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# Phase segregation for binary mixtures of Bose-Einstein Condensates

M. Goldman\*      B. Merlet†

## Abstract

We study the strong segregation limit for mixtures of Bose-Einstein condensates modelled by a Gross-Pitaievskii functional. Our first main result is that in presence of a trapping potential, for different intracomponent strengths, the Thomas-Fermi limit is sufficient to determine the shape of the minimizers. Our second main result is that for asymptotically equal intracomponent strengths, one needs to go to the next order. The relevant limit is a weighted isoperimetric problem. We then study the minimizers of this limit problem, proving radial symmetry or symmetry breaking for different values of the parameters. We finally show that in the absence of a confining potential, even for non-equal intracomponent strengths, one needs to study a related isoperimetric problem to gain information about the shape of the minimizers.

## 1 Introduction

In this paper, we investigate the asymptotic behavior as  $\varepsilon$  goes to zero of the Gross-Pitaievskii functional

$$F_\varepsilon(\eta) = \varepsilon \int_{\mathbf{R}^2} |\nabla \eta_1|^2 + |\nabla \eta_2|^2 + \frac{1}{\varepsilon} \int_{\mathbf{R}^2} \frac{1}{2} \eta_1^4 + \frac{g}{2} \eta_2^4 + K \eta_1^2 \eta_2^2 + (\eta_1^2 + \eta_2^2) V. \quad (1)$$

This functional arises in the study of two component Bose-Einstein condensates. It has been widely studied, both in the physical and mathematical literature (see [24] and the references therein or the book [2]). The potential  $V$  is a trapping potential. In this paper, we will consider for simplicity only the harmonic potential  $V = |x|^2$ . The constant  $g$ , measures the asymmetry between the intracomponent repulsive strengths of each component and  $K$  represents the intercomponent repulsive strength. Without loss of generality, we will take here  $g \geq 1$ . The case  $K < \sqrt{g}$ , where mixing of the two condensates occurs has recently been well understood in [3]. On the other hand, the case  $K > \sqrt{g}$ , where it is expected both experimentally [32, 38], numerically [31, 37] and theoretically [27, 8, 40, 9, 41] that segregation occurs, is maybe not yet so well understood. In the symmetric case  $g = 1$ , it has been proved in [4, 24] that, as expected from the physics literature, the Thomas-Fermi limit i.e. the limit of  $\varepsilon F_\varepsilon$  only imposes segregation but does not give any information about the actual shape of the minimizers. To gain such information, one has to go to the next order since  $F_\varepsilon - \min_\eta F_\varepsilon(\eta)$  converges in the sense of  $\Gamma$ -convergence [13] to a weighted isoperimetric problem. In this paper, we focus on the asymmetric case  $g > 1$  and show that the situation is radically different.

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Let us introduce the Thomas-Fermi energy:

$$E(\rho) = \int_{\mathbf{R}^2} \left[ \frac{1}{2} \rho_1^2 + \frac{g}{2} \rho_2^2 + K \rho_1 \rho_2 + (\rho_1 + \rho_2) V \right]. \quad (2)$$

Our first main theorem is the following:

**Theorem 1.1.** *For every  $\alpha_1, \alpha_2, g, K > 0$ , with  $K > \sqrt{g} > 1$ , there exists a unique minimizer  $\rho^0 = (\rho_1^0, \rho_2^0)$  of (2) under the volume constraints*

$$\int_{\mathbf{R}^2} \rho_1 = \alpha_1 \quad \text{and} \quad \int_{\mathbf{R}^2} \rho_2 = \alpha_2. \quad (3)$$

*This minimizer  $\rho^0$  is radially symmetric (see Lemma 2.1 for explicit formulas for  $\rho^0$  and  $E_0 = E(\rho^0)$ ).*

*Moreover, we have the following stability result: there exists  $C > 0$  (which depends only on  $\alpha_1, \alpha_2$  and  $g$ ) such that if  $\rho$  satisfies the constraints (3) then<sup>1</sup>*

$$\|\rho - \rho^0\|_1 \leq C \sqrt{E(\rho) - E(\rho_0)}. \quad (4)$$

As a consequence of this stability result, we prove that minimizers of  $F_\varepsilon$  converge to  $\sqrt{\rho^0}$ .

**Theorem 1.2.** *There exists  $C > 0$  such that for  $\varepsilon \in (0, 1]$ , any minimizer  $\eta^\varepsilon$  of  $F_\varepsilon$  under the constraints*

$$\int_{\mathbf{R}^2} \eta_i^2 = \alpha_i \quad (5)$$

*satisfies*

$$\left\| \eta^\varepsilon - \left( \sqrt{\rho_1^0}, \sqrt{\rho_2^0} \right) \right\|_2 \leq C \varepsilon^{1/4}. \quad (6)$$

This theorem establishes that in the non-symmetric case, the Thomas-Fermi limit already provides full information on the limiting behavior of the minimizers of (1). It is quite surprising that even without using isoperimetric effects, we are able to obtain strong convergence of the minimizers in the form of (6). Let us point out that the idea of using stability inequalities such as (4) to get (quantitative) convergence results is far from new (see for instance [14, 10]). One crucial point which explains the difference between the asymmetric case and the symmetric one is that here, there is a gap between the two Thomas-Fermi profiles  $\rho_1^0$  and  $\rho_2^0$  in the sense that there exists  $r_0 > 0$  such that  $\text{supp } \rho_1^0 \subset \overline{B(0, r_0)}$  and  $\text{supp } \rho_2^0 \subset \mathbf{R}^2 \setminus B(0, r_0)$  with

$$\inf_{r < r_0} \rho_1^0(r) > \sup_{r > r_0} \rho_2^0(r).$$

We then study the crossover case where  $g = 1 + \varepsilon \xi$  for some  $\xi > 0$ . Let  $\bar{\eta}_\varepsilon$  be the minimizer of

$$G_\varepsilon(\eta) = \varepsilon \int_{\mathbf{R}^2} |\nabla \eta|^2 + \frac{1}{\varepsilon} \int_{\mathbf{R}^2} \frac{1}{2} \eta^4 + V \eta^2$$

under the mass constraint

$$\int_{\mathbf{R}^2} \eta^2 = \bar{\alpha},$$

where  $\bar{\alpha} = \alpha_1 + \alpha_2$ . It is well known [25, 26] that  $\bar{\eta}_\varepsilon^2$  converges when  $\varepsilon$  goes to zero to the Thomas-Fermi profile

$$\bar{\rho} = (R^2 - V)_+$$

where  $R$  is chosen such that  $\int_{\mathbf{R}^2} \bar{\rho} = \bar{\alpha}$ . Building on results obtained in the case  $g = 1$  [24, 4], we get

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<sup>1</sup>we denote by  $\|\cdot\|_p$  the  $L^p$  norm

**Theorem 1.3.** For  $K > 1$ , the functional  $(F_\varepsilon - G_\varepsilon(\bar{\eta}_\varepsilon))$   $\Gamma$ -converges as  $\varepsilon$  goes to zero to

$$\mathcal{G}_\xi(u_1, u_2) = \begin{cases} \sigma_K \int_{\partial E} \bar{\rho}^{3/2} + \xi \int_{E^c} \bar{\rho}^2 & \text{if } u_1 = \bar{\rho}\chi_E, u_2 = \bar{\rho}\chi_{E^c} \text{ and } \int_E \bar{\rho} = \alpha_1, \\ +\infty & \text{otherwise,} \end{cases}$$

where  $\sigma_K$  is defined by the one dimensional optimal transition problem

$$\sigma_K = \inf \left\{ \int_{\mathbf{R}} |\eta'_1|^2 + |\eta'_2|^2 + \frac{1}{2} (\eta_1^2 + \eta_2^2 - 1)^2 + (K-1)\eta_1^2\eta_2^2 : \lim_{-\infty} \eta_1 = 0, \lim_{+\infty} \eta_1 = 1 \right\}.$$

We study the minimizers of the limiting functional  $\mathcal{G}_\xi$  and prove our second main result.

**Theorem 1.4.** The following holds:

- there exists  $\alpha_0 \in (0, \bar{\alpha}/2]$  such that for every  $\alpha_1 \in (\alpha_0, \bar{\alpha} - \alpha_0)$  there exists  $\xi_{\alpha_1}^1$  such that the minimizer of  $\mathcal{G}_\xi$  is not radially symmetric for  $\xi \leq \xi_{\alpha_1}^1$ ,
- for every  $\alpha_1 \in (0, \bar{\alpha})$ , there exists  $\xi_{\alpha_1}^2$  such that the minimizer of  $\mathcal{G}_\xi$  is the centered ball for  $\xi \geq \xi_{\alpha_1}^2$ .

The first part of the theorem is a consequence of a symmetry breaking result from [4]. The second part follows from a combination of two results. The first is a stability result for the functional  $\int_{E^c} (\bar{\rho})^2$ :

**Proposition 1.5.** For every  $\alpha \in (0, \bar{\alpha})$ , there exists  $C = C(\alpha) > 0$  such that for every measurable set  $E \subset \mathbf{R}^2$  with  $\int_E \bar{\rho} = \alpha$ , we have,

$$\int_{E^c} \bar{\rho}^2 - \int_{B_r^c} \bar{\rho}^2 \geq C \left( \int_{E^c \Delta B_r^c} \bar{\rho} \right)^2$$

where  $r$  is such that  $\int_{B_r} \bar{\rho} = \alpha$ .

The second is an estimate on the potential instability of the ball for the weighted isoperimetric problem.

**Proposition 1.6.** For every  $\alpha \in (0, \bar{\alpha})$ , there exists  $c = c(\alpha) > 0$  such that for every set  $E \subset \mathbf{R}^2$  with locally finite perimeter and with  $\int_E \bar{\rho} = \alpha$ , there holds

$$\int_{\partial E} \bar{\rho}^{3/2} - \int_{\partial B_r} \bar{\rho}^{3/2} \geq -c \left( \int_{E \Delta B_r} \bar{\rho} \right)^2, \quad (7)$$

where  $r$  is such that  $\int_{B_r} \bar{\rho} = \alpha$ .

The rigidity result given by Theorem 1.4 is similar in spirit to several rigidity results obtained for variants of isoperimetric problems (see [28, 20, 23] for instance). However, the peculiar aspect here is that this rigidity does not come from the isoperimetric term but rather from the volume term. Nevertheless, the proof of (7) bounding the *instability* of the ball follows the strategy of [15] (see also [1, 20]) to prove quantitative *stability* estimates for isoperimetric problems. The idea is to show first the desired inequality for nearly spherical sets following ideas of [21] and then use the regularity theory for minimal surfaces to reduce oneself to this situation. To the best of our knowledge, it is the first time that this strategy has been implemented to control the *instability* of the ball. Let us also point out that one of the ingredients in our proof is the following isoperimetric inequality:

**Lemma 1.7.** *There exists a constant  $c > 0$  such that for every measurable set  $E \subset \mathbf{R}^2$  satisfying  $\int_E \bar{\rho} \leq \alpha/2$ , there holds*

$$\int_{\partial E} \bar{\rho}^{3/2} \geq c \left( \int_E \bar{\rho} \right)^{5/6}.$$

In recent years, there has been an increasing interest in studying isoperimetric problems with densities (see [35, 20, 17] for instance). However, most of these authors consider either problems where the same density is used for weighting the volume and the perimeter or weights which are increasing at infinity.

In the last part of the paper we come back to the situation  $g > 1$  but consider an infinitely stiff trapping potential. That is, we assume that  $V$  is equal to zero inside some given open set  $\Omega$  and is infinite outside. This is somehow the setting which is considered in [41, 42, 36, 11]. In this case, it is easier to work with slightly different parameters. After a new rescaling and some simple algebraic manipulations (see Section 4), the problem can be seen to be equivalent to minimizing

$$J_\varepsilon(\eta) = \varepsilon \int_\Omega |\nabla \eta_1|^2 + \lambda^2 |\nabla \eta_2|^2 + \frac{1}{\varepsilon} \int_\Omega \frac{1}{2} (\eta_1^2 + \eta_2^2 - 1)^2 + (K - 1) \eta_1^2 \eta_2^2 \quad (8)$$

with the volume constraint

$$\int_\Omega \eta_1^2 = \alpha_1 \quad \text{and} \quad \int_\Omega \eta_2^2 = \alpha_2,$$

for some  $\lambda \leq 1$ ,  $K > 1$  and  $\alpha_1, \alpha_2 \geq 0$  such that  $\alpha_1 + \alpha_2 = |\Omega|$ . The main difficulty in studying the  $\Gamma$ -convergence of (8) to a sharp limit model is to obtain strong compactness for sequences of bounded energy. In the symmetric case  $\lambda = 1$ , which corresponds to  $g = 1$  in (1), one can follow the strategy of [4, 24] and use a nonlinear sigma model representation [27] to rewrite the problem in terms of an amplitude and a phase. In these unknowns, the functional takes a form similar to the celebrated Ambrosio-Tortorelli functional [7] from which one can get compactness. Inspired by recent work on type-I superconductors [16], we prove instead that for every  $\lambda \leq 1$ , the energy (8) directly controls a Modica-Mortola [33] type energy where  $\eta_1$  and  $\eta_2$  are decoupled (see (68)). As a consequence, we recover the compactness of sequences of bounded energy. Besides extending results of [4, 24] to the non-symmetric case, we believe that this more direct approach also has an intrinsic interest. In fact, it seems more natural and gives a better understanding of how the interaction between the two condensates via the term  $\int_\Omega \eta_1^2 \eta_2^2$  gives to the energy a structure of a double well potential. Using then classical arguments from  $\Gamma$ -convergence such as the slicing method [12], we can prove our last main theorem:

**Theorem 1.8.** *When  $\varepsilon \rightarrow 0$ ,  $J_\varepsilon$   $\Gamma$ -converges for the strong  $L^1$  topology to*

$$\mathcal{J}(\eta_1, \eta_2) = \begin{cases} \sigma_{\lambda, K} P(E, \Omega) & \text{if } \eta_1 = \chi_E = 1 - \eta_2 \text{ and } |E| = \alpha_1 \\ +\infty & \text{otherwise,} \end{cases}$$

where, for a set of finite perimeter  $E$  (see [6, 30]),  $P(E, \Omega)$  denotes the relative perimeter of  $E$  inside  $\Omega$  and where  $\sigma_{\lambda, K}$  is defined by the one dimensional optimal transition problem

$$\sigma_{\lambda, K} = \inf \left\{ \int_{\mathbf{R}} |\eta'_1|^2 + \lambda^2 |\eta'_2|^2 + \frac{1}{2} (\eta_1^2 + \eta_2^2 - 1)^2 + (K - 1) \eta_1^2 \eta_2^2 : \lim_{-\infty} \eta_1 = 0, \lim_{+\infty} \eta_1 = 1 \right\}.$$

At last, in the spirit of what was done in [24] in the case  $\lambda = 1$ , we study the asymptotic behavior of  $\sigma_{\lambda, K}$  as  $K$  goes to one (mixing) or  $K$  goes to infinity (strong segregation) and recover

good parts of what is expected from the physics literature [9, 41, 8, 40], namely

$$\lim_{K \rightarrow 1} \frac{\sigma_{\lambda, K}}{\sqrt{K-1}} = \frac{2}{3} \frac{1-\lambda^3}{1-\lambda^2}, \quad \sigma_{\lambda, K} - \sigma_{\lambda, \infty} \underset{\frac{\lambda^2}{K} \downarrow 0}{\sim} - \left( \frac{1}{K^{1/4}} + \frac{\lambda^{1/2}}{K^{1/4}} \right),$$

see Propositions 4.6 and 4.8.

Before closing this introduction, let us point out that most of the results, in particular Theorem 1.1, Theorem 1.3 and Theorem 1.8 can be easily generalized to arbitrary dimension and arbitrary radially symmetric strictly increasing confining potentials.

The structure of the paper is the following. In Section 2, we study the case of different intracomponent strengths in the presence of a confining potential. We then study in Section 3 the crossover case and finally in Section 4, we investigate the case of non-equal intracomponent strengths in the absence of confining potential and in a bounded domain.

## Notation

For  $x \in \mathbf{R}^2$  and  $r > 0$ , we denote by  $B_r(x)$  the ball of radius  $r$  centered at  $x$  and simply write  $B_r$  when  $x = 0$ . Given a set  $E \subset \mathbf{R}^2$ , we let  $\chi_E$  be the characteristic function of  $E$ . For any integer  $k$ , we denote by  $\mathcal{H}^k$  the  $k$ -dimensional Hausdorff measure. The letters,  $c, C$  denote universal constants which can vary from line to line. We also make use of the usual  $o$  and  $O$  notation. The symbols  $\sim, \gtrsim, \lesssim$  indicate estimates that hold up to a positive constant. For instance,  $f \lesssim g$  denotes the existence of a constant  $C > 0$  such that  $f \leq Cg$ . Throughout the paper, with a small abuse of language, we call sequence a family  $(u_\varepsilon)$  of functions labeled by a continuous parameter  $\varepsilon \in (0, 1]$ . A subsequence of  $(u_\varepsilon)$  is any sequence  $(u_{\varepsilon_k})$  such that  $\varepsilon_k \rightarrow 0$  as  $k \rightarrow +\infty$ . For  $1 \leq p \leq +\infty$  and a function  $f$ , we denote the  $L^p$  norm of  $f$  by  $\|f\|_p$ . In the sequel, when it is clear from the context, we omit to indicate the integrating measure. In particular, all integrals involving boundaries are with respect to  $\mathcal{H}^1$  and all integrals involving sets are with respect to the Lebesgue measure.

## 2 The case of different intracomponent strengths

### 2.1 The Thomas-Fermi profile

Let us consider the Thomas-Fermi approximation and prove Theorem 1.1. Let  $\alpha = (\alpha_1, \alpha_2) \in (0, +\infty)^2$ . We denote by  $X$  the set of pairs of measurable functions  $(\rho_1, \rho_2) : \mathbf{R}^2 \rightarrow \mathbf{R}_+^2$  and by  $X_\alpha$  the subset of pairs  $\rho \in X$  satisfying

$$\int_{\mathbf{R}^2} \rho_j = \alpha_j \quad \text{for } j = 1, 2.$$

For  $\rho \in X$  we define the energy

$$E(\rho) = \int_{\mathbf{R}^2} \frac{1}{2} \rho_1^2 + \frac{g}{2} \rho_2^2 + K \rho_1 \rho_2 + V \langle \rho \rangle.$$

where we write  $\langle \rho \rangle = \rho_1 + \rho_2$ . As stated in Theorem 1.1, we study the minimization of  $E$  in the class  $X_\alpha$ : we prove the existence of a unique minimizer for this problem and a stability result. We start by studying the minimization problem in the following subsets: given  $r > 0$ , let

$$X_\alpha^{0,r} = \{ \rho \in X_\alpha : \text{supp}(\rho_1) \subset \overline{B_r}, \text{supp}(\rho_2) \subset \mathbf{R}^2 \setminus B_r \}.$$

and

$$X_\alpha^0 = \bigcup_{r>0} X_\alpha^{0,r}. \quad (9)$$

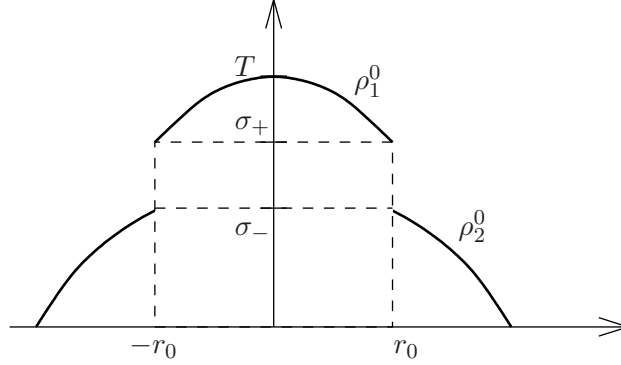


Figure 1: Minimizer of the Thomas-Fermi energy.

