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Asymptotic fields ahead a crack for a class of non linear materials under mode III

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The paper considers the mechanical fields near the tip of a crack deformed by an anti-plane shear at infinity for a class of non linear elastic materials. For brittle material rupture occurs when a maximal stretch is reached. Taking account of this critical value, the crack is replaced by a totally damaged zone of finite thickness named a quasicrack. Inside this domain, the stress is identically zero and the shape of the boundary between damaged and undamaged body is found analytically.

1. Introduction

In fracture mechanics it is usually assumed that the materials far away of the crack tip are undamaged. The concentration of strain near the crack tip leads to initiation of microcracks or microvoids. Hence the constitutive behaviour near the tip is quite different from the behaviour of the bulk. A damage model seems to be a good approach to describe such situations.

The solution of equilibrium of a crack in a linear elastic body is obtained as an outer asymptotic expansion of the displacement according to the distance to the crack tip. Extensions of such analyses have been made in order to take into account of the non linear effects on the asymptotic inner expansion based on finite elasticity or on elasto-plasticity. For the finite anti-plane shear near the tip of a crack in an incompressible hyperelastic solid the displacement field possesses singularities or discontinuities of its

gradient depending on the stress-strain curve (Knowles, 1977; Knowles and Sternberg, 1980, 1981). In these studies, the constitutive behaviour is defined for any amount of shear, which can be unbounded. For elastic power law hardening material in mode III, the HRR field can be recovered (Hutchinson, 1968; Rice, 1967, 1968).

In 1968, Neuber (1968) investigates the stress-concentration under mode III for a notch. The solution is obtained for non linear stress-strain laws monotonically increasing. For a quasi-crack which consists of two parallel straight lines ended by a cycloid, he shows that the stress is uniform along the cycloid.

Different modelization of damage can be considered. Continuous damage has been used to investigate the asymptotic solution of mode III crack in damaged softening material (Wang and Chow, 1992; Zhang et al., 1993). For brittle material rupture occurs when a maximal stretch is reached. When this value is reached, the material is broken and a damaged zone grows inside the body. In the case of elastic-brittle material, a closed form solution for the dynamic propagation of mode-III crack has been obtained (Bui and Ehlacher, 1980). Under quasistatic conditions, the shape of the damaged zone is a quasi-crack (Bui and Ehlacher, 1980). A similar solution is obtained for elastic

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– perfectly plastic brittle materials (Bui, 1980). The thickness of the quasi-crack is determined by the value of the stress intensity factor applied at infinity and the critical value of shear amount.

More recently, brittle hyperelastic material solution for constitutive laws which permit loss of ellipticity has been considered (Stolz, 2010), using hodograph scheme as proposed in Knowles (1977), Knowles and Sternberg (1981), Rice (1967).

The purpose of the paper is to determine the mechanical field ahead of a crack or of the quasicrack, depending on the value of the critical shear. The shape of the damaged zone under anti-plane shear conditions for different class of non-linear elastic brittle material is also investigated.

2. Preliminaries

Consider an isotropic homogeneous non linear elastic body. Further we introduce rectangular cartesian coordinates (x_1, x_2, x_3) . Suppose the interior of the body occupies an open region Ω and consider a deformation corresponding to anti-plane shear which involves displacements w in the direction \mathbf{e}_3 normal to the plane (x_1, x_2)

$$\mathbf{y}(\mathbf{x}) = \mathbf{x} + w(x_1, x_2)\mathbf{e}_3. \quad (2-1)$$

The displacement w is the out-of-plane displacement at point \mathbf{x} of cartesian coordinates x_i , $i = 1, 2$. The modulus of the displacement gradient is denoted by R :

$$\begin{aligned} \nabla \mathbf{w} &= w_{,i} \mathbf{e}_3 \otimes \mathbf{e}_i, \quad R^2 = w_{,i} w_{,i} = \nabla \mathbf{w}^T \cdot \nabla \mathbf{w} \\ &= \|\nabla \mathbf{w}\|^2. \end{aligned} \quad (2-2)$$

