\epsilon,\eta}; \partial_j \phi \right\rangle + \sum_{j=1}^n \left\langle B_j \partial_j W_{\epsilon,\eta}; \phi \right\rangle \right) dx \, dt \\ \to \int_0^T \int_{\mathbb{R}^n} \left( \left\langle \partial_t W_{\epsilon}; \phi \right\rangle + \sum_{j=1}^n \left\langle B_j \partial_j W_{\epsilon}; \phi \right\rangle \right) dx \, dt. \end{split}$$

8

On the other hand, the strong convergence yields

$$\left\langle \frac{P_K(W_{\epsilon,\eta}) - W_{\epsilon,\eta}}{\epsilon}; \phi \right\rangle \to \left\langle \frac{P_K(W_{\epsilon}) - W_{\epsilon}}{\epsilon}; \phi \right\rangle \text{ strongly in } L^1(]0, T[\times \mathbb{R}^n),$$

and consequently, we obtain that

$$\int_{0}^{T} \int_{\mathbb{R}^{n}} \left\langle \partial_{t} W_{\epsilon} + \sum_{j=1}^{n} B_{j} \partial_{j} W_{\epsilon}; \phi \right\rangle dx \, dt = \int_{0}^{T} \int_{\mathbb{R}^{n}} \left\langle \frac{P_{K}(W_{\epsilon}) - W_{\epsilon}}{\epsilon}; \phi \right\rangle dx \, dt.$$
(21)

We next focus on the initial condition. We take  $\phi \in C_c^{\infty}(] - \infty, T[\times \mathbb{R}^n, \mathbb{R}^m)$  (in particular  $\phi(T) = 0$ ). An integration by parts shows that

$$-\int_0^T \int_{\mathbb{R}^n} \langle W_{\epsilon,\eta}; \partial_t \phi \rangle \, dx \, dt + \int_{\mathbb{R}^n} \langle W^0 * \rho_\eta(x); \phi(0,x) \rangle \, dx$$
$$= \int_0^T \int_{\mathbb{R}^n} \langle \partial_t W_{\epsilon,\eta}; \phi \rangle \, dx \, dt.$$

Letting  $\eta$  tend to zero, and using 21 leads to

$$\int_{0}^{T} \int_{\mathbb{R}^{n}} \left( -\langle W_{\epsilon}; \partial_{t} \phi \rangle + \sum_{j=1}^{n} \langle B_{j} \partial_{j} W_{\epsilon}; \phi \rangle \right) dx dt + \int_{\mathbb{R}^{n}} \left\langle W^{0}(x); \phi(0, x) \right\rangle dx = \int_{0}^{T} \int_{\mathbb{R}^{n}} \left\langle \frac{P_{K}(W_{\epsilon}) - W_{\epsilon}}{\epsilon}; \phi \right\rangle dx dt, \quad (22)$$

since  $W^0 * \rho_\eta \to W^0$  strongly in  $L^2_{\text{loc}}(\mathbb{R}^n, \mathbb{R}^m)$ . We now integrate by parts in 21, using the fact that  $W_{\epsilon} \in H^1(0, T, L^2(\mathbb{R}^n, \mathbb{R}^m))$ ,

$$\int_{0}^{T} \int_{\mathbb{R}^{n}} \left( \langle W_{\epsilon}; \partial_{t} \phi \rangle + \sum_{j=1}^{n} \langle B_{j} \partial_{j} W_{\epsilon}; \phi \rangle \right) dx dt + \int_{\mathbb{R}^{n}} \langle W_{\epsilon}(0, x); \phi(0, x) \rangle dx = \int_{0}^{T} \int_{\mathbb{R}^{n}} \left\langle \frac{P_{K}(W_{\epsilon}) - W_{\epsilon}}{\epsilon}; \phi \right\rangle dx dt.$$
(23)

Using the equations 22 and 23, it gives that

$$\int_{\mathbb{R}^n} \langle W_{\epsilon}(0,x); \phi(0,x) \rangle \, dx = \int_{\mathbb{R}^n} \langle W^0(x); \phi(0,x) \rangle \, dx,$$

and consequently the initial condition is satisfied in  $L^2(\mathbb{R}^n, \mathbb{R}^m)$ .