**Lemma 2.1.** For  $K > \sqrt{g} > 1$  and  $r > 0$ ,  $E$  admits a unique minimizer  $\rho^{0,r} = (\rho_1^{0,r}, \rho_2^{0,r})$  in  $X_\alpha^{0,r}$ . Let us set

$$\bar{r} = \min(r, r_1) \quad \text{with } r_1 = \left(\frac{2\alpha_1}{\pi}\right)^{1/4} \quad \text{and then} \quad R_1^2 = \frac{\bar{r}^2}{2} + \frac{r_1^4}{2\bar{r}^2}, \quad R_2^2 = r^2 + \left(\frac{2g\alpha_2}{\pi}\right)^{1/2}.$$

The unique minimizer is given by

$$\rho_1^{0,r}(x) = \begin{cases} (R_1^2 - |x|^2)_+ & \text{if } |x| < r, \\ 0 & \text{if } |x| > r, \end{cases} \quad \rho_2^{0,r}(x) = \begin{cases} 0 & \text{if } |x| < r, \\ \frac{1}{g}(R_2^2 - |x|^2)_+ & \text{if } |x| > r. \end{cases} \quad (10)$$

Moreover, for  $r > 0$  and  $\rho \in X_\alpha^{0,r}$ , there holds

$$E(\rho) - E(\rho^{0,r}) \geq \frac{1}{2} \int_{B_r} (\rho_1 - \rho_1^{0,r})^2 + \frac{g}{2} \int_{B_r^c} (\rho_2 - \rho_2^{0,r})^2 + \int_{B_{R_2}^c} (|x|^2 - R_2^2) \rho_2. \quad (11)$$

Optimizing in  $r > 0$ , the minimum of  $E$  in  $X_\alpha^0$  is reached at  $\rho^0 = \rho^{0,r_0}$  (see Figure 1) with

$$r_0 = r_1 \left( \sqrt{1 + \frac{\alpha_2}{\alpha_1}} - \sqrt{\frac{\alpha_2}{\alpha_1}} \right)^{1/2} \in (0, r_1). \quad (12)$$

In particular, there is a positive gap between  $\rho_2^0$  and  $\rho_1^0$  at the frontier  $\partial B_{r_0}$ :

$$\sigma_+ = \inf_{B_{r_0}} \rho_1^0 = \sqrt{g} \sup \rho_2^0 > \sup \rho_2^0 = \sigma_-. \quad (13)$$

Let us denote  $E^r = \min\{E : X_\alpha^{0,r}\}$  and  $E_0 = E^{r_0} = \min\{E : X_\alpha^0\}$ . The minimal energy is

$$E_0 = E(\rho_0) = \frac{2}{3} \sqrt{\frac{2}{\pi}} \left( (\alpha_1 + \alpha_2)^{3/2} + (\sqrt{g} - 1) \alpha_2^{3/2} \right). \quad (14)$$

Eventually, there exist  $c_1, c_2 > 0$ , depending on  $g$  and  $\alpha$  such that

$$\forall r > 0, \quad E^r - E_0 \geq c_1 |r - r_0|^2, \quad (15)$$

$$\forall r > 0, \quad |r - r_0| \geq c_2 (\|\rho^{0,r} - \rho^0\|_1 + \|\rho^{0,r} - \rho^0\|_2^2). \quad (16)$$

*Proof.*

Let  $r > 0$ . The optimization of  $E$  in  $X_\alpha^{0,r}$  splits into two independent optimization problems. Namely, we set

$$E_1(\rho_1) = \frac{1}{2} \int_{\mathbf{R}^2} \rho_1^2 + \int_{\mathbf{R}^2} V(x)\rho_1, \quad E_2(\rho_2) = \frac{g}{2} \int_{\mathbf{R}^2} \rho_2^2 + \int_{\mathbf{R}^2} V(x)\rho_2.$$

We optimize  $E_1$  in  $Y_{1,\alpha}^r = \{\rho_1 \in L^1(\mathbf{R}^2, \mathbf{R}_+) : \int_{B_r} \rho_1 = \alpha_1, \text{supp } \rho_1 \subset \overline{B_r}\}$  and  $E_2$  in  $Y_{2,\alpha}^r = \{\rho_2 \in L^1(\mathbf{R}^2 \setminus B_{\bar{r}}, [0, +\infty)) : \int_{\mathbf{R}^2 \setminus B_{\bar{r}}} \rho_2 = \alpha_2\}$ . From the associated Euler-Lagrange equations, we see that the minimizers have the form given by (10) with  $R_1, R_2$  fixed by the conditions  $\int_{\mathbf{R}^2} \rho_j = \alpha_j$  for  $j = 1, 2$ . In fact, for  $\rho^{0,r}$  given by (10) and  $\rho \in X_\alpha^{0,r}$ , the energy  $E_1(\rho_1)$  rewrites as

$$E_1(\rho_1) = E_1(\rho_1^{0,r}) + \frac{1}{2} \int_{B_r} (\rho_1 - \rho_1^{0,r})^2 + \int_{B_r \setminus B_{\bar{r}}} (|x|^2 - R_1^2)\rho_1.$$

For  $r \leq r_1$ , we have  $\bar{r} = r$  so that the last term vanishes and for  $r > r_1$ , we have  $\bar{r} = R_1 = r_1$  and the last term is non-negative. Similarly,

$$E_2(\rho_2) = E_2(\rho_2^{0,r}) + \frac{g}{2} \int_{B_{\bar{r}}^c} (\rho_2 - \rho_2^{0,r})^2 + \int_{B_{\mathbf{R}^2}^c} (|x|^2 - R_2^2)\rho_2.$$

Summing these relations, we deduce (11).

Let us now study the variation of  $r \mapsto E^r = \min_{X_\alpha^{0,r}} E = E_1(\rho_1^{0,r}) + E_2(\rho_2^{0,r})$ . Using the notation  $t = (r/r_1)^2$ , we obtain by direct computation,

$$E^r = f(t) + \frac{2}{3} \left( \frac{2g\alpha_2^3}{\pi} \right)^{1/2},$$

with

$$f(t) = \begin{cases} \frac{\pi}{24} r_1^6 \left( 6t + \frac{3}{t} - t^3 \right) + \alpha_2 r_1^2 t & \text{for } t < 1, \\ \frac{\pi}{3} r_1^6 + \alpha_2 r_1^2 t & \text{for } t \geq 1. \end{cases}$$

We have  $f \in C^{1,1}(0, +\infty)$  with  $f'' > 0$  in  $(0, 1)$ ,  $f(t) \rightarrow +\infty$  as  $t \downarrow 0$  and  $f' = f'(1) > 0$  on  $(1, +\infty)$ . Therefore,  $f$  admits a unique minimiser  $t_0 = (r_0/r_1)^2 \in (0, 1)$  and (15) holds true. After some algebraic computations, we get that  $r_0$  is given by (12), we also obtain (13) and (14). Eventually, we easily obtain inequality (16) from the explicit formula (10).  $\square$

Now, let  $\rho \in X_\alpha$ . We start by building a segregated state  $\rho^{seg}$ , i.e  $\rho^{seg}$  satisfies  $\rho_1^{seg} \rho_2^{seg} \equiv 0$  with  $E(\rho^{seg}) \leq E(\rho)$ . For this we add an extra space dimension and consider the set

$$X_\alpha^+ = \left\{ \rho^+ = (\rho_1^+, \rho_2^+) \in L^1(\mathbf{R}^2 \times [0, 1], \mathbf{R}_+^2) : \int_{\mathbf{R}^2 \times [0, 1]} \rho_j^+ = \alpha_j \text{ for } j = 1, 2 \right\}.$$

We consider  $X_\alpha$  as a subset of  $X_\alpha^+$  by using the injection  $j^+$  defined as  $j^+(\rho)(x, t) = \rho(x)$  for  $\rho \in X_\alpha$ . For  $\rho^+ \in X_\alpha^+$ , we define the energy

$$E(\rho^+) = \int_{\mathbf{R}^2 \times [0, 1]} \frac{1}{2} (\rho_1^+)^2 + \frac{g}{2} (\rho_2^+)^2 + \int_{\mathbf{R}^2 \times [0, 1]} K \rho_1^+ \rho_2^+ + \int_{\mathbf{R}^2 \times [0, 1]} V(x) \langle \rho^+ \rangle(x, t) dx dt.$$

This extends the former definition of  $E$  in the sense that  $E(j^+(\rho)) = E(\rho)$  for  $\rho \in X_\alpha$  (in the sequel we identify  $j^+(\rho)$  and  $\rho$ ). We also define a left inverse  $\Pi$  of  $j^+$  by  $\Pi(\rho^+)(x) = \int_0^1 \rho^+(x, t) dt$ . Now let  $\rho \in X_\alpha$ , we define  $\rho^{seg}$  as

$$\rho^{seg} \in \operatorname{argmin} \{ E(\rho^+) : \rho^+ \in X_\alpha^+, \Pi(\rho^+) = \rho \}.$$



**Lemma 2.2.** *The state  $\rho^{seg}$  is well defined and satisfies  $\rho_1^{seg} \rho_2^{seg} \equiv 0$  and*

$$E(\rho^{seg}) \leq E(\rho) \quad \text{with equality if and only if } \rho_1 \rho_2 \equiv 0. \quad (17)$$

Moreover, there holds,

$$\|\rho - \rho_0\|_1 \leq \|\rho^{seg} - \rho_0\|_{L^1(\mathbf{R}^2 \times [0,1])}. \quad (18)$$

*Proof.* We look for  $\rho^{seg} \in X_\alpha^+$  such that  $\Pi(\rho^{seg}) = \rho$  with minimal energy. We can write

$$\rho_1^{seg}(x, t) = \langle \rho \rangle(x) \varphi_1(x, t), \quad \rho_2^{seg}(x, t) = \langle \rho \rangle(x) \varphi_2(x, t).$$

where  $\varphi_1, \varphi_2$  satisfy

$$\varphi_1, \varphi_2 \geq 0, \quad \int_0^1 \varphi_1(x, t) dt = \frac{\rho_1}{\langle \rho \rangle}(x), \quad \int_0^1 \varphi_2(x, t) dt = \frac{\rho_2}{\langle \rho \rangle}(x). \quad (19)$$

Under these constraints, for almost every  $x \in \text{supp } \rho$ , the pair  $(\varphi_1(x, \cdot), \varphi_2(x, \cdot))$  minimizes the local energy

$$e(\varphi) = \int_0^1 \left\{ \frac{1}{2} \varphi_1(t)^2 + \frac{g}{2} \varphi_2(t)^2 + K \varphi_1(t) \varphi_2(t) \right\} dt,$$

Let us now write,

$$e(\varphi) = \frac{1}{2} \int_0^1 (\varphi_1 + \sqrt{g} \varphi_2)^2 + (K - \sqrt{g}) \int_0^1 \varphi_1 \varphi_2.$$

Since  $K > \sqrt{g}$ , we see that the minimizers of  $e$  under (19) are the pairs of measurable functions  $(\varphi_1, \varphi_2)$  satisfying (19) and such that for almost every  $(x, t) \in \mathbf{R}^2 \times [0, 1]$ ,

$$\varphi_1(x, t) \varphi_2(x, t) \equiv 0 \quad \text{and} \quad \varphi_1(x, t) + \sqrt{g} \varphi_2(x, t) = \frac{\rho_1}{\langle \rho \rangle}(x) + \sqrt{g} \frac{\rho_2}{\langle \rho \rangle}(x) =: h(x).$$

To fix ideas, we can choose  $\varphi_1, \varphi_2$  of the form  $\varphi_1(x, t) = h(x) \chi_{[0, \theta(x)]}(t)$ ,  $\varphi_2(x, t) = \frac{h(x)}{\sqrt{g}} \chi_{[\theta(x), 1]}$ , with  $\theta(x) = \rho_1(x) / [\rho_1(x) + \sqrt{g} \rho_2(x)]$ . By construction, we have

$$e(\varphi_1(x, \cdot), \varphi_2(x, \cdot)) \leq e\left(\frac{\rho_1}{\langle \rho \rangle}(x), \frac{\rho_2}{\langle \rho \rangle}(x)\right)$$

with equality if and only if  $\rho_1(x) \rho_2(x) = 0$ . Multiplying by  $\langle \rho \rangle^2(x)$  and integrating, this yields

$$\int_{\mathbf{R}^2 \times [0,1]} \frac{1}{2} (\rho_1^{seg})^2 + \frac{g}{2} (\rho_2^{seg})^2 + K \rho_1^{seg} \rho_2^{seg} \leq \int_{\mathbf{R}^2} \frac{1}{2} (\rho_1)^2 + \frac{g}{2} (\rho_2)^2 + K \rho_1 \rho_2.$$

The remaining term of the energy does not change. Indeed, since  $\Pi(\rho^{seg}) = \rho$ , we have

$$\int_{\mathbf{R}^2 \times [0,1]} V(x) \langle \rho^{seg} \rangle(x, t) dx dt = \int_{\mathbf{R}^2} V(x) \left[ \int_0^1 \langle \rho^{seg} \rangle(x, t) dt \right] dx = \int_{\mathbf{R}^2} V(x) \langle \rho \rangle(x) dx.$$

This establishes (17). Eventually, inequality (18) follows from the triangular inequality:

$$\|\rho - \rho_0\|_1 = \int_{\mathbf{R}^2} \left| \int_0^1 \rho^{seg}(x, s) - \rho_0(x) ds \right| dx \leq \|\rho^{seg} - \rho_0\|_1.$$

This ends the proof of the lemma. □

Next, we build successively from  $\rho^{seg}$  two segregated states  $\rho^>$ ,  $\rho^{sym}$  such that  $\rho^> \in X_\alpha^+$ ,  $\rho^{sym} \in X_\alpha^0$  (see (9)) and  $E(\rho^{seg}) \geq E(\rho^>) \geq E(\rho^{sym})$ . More precisely, we will have

$$\begin{aligned} \rho_1^> \rho_2^> &\equiv 0, \quad \langle \rho^> \rangle = \langle \rho^{seg} \rangle, \quad \inf_{\{\rho_1^> > 0\}} \rho_1^> \geq \sup \rho_2^>, \\ \rho^{sym} &\in X_\alpha^0, \quad \forall \lambda > 0, j = 1, 2, |\{\rho_j^{sym} > \lambda\}| = |\{\rho_j^> > \lambda\}|. \end{aligned} \quad (20)$$

Together with Lemma 2.1 and Lemma 2.2 this will establish that  $\rho^0$  minimizes  $E$  in  $X_\alpha$ . Eventually, we will take into account the energy loss at each step of the construction in order to establish the stability result (4) (proving in particular the uniqueness of the minimizer).

First, since  $\langle \rho^{seg} \rangle \geq 0$  and  $\int_{\mathbf{R}^2 \times [0,1]} \langle \rho^{seg} \rangle = \alpha_1 + \alpha_2$ , there exists a measurable subset  $A \subset \mathbf{R}^2 \times [0, 1]$  such that

$$\int_A \langle \rho^{seg} \rangle = \alpha_1, \quad \int_{A^c} \langle \rho^{seg} \rangle = \alpha_2 \quad \text{and} \quad \inf_A \langle \rho^{seg} \rangle \geq \sup_{A^c} \langle \rho^{seg} \rangle.$$

**Lemma 2.3.** *Let us define  $\rho^> \in X_\alpha^+$  as  $\rho^> = (\rho_1^>, \rho_2^>) = \langle \rho^{seg} \rangle (\chi_A(1, 0) + \chi_{A^c}(0, 1))$ . By construction,  $\rho^> \in X_\alpha^+$  and (20) holds true. Moreover we have the following identities.*

$$\|\rho^{seg} - \rho^>\|_1 = \sqrt{2} \int_{A \setminus \{\rho_1^{seg} > 0\}} \langle \rho^> \rangle + \sqrt{2} \int_{A^c \setminus \{\rho_1^{seg} = 0\}} \langle \rho^> \rangle = 2\sqrt{2} \int_{A \setminus \{\rho_1^{seg} > 0\}} \langle \rho^> \rangle. \quad (21)$$

$$E(\rho^{seg}) - E(\rho^>) = \frac{g-1}{2} \left[ \int_{A \setminus \{\rho_1^{seg} > 0\}} \langle \rho^> \rangle^2 - \int_{A^c \setminus \{\rho_1^{seg} = 0\}} \langle \rho^> \rangle^2 \right] \geq 0. \quad (22)$$

*Proof.* The properties of  $\rho^>$  follow from that of  $\rho^{seg}$ . By definition, we have

$$\rho^{seg} - \rho^> = \chi_{A \setminus \{\rho_1^{seg} > 0\}} \langle \rho^> \rangle (-1, 1) + \chi_{A^c \setminus \{\rho_1^{seg} = 0\}} \langle \rho^> \rangle (1, -1),$$

from which we deduce the first equality of (21). Now notice that by construction, there holds

$$\int_{A \setminus \{\rho_1^{seg} > 0\}} \langle \rho^> \rangle = \int_{A^c \setminus \{\rho_1^{seg} = 0\}} \langle \rho^> \rangle. \quad (23)$$

This implies the second equality of (21). The identity (22) also directly follows from the definition of  $\rho^>$ . Eventually, since  $\langle \rho^> \rangle(x, t) \geq \langle \rho^> \rangle(y, s)$  for every  $(x, t), (y, s) \in A \times A^c$ , we deduce from (23) that  $E(\rho^{seg}) - E(\rho^>) \geq 0$ .  $\square$

Now we define  $\rho^{sym}$  as the non-increasing rearrangement of  $\rho^>$ . More precisely, for  $x \in \mathbf{R}^2$ , we set

$$\rho_1^{sym}(x) = \begin{cases} \mu(x) & \text{if } |x| < r_\star \\ 0 & \text{if } |x| > r_\star, \end{cases} \quad \rho_2^{sym}(x) = \begin{cases} 0 & \text{if } |x| < r_\star \\ \mu(x) & \text{if } |x| > r_\star, \end{cases}$$

where  $\mu : \mathbf{R}^2 \rightarrow [0, +\infty)$  is the unique radial non-increasing function such that for every  $\lambda > 0$ ,

$$|\{x \in \mathbf{R}^2 : \mu(x) > \lambda\}| = |\{(x, t) \in \mathbf{R}^2 \times [0, 1] : \langle \rho^> \rangle(x, t) > \lambda\}|,$$

and  $r_\star > 0$  is fixed by the condition  $\rho^{sym} \in X_\alpha$ .

