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Stability/Instability of Periodic Solutions and Chaos in Physical Models of Musical Instruments

Xavier Rodet

ICMC 94, Aarhus (Denemark) 1994

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Abstract

We study multiple solutions and chaos in simple physical models and their stabilities. A path toward a unique solution can be based upon the low pass character of the linear element. Chaotic signals are shown to cover a large space ranging from a sound perceived as harmonic without noise up to an essentially noisy sound. This is a very important feature for sound synthesis since the synthesis of noise and irregularities of traditional instruments has always been difficult and unsatisfactory. Methods are proposed to find the solutions and their stabilities, and to obtain controllable systems with periodic and chaotic behaviors.

1. Introduction

Real instruments and computer simulations, usually called physical models, often have more than one oscillating solution for a given setting of their parameters [Valimaki et al. 93], [Idogawa 92], [Rodet 93c]. On one hand, it can be argued that this is part of the richness of a musical instrument. On the other hand, unpredictable solutions can render the usage of a physical model rather difficult in a real-time musical performance. Only stable solutions should be considered because unstable ones do not persist. Therefore, the first goal of this paper is to study the multiple solutions of simple physical models and their stabilities. We explain that a path toward such a goal could eventually be based upon the low pass character of the linear element. As a second goal, it would be an interesting achievement to design a system which would model the usual playing behavior of an instrument but could avoid the other behaviors if requested. By the other behaviors, we mean those corresponding to stable solutions other than the usual playing solution. Finally, our last goal is to control the ratio of non sinusoidal components (noise) induced by chaos in synthetic signals, opening a new field of fascinating research and applications. Many systems which have been used as models of physical instruments for music synthesis can produce chaotic signals in some regions of their parameter space. This is related to instability since chaos often appears when all periodic solutions become unstable. The most innovative direction is probably in the area of the control of chaotic behavior of the models and thus of chaotic sounds. In the past it may have seemed that chaotic sounds could not be of any musical interest. To our great surprise we have found the contrary: these signals exhibit very interesting properties, such as a clearly perceived pitch or an intermittent type behavior. We have also found that a signal which is mathematically chaotic can be perceived in a very different way. Actually, chaos can happen within a more or less small neighborhood of a periodic trajectory. It can cover a large space, from a sound perceived as harmonic without noise, up to an essentially noisy sound. This is a very important feature for sound synthesis since the introduction in synthesis of noise and irregularities of natural instruments has always been difficult and unsatisfactory. Our models, by themselves, effectively generate a noise component that is so important for the sound quality of musical instruments. In more general terms it appears that one should not merely "build" models and deliver them to musicians. It is indispensable to go into the understanding of the models, to conceive abstractions of them and to propose explanations helpful to the users. In particular, this comprehension is indispensable for elaborating the control of synthesis models, which are at the same time efficient and musically pertinent.

2. Uniqueness of solutions and low pass feedback loop

We have mentioned in the introduction our interest for the design of a system which would model the usual playing behavior of an instrument but could avoid the other behaviors if requested. For the feedback systems studied here it is conceivable that a path toward such a goal could be based upon the low pass character of the linear element. In [Bergen 71], the low pass character above the oscillation frequency appears among the sufficient conditions for oscillation according to the describing function method. However it does not guarantee the uniqueness of the solution. The low pass character appears also in the formulation of the Hopf theorem in the frequency domain [Mees 79], [Rodet 93c] which gives sufficient conditions for oscillation and uniqueness of the solution.

Similarly, the linear part of Chua's circuit [Chua 90], at least for some values of the parameters, can be viewed as a band pass filter, i.e. a low-pass filter above the frequency of a pair of conjugate poles, this frequency being approximately the oscillation frequency. This suggests some relation with the dominant mode of an acoustic instrument [Rodet 93b]. This idea has also been evoked in [Wawrzynek 83] but was not fully developed. It can be shown [Rodet 93e] that there is a precise correspondence between Chua's circuit and a model of an acoustic instrument where the feedback loop is limited to the first mode(s) (the first mode could easily be replaced here by the dominant mode in the same way). Naturally, the system so obtained has a unique oscillatory solution in a certain range of parameter values. Fig. 1 shows a possible implementation and the control of each partial through a second order section.