It remains to show the uniqueness. Let us consider two solutions  $W_{\epsilon}$  and  $\tilde{W}_{\epsilon}$  associated with the same initial condition  $W^0$ . Using the partial differential equation 18, we obtain that

$$\partial_t \left( W_\epsilon - \tilde{W}_\epsilon \right) + \sum_{j=1}^n B_j \partial_j \left( W_\epsilon - \tilde{W}_\epsilon \right) = \frac{P_K(W_\epsilon) - W_\epsilon - P_K(\tilde{W}_\epsilon) + \tilde{W}^\epsilon}{\epsilon}.$$

As already observed, we know that for a.e.  $t \in [0, T]$ ,

$$\int_{\mathbb{R}^n} \sum_{j=1}^n \left\langle B_j \partial_j W_{\epsilon}(t) - \tilde{W}_{\epsilon}(t); W_{\epsilon}(t) - \tilde{W}_{\epsilon}(t) \right\rangle dx = 0,$$

and also

$$\frac{1}{2}\frac{d}{dt}\left\|W_{\epsilon}(t)-\tilde{W}_{\epsilon}(t)\right\|_{L^{2}_{x}}^{2}=\left\langle\partial_{t}W_{\epsilon}(t)-\partial_{t}\tilde{W}_{\epsilon}(t)\left|W_{\epsilon}(t)-\tilde{W}_{\epsilon}(t)\right\rangle.$$

Consequently, we have that

$$\frac{1}{2} \frac{d}{dt} \left\| W_{\epsilon}(t) - \tilde{W}_{\epsilon}(t) \right\|_{L^{2}_{x}}^{2} = \left\langle \frac{P_{K}(W_{\epsilon})(t) - W_{\epsilon}(t) - P_{K}(\tilde{W}_{\epsilon})(t) + \tilde{W}_{\epsilon}(t)}{\epsilon} \right| W_{\epsilon}(t) - \tilde{W}_{\epsilon}(t) \right\rangle \\
\leq \frac{2}{\epsilon} \left\| W_{\epsilon}(t) - \tilde{W}_{\epsilon}(t) \right\|_{L^{2}_{x}}^{2}.$$

since the projection is 1-Lipschitz. Gronwall's Lemma thus implies that  $W_{\epsilon} = W_{\epsilon}$  since they satisfy the same initial condition. As a consequence of the uniqueness, we deduce that there is no need to extract a subsequence from  $(W_{\epsilon,\eta})_{\eta>0}$  to get the convergences as  $\eta \to 0$ .

4. Convergence of the relaxed formulation. In this section, we first show that the solution  $W_{\epsilon}$  to the relaxation problem 18 satisfies the inequality of Definition 1.1, and then we prove that we can pass to the limit in this inequality to get a solution to the initial problem 1.

**Lemma 4.1.** Let  $W_{\epsilon}$  be the unique solution to 18. For all  $\kappa \in K$  and for all  $\phi \in W^{1,\infty}((-\infty,T) \times \mathbb{R}^n, \mathbb{R}^+)$  with compact support in  $] - \infty, T[\times \mathbb{R}^n, \text{ one has}]$ 

$$\int_{0}^{T} \int_{\mathbb{R}^{n}} \left( |W_{\epsilon} - \kappa|^{2} \partial_{t} \phi + \sum_{j=1}^{n} \langle W_{\epsilon} - \kappa; B_{j}(W_{\epsilon} - \kappa) \rangle \, \partial_{j} \phi \right) dt \, dx + \int_{\mathbb{R}^{n}} |W^{0}(x) - \kappa|^{2} \phi(0, x) \, dx \ge 0.$$
(24)

*Proof.* Since  $W_{\epsilon}$  is a solution to 18, we know that

$$\int_0^T \int_{\mathbb{R}^n} \left\langle \partial_t W_\epsilon + \sum_{j=1}^n B_j \partial_j W_\epsilon - \frac{P_K(W_\epsilon) - W_\epsilon}{\epsilon}; W_\epsilon - \kappa \right\rangle dx \, dt = 0.$$