**Lemma 2.4.** *We have  $\rho \in X_\alpha^0$  and*

$$E(\rho^>) - E(\rho^{sym}) = \int_{\mathbf{R}^2 \times [0,1]} |x|^2 (\langle \rho^> \rangle(x, t) - \langle \rho^{sym} \rangle(x)) dx dt \geq 0. \quad (24)$$

*Proof.* By construction  $\rho^{sym} \in X_\alpha^{0,r_\star}$ . The identity for the energy follows from the fact that  $\rho_j^{sym}$  is a rearrangement of  $\rho_j^\triangleright$  for  $j = 1, 2$ . Since  $\langle \rho^{sym} \rangle$  is a non-increasing rearrangement of  $\langle \rho^\triangleright \rangle$ , we have  $E(\rho^\triangleright) \geq E(\rho^{sym})$ .  $\square$

At this point, we have that for every  $\rho \in X_\alpha$ ,  $E(\rho) \geq E(\rho^{sym})$  and since  $\rho^{sym} \in X_\alpha^0$ , we deduce from Lemma 2.1 that  $\rho^0$  minimizes  $E$  in  $X_\alpha$ .

We now come back to the above constructions in reverse order to establish the stability result (4).

First, let us notice that for every  $\rho \in X_\alpha$ , we have  $\|\rho\|_1 = \alpha_1 + \alpha_2$ , so that it is enough to establish (4) for  $\rho \in X_\alpha$  satisfying

$$\delta E = E(\rho) - E_0 \leq E_0. \quad (25)$$

Let  $\rho \in X_\alpha$  such that (25) holds. We estimate  $\|\rho - \rho^0\|_1$  by estimating successively  $\|\rho^{sym} - \rho^0\|_1$ ,  $\|\rho^\triangleright - \rho^{sym}\|_1$ ,  $\|\rho^{seg} - \rho^\triangleright\|_1$  and  $\|\rho - \rho^{seg}\|_1$ .

*Step 1. Estimating  $\|\rho^{sym} - \rho^0\|_1$ .*

First, from (15),(16) of Lemma 2.1, we get,

$$|r_\star - r_0| \lesssim \sqrt{\delta E} \quad \text{and} \quad \|\rho^{0,r_\star} - \rho^0\|_1 + \|\rho^{0,r_\star} - \rho^0\|_2^2 \lesssim \delta E. \quad (26)$$

Next, from (25) and using the notation of Lemma 2.1 with  $r = r_\star$ , there exists  $R > 0$  only depending on  $\alpha$  and  $g$  such that  $R \geq 2R_2$ . By the Cauchy-Schwarz inequality and the inequality  $|x|^2 - R_2^2 \geq 3R_2^2/4$  in  $B_R^c$ , we get

$$\|\rho^{sym} - \rho^{0,r_\star}\|_1 = \int_{B_R} |\rho^{sym} - \rho^{0,r_\star}| + \int_{B_R^c} \rho_2^{sym} \leq \sqrt{\pi}R \|\rho^{sym} - \rho^{0,r_\star}\|_2 + \frac{4}{3R^2} \int_{B_R^c} (|x|^2 - R_2^2) \rho_2^{sym}.$$

Using (11), this leads to  $\|\rho^{sym} - \rho^{0,r_\star}\|_1 \lesssim \sqrt{\delta E} + \delta E \lesssim \sqrt{\delta E}$  and with (26), we get

$$\|\rho^{sym} - \rho^0\|_1 \lesssim \sqrt{\delta E}. \quad (27)$$

*Step 2. Estimating  $\|\rho^\triangleright - \rho^{sym}\|_1$ .*

Starting from (24) and using Fubini formula, we obtain

$$E(\rho^\triangleright) - E(\rho^{sym}) = \int_0^{+\infty} \left[ \int_{\{\langle \rho^\triangleright \rangle > \lambda\}} |x|^2 dx dt - \int_{\langle j+(\rho^{sym}) \rangle > \lambda} |x|^2 dx dt \right] d\lambda \quad (28)$$

By definition,  $\{\langle \rho^{sym} \rangle > \lambda\}$  has the form  $B \times [0, 1]$  where  $B$  is the ball of volume  $|\{\langle \rho^\triangleright \rangle > \lambda\}|$ . We make use of the following result.

**Lemma 2.5.** *There exists  $c > 0$  such that for every  $r > 0$  and every measurable set  $E \subset \mathbf{R}^2 \times [0, 1]$  with  $|E| = |B_r|$  we have*

$$\int_E |x|^2 dx dt - \int_{B_r \times [0,1]} |x|^2 dx dt \geq c|E\Delta(B_r \times [0,1])|^2. \quad (29)$$

*Proof.* By rescaling it is enough proving (29) for  $r = 1$ . Notice first that

$$\int_E |x|^2 - \int_{B \times [0,1]} |x|^2 = \int_{E \setminus (B \times [0,1])} (|x|^2 - 1) + \int_{(B \times [0,1]) \setminus E} (1 - |x|^2).$$

Letting  $m = |E \setminus (B \times [0,1])| = |(B \times [0,1]) \setminus E|$ , we have

$$\int_{E \setminus (B \times [0,1])} (|x|^2 - 1) \geq \min_{F \subset B^c \times [0,1], |F|=m} \int_F (|x|^2 - 1) = \int_{(B_{(\frac{m}{\pi}+1)^{1/2}} \setminus B) \times [0,1]} (|x|^2 - 1) \geq cm^2$$

where we used that for  $0 \leq m \leq \pi$ ,

$$\int_{B_{(\frac{m}{\pi}+1)^{1/2}} \setminus B} (|x|^2 - 1) = \pi \left( \frac{2}{3} \left[ \left( \frac{m}{\pi} + 1 \right)^{3/2} - 1 \right] - \frac{m}{\pi} \right) \geq cm^2$$

since  $\frac{2}{3} \left[ (t+1)^{3/2} - 1 \right] - t$  is strictly convex on  $[0, 1]$ . Using the analogous estimate on  $(B \times [0, 1]) \setminus E$ , we obtain (29).  $\square$

Plugging now (29) into (28), we obtain

$$E(\rho^>) - E(\rho^{sym}) \geq c \int_0^{+\infty} |\{\langle \rho^> \rangle > \lambda\} \Delta \{\langle j^+(\rho^{sym}) \rangle > \lambda\}|^2 d\lambda. \quad (30)$$

Notice that if  $u, v \geq 0$  with  $|\{u > \lambda\}| = |\{v > \lambda\}|$  for every  $\lambda \geq 0$ ,

$$\begin{aligned} \|u - v\|_1 &= 2 \int_{\{u > v\}} (u - v) = 2 \int_{\mathbf{R}^2 \times \mathbf{R}^+} \chi_{\{u > \lambda\}}(x) \chi_{\{v < \lambda\}}(x) dx dt \\ &= 2 \int_0^{+\infty} |\{u > \lambda\} \setminus \{v < \lambda\}| d\lambda = \int_0^{+\infty} |\{u > \lambda\} \Delta \{v > \lambda\}| d\lambda. \end{aligned}$$

By setting  $T := \max\langle \rho^0 \rangle = \rho_1^0(0)$ , this yields,

$$\|\min(T, \langle \rho^> \rangle) - \min(T, \langle \rho^{sym} \rangle)\|_1 = \int_0^T |\{\langle \rho^> \rangle > \lambda\} \Delta \{\langle \rho^{sym} \rangle > \lambda\}| d\lambda.$$

Using the Cauchy-Schwarz inequality, we deduce from (30),

$$\|\min(T, \langle \rho^> \rangle) - \min(T, \langle \rho^{sym} \rangle)\|_1 \leq \frac{\sqrt{T}}{\sqrt{c}} \sqrt{\delta E}. \quad (31)$$

Recall now that  $\sigma_- = \sup \rho_2^0$ ,  $\sigma_+ = \inf_{B_{r_0}} \rho_1^0$ , and that from (13), we have  $\sigma_+ = \sqrt{g} \sigma_- > \sigma_-$ . For  $\rho \in X_\alpha$  or  $\rho \in X_\alpha^+$  let us define  $P\rho = (P\rho_1, P\rho_2)$  (see Figure 2) as

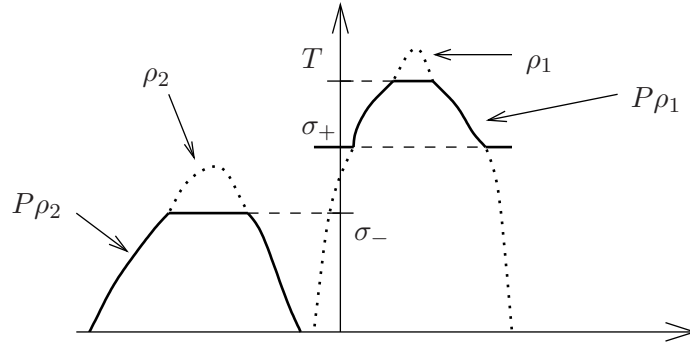


Figure 2:  $P\rho$ .

$$P\rho_1(x, t) = \begin{cases} \min(T, \max(\sigma_+, \rho_1(x, t))) & \text{if } \rho_1(x, t) > 0, \\ 0 & \text{if } \rho_1(x, t) = 0, \end{cases} \quad P\rho_2(x, t) = \min(\sigma_-, \rho_2(x, t)),$$

so that  $\sigma_+ \leq (P\rho)_1 \leq T$  in  $\{(P\rho)_1 > 0\}$  and  $(P\rho)_2 \leq \sigma_-$  in  $\{(P\rho)_1 = 0\}$ . Since  $\rho^{sym}$  is a rearrangement of  $\rho^>$ , we have  $\|\rho^> - P\rho^>\|_1 = \|\rho^{sym} - P\rho^{sym}\|_1$ . On the other hand, we have for  $x$  in  $B_{r_0}$ ,  $\sigma_+ \leq \rho_1^0(x) \leq T$  so that

$$|\rho_1^{sym}(x) - P\rho_1^{sym}(x)| \leq |\rho_1^{sym}(x) - \rho_1^0(x)| \quad \forall x \in B_{r_0},$$

and  $\rho_2^0 \leq \sigma_-$  on  $\mathbf{R}^2$ , hence

$$|\rho_2^{sym}(x) - P\rho_2^{sym}(x)| \leq |\rho_2^{sym}(x) - \rho_2^0(x)| \quad \forall x \in \mathbf{R}^2.$$

Therefore, if  $r_\star < r_0$ ,

$$\begin{aligned} \|\rho^{sym} - P\rho^{sym}\|_1 &= \int_{B_{r_\star}} |\rho_1^{sym} - (P\rho^{sym})_1| + \int_{B_{r_\star}^c} |\rho_2^{sym} - (P\rho^{sym})_2| \\ &\leq \int_{B_{r_\star}} |\rho_1^{sym} - \rho_1^0| + \int_{B_{r_\star}^c} |\rho_2^{sym} - \rho_2^0| \leq \|\rho^{sym} - \rho^0\|_1, \end{aligned}$$

whereas for  $r_\star \geq r_0$ ,

$$\begin{aligned} \|\rho^{sym} - P\rho^{sym}\|_1 &= \int_{B_{r_0}} |\rho_1^{sym} - (P\rho^{sym})_1| + \int_{B_{r_\star} \setminus B_{r_0}} |\rho_1^{sym} - (P\rho^{sym})_1| + \int_{B_{r_\star}^c} |\rho_2^{sym} - (P\rho^{sym})_2| \\ &\leq \int_{B_{r_0}} |\rho_1^{sym} - \rho_1^0| + \int_{B_{r_\star} \setminus B_{r_0}} |\rho_1^{sym}| + \sigma_+ |B_{r_\star} \setminus B_{r_0}| + \int_{B_{r_\star}^c} |\rho_2^{sym} - \rho_2^0| \\ &\leq \|\rho^{sym} - \rho^0\|_1 + C|r_0 - r_\star|. \end{aligned}$$

From (27), we get

$$\|\rho^\rangle - P\rho^\rangle\|_1 = \|\rho^{sym} - P\rho^{sym}\|_1 \lesssim \sqrt{\delta E}. \quad (32)$$

Now, for  $(x, t) \in \mathbf{R}^2 \times [0, 1]$ , if  $\rho_1^\rangle(x, t)\rho_2^{sym}(x) > 0$  or  $\rho_1^{sym}(x, t)\rho_2^\rangle(x) > 0$ , we have  $|\langle P\rho^\rangle(x, t) - \langle P\rho^{sym}(x) \rangle| \geq \sigma_+ - \sigma_-$ , hence,

$$|P\rho^\rangle(x, t) - P\rho^{sym}(x)| \leq 2T \leq \frac{2T}{\sigma_+ - \sigma_-} |\langle P\rho^\rangle(x, t) - \langle P\rho^{sym}(x) \rangle|.$$

In the other cases, we have  $|P\rho^\rangle(x, t) - P\rho^{sym}(x)| = |\langle P\rho^\rangle(x, t) - \langle P\rho^{sym}(x) \rangle|$ . Consequently, there holds

$$\|P\rho^\rangle - P\rho^{sym}\|_1 \lesssim \|\langle P\rho^\rangle - \langle P\rho^{sym} \rangle\|_1.$$

Using (31) and (32) to estimate the right-hand side, we obtain,

$$\begin{aligned} \|P\rho^\rangle - P\rho^{sym}\|_1 &\lesssim \|\min(T, \langle \rho^\rangle) - \min(T, \langle \rho^{sym} \rangle)\|_1 + \|\min(T, \langle \rho^\rangle) - \langle P\rho^\rangle\|_1 \\ &\quad + \|\min(T, \langle \rho^{sym} \rangle) - \langle P\rho^{sym} \rangle\|_1 \\ &\lesssim \|\min(T, \langle \rho^\rangle) - \min(T, \langle \rho^{sym} \rangle)\|_1 + \|\langle \rho^{sym} \rangle - \langle P\rho^{sym} \rangle\|_1 \\ &\lesssim \sqrt{\delta E}. \end{aligned}$$

Together, with (32), we get,

$$\|\rho^\rangle - \rho^{sym}\|_1 \lesssim \sqrt{\delta E}. \quad (33)$$

*Step 3. Estimating  $\|\rho^{seg} - \rho^\rangle\|_1$ .*

Using the notation of Lemma 2.3, we obtain from (22)

$$\begin{aligned} E(\rho^{seg}) - E(\rho^\rangle) &= \frac{g-1}{2} \left[ \int_{A \setminus \{\rho_1^{seg} > 0\}} \langle \rho^\rangle^2 - \int_{A^c \setminus \{\rho_1^{seg} = 0\}} \langle \rho^\rangle^2 \right] \\ &\geq \frac{g-1}{2} \left[ \sigma_+ \int_{A \setminus \{\rho_1^{seg} > 0\}} \langle \rho^\rangle - \sigma_- \int_{A^c \setminus \{\rho_1^{seg} = 0\}} \langle \rho^\rangle \right] \\ &\quad - \frac{g-1}{2} \left[ \int_{A \setminus \{\rho_1^{seg} > 0\}} \langle \rho^\rangle (\sigma_+ - \langle \rho^\rangle)_+ + \int_{A^c \setminus \{\rho_1^{seg} = 0\}} \langle \rho^\rangle (\langle \rho^\rangle - \sigma_-)_+ \right]. \end{aligned}$$

Using (21), we obtain the estimate

$$\|\rho^{seg} - \rho^\triangleright\|_1 \leq \frac{\sqrt{2}}{\sigma_+ - \sigma_-} \left( \frac{2}{g-1} \delta E + \int_A \langle \rho^\triangleright \rangle (\sigma_+ - \langle \rho^\triangleright \rangle)_+ + \int_{A^c} \langle \rho^\triangleright \rangle (\langle \rho^\triangleright \rangle - \sigma_-)_+ \right). \quad (34)$$

Next, using the fact that  $\rho^\triangleright = (\langle \rho^\triangleright \rangle, 0)$  in  $A$ , that  $\rho_1^{sym}$  is a rearrangement of  $\rho_1^\triangleright$  and that  $\rho_1^0 \geq \sigma_+$  in  $B_{r_0}$ , we have,

$$\begin{aligned} \int_A \langle \rho^\triangleright \rangle (\sigma_+ - \langle \rho^\triangleright \rangle)_+ &\leq \sigma_+ \int_A (\sigma_+ - \langle \rho^\triangleright \rangle)_+ = \sigma_+ \int_{B_{r_\star}} (\sigma_+ - \rho_1^{sym})_+ \\ &\lesssim \|\rho_1^{sym} - \rho_1^0\|_1 + |r_0 - r_\star| \stackrel{(27)}{\lesssim} \sqrt{\delta E}. \end{aligned} \quad (35)$$

Similarly,

$$\begin{aligned} \int_{A^c} \langle \rho^\triangleright \rangle (\langle \rho^\triangleright \rangle - \sigma_-)_+ &\leq \sigma_- \int_{A^c} (\langle \rho^\triangleright \rangle - \sigma_-)_+ + \int_{A^c} (\langle \rho^\triangleright \rangle - \sigma_-)_+^2 \\ &= \sigma_- \int_{B_{r_\star}^c} (\rho_2^{sym} - \sigma_-)_+ + \int_{B_{r_\star}^c} (\rho_2^{sym} - \sigma_-)_+^2 \\ &\lesssim \|\rho_2^{sym} - \rho_2^0\|_1 + \|\rho_2^{sym} - \rho_2^0\|_2^2 \stackrel{(27)}{\lesssim} \sqrt{\delta E}. \end{aligned} \quad (36)$$

We deduce from estimates (34)(35)(36),

$$\|\rho^{seg} - \rho^\triangleright\|_1 \lesssim \sqrt{\delta E}. \quad (37)$$

*Step 4: Conclusion.*

Eventually, using (18) of Lemma 2.2 and (27)(33)(37), we get

$$\|\rho - \rho_0\|_1 \leq \|\rho^{seg} - \rho_0\|_1 \leq \|\rho^{seg} - \rho^\triangleright\|_1 + \|\rho^\triangleright - \rho^{sym}\|_1 + \|\rho^{sym} - \rho_0\|_1 \lesssim \sqrt{\delta E}.$$

This establishes (4) and ends the proof of Theorem 1.1.

**Remark 2.6.** One cannot hope for a stronger inequality of the form

$$\|\rho - \rho^0\|_1 \lesssim \delta E \quad \text{or} \quad \|\rho - \rho^0\|_2^2 \lesssim \delta E.$$

since one can easily get a contradiction by exchanging  $\rho_1^0$  and  $\rho_2^0$  on small balls close to  $\partial B_{r_0}$ .

## 2.2 Approximation of the Thomas-Fermi limit by the Gross-Pitaievskii functional

We prove that minimizers of  $F_\varepsilon$  converge to  $\sqrt{\rho^0}$  as stated in Theorem 1.2.