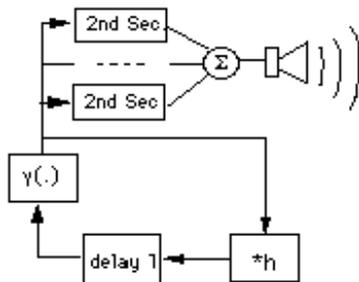


Fig. 1. Feedback loop limited to the first mode(s).

3. NLO systems with chaotic behavior

Many systems which have been used as models of physical instruments for music synthesis can produce chaotic signals in some regions of their parameter space. Chua's circuit [Chua 90] is a very simple circuit which exhibits a surprisingly large variety of bifurcations and chaos. As mentioned in [Madan 92] and [Rodet 92a], signals with period-1, period-2, etc..., can be obtained and lead to *harmonic* sounds. In the case of chaotic signals that can be obtained with some other values, the corresponding sound can be qualified as *noisy*. Some short-time spectra of periodic and chaotic signals from Chua's circuit are shown in [Rodet 93b]. For certain parameter values noisy signals are obtained together with some harmonic sinusoidal components.

Some properties and musical uses of the Time Delayed Chua's circuit [Sharkovsky 93] have been presented in [Rodet 92c] and [Rodet 93d]. Compared to the original Chua's circuit, the time delayed version includes a delay line which is directly related to the structure and physics of many classical musical instruments. This delay allows for much better control of the circuit in terms of musical usage. The delay can also be seen as a stabilizing feedback loop applied to the original circuit. The Time Delayed Chua's Circuit exhibits a very interesting variety of sounds which we have analysed in [Rodet 93c]. This is due to the combination of the rich dynamics of the nonlinear map together with the number of states represented by the delay line. In particular, Sharkovsky et al. [1993] have shown a remarkable period-adding property. In some regions the system has a stable limit cycle with periods 2, 3, 4, etc.... In between every two consecutive stable regions the system exhibits a chaotic behavior. We also observe the simultaneous presence of sinusoidal components and noise in the signal. By a proper affine change of variables, the invariant interval of the nonlinear function used in the circuit can be set to the interval $[0,1]$. For certain parameter values, the function is composed of two segments only in the invariant interval

[Rodet 93a] with slopes l_1 and l_2 . When $l_1 = 1.025$ and l_2 is varied from 1 to 2 and then to 40, we observe the progressive appearance of non sinusoidal components superimposed on the harmonic sinusoidal components (Fig. 2).

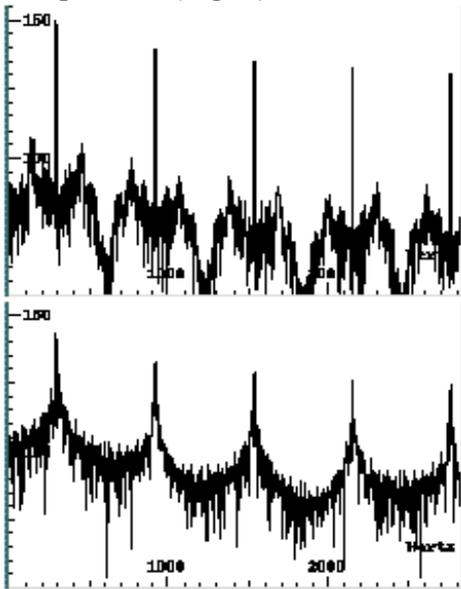


Fig. 2. Spectra showing presence of sinusoidal components and noise, $l_2 = 1.7$ (top) and 30 (bottom).

This property is very promising since it opens the possibility of a precise control of the proportion of noise added to the harmonic components, which is essential for a musical usage. This is the typical property that we want to extend to nonlinear functions and systems beyond the very specific two segment function of Sharkovsky's case.

We focus here on the class of systems composed of an instantaneous nonlinearity and a linear feedback loop, which is a valid *basic* model of reed instruments when the mass of the reed is considered negligible. But in the case of the trumpet [Rodet 92b] for example, this assumption cannot be maintained: as we noticed in [Rodet 93c], the clarinet and trumpet would then be represented by the same model. On the contrary, in [Rodet & Steinecke 94a] we propose a *minimal* one mass model of the trumpet in the form of a two-loop system. This system can also exhibit chaotic behaviors.