By the first order characterization of the projection, one has

$$\left\langle \frac{P_K(W_\epsilon) - W_\epsilon}{\epsilon}; W_\epsilon - \kappa \right\rangle (t, x) \le 0$$

for a.e.  $(t,x) \in [0,T] \times \mathbb{R}^n$ . On the other hand, since  $W_{\epsilon} \in H^1(0,T; L^2(\mathbb{R}^n,\mathbb{R}^m))$ and  $W_{\epsilon} \in L^2(0,T; H^1(\mathbb{R}^n,\mathbb{R}^m)))$  we can integrate by parts to obtain the desired result.

**Remark 1.** Let us stress that, although the function  $W_{\epsilon}$  satisfies the same inequality than the weak constrained solution, it is not a weak constrained solution in the sense of Definition 1.1 because it does not *a priori* belong to *K*.

To get a weak constrained solution from the sequence of solutions  $(W_{\epsilon})_{\epsilon>0}$  to the relaxation problem 18, we need to pass to the limit as  $\epsilon \to 0$  in the previous inequality. This is the purpose of the following result.

**Theorem 4.2.** For every bounded open set  $\omega \subset \mathbb{R}^n$ , the sequence  $(W_{\epsilon})_{\epsilon>0}$  converges strongly in  $L^2((0,T) \times \omega, \mathbb{R}^m)$  to some function W which is a weak constrained solution to problem 1.

*Proof.* Let  $\omega$  be an open bounded subset of  $\mathbb{R}^n$ . We are going to prove the existence of a subsequence of  $(W_{\epsilon})_{\epsilon>0}$  (associated with the same initial data  $W^0$ ) which converges in  $L^2(0, T, L^2(\omega, \mathbb{R}^m))$ . We will use the following compactness criterion (see [10]).

**Theorem 4.3.** Let B be a Banach space. A subset F of  $L^2(0,T;B)$  is relatively compact if and only if both conditions are fulfilled:

- the set  $\left\{ \int_{t_1}^{t_2} f(t) dt : f \in F \right\}$  is relatively compact in B for all  $0 < t_1 < t_2 < T$ ;
- we have

where  $\tau_h f$ 

$$\sup_{f \in F} \|\tau_h f - f\|_{L^2(0, T-h; B)} \xrightarrow[h \to 0]{} 0,$$
  
:  $(t, x) \mapsto \tau_h f(t, x) := f(t+h, x).$ 

We are going to apply this result to  $F = (W_{\epsilon|\omega})_{\epsilon}$  where  $W_{\epsilon|\omega}$  is the restriction to  $[0,T] \times \omega$  of  $W_{\epsilon}$ . We first show that the set  $\mathcal{F} = \left\{ \int_{t_1}^{t_2} W_{\epsilon|\omega}(t,\cdot) dt : \epsilon > 0 \right\}$  is relatively compact in  $L^2(\omega, \mathbb{R}^m)$  for all  $0 < t_1 < t_2 < T$ . We first observe that  $\mathcal{F}$  is bounded in  $L^2(\omega, \mathbb{R}^m)$  by  $(t_2 - t_1) \|W^0\|_{L^2_x}$  thanks to estimate 20. To show that  $\mathcal{F}$  is relatively compact in  $L^2(\omega, \mathbb{R}^m)$ , it is enough to check the validity the Riesz-Fréchet-Kolmogorov compactness criterion (see [2] remark 13 page 74), *i.e.*,

$$\lim_{h \to 0} \sup_{\epsilon > 0} \left\| \int_{t_1}^{t_2} (W_{\epsilon}(t, x+h) - W_{\epsilon}(t, x)) \, dt \right\|_{L^2_x} = 0.$$
(25)