*Proof of Theorem 1.2.* Let  $\eta^\varepsilon$  be minimizers of  $F_\varepsilon$  for  $\varepsilon \in (0, 1)$ . Regularizing  $(\sqrt{\rho_1^0}, \sqrt{\rho_2^0})$ , one can easily construct a competitor  $\tilde{\eta}_\varepsilon$  with  $F_\varepsilon(\tilde{\eta}_\varepsilon) - E_0/\varepsilon \leq C$ . In particular, for minimizers  $\eta^\varepsilon$ , there holds

$$\frac{E((\eta_1^\varepsilon)^2, (\eta_2^\varepsilon)^2) - E_0}{\varepsilon} \leq F_\varepsilon(\eta^\varepsilon) - \frac{E_0}{\varepsilon} \leq C$$

so that (4) implies

$$\left\| \eta^\varepsilon - \left( \sqrt{\rho_1^0}, \sqrt{\rho_2^0} \right) \right\|_2 \leq C (E((\eta_1^\varepsilon)^2, (\eta_2^\varepsilon)^2) - E_0)^{1/4} \leq C \varepsilon^{1/4}.$$

□

### 3 The crossover case

We now study what happens when  $g = 1 + \xi\varepsilon$  for some  $\xi > 0$ . For  $\eta$  satisfying (5), the energy then reads

$$F_\varepsilon(\eta_1, \eta_2) = \varepsilon \int_{\mathbf{R}^2} |\nabla\eta_1|^2 + |\nabla\eta_2|^2 + \frac{1}{\varepsilon} \left[ \int_{\mathbf{R}^2} \frac{1}{2}\eta_1^4 + \frac{1}{2}\eta_2^4 + K\eta_1^2\eta_2^2 + V(\eta_1^2 + \eta_2^2) \right] + \frac{\xi}{2} \int_{\mathbf{R}^2} \eta_2^4.$$

Let us first rewrite the energy in a more convenient way. For this, let  $\bar{\eta}_\varepsilon$  be the minimizer of

$$G_\varepsilon(\eta) = \varepsilon \int_{\mathbf{R}^2} |\nabla\eta|^2 + \frac{1}{\varepsilon} \int_{\mathbf{R}^2} \left[ \frac{1}{2}\eta^4 + V\eta^2 \right]$$

under the constraint

$$\int_{\mathbf{R}^2} \eta^2 = \bar{\alpha},$$

where  $\bar{\alpha} = \alpha_1 + \alpha_2$ . It is well known [25, 26] that  $\bar{\eta}_\varepsilon^2$  converges to

$$\bar{\rho} = (R^2 - V(x))_+$$

where  $R$  is such that  $\int_{\mathbf{R}^2} \bar{\rho} = 1$ . We denote by  $\mathcal{D}$  the support of  $\bar{\rho}$  (which is  $B_R$  when  $V = |x|^2$ ). We first rewrite the energy in a more convenient form.

**Proposition 3.1.** *For  $u = (u_1, u_2)$  a pair of non-negative functions, let*

$$\tilde{F}_\varepsilon(u) = \varepsilon \int_{\mathbf{R}^2} \bar{\eta}_\varepsilon^2 (|\nabla u_1|^2 + |\nabla u_2|^2) + \frac{1}{\varepsilon} \int_{\mathbf{R}^2} \left( \frac{1}{2} \bar{\eta}_\varepsilon^4 (1 - (u_1^2 + u_2^2))^2 + (K-1) \bar{\eta}_\varepsilon^4 u_1^2 u_2^2 \right)$$

then if  $\eta = \bar{\eta}_\varepsilon u$ , there holds

$$F_\varepsilon(\eta) = G_\varepsilon(\bar{\eta}_\varepsilon) + \tilde{F}_\varepsilon(u) + \int_{\mathbf{R}^2} \frac{\xi}{2} \bar{\eta}_\varepsilon^4 u_1^4$$

*Proof.* The proof follows as in [4]. We use the Lassoued-Mironescu trick [29] and write  $\eta_1 = \bar{\eta}_\varepsilon u_1$ ,  $\eta_2 = \bar{\eta}_\varepsilon u_2$  to get

$$|\nabla\eta_1|^2 + |\nabla\eta_2|^2 = (u_1^2 + u_2^2) |\nabla\bar{\eta}_\varepsilon|^2 + \bar{\eta}_\varepsilon \nabla\bar{\eta}_\varepsilon \cdot \nabla(u_1^2 + u_2^2) + \bar{\eta}_\varepsilon^2 (|\nabla u_1|^2 + |\nabla u_2|^2) \quad (38)$$

The function  $\bar{\eta}_\varepsilon$  solves the Euler-Lagrange equation

$$-\varepsilon \Delta \bar{\eta}_\varepsilon + \frac{1}{\varepsilon} (\bar{\eta}_\varepsilon^3 + V(x)\bar{\eta}_\varepsilon) = \lambda_\varepsilon \bar{\eta}_\varepsilon \quad (39)$$

where  $\lambda_\varepsilon$  is some constant. Multiplying the equation (39) by  $\bar{\eta}_\varepsilon (u_1^2 + u_2^2)$ , integrating and using integration by parts we get

$$\varepsilon \int_{\mathbf{R}^2} (u_1^2 + u_2^2) |\nabla\bar{\eta}_\varepsilon|^2 + \bar{\eta}_\varepsilon \nabla\bar{\eta}_\varepsilon \cdot \nabla(u_1^2 + u_2^2) = \lambda_\varepsilon \bar{\alpha} - \frac{1}{\varepsilon} \int_{\mathbf{R}^2} \bar{\eta}_\varepsilon^4 (u_1^2 + u_2^2) + V \bar{\eta}_\varepsilon^2 (u_1^2 + u_2^2).$$

On the other hand, multiplying (39) by  $\bar{\eta}_\varepsilon$  and integrating, we find

$$\lambda_\varepsilon \bar{\alpha} = G_\varepsilon(\bar{\eta}_\varepsilon) + \frac{1}{2\varepsilon} \int_{\mathbf{R}^2} \bar{\eta}_\varepsilon^4$$

so that (38) leads to

$$\int_{\mathbf{R}^2} |\nabla\eta_1|^2 + |\nabla\eta_2|^2 = \int_{\mathbf{R}^2} \varepsilon \bar{\eta}_\varepsilon^2 (|\nabla u_1|^2 + |\nabla u_2|^2) + \frac{1}{\varepsilon} \left( \frac{1}{2} \bar{\eta}_\varepsilon^4 (1 - 2(u_1^2 + u_2^2)) - V \bar{\eta}_\varepsilon^2 (u_1^2 + u_2^2) \right) + G_\varepsilon(\bar{\eta}_\varepsilon)$$

and therefore

$$\begin{aligned}
F_\varepsilon(\eta_1, \eta_2) &= G_\varepsilon(\bar{\eta}_\varepsilon) + \int_{\mathbf{R}^2} \frac{\xi}{2} \bar{\eta}_\varepsilon^4 u_1^4 + \varepsilon \bar{\eta}_\varepsilon^2 (|\nabla u_1|^2 + |\nabla u_2|^2) \\
&\quad + \frac{1}{\varepsilon} \left( \frac{1}{2} \bar{\eta}_\varepsilon^4 (1 - 2(u_1^2 + u_2^2) + 2u_1^2 u_2^2 + (u_1^4 + u_2^4) + (K-1) \bar{\eta}_\varepsilon^4 u_1^2 u_2^2) \right) \\
&= G_\varepsilon(\bar{\eta}_\varepsilon) + \int_{\mathbf{R}^2} \frac{\xi}{2} \bar{\eta}_\varepsilon^4 u_1^4 + \varepsilon \int_{\mathbf{R}^2} \bar{\eta}_\varepsilon^2 (|\nabla u_1|^2 + |\nabla u_2|^2) \\
&\quad + \frac{1}{\varepsilon} \int_{\mathbf{R}^2} \left( \frac{1}{2} \bar{\eta}_\varepsilon^4 (1 - (u_1^2 + u_2^2))^2 + (K-1) \bar{\eta}_\varepsilon^4 u_1^2 u_2^2 \right) \\
&= G_\varepsilon(\bar{\eta}_\varepsilon) + \tilde{F}_\varepsilon(u_1, u_2) + \int_{\mathbf{R}^2} \frac{\xi}{2} \bar{\eta}_\varepsilon^4 u_1^4,
\end{aligned}$$

which completes the proof.  $\square$

For  $E$  a set of locally finite perimeter in  $\mathcal{D}$  (see [6, 30]), let

$$\mathcal{F}(E) = \int_{\partial E} \bar{\rho}^{3/2} \quad \text{and} \quad \mathcal{V}(E) = \int_E \bar{\rho}.$$

It is proved in [24] (see also [4]) that for all  $p < \infty$ ,  $\tilde{F}_\varepsilon$   $L^p - \Gamma$  converges to the functional

$$\mathcal{G}_0(u) = \begin{cases} \sigma_K \mathcal{F}(E) & \text{if } u_1 = \chi_E, u_2 = \chi_{E^c} \text{ and } \mathcal{V}(E) = \alpha_1, \\ +\infty & \text{otherwise.} \end{cases}$$

where  $\sigma_K$  is defined by the one dimensional optimal transition problem

$$\sigma_K = \inf \left\{ \int_{\mathbf{R}} |\eta_1'|^2 + |\eta_2'|^2 + \frac{1}{2} (\eta_1^2 + \eta_2^2 - 1)^2 + (K-1) \eta_1^2 \eta_2^2 : \lim_{-\infty} \eta_1 = 0, \lim_{+\infty} \eta_1 = 1 \right\}.$$

Since  $\frac{\xi}{2} \int \bar{\eta}_\varepsilon^4 u_1^4$  is a continuous perturbation of  $\tilde{F}_\varepsilon(u_1, u_2)$ , we immediately obtain the following result.

**Theorem 3.2.** *For every  $p < \infty$ , the functional  $\tilde{F}_\varepsilon(u_1, u_2) + \frac{\xi}{2} \int \bar{\eta}_\varepsilon^4 u_1^4$ ,  $L^p - \Gamma$  converges to*

$$\mathcal{G}_\xi(u) = \begin{cases} \sigma_K \mathcal{F}(E) + \frac{\xi}{2} \int_{E^c} \bar{\rho}^2 & \text{if } u_1 = \chi_E, u_2 = \chi_{E^c} \text{ and } \mathcal{V}(E) = \alpha_1, \\ +\infty & \text{otherwise.} \end{cases}$$

Up to dividing  $\mathcal{G}_\xi$  by  $\sigma_K$  and modifying  $\xi$ , we can assume that  $\sigma_K = 1$ . If  $u_1 = \chi_E, u_2 = \chi_{E^c}$ , we will, by a slight abuse of notation denote  $\mathcal{G}_\xi(u)$  by  $\mathcal{G}_\xi(E)$ . We now want to study the minimizers of  $\mathcal{G}_\xi$  (whose existence follows from the Direct Method). As in [20], by making a spherical symmetrization, we can restrict the analysis to spherically symmetric sets.

**Proposition 3.3.** *For every minimizer  $E_\xi$  of  $\mathcal{G}_\xi$ , there exists an half line  $\ell$  starting from zero such that for every  $r > 0$ ,  $\partial B_r \cap E_\xi$  is an arc of circle centered in  $\ell$ . Moreover,  $\partial E_\xi \cap \mathcal{D}$  is a  $C^\infty$  hypersurface.*

*Proof.* Let  $E_\xi$  be a minimizer of  $\mathcal{G}_\xi$ . The symmetry follows as in [20, Th. 3.2] since spherical symmetrization reduces  $\mathcal{F}$  and leaves  $\int_{E^c} \bar{\rho}^2$  invariant. The regularity of  $\partial E_\xi$  is a consequence of the regularity theory for minimal surfaces. Indeed, since  $\bar{\rho}$  is locally bounded away from zero in  $\mathcal{D}$ , any minimizer of  $\mathcal{G}_\xi$  is locally a quasi-minimizer of the perimeter (see [20, Th. 3.2] again or [30, 18] for instance). From this, one can infer  $C^{1,\alpha}$  regularity of  $E_\xi$ . Since  $\bar{\rho}$  is smooth in  $\mathcal{D}$  further regularity follows.  $\square$



When  $\xi$  is small, the perimeter term is dominant. In this case, we are in a situation similar to the one studied in [4, 24].

**Proposition 3.4.** *There exists  $\alpha_0 \in [0, 1/2)$  such that for every  $\alpha_1 \in (\alpha_0, \bar{\alpha} - \alpha_0]$ , there exists  $\xi_0(\alpha_1)$  such that for every  $\xi \leq \xi_0(\alpha_1)$ , the minimizer of  $\mathcal{G}_\xi$  is not radially symmetric.*

*Proof.* It is proved in [4] that for every such  $\alpha_1$ , the minimizer of  $\mathcal{F}$  under volume constraint is not radially symmetric and thus there exists  $E_{\alpha_1}$  with

$$\mathcal{F}(E_{\alpha_1}) < \inf_{\mathcal{V}(F)=\alpha_1} \{ \mathcal{F}(F) : F \text{ radially symmetric} \}.$$

but since

$$\mathcal{G}_\xi(E_{\alpha_1}) = \mathcal{F}(E_{\alpha_1}) + \xi \int_{E_{\alpha_1}^c} \bar{\rho}^2$$

it is clear that

$$\mathcal{G}_\xi(E_{\alpha_1}) < \inf_{\mathcal{V}(F)=\alpha_1} \{ \mathcal{F}(F) : F \text{ radially symmetric} \} < \inf \{ \mathcal{G}_\xi(F) : F \text{ radially symmetric} \}$$

for  $\xi$  small enough.  $\square$

We now study the situation of large  $\xi$ . Our main result is a rigidity result stating that for large (but not infinite)  $\xi$ , the unique minimizer is the centered ball.

**Theorem 3.5.** *For every  $\alpha_1 \in (0, \bar{\alpha})$ , there exists  $\xi_1(\alpha)$  such that for every  $\xi \geq \xi_1(\alpha)$ , the unique minimizer of  $\mathcal{G}_\xi$  is the centered ball  $\tilde{B}$  such that  $\mathcal{V}(\tilde{B}) = \alpha_1$ .*

In the rest of the section,  $\alpha_1$  is fixed. In order to ease notation, we assume that the unit ball is such that  $\mathcal{V}(B) = \alpha_1$  (the general case follows by dilation). Theorem 3.5 follows from a combination of two results. The first one is a stability result for the volume term  $\int_{E^c} \bar{\rho}^2$ :

**Proposition 3.6.** *There exists  $c > 0$  such that for every set  $E$  with  $\mathcal{V}(E) = \mathcal{V}(B)$ ,*

$$\int_{E^c} \bar{\rho}^2 - \int_{B^c} \bar{\rho}^2 \geq c \left( \int_{E^c \Delta B^c} \bar{\rho} \right)^2 \quad (40)$$

*Proof.* The proof resembles the proof of (29). Let  $E$  be such that  $\mathcal{V}(E) = \mathcal{V}(B)$  then

$$\int_{E^c} \bar{\rho}^2 - \int_{B^c} \bar{\rho}^2 = \int_{E^c \cap B} \bar{\rho}(1 - |x|^2) + \int_{B^c \cap E} \bar{\rho}(|x|^2 - 1)$$

Let

$$\bar{V} = \int_{E^c \cap B} \bar{\rho}$$

and  $\delta$  be such that

$$\int_{B \setminus B_{1-\delta}} \bar{\rho} = \bar{V}.$$

Letting  $F_\delta = B \setminus B_{1-\delta}$ , as in Lemma (2.5), we have

$$\int_{E^c \cap B} \bar{\rho}(1 - |x|^2) - \int_{F_\delta} \bar{\rho}(1 - |x|^2) \geq \left( \inf_{(E^c \cap B) \setminus F_\delta} (1 - |x|^2) - \sup_{F_\delta \setminus E^c} (1 - |x|^2) \right) \int_{F_\delta \setminus E^c} \bar{\rho} \geq 0$$

and therefore,  $F_\delta$  minimizes  $\int_G \bar{\rho}(1 - |x|^2)$  among sets  $G \subset B$  with  $\int_G \bar{\rho} = \bar{V}$  so that

$$\int_{E^c \cap B} \bar{\rho}(1 - |x|^2) \geq \int_{F_\delta} \bar{\rho}(1 - |x|^2) \geq c\bar{V}^2 \quad (41)$$

Similarly, letting  $F_{\bar{\delta}} = B_{1+\bar{\delta}} \cap B^c$  with  $\int_{F_{\bar{\delta}}} \bar{\rho} = \bar{V} = \int_{B^c \cap E} \bar{\rho}$ , then  $F_{\bar{\delta}}$  minimizes  $\int_G \bar{\rho}(|x|^2 - 1)$  among  $G \subset B^c$  with  $\int_G \bar{\rho} = \bar{V}$  and thus

$$\int_{B^c \cap E} \bar{\rho}(|x|^2 - 1) \geq \int_{F_{\bar{\delta}}} \bar{\rho}(|x|^2 - 1) \geq C\bar{V}^2.$$

Together with (41), this gives (40).  $\square$

The second is an estimate on the possible instability of the ball for  $\mathcal{F}$ .

**Proposition 3.7.** *There exist  $\varepsilon > 0$  and  $C > 0$  such that for every set  $E$  with  $\mathcal{V}(E) = \mathcal{V}(B)$  and  $\int_{E \Delta B} \bar{\rho} \leq \varepsilon$*

$$\mathcal{F}(E) - \mathcal{F}(B) \geq -C \left( \int_{E \Delta B} \bar{\rho} \right)^2. \quad (42)$$

Since the proof of Proposition 3.7 is long and involved, we postpone it. Let us show first how Proposition 3.6 and Proposition 3.7 yield together Theorem 3.5. Let  $E_\xi$  be a minimizer of  $\mathcal{G}_\xi$  then using  $B$  as competitor, we obtain thanks to (40) and (42),

$$\begin{aligned} \left( \int_{E_\xi \Delta B} \bar{\rho} \right)^2 &\leq \frac{1}{c} \left( \int_{E_\xi^c} \bar{\rho}^2 - \int_{B^c} \bar{\rho}^2 \right) = \frac{1}{c\xi} ([\mathcal{G}_\xi(E_\xi) - \mathcal{F}(E_\xi)] - [\mathcal{G}_\xi(B) - \mathcal{F}(B)]) \\ &\leq \frac{1}{c\xi} (\mathcal{F}(B) - \mathcal{F}(E_\xi)) \leq \frac{C}{c\xi} \left( \int_{E_\xi \Delta B} \bar{\rho} \right)^2. \end{aligned}$$

This implies  $\left( \int_{E_\xi \Delta B} \bar{\rho} \right)^2 = 0$  for  $\xi > C/c$  and concludes the proof of Theorem 3.5.

Before going into the proof of Proposition 3.7, let us comment a bit on the statement and give the strategy for proving it. As explained below, in general, we do not expect the ball to be a local minimizer of  $\mathcal{F}$  (and in particular, we cannot expect (42) to be true with a plus sign on the right-hand side). However, (42) shows that in some sense, the Hessian of  $\mathcal{F}$  at the ball is bounded from below. The proof of (42) follows the strategy of [15] for proving the quantitative isoperimetric inequality (see also [1, 20]). Inequality (42) is first shown for nearly spherical sets, borrowing ideas from [21]. The proof is then finished by arguing by contradiction, constructing a sequence  $(E_n)$  converging to the ball and contradicting the inequality. Using a Selection Principle and regularity theory for minimal surfaces, it is possible to replace  $(E_n)$  by a better sequence  $(F_n)$ , still contradicting (42) but converging in a much stronger way to the ball. Since for  $n$  large enough  $F_n$  are nearly spherical, we reach a contradiction and Proposition 3.7 is proved.

We thus start by proving (42) for nearly spherical sets (in the sense of Fuglede [21]).