For speech production, various models have been proposed. The most common is a *two mass model of vocal folds* [Flanagan 72]. Using the impedance of the opening in the glottis and the driving point impedance of the vocal tract, air flow pressure and velocity in the glottis are computed and used to drive two coupled non-linear oscillators representing a vibrating vocal fold. This model usually exhibits bifurcations and chaos often related to the desynchronisation of lower order modes, e.g. of lateral and vertical motion, or to a left-right asymmetry [Steinecke & Herzel 93], [Rodet & Steinecke 94b].

4. Stability of 2-periodic solutions of a

class of systems We consider the following equations which $x_n = h * g(x_{n-T})$ (1), or $x_n = g(h * x_{n-T})$ (2), where h is the impulse response of a filter in the linear feedback loop, $*$ is the convolution operator and $g(\cdot)$ is the nonlinear function representing for instance the influence of the reed in a basic clarinet model [Rodet 92a]. The system can be decomposed into an instantaneous nonlinearity $g(\cdot)$ and a linear element including h and a delay. This defines the class of single feedback loop systems with an instantaneous nonlinearity of which we can easily determine the stability and some oscillation properties [Rodet 93c]. Note that the only restriction on the linear element is that its impulse response be *stable*. In particular the continuous transfer function of the linear element does not need to be a rational function and thus can include delays. Many systems can be redesigned to fall into this class. Equation (1) can also be written:

$$x_n = \sum_{m=-m_1}^{m_2} h_m \gamma(x_{n-m-T}) \quad (3)$$

(3) defines a map $Q: \mathbb{R}^{m_2+T+1} \rightarrow \mathbb{R}^{m_2+T+1}$. Since we are dealing with physical systems, it is natural to suppose that $\{h_m\}$ is causal, $m_1 < T$ and finite and $m_2 < \infty$. The sample n depends on the past samples $n-$

$m_2 - T$ to $n + m_1 - T$. Therefore the set of samples from $n - m_2$ to $n + T - 1$ depends only on the values $x_{n - m_2 - T}$ to $x_{n + m_1 - 1}$, themselves depending on the values $x_{n - m_2 - T}$ to $x_{n - 1}$ only which defines the map Q . Such a system may have many periodic solutions. Since they are the most important for the normal playing conditions of a musical instrument, we study here only 2-periodic solutions, i.e. fixed points of $Y = Q \circ Q$ with period length about $2T$. But finding the 2-periodic solutions of Q and their stabilities in general is not easy. Therefore we are going to introduce another system. Let $\{s_n\}$ be a 2-periodic solution of (3). Assume now that $\{h_n\}$ is even symmetric. We have noticed in [Rodet 93c] that in this case the period 2 is of length $2T$ exactly, i.e. such that $s_n = s_{n+2T}$ for all integer n . Let us take the Fourier series of one period of s_n and of $g(s_n)$ and call $S = \{S_k\}$ and $G(S) = \{G_k(S)\}$ respectively their Fourier coefficients which are vectors of length $2T$. Then for $\{s_n\}$ equation (3) can be written:

$$S_k = (-1)^k H_k G_k(S) \quad (4),$$

where the term $(-1)^k$ accounts for the time delay T in (3), and $H = \{H_k\}$ is the Fourier series taken after time aliasing of $\{h_n\}$ on an interval of length $2T$ and overlap-adding.

From equation (4) we define a map $M: B_{2T} \rightarrow B_{2T}$ where $B \subset C$ is the set of Fourier series of real periodic discrete signals of period length $2T$:

$$M_k(S) = (-1)^k H_k G_k(S), \quad k=1,2,\dots,2T \quad (5)$$

where $G_k(S)$ is the k th Fourier coefficient of $g(s)$ when s is the inverse Fourier series of S . In the time domain, we call $P: R_{2T} \rightarrow R_{2T}$ the map associated with M and defined by:

$$P(s) = \text{Fourier}^{-1}(M(S)) \quad (6)$$

where *Fourier*⁻¹ designates the inverse Fourier series, or equivalently:

$$P(s) = h * g(s - T) \quad (7),$$

where $h = \text{Fourier}^{-1}(H_k)$ and $*$ is the circular convolution. The reason for introducing the maps M and P is that they have the same fixed points as $Y = Q \circ Q$. They are simpler than Y in the sense that they have smaller dimension and that they map exactly the signal of one time interval of length $2T$ into the signal of the next time interval. The dynamical system that they define is different from the original one and we call it the *period-after-period* (p.a.p.) system as opposed to the original one (1). Let DY_x be the derivative of Y at the point $x \in R_{m_2+T+1}$. The stability of a fixed point s of Y is given by the eigenvalues of DY_s . The problem is therefore to find all the fixed points of Y or equivalently of P and M .