Note that  $W_{\epsilon,h} := W_{\epsilon}(\cdot, \cdot + h)$  is a solution to the problem 18 associated with the initial condition  $W^0(\cdot + h)$ . Consequently, since the projection is 1-Lipschitz, we have for all  $\phi \in W^{1,\infty}((-\infty,T) \times \mathbb{R}^n, \mathbb{R}^+)$  with compact support in  $]-\infty, T[\times \mathbb{R}^n,$ 

$$\int_0^T \int_{\mathbb{R}^n} \left\langle \partial_t (W_{\epsilon,h} - W_{\epsilon}) + \sum_{j=1}^n B_j \partial_j (W_{\epsilon,h} - W_{\epsilon}), W_{\epsilon,h} - W_{\epsilon} \right\rangle \varphi \, dt \, dx$$
$$= \frac{1}{\epsilon} \int_0^T \int_{\mathbb{R}^n} \left\langle P_K(W_{\epsilon,h}) - P_K(W_{\epsilon}) - (W_{\epsilon,h} - W_{\epsilon}), W_{\epsilon,h} - W_{\epsilon} \right\rangle \varphi \, dt \, dx \le 0,$$

which implies that

$$\int_{0}^{T} \int_{\mathbb{R}^{n}} \left( |W_{\epsilon,h} - W_{\epsilon}|^{2} \partial_{t} \varphi + \sum_{i=1}^{n} \langle W_{\epsilon,h} - W_{\epsilon}, B_{i}(W_{\epsilon,h} - W_{\epsilon}) \rangle \, \partial_{x_{i}} \varphi \right) dt \, dx + \int_{\mathbb{R}^{n}} |W^{0} - W_{h}^{0}|^{2}(x) \varphi(0,x) \, dx \ge 0, \quad (26)$$

where  $W_h^0 = W^0(\cdot + h)$ . Let r > 0, we define the function  $\varphi$  as

$$\varphi(t,x) = \begin{cases} \frac{T-t}{T} + \frac{r-|x|}{nLT}, & \text{if } t \in [0,T] \text{ and } r \le |x| \le r + nL(T-t), \\ \frac{T-t}{T}, & \text{if } t \in [0,T] \text{ and } x \in B(0,r), \\ 0, & \text{otherwise,} \end{cases}$$

where L is the maximum of the spectral radii of the matrices  $B_i$ . We claim that for a.e.  $(t, x) \in [0, T] \times \mathbb{R}^n$ 

$$\left(|W_{\epsilon,h} - W_{\epsilon}|^{2}\partial_{t}\varphi + \sum_{i=1}^{n} \langle W_{\epsilon,h} - W_{\epsilon}, B_{i}(W_{\epsilon,h} - W_{\epsilon}) \rangle \,\partial_{x_{i}}\varphi\right)(t,x) \leq 0.$$

This inequality is obviously satisfied as soon as  $x \in B(0, r)$  and  $t \in [0, T]$ . In the case where  $r \leq |x| \leq r + nL(T - t)$ , we get for all  $1 \leq i \leq n$ 

$$\langle W_{\epsilon,h} - W_{\epsilon}, B_i(W_{\epsilon,h} - W_{\epsilon}) \rangle (t,x) \ge -L|W_{\epsilon,h} - W_{\epsilon}|^2(t,x).$$

Consequently multiplying by  $\partial_{x_i}\varphi$ , it yields

$$\left(\sum_{i=1}^{n} \left\langle W_{\epsilon,h} - W_{\epsilon}, B_i(W_{\epsilon,h} - W_{\epsilon}) \right\rangle \partial_{x_i} \varphi\right)(t,x) \le -|W_{\epsilon,h} - W_{\epsilon}|^2(t,x) \partial_t \varphi(t,x),$$

and the inequality is true also in that case. According to 26, we obtain that

$$\int_{\mathbb{R}^n} |W^0 - W_h^0|^2(x)\varphi(0,x) \, dx$$
  
$$\geq -\int_0^T \int_{B(0,r)} \left( |W_{\epsilon,h} - W_\epsilon|^2 \partial_t \varphi + \sum_{i=1}^n \left\langle W_{\epsilon,h} - W_\epsilon, B_i(W_{\epsilon,h} - W_\epsilon) \right\rangle \, \partial_{x_i} \varphi \right) \, dt \, dx.$$