**Proposition 3.8.** *There exist  $\varepsilon_0 > 0$  and  $C_0 > 0$  such that if  $\varepsilon \leq \varepsilon_0$ , every set  $\partial E = \{(1+u(x))x : x \in \partial B\}$  with  $\|u\|_{Lip} \leq \varepsilon$  and  $\mathcal{V}(E) = \mathcal{V}(B)$  satisfies*

$$\mathcal{F}(E) - \mathcal{F}(B) \geq -C_0 \left( \int_{E \Delta B} \bar{\rho} \right)^2.$$

*Proof.* Recall that  $\bar{\rho}(x) = (R^2 - |x|^2)_+$  with  $R > 1$ . The condition  $\int_E \bar{\rho} = \int_B \bar{\rho}$  can be written as

$$\int_{\partial B} \frac{1}{2} R^2 ((1+u)^2 - 1) - \frac{1}{4} ((1+u)^4 - 1) = 0$$

which leads to

$$\int_{\partial B} u = \frac{1}{2} \frac{3 - R^2}{R^2 - 1} \int_{\partial B} u^2 + o\left(\|u\|_{L^2(\partial B)}^2\right). \quad (43)$$

We can now compute the energy

$$\begin{aligned} \frac{1}{(R^2 - 1)^{3/2}} (\mathcal{F}(E) - \mathcal{F}(B)) &= \int_{\partial B} \left(1 - \frac{2u + u^2}{R^2 - 1}\right)^{3/2} (1 + u) \left(1 + \frac{|\nabla u|^2}{(1 + u)^2}\right)^{1/2} - 1 \\ &= \int_{\partial B} u \left(1 - \frac{3}{R^2 - 1}\right) + \frac{3}{2} \frac{4 - 3R^2}{(R^2 - 1)^2} u^2 + \frac{1}{2} \|\nabla u\|_{L^2(\partial B)}^2 + o\left(\|u\|_{L^2(\partial B)}^2\right) \end{aligned}$$

Using (43), this turns into

$$\begin{aligned} \frac{1}{(R^2 - 1)^{3/2}} (\mathcal{F}(E) - \mathcal{F}(B)) &= \frac{1}{2} \int_{\partial B} -\frac{R^2(2 + R^2)}{(R^2 - 1)^2} u^2 + \|\nabla u\|_{L^2(\partial B)}^2 + o\left(\|u\|_{L^2(\partial B)}^2\right) \\ &\geq \frac{1}{2} \|\nabla u\|_{L^2(\partial B)}^2 - C \|u\|_{L^2(\partial B)}^2. \end{aligned} \quad (44)$$

We now claim that for every  $\delta > 0$ , there exists  $\Lambda_\delta > 0$  such that

$$\int_{\partial B} u^2 \leq \delta \int_{\partial B} |\nabla u|^2 + \Lambda_\delta \left(\int_{\partial B} |u|\right)^2. \quad (45)$$

Indeed, denoting by  $\bar{u} = \frac{1}{2\pi} \int_{\partial B} u$  and using Sobolev embedding and Young inequality, we compute

$$\begin{aligned} \int_{\partial B} u^2 &= \int_{\partial B} (u - \bar{u})u + 2\pi(\bar{u})^2 \leq \|u - \bar{u}\|_{L^\infty(\partial B)} \|u\|_{L^1(\partial B)} + (2\pi)^{-1} \|u\|_{L^1(\partial B)}^2 \\ &\lesssim \|\nabla u\|_{L^2(\partial B)} \|u\|_{L^1(\partial B)} + \|u\|_{L^1(\partial B)}^2 \\ &\leq \delta \|\nabla u\|_{L^2(\partial B)}^2 + \Lambda_\delta \|u\|_{L^1(\partial B)}^2. \end{aligned}$$

This proves (45). Using (45) in (44), we obtain,

$$\mathcal{F}(E) - \mathcal{F}(B) \geq -C \left(\int_{\partial B} |u|\right)^2.$$

Now, since for nearly spherical sets, there holds

$$\int_{E \Delta B} \bar{\rho} \simeq |E \Delta B| \simeq \int_{\partial B} |u|,$$

we can finally conclude the proof.  $\square$

**Remark 3.9.** In the proof of Proposition 3.8, if we assume that  $E$  is centered, that is

$$\int_E \bar{\rho}(x)x = 0,$$

we could do as in [21, 15] and decompose  $u$  in Fourier series on  $\partial B$  to get

$$u = \sum_k u_k Y_k$$

with  $Y_0 = 1$  and  $Y_1 = x \cdot \nu$  for an appropriately chosen  $\nu \in \partial B$ . So that

$$\int_{\partial B} \left(\frac{R^2}{3}(1 + u)^3 - \frac{1}{5}(1 + u)^5 - \left(\frac{R^2}{3} - \frac{1}{5}\right)\right) Y_1 = \int_E \bar{\rho}(x)x \cdot \nu - \left(\frac{R^2}{3} - \frac{1}{5}\right) \int_{\partial B} x \cdot \nu = 0.$$

We find thanks to (43)

$$u_0 = \int_{\partial B} u = O\left(\|u\|_{L^2(\partial B)}^2\right) \quad u_1 = \int_{\partial B} u Y_1 = O\left(\|u\|_{L^2(\partial B)}^2\right).$$

From this, letting  $\mu_k = k^2$  being the  $k$ -th eigenvalue of the Laplacian on  $\partial B$ ,

$$\int_{\partial B} -\frac{R^2(2+R^2)}{(R^2-1)^2} u^2 + |\nabla u|^2 = \sum_{k \geq 2} u_k^2 \left(\mu_k - \frac{R^2(2+R^2)}{(R^2-1)^2}\right) + o\left(\|u\|_2^2\right).$$

Since  $\mu_k \geq 4$  for  $k \geq 2$ , and since

$$4 - \frac{R^2(2+R^2)}{(R^2-1)^2} > 0 \quad \iff \quad \left[ R < \frac{1}{\sqrt{3}}(5 - \sqrt{13})^{1/2} \quad \text{or} \quad R > \frac{1}{\sqrt{3}}(5 + \sqrt{13})^{1/2} \right],$$

we expect the ball to be unstable for  $R \in ((5 - \sqrt{13})^{1/2}/\sqrt{3}, (5 + \sqrt{13})^{1/2}/\sqrt{3})$  and unstable otherwise.

In order to go further, we need an isoperimetric inequality which is, we believe, of independent interest.

**Lemma 3.10.** *There exists  $c = c(\alpha) > 0$  such that for every set  $E$  with  $\mathcal{V}(E) \leq \alpha/2$ , we have*

$$\mathcal{F}(E) \geq c \mathcal{V}(E)^{5/6}. \quad (46)$$

*Proof.* Up to a dilation, we assume here that  $R = 1$  that is  $\bar{\rho}(x) = (1 - |x|^2)_+$ . Since the estimate is sublinear, it is moreover enough to prove it for connected sets. Let us first show that we can assume that  $E \subset B_{1/2}^c$ . Indeed, if this is not the case then there exists  $x \in \partial E \cap B_{1/2}$ . Then, there are two possibilities. Either  $E \subset B_{3/4}$  and the classical isoperimetric inequality already gives (46) or there is  $y \in \partial E \cap B_{3/4}^c$ . Since  $\partial E$  is connected it contains a path from  $x \in B_{1/2}$  to  $y \in B_{3/4}^c$  and  $\mathcal{F}(E)$  is bounded from below by  $\int_{1/2}^{3/4} (1 - s^2) = 29/192$ . Again, this gives (46). As in Proposition 3.3, we can make a spherical rearrangement and assume further that  $E$  is spherically symmetric.

We now transform our problem in order to work on a periodic strip. Let  $S^1$  be the unit torus and consider the diffeomorphism  $\phi : B_1 \setminus B_{1/2} \rightarrow S^1 \times (0, 1/2)$  given by

$$\phi(\theta, r) = \left( \frac{\theta}{2\pi}, 1 - r \right).$$

For  $F \subset S^1 \times (0, 1/2)$ , we let

$$\mathcal{V}_1(F) = \int_0^1 \int_0^{1/2} \chi_F(x, y) y \, dy dx \quad \text{and} \quad \mathcal{F}_1(F) = \int_{\partial F} y^{3/2}$$

then, since for  $E \subset B_1 \setminus B_{1/2}$ ,

$$\mathcal{V}(E) \sim \mathcal{V}_1(\phi(E)) \quad \text{and} \quad \mathcal{F}(E) \sim \mathcal{F}_1(\phi(E)),$$

we are left to prove

$$\mathcal{V}_1(F) \lesssim \mathcal{F}_1(F)^{6/5} \quad \forall F \subset S^1 \times (0, 1/2). \quad (47)$$

Notice also that since the sets  $E \subset B_1 \setminus B_{1/2}$  we started with were spherically symmetric, we can further assume that for every  $\bar{y} \in (0, 1/2)$ ,  $F \cap \{y = \bar{y}\}$  is a segment centered in  $0 \times \{\bar{y}\}$  for

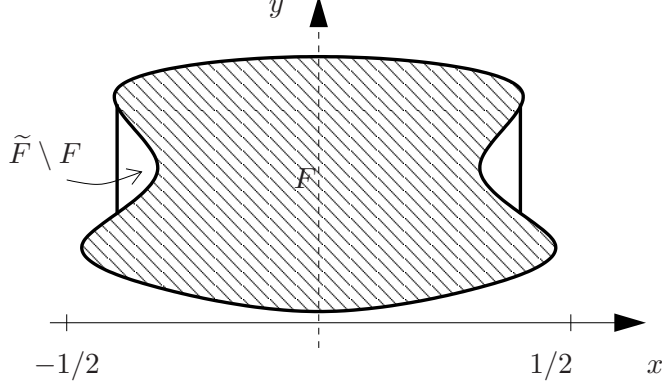


Figure 3: Convexification step in the  $y$ -direction.

instance. We then make a convexification step. We define  $\tilde{F}$  as the smallest set which contains  $F$  and which is  $y$ -convex, that is for every  $x \in S^1$  the set  $\{y > 0 : (x, y) \in \tilde{F}\}$  is a segment (see Figure 3). We have

$$\mathcal{V}_1(\tilde{F}) \geq \mathcal{V}_1(F) \quad \text{and} \quad \mathcal{F}_1(\tilde{F}) \leq \mathcal{F}_1(F).$$

Indeed, the first inequality comes from  $F \subset \tilde{F}$  and the second from the fact that for every  $x \in S^1$ , the shortest (weighted) path between the points  $(x, y_1)$  and  $(x, y_2)$  is the straight segment. It is therefore enough to prove (47) for the sets  $\tilde{F}$ . After these two symmetrizations, the set  $\tilde{F}$  is contained between two graphs  $y_1 : (-T, T] \rightarrow (0, 1/2)$  and  $y_2 : (-T, T] \rightarrow (0, 1/2)$  for some  $0 < T \leq 1/2$  i.e.

$$\tilde{F} = \{(x, y) : -T < x \leq T, y_1(x) \leq y \leq y_2(x)\}.$$

Moreover,  $y_1$  and  $y_2$  are even,  $y_1$  is non-decreasing and  $y_2$  is non-increasing in  $[0, T]$ . Using this parameterization, we have

$$\mathcal{V}_1(\tilde{F}) = \frac{1}{2} \int_{-T}^T [y_2^2 - y_1^2] \quad \text{and} \quad \mathcal{F}_1(\tilde{F}) = \int_{-T}^T y_1^{3/2} \sqrt{1 + |Dy_1|^2} + \int_{-T}^T y_2^{3/2} \sqrt{1 + |Dy_2|^2},$$

where for a function  $y$  of locally bounded variation,

$$\mathcal{F}_1(y) = \int_{-T}^T y^{3/2} \sqrt{1 + |Dy|^2} := \int_{-T}^T y^{3/2} \sqrt{1 + |y'|^2} + \sum_{x \in J_y} |y(x^+)^{5/2} - y(x^-)^{5/2}| + \int_{-T}^T y^{3/2} d|D^c y|,$$

see [6, Th. 5.54]. Notice that we can assume that  $y_1(\pm T) = y_2(\pm T)$ . Indeed, if  $T = 1/2$ , then  $\mathcal{F}_1(\tilde{F}) \gtrsim y_2(1/2)^{3/2}$ . In this case, we can add a vertical cut between  $y_2(1/2)$  and  $y_1(1/2)$ . The additional contribution to the weighted perimeter is of order of  $y_2(1/2)^{5/2} - y_1(1/2)^{5/2}$  which is controlled by  $\mathcal{F}_1(\tilde{F})$ . Eventually, since  $\mathcal{F}_1$  is the relaxed functional of its restriction to  $C^1$  paths with respect to  $L^1$  convergence [6, Th. 5.54], we can further assume that  $y_1$  and  $y_2$  are smooth. After, these symmetrisation and regularisation steps, we can write,

$$\mathcal{F}_1(\tilde{F}) \sim \int_{-T}^T y_2^{3/2} + \int_{-T}^T y_1^{3/2} + y_2(0)^{5/2} - y_1(0)^{5/2}. \quad (48)$$

We then consider two cases. First, if  $y_1(0) \ll y_2(0)$ , then (48) yields

$$\mathcal{F}_1(\tilde{F}) \gtrsim \int_{-T}^T y_2^{3/2} + y_2(0)^{5/2}$$

from which we deduce

$$\mathcal{V}_1(\tilde{F}) \lesssim \int_{-T}^T y_2^2 \leq y_2(0)^{1/2} \int_{-T}^T y_2^{3/2} \lesssim \mathcal{F}_1(\tilde{F})^{1/5} \mathcal{F}_1(\tilde{F}) \lesssim \mathcal{F}_1(\tilde{F})^{6/5}.$$

Then, in the case  $y_1(0) \sim y_2(0)$ , by convexity  $y_2(0)^{5/2} - y_1(0)^{5/2} \geq \frac{5}{2}y_1(0)^{3/2}(y_2(0) - y_1(0))$  so that (48) becomes

$$\mathcal{F}_1(\tilde{F}) \gtrsim T y_2^{3/2}(0) + y_2(0)^{3/2}(y_2(0) - y_1(0)).$$

We infer that

$$\begin{aligned} \mathcal{V}_1(\tilde{F}) &= \int_{-T}^T (y_2 - y_1)(y_2 + y_1) \leq 4T y_2(0)(y_2(0) - y_1(0)) \\ &\leq 4T y_2(0) y_2(0)^{4/5} (y_2(0) - y_1(0))^{1/5} = 4T y_2(0)^{3/2} \left( y_2(0)^{3/2} (y_2(0) - y_1(0)) \right)^{1/5} \\ &\lesssim \mathcal{F}_1(\tilde{F}) \mathcal{F}_1(\tilde{F})^{1/5} \lesssim \mathcal{F}_1(\tilde{F})^{6/5}, \end{aligned}$$

which concludes the proof of (47). □

**Remark 3.11.** The exponent  $5/6$  in (46) can be easily seen to be optimal by considering as competitor a small ball touching the boundary of  $B$ .

We can now prove the following  $\varepsilon$ -regularity result:

**Proposition 3.12.** *Let  $\Lambda > 0$ . Then, there exists  $\varepsilon_1 > 0$  such that if  $E$  is a  $\Lambda$ -minimizer of  $\mathcal{F}$ , i.e. for every  $G$ ,*

$$\mathcal{F}(E) \leq \mathcal{F}(G) + \Lambda \int_{E \Delta G} \bar{\rho}$$

and if  $\int_{E \Delta B} \bar{\rho} \leq \varepsilon_1$  then  $E$  is nearly spherical i.e.  $\partial E = \{(1 + u(x))x : x \in \partial B\}$  and  $\|u\|_{C^{1,\alpha}} \leq \varepsilon_0$  (where  $\varepsilon_0$  is the one defined in Proposition 3.8).

*Proof.* Fix  $\delta > 0$  then inside  $B_{1+\delta}$  by classical  $\varepsilon$ -regularity results for quasi-minimizers of the perimeter (see [18, Th. 6.1] or [30, Th. II.6.3] for instance as well as [15, Lem. 3.6]), if  $\int_{(E \cap B_{1+\delta}) \Delta B} \bar{\rho} \leq \varepsilon_1$  then  $\partial E \cap B_{1+\delta}$  is a small  $C^{1,\alpha}$  perturbation of  $\partial B$  in particular, if  $\int_{E \Delta B} \bar{\rho} \leq \varepsilon_1$ ,  $\partial E \cap B_{1+\delta} \subset B_{1+\frac{\delta}{2}}$  and  $E$  can be written as  $E = E_1 \cup E_2$  where  $E_1$  is nearly spherical,  $E_2 \subset B_{1+\delta}^c$ . By testing the  $\Lambda$ -minimality of  $E$  against  $E_1$ , we find by (46)

$$c\mathcal{V}(E_2)^{5/6} \leq \mathcal{F}(E_2) \leq \Lambda\mathcal{V}(E_2)$$

and thus if  $\mathcal{V}(E_2) \neq 0$ ,

$$\left(\frac{c}{\Lambda}\right)^6 \leq \mathcal{V}(E_2) \leq \varepsilon_1$$

which is absurd for  $\varepsilon_1$  small enough. □

We will also need the following simple lemma which is a weak version of (42).

**Lemma 3.13.** *There exists  $\Lambda_1 > 0$  such that for every set  $E$ ,*

$$\mathcal{F}(E) - \mathcal{F}(B) \geq -\Lambda_1 \int_{E \Delta B} \bar{\rho}. \quad (49)$$

*Proof.* Let  $v$  be a vector field with  $\text{supp } v \subset B_R$ ,  $|v| \leq 1$ ,  $v = x$  on  $\partial B$  and  $\|\text{div } v\|_\infty \leq C$  and let

$$w = \frac{3}{2}\bar{\rho}^{-1/2}\nabla\bar{\rho} \cdot v + \bar{\rho}^{1/2}\text{div } v,$$

so that  $\text{div}(\bar{\rho}^{3/2}v) = \bar{\rho}w$ . Then, for every set  $E$ , denoting by  $\nu^E$  the outward normal to  $E$ ,

$$\begin{aligned} \mathcal{F}(E) - \mathcal{F}(B) &\geq \int_{\partial E} \bar{\rho}^{3/2}v \cdot \nu^E - \int_{\partial B} \bar{\rho}^{3/2}v \cdot \nu^B \\ &= \int_E \bar{\rho}w - \int_B \bar{\rho}w \geq -\|w\|_\infty \int_{E\Delta B} \bar{\rho}. \end{aligned}$$

□

We can finally prove (42).

*Proof.* We argue by contradiction. Assume that there exists a sequence of measurable sets  $(E_n)$  with  $\int_{E_n\Delta B} \bar{\rho} \rightarrow 0$  and

$$\mathcal{F}(E_n) + 2C_0 \left( \int_{E_n\Delta B} \bar{\rho} \right)^2 \leq \mathcal{F}(B), \quad (50)$$

where  $C_0$  is the constant given by Proposition 3.8. Let  $\varepsilon_n = \int_{E_n\Delta B} \bar{\rho}$  and for  $\Lambda_1 > 0$  given by Lemma 3.13 and  $\Lambda_2 > 0$ , let  $F_n$  be a minimizer of

$$\mathcal{F}(F) + 2\Lambda_1 \left( \left[ \varepsilon_n - \int_{F\Delta B} \bar{\rho} \right]^2 + \varepsilon_n \right)^{1/2} + \Lambda_2 |\mathcal{V}(F) - \mathcal{V}(B)|. \quad (51)$$

*Step 1.* We claim that if  $\Lambda_2$  is large enough,  $\mathcal{V}(F_n) = \mathcal{V}(B)$  for all  $n$ .