5. Case of a polynomial nonlinearity

The case where g is a polynomial has interesting properties. The first one is that it is easy to implement and can be efficiently computed on a digital processor. The control of the important characteristics of g is easy. The slope of g at a fixed point, assumed to be at the origin, controls the oscillating/non oscillating behavior of the system [Rodet 93b]. This slope is easily changed by a modification of the linear term of the polynomial. Another property is that the equation (1) or (2) leads to a system of $m_2 + T - 1$ polynomial equations in $m_2 + T - 1$ variables, and that the map P becomes a system of $2T$ polynomials in $2T$ variables. But this is also true for the map M associated with P in the Fourier domain: Note first that the Fourier series of x^2 , $x \in R_{2T}$, is the circular convolution $X * X$ which is a polynomial in the variables $\{X_k, k=1,2,\dots,2T\}$. Then $G_k(X)$ is also a polynomial in the same variables and (5) for $k=1,2,\dots,2T$ defines a system of $2T$ polynomial equations.

The portion of g which we are interested in for easy control of oscillations is that of negative slope around the origin and greater slope further from the origin [Rodet 92a]. A drawback of the choice of a polynomial g is that it is not possible to avoid the existence of other points of functioning than the origin, which is a serious inconvenience for musical usage [Rodet 93d]. We will see in section 10 that a rational function allows us to avoid that difficulty and can be studied with the same tools which we are using for the polynomials.

6. Solutions of the p.a.p. system: a homotopy method

In general, it is possible to find the solutions of a system of polynomial equations by a homotopy method [Chow 79]. For two maps M_0 and $M_1: R_{2T} \rightarrow R_{2T}$, a homotopy is a smooth map $F: R_{2T} \times [0,1] \rightarrow R_{2T}$,

such that $F(\cdot, 0)$ is $M_0(\cdot)$ and $F(\cdot, 1)$ is $M_1(\cdot)$. More precisely, we are looking for the zeros of the system of polynomials $M_1(s) = s - P(s)$. Suppose that $M_0(s)$ has known zeros, then under certain conditions, it is possible to follow the trajectories of the zeros of $F(s, r)$ starting from the zeros of $M_0(s)$ and going to those of $M_1(s)$ when r is varied from 0 to 1. However this is possible provided that no such zero escape to infinity for some value of r in the interval $[0, 1]$ and that 0 is a regular value of F on $\mathbb{R}^{2T} \times [0, 1]$. The first requirement is easily fulfilled by making sure that the coefficients of the highest terms of F in s_n do not vanish. The second requirement guarantees that $F^{-1}(0)$ is a submanifold of dimension 1, i.e. consists only of arcs from a zero of $M_0(s)$ to a zero of $M_1(s)$, but it may not be fulfilled for some trajectories in our case. In our problem, an interesting choice for M_0 is the one for which, in the equation (3), $h_0 = 1$ and $h_m = 0$ for $m > 0$, $M_0(s) = s - g(s)$, i.e. there is no filtering in the feedback loop. For simplicity we assume from now on that $g(\cdot)$ is odd symmetric. Let W be the set of solutions of:

$$g(u) = -u, u \in \mathbb{R}. \quad (8)$$

Among the solutions of $M_0(s) = s - g(s) = 0$, $s \in \mathbb{R}^{2T}$, we retain only the vectors s such that $s_n \in W$ for $n = 1, 2, \dots, 2T$. The other solutions imply a value $x \in \mathbb{R}$, $x \neq 0$, such that $g(x) = x$ which we want to avoid since it corresponds to non-zero fixed points of the original system, which give no sound. Note that for the solutions which we retain $s_n = -s_{n-T}$ for $n = T+1, T+2, \dots, 2T$. When a signal s_n has this odd symmetry of the period with respect to its middle, we say that the signal is *odd harmonic* (o.h.) because its even harmonics are null. The initial solutions for the homotopy are easy to build since they are the combinations of values from W . Another reason for this choice of M_0 is as follows. The homotopy can simply be chosen as:

$$F(s, r) = (1-r)M_0(s) + rM_1(s). \quad (9)$$

Therefore, when varying r from 0 to 1, we follow the evolution of the solutions according to the increase of low-pass filtering in the feedback loop, which is one of the main goals we mentioned earlier.