Thanks to the definition of  $\varphi$ , we get

$$\int_0^T \int_{B(0,r)} |W_{\epsilon,h} - W_{\epsilon}|^2 \, dx \, dt \le T \int_{B(0,r+nLT)} |W^0 - W_h^0|^2 \, dx,$$

and the regularity of  $W_0$  together with [2, Proposition 9.3] yields

$$\int_0^T \int_{B(0,r)} |W_{\epsilon,h} - W_{\epsilon}|^2 \, dx \, dt \le T |h|^2 \int_{\mathbb{R}^n} |\nabla W^0|^2 \, dx.$$

Therefore, 25 holds, and consequently the set  $\mathcal{F}$  is relatively compact in  $L^2(\omega, \mathbb{R}^m)$  for all  $0 < t_1 < t_2 < T$ .

It remains to show that

$$\lim_{h \to 0} \sup_{\epsilon > 0} \|\tau_h W_{\epsilon} - W_{\epsilon}\|_{L^2(0, T-h; L^2(\omega, \mathbb{R}^m))} 0.$$

For all  $\phi \in W^{1,\infty}(]-\infty, T-h[\times \mathbb{R}^n, \mathbb{R}^+)$  with compact support in  $]-\infty, T-h[\times \mathbb{R}^n,$  one has

$$\int_{0}^{T-h} \int_{\mathbb{R}^{n}} \left\langle \partial_{t}(\tau_{h}W_{\epsilon} - W_{\epsilon}), (\tau_{h}W_{\epsilon} - W_{\epsilon}) \right\rangle \varphi \, dt \, dx \\ + \int_{0}^{T-h} \int_{\mathbb{R}^{n}} \left\langle \sum_{j=1}^{n} B_{j} \partial_{j}(\tau_{h}W_{\epsilon} - W_{\epsilon}), (\tau_{h}W_{\epsilon} - W_{\epsilon}) \right\rangle \varphi \, dt \, dx \\ = \frac{1}{\epsilon} \int_{0}^{T-h} \int_{\mathbb{R}^{n}} \left\langle P_{K}(\tau_{h}W_{\epsilon}) - P_{K}(W_{\epsilon}) - (\tau_{h}W_{\epsilon} - W_{\epsilon}), \tau_{h}W_{\epsilon} - W_{\epsilon} \right\rangle \varphi \, dt \, dx \le 0,$$

since the projection is 1-Lipschitz. Arguing as before, we obtain

$$\int_{0}^{T-h} \int_{\mathbb{R}^{n}} |\tau_{h} W_{\epsilon} - W_{\epsilon}|^{2} \partial_{t} \varphi \, dt \, dx$$
$$+ \int_{0}^{T-h} \int_{\mathbb{R}^{n}} \sum_{i=1}^{n} \langle \tau_{h} W_{\epsilon} - W_{\epsilon}, B_{i} (\tau_{h} W_{\epsilon} - W_{\epsilon}) \rangle \, \partial_{x_{i}} \varphi \, dt \, dx$$
$$+ \int_{\mathbb{R}^{n}} |W_{\epsilon}(h, x) - W^{0}(x)|^{2} \varphi(0, x) \, dx \ge 0.$$

12

0

Using a similar test function

$$\varphi(t,x) = \begin{cases} \frac{T-h-t}{T-h} + \frac{r-|x|}{nL(T-h)}, & \text{if } t \in [0,T-h] \text{ and } r \le |x| \le r + nL(T-t), \\ \frac{T-h-t}{T-h}, & \text{if } t \in [0,T-h] \text{ and } x \in B(0,r), \\ 0, & \text{otherwise}, \end{cases}$$

we get that for a.e.  $(t, x) \in [0, T - h] \times \mathbb{R}^n$ ,

$$\left(|\tau_h W_{\epsilon} - W_{\epsilon}|^2 \partial_t \varphi + \sum_{i=1}^n \left\langle \tau_h W_{\epsilon} - W_{\epsilon}, B_i(\tau_h W_{\epsilon} - W_{\epsilon}) \right\rangle \partial_{x_i} \varphi \right)(t, x) \le 0,$$

and then

$$\int_{0}^{T-h} \int_{B(0,r)} |\tau_{h} W_{\epsilon} - W_{\epsilon}|^{2} dx dt \\
\leq (T-h) \int_{B(0,r+nL(T-h))} |W_{\epsilon}(h,x) - W^{0}(x)|^{2} \varphi(0,x) dx \\
\leq T \int_{B(0,r+nLT)} |W_{\epsilon}(h,x) - W^{0}(x)|^{2} dx. \quad (27)$$

The conclusion then follows from the following result whose proof is very close to that of [3, Proposition 7].