To prove this, we follow the approach of [19] (see also [22] for another approach). Assume by contradiction that the claim does not hold, then there exist sequences of positive numbers  $\varepsilon'_k \rightarrow 0$  and  $\Lambda_{2,k} \rightarrow +\infty$ , and a sequence of measurable sets  $G_k \subset \mathbf{R}^2$  such that  $G_k$  minimizes

$$\mathcal{F}(G) + 2\Lambda_1 \left( \left[ \varepsilon'_k - \int_{G\Delta B} \bar{\rho} \right]^2 + \varepsilon'_k \right)^{1/2} + \Lambda_{2,k} |\mathcal{V}(G) - \mathcal{V}(B)| \quad (52)$$

and for instance  $\mathcal{V}(G_k) < \mathcal{V}(B)$  (the other case is similar). In order to get a contradiction, we build a new sequence  $\tilde{G}_k$  such that  $\mathcal{V}(\tilde{G}_k) = \mathcal{V}(B)$  and

$$\mathcal{F}(\tilde{G}_k) + 2\Lambda_1 \left( \left[ \varepsilon'_k - \int_{\tilde{G}_k\Delta B} \bar{\rho} \right]^2 + \varepsilon'_k \right)^{1/2} \lesssim |\mathcal{V}(G_k) - \mathcal{V}(B)|.$$

The construction is a bit delicate since we want all the constants to be uniform in  $\varepsilon'_k$ . If this were not the case, one could have simply used Almgren's construction (see [30]).

First, testing the energy (52) with  $B$ , we find that  $\mathcal{F}(G_k) \leq C$  and therefore, up to extraction,  $G_k$  converges in  $L^1_{loc}(\mathcal{D})$  to some  $G_\infty$ . Moreover, since  $|\mathcal{V}(G_k) - \mathcal{V}(B)| \leq \frac{C}{\Lambda_{2,k}}$ ,  $G_\infty$  satisfies  $\mathcal{V}(G_\infty) = \mathcal{V}(B)$ . The set  $G_\infty$ , minimizes

$$\mathcal{F}(G) + 2\Lambda_1 \int_{G\Delta B} \bar{\rho}$$

under the constraint  $\mathcal{V}(G) = \mathcal{V}(B)$  (this can be seen for instance by a  $\Gamma$ -convergence argument). Therefore, by (49),  $G_\infty = B$ . Notice that arguing as in Proposition 3.3, we get that the sets  $G_k$

are  $C^{1,\alpha}$ . Now, let us fix  $\delta > 0$  and  $r \ll 1$ , let us choose  $\tilde{x}_0 \in \partial B$  and let us set  $x_0 = (1 + \frac{r}{2})\tilde{x}_0$  and  $\bar{C} = (1/2) \int_{B \cap B_r(x_0)} \bar{\rho}$ . Then since  $B_{r/2}(x_0) \cap B = \emptyset$ , there hold

$$\int_{G_k \cap B_{r/2}(x_0)} \bar{\rho} \leq \delta \quad \text{and} \quad \int_{G_k \cap B_r(x_0)} \bar{\rho} > \bar{C} \quad (53)$$

for  $k$  large enough. Let  $0 < \sigma_k < 1/2$  be a sequence to be fixed later and consider the bilipschitz maps:

$$\Phi_k(x_0 + x) = \begin{cases} (1 - 3\sigma_k)x & \text{if } |x| \leq \frac{r}{2}, \\ x + \sigma_k(1 - \frac{r^2}{|x|^2})x & \text{if } \frac{r}{2} \leq |x| < r, \\ x & \text{if } |x| \geq r, \end{cases}$$

and let  $\tilde{G}_k = \Phi_k(E_k)$ .

*Step 1.1.* We first prove that

$$\mathcal{F}(G_k) - \mathcal{F}(\tilde{G}_k) \geq -C\sigma_k \mathcal{F}(G_k). \quad (54)$$

Following the notation of [19], we let for  $x \in \partial G_k$ ,  $T_{k,x}(\tau) = \nabla \Phi_k(x) \circ \tau$  for  $\tau \in \pi_{k,x}$  (where  $\pi_{k,x}$  is the tangent space to  $\partial G_k$  at  $x$ ) and

$$J_1 T_{k,x} = \sqrt{\det(T_{k,x}^* \circ T_{k,x})}$$

be the one Jacobian of  $T_{k,x}$  so that

$$\mathcal{F}(\tilde{G}_k) = \int_{\partial \tilde{G}_k} \bar{\rho}^{3/2}(\Phi_k(x)) J_1 T_{k,x} d\mathcal{H}^1.$$

In particular, it is proved in [19] that  $J_1 T_{k,x} < 1$  in  $B_{r/2}(x_0)$  and

$$J_1 T_{k,x} \leq 1 + 5\sigma_k$$

in  $C_r(x_0) = B_r(x_0) \setminus B_{r/2}(x_0)$ . We can now decompose,

$$\begin{aligned} \mathcal{F}(G_k) - \mathcal{F}(\tilde{G}_k) &= \int_{\partial G_k \cap B_r(x_0)} \bar{\rho}^{3/2} - \int_{\partial G_k \cap B_r(x_0)} \bar{\rho}^{3/2}(\Phi_k(x)) J_1 T_{k,x} d\mathcal{H}^1 \\ &= \int_{\partial G_k \cap C_r(x_0)} \bar{\rho}^{3/2} - \int_{\partial G_k \cap C_r(x_0)} \bar{\rho}^{3/2}(\Phi_k(x)) J_1 T_{k,x} d\mathcal{H}^1 \\ &\quad + \int_{\partial G_k \cap B_{r/2}(x_0)} \bar{\rho}^{3/2} - \int_{\partial G_k \cap B_{r/2}(x_0)} \bar{\rho}^{3/2}(\Phi_k(x)) J_1 T_{k,x} d\mathcal{H}^1. \end{aligned}$$

But since in  $B_r(x_0)$ ,  $|\bar{\rho}^{3/2}(x) - \bar{\rho}^{3/2}(\Phi_k(x))| \leq C\bar{\rho}^{3/2}(x)\sigma_k$ ,

$$\begin{aligned} \int_{\partial G_k \cap C_r(x_0)} \bar{\rho}^{3/2} - \bar{\rho}^{3/2}(\Phi_k(x)) J_1 T_{k,x} d\mathcal{H}^1 &= \int_{\partial G_k \cap C_r(x_0)} (1 - J_1 T_{k,x}) \bar{\rho}^{3/2} \\ &\quad + J_1 T_{n,1} (\bar{\rho}^{3/2}(x) - \bar{\rho}^{3/2}(\Phi_k(x))) d\mathcal{H}^1 \\ &\geq -C\sigma_k \int_{\partial G_k \cap C_r(x_0)} \bar{\rho}^{3/2}. \end{aligned}$$

Similarly,

$$\begin{aligned} \int_{\partial G_k \cap B_{r/2}(x_0)} \bar{\rho}^{3/2} - \bar{\rho}^{3/2}(\Phi_k(x)) J_1 T_{k,x} d\mathcal{H}^1 &= \int_{\partial G_k \cap B_{r/2}(x_0)} (1 - J_1 T_{k,x}) \bar{\rho}^{3/2} \\ &\quad + J_1 T_{n,1} (\bar{\rho}^{3/2}(x) - \bar{\rho}^{3/2}(\Phi_k(x))) d\mathcal{H}^1 \\ &\geq -C\sigma_k \int_{\partial G_k \cap B_{r/2}(x_0)} \bar{\rho}^{3/2}. \end{aligned}$$



Hence, (54) follows.

*Step 1.2.* We now prove that for some  $\kappa > 0$ ,

$$\mathcal{V}(\tilde{G}_k) - \mathcal{V}(G_k) \geq \kappa\sigma_k. \quad (55)$$

If  $J\Phi_k$  denotes the Jacobian of  $\Phi_k$ , it is shown in [19] that

$$J\Phi_k \leq 1 + 8\sigma_k \quad (56)$$

and that in  $C_r(x_0)$ ,

$$J\Phi_k \geq 1 + c\sigma_k \quad (57)$$

for some  $c > 0$ . We can decompose

$$\begin{aligned} \int_{\tilde{G}_k} \bar{\rho} - \int_{G_k} \bar{\rho} &= \int_{G_k \cap B_r(x_0)} \bar{\rho}(\Phi_k(x)) J\Phi_k(x) - \bar{\rho}(x) \\ &= \int_{G_k \cap C_r(x_0)} \bar{\rho}(\Phi_k(x)) J\Phi_k(x) - \bar{\rho}(x) + \int_{G_k \cap B_{r/2}(x_0)} \bar{\rho}(\Phi_k(x)) J\Phi_k(x) - \bar{\rho}(x). \end{aligned}$$

Thanks to (56) and (57), there holds

$$\begin{aligned} \int_{G_k \cap C_r(x_0)} \bar{\rho}(\Phi_k(x)) J\Phi_k(x) - \bar{\rho}(x) &= \int_{G_k \cap C_r(x_0)} (\bar{\rho}(\Phi_k(x)) - \bar{\rho}(x)) J\Phi_k(x) + (J\Phi_k - 1)\bar{\rho} \\ &\geq c\sigma_k \int_{G_k \cap C_r(x_0)} \bar{\rho} - 2 \int_{G_k \cap C_r(x_0)} |\bar{\rho}(\Phi_k(x)) - \bar{\rho}(x)| \\ &= \int_{G_k \cap C_r(x_0)} \bar{\rho} \left( c\sigma_k - 2 \left| \frac{\bar{\rho}(\Phi_k(x)) - \bar{\rho}(x)}{\bar{\rho}(x)} \right| \right). \end{aligned}$$

Since  $\bar{\rho}(x) = (R^2 - |x|^2)_+$ , we see from the definition of  $\Phi_k$  that in  $C_r(x_0)$ ,

$$\left| \frac{\bar{\rho}(\Phi_k(x)) - \bar{\rho}(x)}{\bar{\rho}(x)} \right| \leq C\sigma_k r$$

hence we can choose  $r$  small enough so that  $2C\sigma_k r \leq c\sigma_k/2$ . So that

$$\int_{G_k \cap C_r(x_0)} \bar{\rho}(\Phi_k(x)) J\Phi_k(x) - \bar{\rho}(x) \geq \kappa_1 \sigma_k \int_{G_k \cap C_r(x_0)} \bar{\rho}, \quad (58)$$

for some  $\kappa_1 > 0$ . Similarly, thanks to (57),

$$\begin{aligned} \int_{G_k \cap B_{r/2}(x_0)} \bar{\rho}(\Phi_k(x)) J\Phi_k(x) - \bar{\rho}(x) &= \int_{G_k \cap B_{r/2}(x_0)} (\bar{\rho}(\Phi_k(x)) - \bar{\rho}(x)) J\Phi_k(x) + (J\Phi_k - 1)\bar{\rho} \\ &\geq -\kappa_2 \sigma_k \int_{G_k \cap B_{r/2}(x_0)} \bar{\rho}, \end{aligned}$$

for some  $\kappa_2 > 0$ . Combining this with (58) and (53) gives

$$\mathcal{V}(\tilde{G}_k) - \mathcal{V}(G_k) \geq \sigma_k \left( \kappa_1 \int_{G_k \cap B_r(x_0)} \bar{\rho} - \kappa_2 \int_{G_k \cap B_{r/2}(x_0)} \bar{\rho} \right) \geq (\kappa_1 \bar{C} - \kappa_2 \delta) \sigma_k,$$

so that (55) holds if  $\delta$  is small enough.

*Step 1.3.* Since  $\mathcal{V}(G_k) < \mathcal{V}(B)$  and  $\mathcal{V}(G_k) \rightarrow \mathcal{V}(B)$ , (55) and the continuity of the map  $\sigma_k \rightarrow \mathcal{V}(\tilde{G}_k)$ , show that we can find  $\sigma_k \rightarrow 0$  such that  $\mathcal{V}(\tilde{G}_k) = \mathcal{V}(B)$ . From this we get a contradiction. Indeed, by minimality of  $G_k$ ,

$$\begin{aligned} \mathcal{F}(G_k) + 2\Lambda_1 \left( \left[ \int_{G_k \Delta B} \bar{\rho} - \varepsilon'_k \right]^2 + \varepsilon'_k \right)^{1/2} + \Lambda_{2,k} |\mathcal{V}(G_k) - \mathcal{V}(B)| \\ \leq \mathcal{F}(\tilde{G}_k) + 2\Lambda_1 \left( \left[ \int_{\tilde{G}_k \Delta B} \bar{\rho} - \varepsilon'_k \right]^2 + \varepsilon'_k \right)^{1/2} \\ \leq \mathcal{F}(G_k) + C\sigma_k + 2\Lambda_1 \left( \left[ \int_{\tilde{G}_k \Delta B} \bar{\rho} - \varepsilon'_k \right]^2 + \varepsilon'_k \right)^{1/2}, \end{aligned}$$

where in the last line we used (54). Since the function  $x \rightarrow ((x - \varepsilon'_k)^2 + \varepsilon'_k)^{1/2}$  is 1-Lipschitz, we obtain, recalling (55),

$$\Lambda_{2,k}\sigma_k \lesssim \sigma_k + \left| \int_{\tilde{G}_k \Delta B} \bar{\rho} - \int_{G_k \Delta B} \bar{\rho} \right|.$$

Arguing as for (55), we can prove that

$$\left| \int_{\tilde{G}_k \Delta B} \bar{\rho} - \int_{G_k \Delta B} \bar{\rho} \right| \lesssim \sigma_k,$$

from which we obtain

$$\Lambda_{2,k}\sigma_k \lesssim \sigma_k.$$

This is not possible since  $\Lambda_{2,k} \rightarrow +\infty$ .

*Step 2.* Going back to  $F_n$ , minimizers of (51), we have that  $F_n$  converge to some  $F_\infty$  minimizing

$$\mathcal{F}(F) + 2\Lambda_1 \int_{F \Delta B} \bar{\rho} + \Lambda_2 |\mathcal{V}(F) - \mathcal{V}(B)|$$

that is by (49),  $F_\infty = B$ . The  $F_n$  are  $\Lambda$ -minimizers of  $\mathcal{F}$ . Indeed, for every  $E$ ,

$$\mathcal{F}(F_n) \leq \mathcal{F}(E) + 2\Lambda_1 \left| \int_{E \Delta B} \bar{\rho} - \int_{F_n \Delta B} \bar{\rho} \right| + \Lambda_2 \left| \int_{F_n} \bar{\rho} - \int_E \bar{\rho} \right| \leq \mathcal{F}(E) + (2\Lambda_1 + \Lambda_2) \int_{E \Delta F_n} \bar{\rho}.$$

Therefore, by Proposition 3.12 and Proposition 3.8, for  $n$  large enough,

$$\mathcal{F}(F_n) - \mathcal{F}(B) \geq -C_0 \left( \int_{F_n \Delta B} \bar{\rho} \right)^2. \quad (59)$$

*Step 3.* Let  $\gamma_n = \int_{F_n \Delta B} \bar{\rho}$ . Then, by minimality of  $F_n$ , there holds (using  $E_n$  as a competitor and recalling (50)),

$$\mathcal{F}(F_n) + 2\Lambda_1 ((\gamma_n - \varepsilon_n)^2 + \varepsilon_n)^{1/2} \leq \mathcal{F}(E_n) + 2\Lambda_1 \varepsilon_n^{1/2} \leq \mathcal{F}(B) + 2\Lambda_1 \varepsilon_n^{1/2} - 2C_0 \varepsilon_n^2$$

which combined with (59) gives

$$2\Lambda_1 ((\gamma_n - \varepsilon_n)^2 + \varepsilon_n)^{1/2} - C_0 \gamma_n^2 \leq -2C_0 \varepsilon_n^2 + 2\Lambda_1 \varepsilon_n^{1/2}. \quad (60)$$

From this we obtain

$$2\Lambda_1 ((\gamma_n - \varepsilon_n)^2 + \varepsilon_n)^{1/2} \leq C_0\gamma_n^2 + 2\Lambda_1\varepsilon_n^{1/2}.$$

Dividing by  $2\Lambda_1$  and taking the square of both sides, we get

$$(\gamma_n - \varepsilon_n)^2 + \varepsilon_n \leq \frac{C_0^2}{4\Lambda_1^2}\gamma_n^4 + \frac{C_0}{\Lambda_1}\gamma_n^2\varepsilon_n^{1/2} + \varepsilon_n$$

subtracting  $\varepsilon_n$  and dividing by  $\gamma_n^2$  we deduce

$$\left(1 - \frac{\varepsilon_n}{\gamma_n}\right)^2 \leq \frac{C_0^2}{4\Lambda_1^2}\gamma_n^2 + 2\frac{C_0}{\Lambda_1}\varepsilon_n^{1/2}$$

hence  $1 - \frac{\varepsilon_n}{\gamma_n} \rightarrow 0$ . Going back to (60), we obtain

$$2\Lambda_1\varepsilon_n^{1/2} \left( \left[ \varepsilon_n \left( \frac{\gamma_n}{\varepsilon_n} - 1 \right)^2 + 1 \right]^{1/2} - 1 \right) \leq -C_0\varepsilon_n^2 \left( 2 - \frac{\gamma_n^2}{\varepsilon_n^2} \right)$$

from which we get a contradiction since for  $n$  large enough the left-hand side is positive while the right-hand side is negative. This ends the proof of Proposition 3.7.  $\square$

## 4 The case without confining potential

### 4.1 The functional

We finally go back to the situation where  $g > 1$  but where there is no trapping potential. We thus consider for a fixed bounded open set  $\Omega \subset \mathbf{R}^2$

$$F_\varepsilon(\eta) = \varepsilon \int_\Omega |\nabla\eta_1|^2 + |\nabla\eta_2|^2 + \frac{1}{\varepsilon} \int_\Omega \frac{1}{2}\eta_1^4 + \frac{g}{2}\eta_2^4 + K\eta_1^2\eta_2^2$$

with the constraints

$$\int_\Omega \eta_i^2 = \alpha_i. \tag{61}$$

Letting

$$\gamma = (\alpha_1 + \alpha_2 g^{1/2}),$$

it can be easily seen that minimizers of the Thomas-Fermi energy

$$E(\bar{\rho}) = \int_\Omega \frac{1}{2}\bar{\rho}_1^2 + \frac{g}{2}\bar{\rho}_2^2 + K\bar{\rho}_1\bar{\rho}_2$$

under the mass constraint

$$\int_\Omega \bar{\rho}_i = \alpha_i$$

are given by

$$\bar{\rho}_1 = \frac{\gamma}{|\Omega|}\chi_E, \quad \bar{\rho}_2 = \frac{\gamma}{|\Omega|g^{1/2}}\chi_{\Omega \setminus E}$$

for any set  $E \subset \Omega$  with  $|E| = \frac{|\Omega|\alpha_1}{\gamma}$ . The minimal energy is then

$$E_0 = E(\bar{\rho}) = \frac{\gamma^2}{2|\Omega|}$$

This motivates the change of variables

$$\tilde{\eta}_1 = \left( \frac{\gamma}{|\Omega|} \right)^{-1/2} \eta_1, \quad \tilde{\eta}_2 = \left( \frac{\gamma}{|\Omega|g^{1/2}} \right)^{-1/2} \eta_2, \quad \tilde{x} = \left( \frac{\gamma}{|\Omega|} \right)^{1/2} x, \quad J_\varepsilon(\tilde{\eta}) = \frac{|\Omega|}{\gamma} F_\varepsilon(\eta) - \frac{\gamma}{2\varepsilon}$$

yielding

$$J_\varepsilon(\tilde{\eta}) = \varepsilon \int_{\tilde{\Omega}} |\nabla \tilde{\eta}_1|^2 + g^{-1/2} |\nabla \tilde{\eta}_2|^2 + \frac{1}{\varepsilon} \int_{\tilde{\Omega}} \frac{1}{2} (\tilde{\eta}_1^2 + \tilde{\eta}_2^2 - 1)^2 + (\tilde{K} - 1) \tilde{\eta}_1^2 \tilde{\eta}_2^2$$

where  $1 < \tilde{K} = \frac{K}{g^{1/2}}$ , under the mass constraint

$$\int_{\tilde{\Omega}} \tilde{\eta}_i^2 = \tilde{\alpha}_i$$

where  $\tilde{\alpha}_1 = \alpha_1$  and  $\tilde{\alpha}_2 = g^{1/2} \alpha_2$  so that  $\tilde{\alpha}_1 + \tilde{\alpha}_2 = \gamma = |\tilde{\Omega}|$ . Forgetting the tildas and letting  $1 > \lambda^2 = g^{-1/2}$ , we finally obtain that the original minimization problem is equivalent to minimizing

$$J_\varepsilon(\eta) = \varepsilon \int_{\Omega} |\nabla \eta_1|^2 + \lambda^2 |\nabla \eta_2|^2 + \frac{1}{\varepsilon} \int_{\Omega} \frac{1}{2} (\eta_1^2 + \eta_2^2 - 1)^2 + (K - 1) \eta_1^2 \eta_2^2$$

under the volume constraint (61).