7. Example of the homotopy method

Let us detail the application of the method on a very simple example. Let $T=3$ and $g(x) = x^3 + a_1x$ for $a_1 = 1.1$. We also choose the most simple low-pass filter, $h_{-1} = h_1 = 0.25$, $h_0 = 1 - 2h_1$, otherwise $h_m = 0$. More precisely, this means that when the homotopy variable r goes from 0 to 1, (h_0, h_1) goes from $(1, 0)$, i.e. no filtering, to $(0.5, 0.25)$, progressively applying more and more low-pass filtering. The solutions of $x^3 + a_1x = -x$ are $\{0, \pm\sqrt{-a_1 - 1}\} = W$. The initial solutions which we retain for the homotopy are obtained as the o.h. signals with sample values in W , so that $s - g(s) = 0$. Two solutions are equivalent if they are identical by sign inversion or time delay. Then, apart from the identically null solution which is irrelevant, there are only 5 equivalence classes of solutions. These are shown in Fig. 3 and are designated by the letters A, B, C, D , and E .

Starting from one of these initial solutions and from $r=0$, we increase r by a small Δr , estimate the new solution by the Newton method from the previous solution and iterate this procedure until $r=1$. Using prefixed superscripts to index successive steps of the algorithm, let $0s$ be the initial solution, rs the solution estimated for the value r of the homotopy variable. The solution $r+\Delta r s$ is obtained as follows. Let $z_0 = rs$ be the initial value for the Newton method, and DF the differential of F . We iterate:

$$z_{k+1} = z_k - [DF(z_k, r+\Delta r)]^{-1} F(z_k, r+\Delta r), \quad (10)$$

until when $z_{k+1} - z_k$ is sufficiently small. Then $r+\Delta r s$ gets the last value z_{k+1} . It is easy to derive expressions for the coefficients of the polynomials of F and to show that these coefficients never vanish for r in $[0, 1]$ so that no solutions escape to infinity.

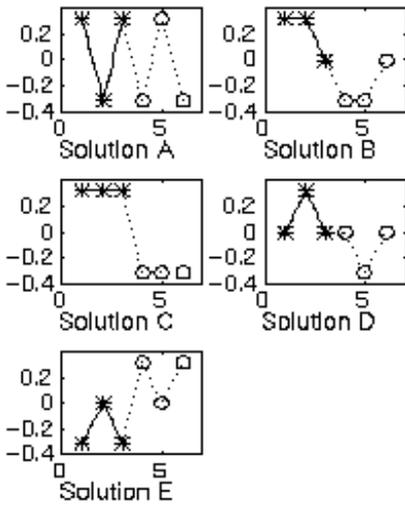


Fig. 3. Equivalence classes of initial solutions

Fig. 4 shows the successive values of rs for the equivalence class C , $a_1 = -1.35$ and r from 0 to 1. Note that $1s$ is a solution of the original system (1), which is otherwise not easy to find in general. However, in case of class A the numerical estimation procedure does not succeed to go further than $r = 0.965$ for $a_1 = -1.1$. The reason is that the arcs issued from two different initial solutions join together at that value of r and disappear for larger values as explained in section 9. But as long as we can follow a real solution s , we can find its stability by computing the eigenvalues of DY_s . Since $Y = Q \cdot Q$ and the solutions under study are o. h., it is sufficient to compute the eigenvalues of Q and to compare them to unity. Solution A for instance rapidly becomes unstable.