**Lemma 4.4.** For all  $\xi \in C_c^{\infty}(\mathbb{R}^n, \mathbb{R}^+)$ , one has

$$\lim_{h \to 0} \sup_{\epsilon > 0} \int_{\mathbb{R}^n} |W_\epsilon(h, x) - W^0(x)|^2 \xi(x) \, dx = 0.$$

According to Theorem 4.3 and estimate 20, the sequence  $(W_{\epsilon})_{\epsilon>0}$  admits a subsequence (not relabeled) which converges strongly in  $L^2([0,T]; L^2(\omega, \mathbb{R}^m))$  to some  $\tilde{W}$  and weakly in  $L^2([0,T]; L^2(\mathbb{R}^n, \mathbb{R}^m))$  to some  $W \in L^2([0,T]; L^2(\mathbb{R}^n, \mathbb{R}^m))$ . By uniqueness of the limit, we infer that  $\tilde{W} = W \in L^2([0,T]; L^2(\mathbb{R}^n, \mathbb{R}^m))$ . Let us take a test function  $\phi \in \mathcal{C}^{\infty}_c([0,T] \times \mathbb{R}^n)$  in 19. Multiplying this inequality by  $\epsilon$  and passing to the limit as  $\epsilon \to 0$  yields  $W = P_K(W)$  a.e. in  $]0, T[\times \mathbb{R}^n$  which shows that  $W \in L^2([0,T]; L^2(\mathbb{R}^n, K))$ . Finally, passing to the limit as  $\epsilon \to 0$  in 24 shows that W is a solution in the sense of Definition 1.1 to the problem 1. Note finally that, by uniqueness of the solution to 1 (see [3, Lemma 9]), there is no need to extract a subsequence to get the above convergences as  $\epsilon \to 0$ .

The construction of the solution W to 1 rests on the assumption that the initial data  $W_0 \in H^1(\mathbb{R}^n, K)$ . Let us now explain how to construct a solution W when  $W_0$  only belongs to  $L^2(\mathbb{R}^n, K)$ . We use here the following result whose proof can be found in [3]

**Theorem 4.5.** Let  $W^0$  and  $\tilde{W}^0 \in H^1(\mathbb{R}^n, K)$ . We denote by W (resp.  $\tilde{W}$ ) the solution in  $L^2(0,T; L^2(\mathbb{R}^n, K))$  to problem 1 in the sense of Definition 1.1 associated with  $W^0$  (resp.  $\tilde{W^0}$ ). Then, W and  $\tilde{W}$  belong to  $\mathcal{C}([0,T]; L^2(\mathbb{R}^n, K))$ , and, in addition, we have the following estimate

$$\forall t \in [0,T], \forall r > 0, \quad \left\| W(t,\cdot) - \tilde{W}(t,\cdot) \right\|_{L^2(B(0,r))} \le \left\| W^0 - \tilde{W^0} \right\|_{L^2(B(0,r+nLT))},$$

where L is the maximum of the spectral radii of the matrices  $B_i$ .

By mollification, let us construct a sequence  $(W_k^0)_{k \in \mathbb{N}}$  such that  $W_k^0 \in H^1(\mathbb{R}^n, K)$ for all  $k \in \mathbb{N}$ , which converges to  $W^0$  in  $L^2(\mathbb{R}^n, K)$ . The estimates of Theorem 4.5 imply that

$$\sup_{t \in [0,T]} \|W_k(t,\cdot) - W_l(t,\cdot)\|_{L^2(\mathbb{R}^n)} \le \|W_k^0 - W_l^0\|_{L^2(\mathbb{R}^n)},$$
(28)