## 4.2 The one dimensional transition problem

Let us introduce, the following energy defined for  $\eta = (\eta_1, \eta_2) \in W_{loc}^{1,2}(\mathbf{R}, \mathbf{R}^2)$ ,

$$E_{\lambda,K}(\eta) = \int_{\mathbf{R}} |\eta'_1|^2 + \lambda^2 |\eta'_2|^2 + W_K(\eta_1, \eta_2),$$

where for  $s, t \geq 0$ , we introduced the potential

$$W_K(s, t) = \frac{1}{2} (1 - s^2 - t^2)^2 + (K - 1) s^2 t^2.$$

We consider the minimization problem,

$$\sigma_{\lambda,K} = \inf \left\{ E_{\lambda,K}(\eta) : \eta = (\eta_1, \eta_2) \in W_{loc}^{1,2}(\mathbf{R}, \mathbf{R}_+^2), \lim_{-\infty} \eta_1 = 0, \lim_{+\infty} \eta_1 = 1 \right\}. \quad (62)$$

Let us show that problem (62) admits a minimizer. The result also follows from [5, Th. 2.1] with a different proof.

**Proposition 4.1.** *There exist minimizing pairs to  $\sigma_{\lambda,K}$ . Every such minimizing pair is smooth. Moreover, the following equipartition of energy holds:*

$$|\eta'_1|^2 + \lambda^2 |\eta'_2|^2 = W_K(\eta_1, \eta_2).$$

*Proof.* We establish the existence of a minimizing pair  $\eta = (\eta_1, \eta_2)$  by the Direct Method of the calculus of variations. The required compactness and semi-continuity result is stated in Lemma 4.2. The smoothness of  $\eta$  is the consequence of the Euler-Lagrange equations

$$\begin{pmatrix} -2\eta''_1 \\ -2\lambda^2 \eta''_2 \end{pmatrix} + \nabla W_K(\eta_1, \eta_2) = 0, \quad \text{in } \mathbf{R}.$$

For the equipartition of energy, we take the dot product of the Euler-Lagrange equations with  $(\eta'_1, \eta'_2)^T$ . Integrating, we see that the quantity  $-|\eta'_1|^2 - \lambda^2 |\eta'_2|^2 + W_K(\eta_1, \eta_2)$  does not depend on  $x$ . Using the conditions at infinity, we conclude that  $|\eta'_1|^2 + \lambda^2 |\eta'_2|^2 = W_K(\eta_1, \eta_2)$  on  $\mathbf{R}$ .  $\square$

**Lemma 4.2.** Let  $(\eta^k)_{k \geq 0} \subset W_{loc}^{1,2}(\mathbf{R}, \mathbf{R}_+^2)$  be such that there exists  $C_0 \geq 0$  such that for  $k \geq 0$ ,

$$\liminf_{x \rightarrow -\infty} \eta_1^k < 1/4, \quad \limsup_{x \rightarrow +\infty} \eta_1^k > 3/4 \quad \text{and} \quad E_{\lambda, K}(\eta^k) \leq C_0.$$

Then, there exist  $\eta = (\eta_1, \eta_2) \in W_{loc}^{1,2}(\mathbf{R}, \mathbf{R}^2)$ , a subsequence (still denoted by  $(\eta^k)$ ) and a sequence  $(z_k) \subset \mathbf{R}$  such that

$$\eta^k(\cdot - z_k) \rightarrow \eta \quad \text{uniformly on any bounded subset of } \mathbf{R}.$$

Moreover,

$$\lim_{x \rightarrow -\infty} \eta = (0, 1), \quad \lim_{x \rightarrow +\infty} \eta = (1, 0), \quad (63)$$

and

$$E_{\lambda, K}(\eta) \leq \liminf_{k \rightarrow +\infty} E_{\lambda, K}(\eta^k).$$

To prepare for the proof, we start by noticing that (similarly to the Ginzburg-Landau functional for Type-I superconductors [16]) the Gross-Pitaevskii energy controls a classical double well potential. This is the key ingredient in order to get compactness for sequences of bounded energy. Let us introduce the relaxed potential,

$$w_K(s) = \inf_{t \in \mathbf{R}} W_K(s, t) = \begin{cases} \frac{1}{2}(1-s^2)^2 - \frac{1}{2}(1-Ks^2)^2 & \text{if } 0 \leq s < K^{-1/2}, \\ \frac{1}{2}(1-s^2)^2 & \text{if } s \geq K^{-1/2}. \end{cases} \quad (64)$$

The function  $w_K$  is a standard double-well potential (see Figure 4). In particular, since there

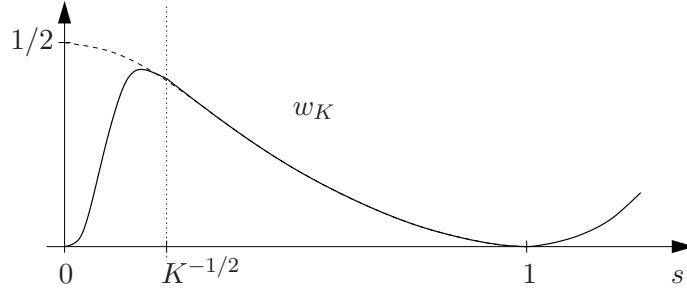


Figure 4: Relaxed potential  $w_K(s)$  and limit potential  $w(s) = (1-s^2)^2$  (dashed line).

holds  $w_K(s) = w_K(t) \leq W_K(s, t)$ , we have for  $\eta = (\eta_1, \eta_2) \in W_{loc}^{1,2}(\mathbf{R}, \mathbf{R}_+^2)$

$$\int_{\mathbf{R}} |\eta_1'|^2 + w_K(\eta_1) \leq E_{\lambda, K}(\eta) \quad \text{and} \quad \int_{\mathbf{R}} \lambda^2 |\eta_2'|^2 + w_K(\eta_2) \leq E_{\lambda, K}(\eta). \quad (65)$$

*Proof of Lemma 4.2.* Let us consider a sequence  $\eta^k = (\eta_1^k, \eta_2^k)$  and  $C_0 \geq 0$  satisfying the hypotheses of the lemma. Let us fix  $k \geq 0$  and let us consider  $y < z$  such that either,  $\eta_1^k(y) = 1/4$ ,  $\eta_1^k(z) = 3/4$  or  $\eta_1^k(y) = 3/4$ ,  $\eta_1^k(z) = 1/4$ . Using the Modica-Mortola trick, we see that

$$\int_y^z |(\eta_1^k)'|^2 + w_K \eta_1^k \geq 2 \int_y^z \sqrt{w_K \eta_1^k} |(\eta_1^k)'| \geq 2 \int_{1/4}^{3/4} \sqrt{w_K(s)} ds = \delta.$$

Since  $\delta > 0$ , taking into account (65) and the bound  $E_{\lambda, K}(\eta^k) \leq C_0$ , we deduce that there exist an odd integer  $n_k \in [1, C_0/\delta]$  and sequence of intervals

$$I_0^k = (-\infty, x_1^k), \quad I_1^k = (x_1^k, x_2^k), \quad \dots, \quad I_{n_k-1}^k = (x_{n_k-1}^k, x_{n_k}^k), \quad I_{n_k}^k = (x_{n_k}^k, +\infty),$$

such that  $\eta_1^k(x_j^k) = 1/2$  for  $j = 1, \dots, n_k$  and there exists  $y_0^k, \dots, y_{n_k}^k$  with  $y_j \in I_j^k$  such that

$$\begin{cases} \eta_1^k(y_j^k) = 1/4 & \text{and } \eta_1^k < 3/4 \text{ in } I_j^k & \text{if } j \text{ is even,} \\ \eta_1^k(y_j^k) = 3/4 & \text{and } \eta_1^k > 1/4 \text{ in } I_j^k & \text{if } j \text{ is odd.} \end{cases}$$

See the example of Figure 5. Now, up to extraction, we assume that  $n_k = n$  does not depend

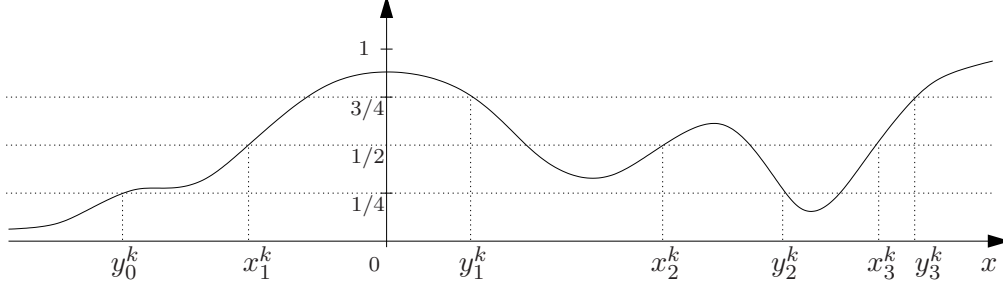


Figure 5: Example of construction of the intervals  $I_j^k$ .

on  $k$ . Now, if the sequence  $(x_2^k - x_1^k)_k$  is not bounded, we can extract a subsequence such that  $(x_2^k - x_1^k) \rightarrow \infty$ . Repeating the process, we can assume that the following property holds true: there exist  $R \geq 0$  and a partition of  $\{1, \dots, n\} = \{a_0, \dots, b_0\} \cup \{a_1, \dots, b_1\} \cup \dots \cup \{a_m, \dots, b_m\}$  with  $a_0 = 1$ ,  $b_m = n$  and  $a_{j+1} = b_j + 1$  for  $j = 0, \dots, m-1$ , such that,

$$\forall l \in \{0, \dots, m\} \quad 0 < x_{b_l}^k - x_{a_l}^k \leq R, \quad \forall l \in \{0, \dots, m-1\} \quad x_{a_{l+1}}^k - x_{a_l}^k \xrightarrow{k \rightarrow \infty} +\infty.$$

Since  $a_0 = 1$  and  $b_m = n$  are odd there exists  $l_* \in \{0, \dots, m\}$  such that  $a_{l_*}$  and  $b_{l_*}$  are odd. We set  $z_k = x_{a_{l_*}}^k$  and  $\tilde{\eta}_1^k = \eta_1^k(\cdot - z_k)$ . By construction, there exists a sequence  $(R_k) \subset [R, +\infty)$  with  $R_k \rightarrow +\infty$  such that for  $k \geq 0$ ,

$$\tilde{\eta}_1^k < 3/4 \quad \text{in } [-R_k, 0], \quad \tilde{\eta}_1^k(0) = 1/2 \quad \text{and} \quad \tilde{\eta}_1^k > 1/4 \quad \text{in } [R, R_k]. \quad (66)$$

Now, from the energy bound  $E_{\lambda, K}(\eta^k) \leq C_0$ , we see that  $(\tilde{\eta}_1^k)$  is bounded in  $W_{loc}^{1,2}(\mathbf{R})$ . Up to extraction, there exists  $\eta_1 \in W_{loc}^{1,2}(\mathbf{R})$  such that  $\tilde{\eta}_1^k(\cdot - z_k) = \tilde{\eta}_1^k \rightarrow \eta_1$  locally uniformly. Moreover, from (66), we have

$$\limsup_{-\infty} \eta_1 \leq 3/4, \quad \eta_1(0) = 1/2 \quad \text{and} \quad \liminf_{+\infty} \eta_1 \geq 1/4. \quad (67)$$

Similarly, up to extraction, there exists  $\eta_2 \in W_{loc}^{1,2}(\mathbf{R})$  such that  $\eta_2^k(\cdot - z_k) \rightarrow \eta_2$  locally uniformly. By lower semi-continuity of the Dirichlet energy, we have

$$E_{\lambda, K}(\eta_1, \eta_2) \leq \liminf_{k \rightarrow +\infty} E_{\lambda, K}(\eta_1^k, \eta_2^k).$$

To end the proof of Lemma 4.2, we have to establish that  $\eta = (\eta_1, \eta_2)$  satisfies the conditions at infinity (63). Since  $\int_{\mathbf{R}} |\eta_i'|^2$  is finite  $\eta$  admits limits  $\eta^\pm$  at  $\pm\infty$ . From the bound  $\int_{\mathbf{R}} W_K(\eta_1, \eta_2) \leq C$ , we get  $\eta^-, \eta^+ \in \{(0, 1), (1, 0)\}$ . Eventually (67) implies  $\eta^- = (0, 1)$ ,  $\eta^+ = (1, 0)$ , that is (63).  $\square$

### 4.3 The $\Gamma$ -convergence result

In this section we study the  $\Gamma$ -convergence of  $J_\varepsilon$  as  $\varepsilon$  goes to zero.

**Theorem 4.3.** *When  $\varepsilon \rightarrow 0$ ,  $J_\varepsilon$   $\Gamma$ -converges for the strong  $L^1$  topology to*

$$\mathcal{J}(\eta_1, \eta_2) = \begin{cases} \sigma_{\lambda, K} P(E, \Omega) & \text{if } \eta_1 = \chi_E = 1 - \eta_2 \text{ and } |E| = \alpha_1 \\ +\infty & \text{otherwise.} \end{cases}$$

*Proof.* Since  $J_\varepsilon(\min(\eta_1, 1 + \delta), \min(\eta_2, 1 + \delta)) \leq J_\varepsilon(\eta)$  for all  $\delta > 0$  and since the bound  $\int_\Omega \frac{1}{\varepsilon} ((\eta_1^\varepsilon)^2 + (\eta_2^\varepsilon)^2 - 1)^2 \leq C$  implies that if  $\min(\eta_i^\varepsilon, 1 + \delta)$  (strongly) converges to some  $\eta$  then also  $\eta_\varepsilon$  converges to the same  $\eta$ , we can always assume that sequences which are bounded in energy are bounded in  $L^\infty$ .

Now, we use again that the Gross-Pitaievskii potential controls the double well-potential  $w_K$  (see (64)). We have,

$$\int_\Omega w_K(\eta_1) + \int_\Omega w_K(\eta_2) \leq 2 \int_\Omega W_K(\eta_1, \eta_2). \quad (68)$$

Thanks to (68) and the usual Modica-Mortola argument, we then have that from every sequence  $(\eta_1^\varepsilon, \eta_2^\varepsilon)$  of bounded energy, we can extract a subsequence converging strongly in  $L^1$  to some pair  $(\eta_1, \eta_2)$ . Moreover, from the bound on the energy, we get that  $\eta_1(x), \eta_2(x) \in \{0, 1\}$  and  $\eta_1(x)\eta_2(x) = 0$  for almost every  $x \in \mathbf{R}$ . From the volume constraint and the strong convergence, we also deduce that  $\int_\Omega \eta_1^2 = \alpha_1$ . The lower bound inequality is then a standard application of the slicing technique (see [12, 4, 24] for instance). The upper bound is also standard. By approximation it is enough doing the construction for a smooth set  $E$ . Let  $\delta > 0$  be fixed then we can find  $T > 0$  and  $(\eta_1^\delta, \eta_2^\delta)$  with  $\eta_1^\delta(-T) = \eta_2^\delta(T) = 0$  and  $\eta_1^\delta(T) = \eta_2^\delta(-T) = 1$  with

$$\int_{-T}^T |(\eta_1^\delta)'|^2 + \lambda^2 |(\eta_2^\delta)'|^2 + W_K(\eta_1^\delta, \eta_2^\delta) \leq \sigma_{\lambda, K} + \delta$$

Let  $d_E$  be the signed distance to  $\partial E$ .