8. Odd harmonic (o. h.) solutions

We have mentioned in section 6 that the initial solutions of the homotopy have an o. h. symmetry. We now show that this is true for the trajectories A to E . These curves are sets of points where $F(s, r) = 0$. Therefore they are integral curves of the ordinary differential equation:

$$D_s \Phi(s, \rho) \frac{ds}{d\lambda} + D_\rho \Phi(s, \rho) \frac{d\rho}{d\lambda} = 0 \quad (11)$$

where D_s and D_ρ denote partial derivatives and l denotes arc length. If $D_s F(s, r)$ is non singular, then:

$$\frac{ds}{d\lambda} = -[D_s \Phi(s, \rho)]^{-1} D_\rho \Phi(s, \rho) \frac{d\rho}{d\lambda} \quad (12)$$

It is straightforward to verify that when s is o. h., $D_\rho \Phi(s, \rho)$ is o. h. also. $[D_s \Phi(s, \rho)]^{-1}$ preserves this property, which implies that $[D_s \Phi(s, \rho)]^{-1} D_\rho \Phi(s, \rho)$ is o. h. as well. From (12) it is then clear that the curves A to E are composed of o. h. solutions. In consequence, any solution issued from the initial ones is o. h.. We have observed in [Rodet 93f] that a slight breaking of the odd symmetry of g can lead to the appearance of even partials, with large amplitudes if $\{h_n\}$ is not even symmetric.

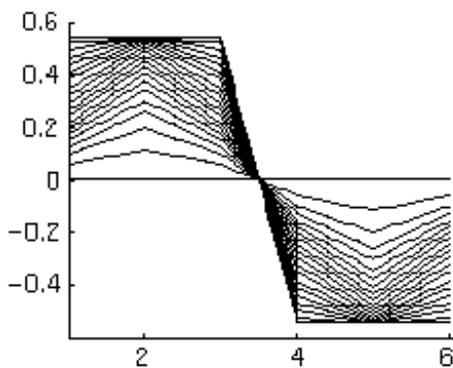


Fig. 4. Successive values of rs for the equivalence class C of solutions, $a_1 = -1.35$ and r from 0 to 1.

9. Direct computation on the example

The method consists of finding the equation of manifolds on which the solutions lie and then finding the intersections of these manifolds as 1-dimensional manifolds in a parametric form so that they can easily be explored for solutions. Here we use the o. h. property of the solutions to find the solutions analytically in the previous example. The system of polynomial equations (7) is:

$$x_1 = h_1 g(x-3) + h_0 g(x-2) + h_{-1} g(x-1) \quad (13)$$

$$x_2 = h_1 g(x-2) + h_0 g(x-1) + h_{-1} g(x_0) \quad (14)$$

$$x_3 = h_1 g(x-1) + h_0 g(x_0) + h_{-1} g(x_1) \quad (15)$$

Using the periodicity of the solution x , the symmetry $h_{-1}=h_1$, and the odd symmetry of g , we can rewrite the equations in x_1 , x_2 and x_3 :

$$x_1 = -h_0 g(x_1) - h_1 g(x_2) + h_1 g(x_3) \quad (16)$$

$$x_2 = -h_1 g(x_1) - h_0 g(x_2) - h_1 g(x_3) \quad (17)$$

$$x_3 = h_1 g(x_1) - h_1 g(x_2) - h_0 g(x_3) \quad (18)$$

This can be written in matrix notation $x = N g(x)$, (19), i.e. $N^{-1}x = g(x)$, (20) where:

$$N^{-1} = \begin{bmatrix} \xi_0 & -\xi_1 & \xi_1 \\ -\xi_1 & \xi_0 & -\xi_1 \\ \xi_1 & -\xi_1 & \xi_0 \end{bmatrix} \quad (21)$$

$$\text{with } \xi_0 = \frac{h_0^2 - h_1^2}{\det(N)}, \text{ and } \xi_1 = \frac{h_0^2 + h_0 h_1}{\det(N)}$$

Using $g(x) = x^3 + a_1 x$, equation (20) leads to

$$x_1^3 - (g_0 - a_1)x_1 + g_1 x_2 - g_1 x_3 = 0 \quad (22)$$

$$x_2^3 + g_1 x_1 - (g_0 - a_1)x_2 + g_1 x_3 = 0 \quad (23)$$

$$x_3^3 - g_1 x_1 + g_1 x_2 - (g_0 - a_1)x_3 = 0 \quad (24)$$

Let $a = g_0 - a_1 - g_1$. Using two equations to eliminate one variable, we get:

$$(x_1 + x_2)(x_1^2 + x_2^2 - x_1 x_2 - a) = 0 \quad (25)$$

$$(x_1 - x_3)(x_1^2 + x_3^2 - x_1 x_3 - a) = 0 \quad (26)$$

The two left hand terms of (25) and (26) define a plane and a cylinder with an ellipsoid basis provided $a > 0$. Each ellipse can be described by a parametric equation with $r = 2\sqrt{a/3}$:

$$x_1 = r \cos(w + p/6), x_2 = r \cos(w - p/6) \quad (27)$$

$$x_1 = r \cos(m + p/6), x_2 = r \cos(m - p/6) \quad (28)$$

The solutions are found on the intersection of the planes and cylinders. The intersection of the cylinders