where  $W_k$  (resp.  $W_l$ ) is the solution to 1 associated with the initial condition  $W_k^0$  (resp.  $W_l^0$ ). It follows that the sequence  $(W_k)_{k\in\mathbb{N}}$  is of Cauchy type in  $L^{\infty}(0,T; L^2(\mathbb{R}^n,\mathbb{R}^m))$ , and therefore it converges strongly in  $L^{\infty}(0,T; L^2(\mathbb{R}^n,\mathbb{R}^m))$  to some function  $W \in L^{\infty}(0,T; L^2(\mathbb{R}^n,\mathbb{R}^m))$ . Thanks to the strong convergence, we find that W satisfies the inequality 2. In addition, since  $W_k = P_K(W_k)$  for all  $k \in \mathbb{N}$ , we deduce that  $W = P_K(W)$  which ensures that  $W \in L^{\infty}(0,T; L^2(\mathbb{R}^n,K))$ . The following result has thus been established.

**Theorem 4.6.** Let  $W^0 \in L^2(\mathbb{R}^n, K)$ , then there exists a unique solution  $W \in L^{\infty}(0, T, L^2(\mathbb{R}^n, K))$  to 1 in the sense of Definition 1.1.

5. Conclusion. In definitive, the relaxed problem 18, that was used in [3] to derive formally a definition of weak solutions of hyperbolic constrained problems, is in fact a rigorous way to construct weak solutions of hyperbolic constrained problems. It is worth noting that this relaxation procedure is deeply related to viscoplastic models. In order to fully apply this theory to mechanical problems, one should consider problems that are posed in bounded spatial domain. To do so, a new formulation of weak solutions to Friedrichs' systems posed in bounded domains is proposed in [7], without constraints. It remains now to investigate the interactions between the boundary conditions which are considered in [7] and the convex constraints.

Acknowledgments. The authors wish to express their thanks to Bruno Després for many stimulating conversations. The first and the third authors would also mention that they have benefited from the unrivaled working atmosphere of the 2D24.

#### REFERENCES

- [1] R. A. Adams and J. J. F. Fournier, Sobolev Spaces, 2<sup>nd</sup> edition, Elsevier, Amsterdam, 2003.
- [2] H. Brézis, Analyse fonctionnelle Masson, Paris, 1983.
- [3] B. Després, F. Lagoutière and N. Seguin, Weak solutions to Friedrichs systems with convex constraints, *Nonlinearity*, 24, (2011), 3055–3081.
- [4] L. C. Evans, Partial Differential Equations, 2<sup>nd</sup> edition, American mathematical society, Providence, 2010.
- [5] K. O. Friedrichs, Symmetric positive linear differential equations, Comm. Pure Appl. Math., 11, (1958), 333?418.
- [6] A. Morando and D. Serre, On the L<sup>2</sup>-well posedness of an initial boundary value problem for the 3D linear elasticity, *Commun. Math. Sci.*, 3, 4, (2005), 575–586.
- [7] C. Mifsud, B. Després and N. Seguin, Dissipative formulation of initial boundary value problems for Friedrichs' systems, 2014. jhal-01074542;
- [8] J-J. Moreau, Proximité et dualité dans un espace hilbertien, Bull. Soc. Math. France, 93, (1965), 273–299.
- [9] A. Nouri and M. Rascle, A global existence and uniqueness theorem for a model problem in dynamic elastoplasticity with isotropic strain-hardening, SIAM J. Math. Anal., 26, 4, (1995), 850–868.
- [10] J. Simon, Compact Sets in the Space  $L^p(0,T,B)$ , Annali Mat. Pura Appl., **CXLVI**, (1987), 65–96.
- P-M. Suquet, Evolution problems for a class of dissipative materials, *Quart. Appl. Math.*, 38, 4, (1980), 391–414.

[12] P-M. Suquet, Sur les équations de la plasticité: existence et régularité des solutions, J. Mcanique, 20, 1, (1981), 3–39.

Received xxxx 20xx; revised xxxx 20xx.

*E-mail address*: jean-francois.babadjian@upmc.fr *E-mail address*: mifsud@ljll.math.upmc.fr *E-mail address*: seguin@upmc.fr