We then let

$$\eta^\varepsilon(x) = \eta^\delta \left( \frac{d_E(x)}{\varepsilon} \right).$$

Using the coarea formula it can be seen that  $\eta^\varepsilon$  converges strongly to  $(\chi_E, 1 - \chi_E)$  and that

$$\limsup_{\varepsilon \rightarrow 0} J_\varepsilon(\eta^\varepsilon) \leq (\sigma_{\lambda, K} + \delta) P(E, \Omega).$$

Letting finally

$$\tilde{\eta}^\varepsilon = \left( \frac{\sqrt{\alpha_1}}{\|\eta_1^\varepsilon\|_2} \eta_1^\varepsilon, \frac{\sqrt{\alpha_2}}{\|\eta_2^\varepsilon\|_2} \eta_2^\varepsilon \right),$$

we have that by definition  $\tilde{\eta}^\varepsilon$  satisfies the mass constraint. Moreover, using that  $\|\eta_i^\varepsilon\|_2 = \alpha_i + O(\varepsilon)$  (and therefore,  $\tilde{\eta}_i^\varepsilon = \eta_i(1 + O(\varepsilon))$ ) we have

$$\limsup_{\varepsilon \rightarrow 0} J_\varepsilon(\tilde{\eta}^\varepsilon) \leq (\sigma_{\lambda, K} + \delta) P(E, \Omega). \quad (69)$$

Indeed, this follows from

$$\begin{aligned} |\nabla \tilde{\eta}_1^\varepsilon|^2 + \lambda |\nabla \tilde{\eta}_2^\varepsilon|^2 &= (1 + O(\varepsilon)) (|\nabla \eta_1^\varepsilon|^2 + \lambda^2 |\nabla \eta_2^\varepsilon|^2) \\ (\tilde{\eta}_1^\varepsilon)^2 (\tilde{\eta}_2^\varepsilon)^2 &= (1 + O(\varepsilon)) (\eta_1^\varepsilon)^2 (\eta_2^\varepsilon)^2 \end{aligned}$$

and

$$\int_\Omega ((\tilde{\eta}_1^\varepsilon)^2 + (\tilde{\eta}_2^\varepsilon)^2 - 1)^2 = \int_\Omega ((\eta_1^\varepsilon)^2 + (\eta_2^\varepsilon)^2 - 1)^2 + O(\varepsilon^2).$$

This last inequality comes from the fact that  $A_\varepsilon = \{|d_E| \leq T\varepsilon\}$  satisfies  $|A_\varepsilon| = O(\varepsilon)$ , and that outside  $A_\varepsilon$ , we have  $((\eta_1^\varepsilon)^2 + (\eta_2^\varepsilon)^2 - 1)^2 = 0$  and  $\eta_i^\varepsilon \in \{0, 1\}$ . From this we have

$$\begin{aligned} \int_{\Omega} ((\tilde{\eta}_1^\varepsilon)^2 + (\tilde{\eta}_2^\varepsilon)^2 - 1)^2 &= \int_{A_\varepsilon} ((\tilde{\eta}_1^\varepsilon)^2 + (\tilde{\eta}_2^\varepsilon)^2 - 1)^2 + \int_{A_\varepsilon^c} ((\tilde{\eta}_1^\varepsilon)^2 + (\tilde{\eta}_2^\varepsilon)^2 - 1)^2 \\ &\leq \int_{A_\varepsilon} ((\eta_1^\varepsilon)^2 + (\eta_2^\varepsilon)^2 - 1)^2 + |A_\varepsilon|O(\varepsilon) + O(\varepsilon^2) \\ &\leq \int_{A_\varepsilon} ((\eta_1^\varepsilon)^2 + (\eta_2^\varepsilon)^2 - 1)^2 + O(\varepsilon^2). \end{aligned}$$

Since  $\delta$  is arbitrary in (69), this concludes the proof of the upper bound.  $\square$

**Remark 4.4.** With a minor adaptation of this proof (see [34, 39]), one could also deal with Dirichlet boundary conditions (that is impose  $\eta_i^\varepsilon = 0$  on  $\partial\Omega$ ). Letting

$$\gamma_i = \inf \left\{ \int_0^{+\infty} |\eta'_1|^2 + \lambda^2 |\eta'_2|^2 + \frac{1}{2} (\eta_1^2 + \eta_2^2 - 1)^2 + (K-1)\eta_1^2 \eta_2^2 : \eta_1(0) = \eta_2(0) = 0, \lim_{+\infty} \eta_i = 1 \right\},$$

we would obtain as  $\Gamma$ -limit (at least for  $\partial\Omega$  of class  $C^2$ ).

$$\mathcal{J}_{Dir}(\eta_1, \eta_2) = \begin{cases} \sigma_{\lambda, K} P(E, \Omega) + \gamma_1 \mathcal{H}^1(\partial\Omega \cap \partial E) + \gamma_2 \mathcal{H}^1(\partial\Omega \cap \partial E^c) & \text{if } \eta_1 = \chi_E = 1 - \eta_2 \\ & \text{and } |E| = \alpha_1 \\ +\infty & \text{otherwise.} \end{cases}$$

Notice that in this case, by definition,  $\gamma_i \leq \sigma_{\lambda, K} + \gamma_j$ . Hence, on the macroscopic level, there is always a contact angle. Complete wetting is still possible in the form of an infinitely thin layer of one of the phases (this would show up in the  $\gamma_i$ ). The behavior of  $\gamma_i$  has been recently studied in the physics literature [42].

**Remark 4.5.** When  $K \rightarrow +\infty$  as  $\varepsilon \rightarrow 0$  a refinement of the argument giving (68) gives also the optimal prefactor (see [16, Prop. 6.2]).

#### 4.4 Study of the surface tension

We study the asymptotics of the surface tension  $\sigma_{\lambda, K}$  in the limit regimes  $K \rightarrow 1$  and  $K \rightarrow +\infty$ . We first consider the case  $K \rightarrow 1$  (weak segregation).

**Proposition 4.6.** *There holds:*

$$\lim_{K \rightarrow 1} \frac{\sigma_{\lambda, K}}{\sqrt{K-1}} = \frac{2}{3} \frac{1 - \lambda^3}{1 - \lambda^2}.$$

*Proof.* Letting  $x = (K-1)^{-1/2}y$  in the definition of  $\sigma_{\lambda, K}$ , we have

$$\frac{\sigma_{\lambda, K}}{\sqrt{K-1}} = \inf \left\{ \int_{\mathbf{R}} |\eta'_1|^2 + \lambda^2 |\eta'_2|^2 + \eta_1^2 \eta_2^2 + \frac{1}{2(K-1)} (\eta_1^2 + \eta_2^2 - 1)^2 : \lim_{-\infty} \eta_1 = 0, \lim_{+\infty} \eta_1 = 1 \right\}.$$

Hence, using similar  $\Gamma$ -convergence arguments as above, we get that,

$$\lim_{K \rightarrow 1} \frac{\sigma_{\lambda, K}}{\sqrt{K-1}} = \inf \left\{ \int_{\mathbf{R}} |\eta'_1|^2 + \lambda^2 |\eta'_2|^2 + \eta_1^2 \eta_2^2 : \lim_{-\infty} \eta_1 = 0, \lim_{+\infty} \eta_1 = 1, \eta_2^2 = (1 - \eta_1^2) \right\}.$$



Letting  $\eta_1 = \cos \phi$  for  $\lim_{-\infty} \phi = \frac{\pi}{2}$  and  $\lim_{+\infty} \phi = 0$ , and using that  $a^2 + b^2 \geq 2ab$ , we find

$$\begin{aligned} \lim_{K \rightarrow 1} \frac{\sigma_{\lambda, K}}{\sqrt{K-1}} &\geq 2 \int_{-\infty}^{\infty} |\phi'| (1 - (\cos \phi)^2 (1 - \lambda^2))^{1/2} |\cos \phi \sin \phi| \\ &= 2 \left[ \frac{1}{3(1 - \lambda^2)} (\cos^2(x) \lambda^2 - \cos^2(x) + 1)^{3/2} \right]_0^{\frac{\pi}{2}} \\ &= \frac{2}{3} \frac{1 - \lambda^3}{1 - \lambda^2} \end{aligned}$$

In order to reach equality it is enough to consider the unique solution of

$$\phi' (1 - (\cos \phi)^2 (1 - \lambda^2))^{1/2} = -\cos \phi \sin \phi,$$

with  $\phi(0) = \frac{\pi}{4}$ . The solution is indeed unique and defined on  $\mathbf{R}$  since  $\frac{\cos \phi \sin \phi}{(1 - (\cos \phi)^2 (1 - \lambda^2))^{1/2}}$  is Lipschitz continuous. Moreover, it is decreasing and has the right values at  $\pm\infty$ .  $\square$

**Remark 4.7.** Notice that the value of  $\lim_{K \rightarrow 1} \frac{\sigma_{\lambda, K}}{\sqrt{K-1}}$  exactly coincides with the one found in [9] (see also [41]). Using (68), it is moreover not hard to prove that there exists  $C > 0$  (not depending on  $\lambda$ ), such that

$$C^{-1} \sqrt{K-1} \leq \sigma_{\lambda, K} \leq C \sqrt{K-1}$$

when  $K \rightarrow 1$ .

Eventually, we consider the strong segregation asymptotics,  $K \rightarrow +\infty$  (notice that we recover the same scaling as the one predicted in the physics literature [41]).

**Proposition 4.8.** *There exist  $0 < c \leq C$  such that for  $K > 1$  and  $\lambda > 0$ , there holds*

$$\sigma_{\lambda, \infty} - C \left( \frac{\lambda^{1/2}}{(K-1)^{1/4}} + \frac{1}{K^{1/2}} \right) \leq \sigma_{\lambda, K} \leq \sigma_{\lambda, \infty} - c \left( \frac{1}{K^{1/2}} + \frac{\lambda^{1/2}}{K^{1/4}} \right), \quad (70)$$

with

$$\begin{aligned} \sigma_{\lambda, \infty} &= \inf \left\{ \int_{\mathbf{R}} |\eta_1'|^2 + \lambda^2 |\eta_2'|^2 + \frac{1}{2} (\eta_1^2 + \eta_2^2 - 1)^2 : \lim_{-\infty} \eta_1 = 0, \lim_{+\infty} \eta_1 = 1, \eta_1 \eta_2 \equiv 0 \right\}, \\ &= (1 + \lambda) \frac{2\sqrt{2}}{3}. \end{aligned} \quad (71)$$

*Proof.* Let us establish the identity (71). Since any admissible pair  $(\eta_1, \eta_2)$  with finite energy is continuous and satisfies  $\eta_1(x) \rightarrow 1$  as  $x \rightarrow +\infty$  and  $\eta_2(x) \rightarrow 1$  as  $x \rightarrow -\infty$ , there exists  $x \in \mathbf{R}$  such that  $(\eta_1, \eta_2)(x) = 0$ . Using translation and symmetry, we see that

$$\sigma_{\lambda, \infty} = \gamma_1 + \gamma_\lambda,$$

with

$$\gamma_\lambda = \inf \left\{ \int_0^{+\infty} \lambda^2 |\eta'|^2 + \frac{1}{2} (\eta^2 - 1)^2 : \eta \in W_{loc}^{1,2}([0, +\infty)), \eta(0) = 0, \lim_{+\infty} \eta = 1 \right\}.$$

The classical Modica-Mortola procedure applies to this minimization problem. For  $\lambda > 0$  we have (see *e.g.* Chapter 6 in [13]),

$$\gamma_\lambda = 2\lambda^2 \int_0^1 \sqrt{\frac{(1-s^2)^2}{2\lambda^2}} ds = \frac{2\sqrt{2}}{3} \lambda.$$

Moreover, the minimizer is given by the formula

$$\eta(x) = \tanh\left(\frac{x}{\sqrt{2}\lambda}\right) \quad \text{for } x \geq 0. \quad (72)$$

Let us now establish the upper bound in (70) (right inequality). If  $\lambda \gtrsim K^{-1/2}$  (that is  $\frac{\lambda^{1/2}}{K^{1/4}} \gtrsim \frac{1}{K^{1/2}}$ ), inspired by (72), we consider the profile

$$\eta_1(x) = \begin{cases} 0 & \text{if } x < 0, \\ \tanh\left(\frac{x}{\sqrt{2}}\right) & \text{if } x \geq 0, \end{cases} \quad \eta_2(x) = \begin{cases} \tanh\left(\frac{\delta-x}{\sqrt{2}\lambda}\right) & \text{if } x \leq \delta, \\ 0 & \text{if } x > \delta, \end{cases}$$

The parameter  $\delta > 0$  tunes the width of the overlap between the two species (see Figure 6).

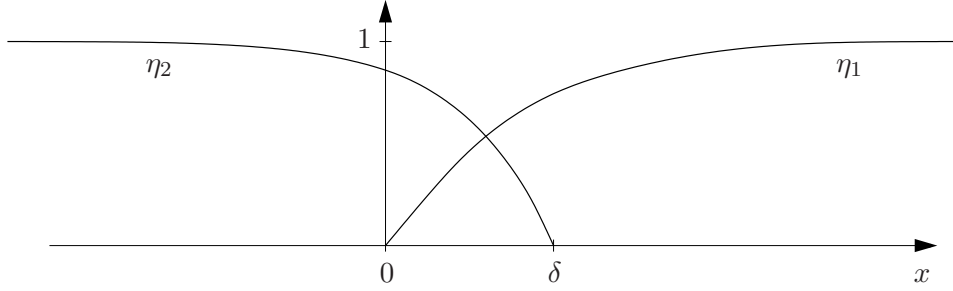


Figure 6: Profile of the competitor for the upper bound

The energy  $f(\delta)$  of this competitor is an upper bound for  $\sigma_{\lambda,K}$ . We compute

$$\begin{aligned} f(\delta) &= E_{\lambda,K}(\eta) = \int_{\mathbf{R}} |\eta_1'|^2 + \lambda^2 |\eta_2'|^2 + \frac{1}{2} (\eta_1^2 + \eta_2^2 - 1)^2 + (K-1)\eta_1^2 \eta_2^2 \\ &= \int_{\mathbf{R}} |\eta_1'|^2 + \lambda^2 |\eta_2'|^2 + \int_{-\infty}^{\delta} \frac{1}{2} (1 - \eta_2^2)^2 + \int_0^{+\infty} \frac{1}{2} (1 - \eta_1^2)^2 + \int_0^{\delta} (K\eta_1^2 \eta_2^2 - \frac{1}{2}) \\ &= \gamma_1 + \gamma_\lambda + \int_0^{\delta} \left( K\eta_1^2 \eta_2^2 - \frac{1}{2} \right) = \sigma_{\lambda,\infty} + \int_0^{\delta} \left( K\eta_1^2 \eta_2^2 - \frac{1}{2} \right). \end{aligned}$$

Using the expressions of  $\eta_1$  and  $\eta_2$  and the concavity of  $t \in \mathbf{R}_+ \mapsto \tanh(t)$ , we get

$$\sigma_{\lambda,K} \leq f(\delta) \leq \sigma_{\lambda,\infty} - \frac{\delta}{2} + \frac{K\delta^5}{4\lambda^2} \int_0^1 s^2(1-s)^2 ds = \sigma_{\lambda,\infty} - \frac{\delta}{2} + \frac{K\delta^5}{120\lambda^2}.$$

Optimizing in  $\delta$ , we obtain with  $\delta = (12\lambda^2/K)^{1/4}$ ,

$$\sigma_{\lambda,K} \leq \sigma_{\lambda,\infty} - \frac{4}{5} \left( \frac{12\lambda^2}{K} \right)^{1/4} = \sigma_{\lambda,\infty} - \frac{4(12)^{1/4}}{5} \frac{\sqrt{\lambda}}{K^{1/4}}.$$

This yields the right inequality of (70) with  $c = \frac{4(12)^{1/4}}{5}$ .

If on the contrary  $\lambda \ll K^{-1/2}$ , we keep  $\eta_1(x) = [\tanh(x/\sqrt{2})]_+$  as above and let

$$\eta_2(x) = \begin{cases} 1 & x \leq 0 \\ 1 - K^{1/2}x & x \in [0, K^{-1/2}] \\ 0 & x \geq K^{-1/2}. \end{cases}$$

The energy can then be estimated by

$$E_{\lambda,K}(\eta_1, \eta_2) \leq \gamma_1 + \lambda^2 K^{1/2} + \int_0^{K^{-1/2}} \left[ \frac{1}{2}(1 - \eta_2^2)^2 + (K - 1)\eta_1^2 \eta_2^2 - \frac{1}{2} \right].$$

Notice now that  $\lambda^2 K^{1/2} \ll K^{-1/2}$  and that since  $\tanh(x/\sqrt{2}) \leq x/\sqrt{2}$ ,

$$\int_0^{K^{-1/2}} [(1 - \eta_2^2)^2 + 2(K - 1)\eta_1^2 \eta_2^2 - 1] \leq \int_0^{K^{-1/2}} [(1 - Kx^2)^2 + K^2 x^4 - 1],$$

which implies,

$$\int_0^{K^{-1/2}} \left[ \frac{1}{2}(1 - \eta_2^2)^2 + (K - 1)\eta_1^2 \eta_2^2 - \frac{1}{2} \right] \leq \int_0^{K^{-1/2}} \frac{K^2}{2} x^4 - Kx^2 = -\frac{7}{10} K^{-1/2}.$$

Putting all these estimates together gives the upper bound

$$\sigma_{\lambda,K} \leq \sigma_{\lambda,\infty} - cK^{-1/2}.$$

We now turn to the proof of the lower bound (left inequality of (70)).

Let us consider an admissible pair  $\eta = (\eta_1, \eta_2) \in W_{loc}^{1,2}(\mathbf{R}, \mathbf{R}^2)$  with finite energy. Without loss of generality, we may assume  $0 < \eta_1(x), \eta_2(x) < 1$  for  $x \in \mathbf{R}$ . By continuity of  $\eta$  and from the conditions at infinity, there exists  $x_0 \in \mathbf{R}$  such that  $\eta_1(x_0) = \lambda\eta_2(x_0) = m$  and a maximal interval  $I_0 = [x_0 - \delta_-, x_0 + \delta_+]$  such that  $m/2 \leq \eta_1, \lambda\eta_2 \leq 2m$  in  $I_0$ . We define  $I_- = (-\infty, x_0 - \delta_-)$  and  $I_+ = (x_0 + \delta_+, +\infty)$  and split the integration in three parts:  $E_{\lambda,K}(\eta) = E_- + E_0 + E_+$ , with

$$E_i = \int_{I_i} |\eta_1'|^2 + \lambda^2 |\eta_2'|^2 + \frac{1}{2} (\eta_1^2 + \eta_2^2 - 1)^2 + (K - 1)\eta_1^2 \eta_2^2, \quad \text{for } i \in \{-, 0, +\}.$$

Using the Modica-Mortola trick, we have

$$E_+ \geq \int_{x_0 + \delta_+}^{+\infty} |\eta_1'|^2 + w_K(\eta_1) \geq 2 \int_{\eta_1(x_0 + \delta_+)}^1 \sqrt{w_K(s)} ds.$$

Using  $\eta_1(x_0 + \delta_+) \leq 2m$  and  $2\sqrt{w_K} \geq \sqrt{2}(1 - s^2) - \sqrt{2}\chi_{[0, K^{-1/2}]}$ , we obtain,

$$E_+ \geq \sqrt{2} \int_{2m}^1 (1 - s^2) ds - \sqrt{2}K^{-1/2} \geq \frac{2\sqrt{2}}{3} - 2\sqrt{2}m - \sqrt{2}K^{-1/2}. \quad (73)$$

Similarly,

$$E_- \geq \frac{2\sqrt{2}\lambda}{3} - 2\sqrt{2}m - \sqrt{2}K^{-1/2}. \quad (74)$$

Now we consider the middle part. Since  $\min\left(\int_{I_0} |\eta_1'|, \int_{I_0} \lambda |\eta_2'|\right) \geq m/2$ , we have by Cauchy-Schwarz inequality,

$$\int_{I_0} |\eta_1'|^2 + \lambda^2 |\eta_2'|^2 \geq \frac{m^2}{4(\delta_+ - \delta_-)}.$$

On the other hand, since  $\eta_1, \lambda\eta_2 \geq m/2$  in  $I_0$ , we have

$$\int_{I_0} \frac{1}{2} (\eta_1^2 + \eta_2^2 - 1)^2 + (K - 1)\eta_1^2 \eta_2^2 \geq \frac{(\delta_+ - \delta_-)(K - 1)m^4}{16\lambda^2}.$$

Optimizing with respect to  $(\delta_+ - \delta_-)$ , we obtain,

$$E_0 = \int_{I_0} |\eta'_1|^2 + \lambda^2 |\eta'_2|^2 + \int_{I_0} \frac{1}{2} (\eta_1^2 + \eta_2^2 - 1)^2 + (K-1)\eta_1^2 \eta_2^2 \geq \frac{\sqrt{K-1}m^3}{4\lambda}. \quad (75)$$

Gathering together (73),(74),(75), we get

$$E_{\lambda,K}(\eta) \geq \frac{2\sqrt{2}}{3}(1+\lambda) - 4\sqrt{2}m + \frac{\sqrt{K-1}m^3}{4\lambda} - 2\sqrt{2}K^{-1/2}.$$

Minimizing with respect to  $m$ , the minimum is reached for  $m = [16\sqrt{2}\lambda/(3\sqrt{K-1})]^{1/2}$ , which yields,

$$\sigma_{\lambda,K} \geq \frac{2\sqrt{2}}{3}(1+\lambda) - \frac{32 \cdot 2^{3/4}}{3^{3/2}} \frac{\lambda^{1/2}}{(K-1)^{1/4}} - \frac{2\sqrt{2}}{K^{1/2}}.$$

This establishes the left inequality of (70) and ends the proof of Proposition 4.8.  $\square$

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