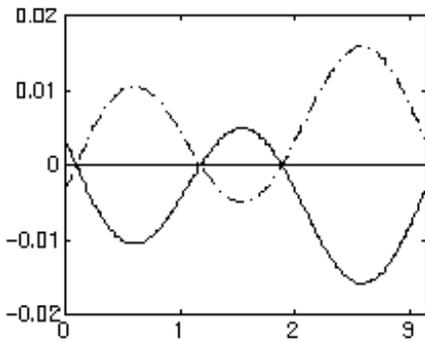


Fig. 5. Left hand term of (22) to (24) versus w .

is a curve obtained with $m = -w$ or $m = w - p/3$. Then we can follow this curve by using the parameter w and evaluating left hand terms of (22) to (24) for zero crossings. The values of these terms versus w from 0 to p are shown on Fig. 5 for $a_1 = -1.1$, $h_1 = 0.98$ and $h_0 = 1 - 2h_1$ (two terms are identical). We observe common zero crossing for three values of w which correspond to the solutions designated previously by A , D and E . The value of the left hand term of (22) versus w is shown on Fig. 6 for $h_1 = 1 - h_1/2$ and $h_0 = 1., 0.98, 0.965, \text{ and } 0.955$. This explains how solutions designated by D and E join together when h_1 reaches 0.965 and then disappear as discussed in section 7. This occurs when a becomes negative because the ellipses collapse, leading to imaginary solutions. The other solutions are easily found in the intersections of the planes with the ellipses or between the planes. Finally we get the classes A to E previously found (Fig. 7).

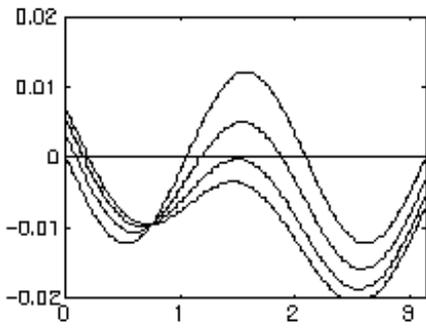


Fig. 6. Left hand term of (22) $h_0 = 1, .98, .965, .955$.

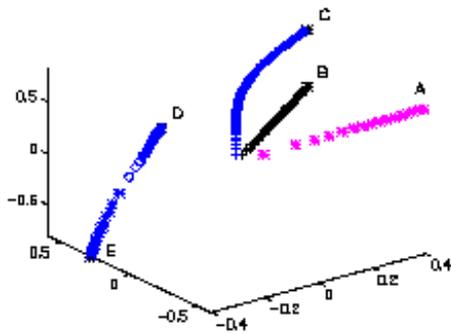


Fig. 7. The five classes of solutions of the example

for $h_0 = 1$ to 0 .

10. A rational function nonlinearity

As we mentioned in section 5, the choice of a polynomial g forces the existence of other points of functioning than the origin, which is serious inconvenience for a musical usage [Rodet 93d]. We present here a rational function nonlinearity which avoids that difficulty and can be studied with the same tools which we are using for the polynomials. The most simple polynomial for our purpose is the one which we have looked after in the previous sections, $g(x) = x^3 + a_1x$. Apart from the origin, it intersects the line $y=x$ at two fixed points:

$$x = \pm \sqrt{1 - a_1} \quad (29)$$

which in general are unstable. Furthermore, as $x \rightarrow \infty$, the slope of $g \rightarrow \infty$, which implies that the system can become unstable and escape to infinity, this is rather annoying for a musical instrument.

We correct the behavior of g by dividing the polynomial by another polynomial of degree less or equal to 3. With an even degree the odd symmetry is kept :

$$g(x) = (x^3 + a_1x)/(1 + b_2x^2) \quad (30)$$

Then the system (1) has no spurious fixed points provided $b_2 > 1$ and can be studied with the same tools as the polynomial case. This function is shown on Fig. 8 for $b_2 = 2$ and $a_1 = -1.7$. To keep things simple, let us

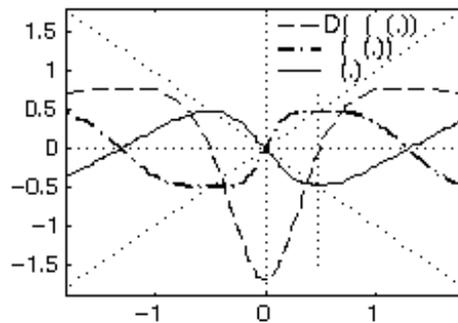


Fig. 8. Rational

consider the case where there is no filtering in the function g for $a_1 = 1.8$ and $b_2 = 2$

feedback loop ($h_0 = 1, h_1 = 0$). Then the first 2-periodic solutions of (1) oscillate between points where $g(x) = -x$:

$$x = \pm \sqrt{(-1-a_1)/(1+b_2)} \quad (31)$$

Such a solution is stable if the derivative of $(g \circ g)(x)$:

$$(g \circ g(x))' = g'(g(x))g'(x) = g'(-x) \cdot g'(x) = -(g'(x))^2 \quad (32)$$

is less than 1 in modulus at these points. This happens when $-4 < a_1 < -1$. Fig. 8 shows $g(x)$, $(g \circ g)(x)$ and $(g \circ g(x))'$ for $a_1 = -1.7$. But for approximately $-6.15 < a_1 < -4$, we get two 2-periodic solutions. Finally chaotic solutions can be obtained for $a_1 < -7.2$ approximately as shown on Fig. 9.

11. Conclusion

We have studied the class of single feedback loop systems with a single instantaneous nonlinearity. It appears that both the number of solutions and the number of stable solutions can be largely reduced by the presence of a low-pass filter in the feedback loop. For a system in the defined class we have defined an equivalent but simpler system named the period-after-period system which has the same 2-periodic solutions. In the case of a polynomial nonlinearity, it is possible to follow the solutions from those of a trivial system, i.e. the one without a filter in the feedback loop, to the solutions of the system with a specific filter. Having determined all the 2-periodic solutions, it is easy to compute their stabilities. Finally we have proposed that the nonlinearity be implemented as a rational function. This avoids any spurious functioning point and guarantees that the oscillation of the system does not grow to infinity. Furthermore, this rational function system exhibits a chaotic behavior with the simultaneous presence of harmonic sinusoidal components and of non sinusoidal components which are heard as *noise* added to the harmonic tone. This is a first step towards the control of physical models.

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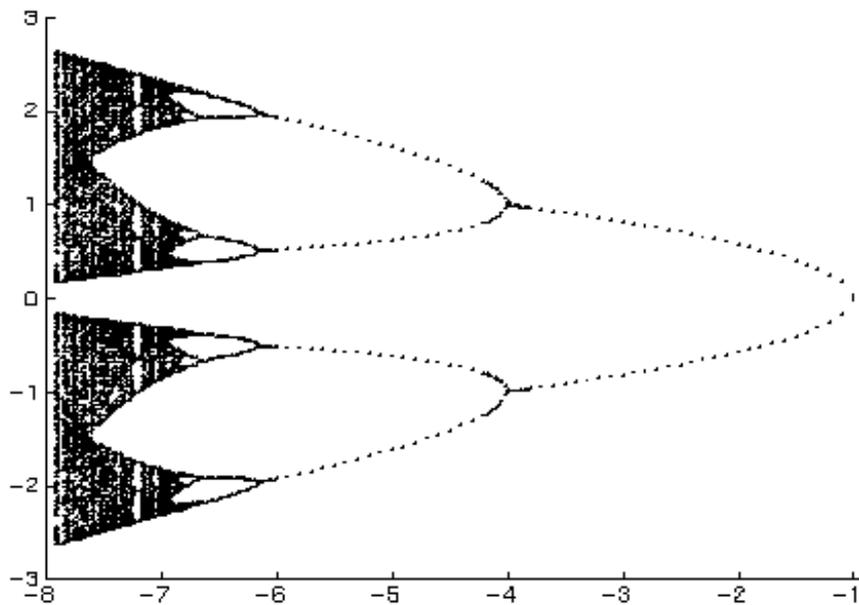


Fig. 9. Bifurcation diagram of the rational function nonlinearity versus parameter a_1 .

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