Suppose further the medium is an incompressible, homogeneous, isotropic elastic solid that possesses a strain energy density \mathcal{E} per unit undeformed volume. The energy is a function $\mathcal{E}(I_1, I_2)$ of the first two fundamental invariants I_1, I_2 of the left (or right) Cauchy–Green tensor. The third fundamental invariant is unity since the material is incompressible. For anti-plane shear $I_1 = I_2 = I = 3 + R^2$ and the strain energy becomes a function $\Psi(I) = \mathcal{E}(I, I)$. The Cauchy stress $\boldsymbol{\sigma}$ satisfies the state equation

$$\boldsymbol{\sigma} = \sum_{i=1}^2 \tau^i (\mathbf{e}_3 \otimes \mathbf{e}_i + \mathbf{e}_i \otimes \mathbf{e}_3), \quad \tau_i = \frac{\partial \Psi}{\partial w_{,i}} = \mu(R) w_{,i}. \quad (2-3)$$

The balance of linear momentum in the absence of body forces leads to the equilibrium equations

$$\text{Div } \boldsymbol{\sigma}^T = 2(\Psi'(I) w_{,i})_{,i} = 0, \quad (2-4)$$

where $i = 1, 2$ and the summation is implied. These differential equations are elliptic under the condition

$$2\Psi' + 4R^2\Psi'' \geq 0. \quad (2-5)$$

This conditions implies that the local stress–strain curve $\tau(R) = 2R\Psi'(I) = R\mu(R)$ is an increasing function of R .

In this paper, we consider a class of constitutive laws defined by material constants $\mu_o, \mu_1, R_o, R_u, R_m, \alpha$:

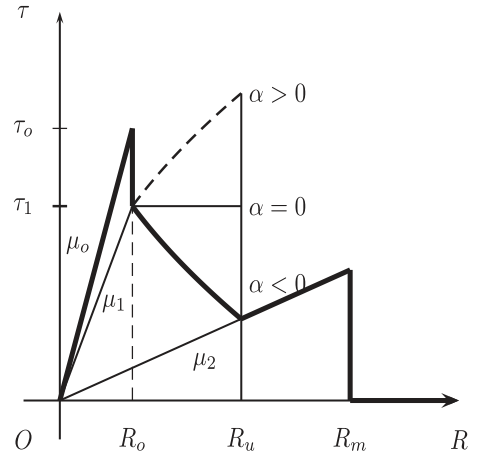


Fig. 1. Typical constitutive behaviour.

$$\begin{cases} R \leq R_o, & \tau = \mu_o R, \\ R_o \leq R \leq R_u, & \tau = \mu_1 R_o \left(\frac{R}{R_o}\right)^\alpha, \\ R_u \leq R_m, & \tau = \mu_1 R_o \left(\frac{R_u}{R_o}\right)^\alpha \frac{R}{R_u} = \mu_2 R, \\ R_m \leq R, & \tau = 0. \end{cases} \quad (2-6)$$

For $R \leq R_o$ the Eq. (2-4) are elliptic. For $R_o \leq R \leq R_u$ ellipticity is ensured if $\alpha \geq 0$ otherwise the Eq. (2-4) are hyperbolic.

This class of constitutive law contains many type of classical laws used by different authors: (See Fig. 1).

- *Elastic-brittle material*, Bui and Ehrlacher (1980). This classical law is described by $R_o = R_u = R_m$.
- *Power-law brittle material*, Neuber (1968), Stolz (2010). $\mu_1 = \mu_o, R_u = R_m, \alpha \geq 0$.
- *Perfectly plastic materials*, Rice (1967), Wang and Chow (1992). $\alpha = 0, R_m = R_u = \infty$
- *Special material*, Abeyaratne (1981). $\mu_u = \mu_o, \mu_2 = \mu_o (R_o/R_u)^{3/2}, R_m = \infty$
- *Non linear softening behaviour*, Knowles and Sternberg (1981). $\mu_1 = \mu_o, \alpha = -.5, R_m = \infty$.

For brittle materials rupture occurs at a finite amount of shear R_m . The goal of this article is to study the influence of R_m on the solution of the problem of equilibrium.

3. Hodograph transformation

To determine the mechanical fields around the damaged zone, the hodograph transformation is useful as pointed by Hult and McClintock (1956), Knowles and Sternberg (1980, 1981), Rice (1967). Other technics can be used (Bui and Ehrlacher, 1980; Neuber, 1968). In the hodograph transformation the components of the gradient $\nabla \mathbf{w}$ become the new independent variables. We consider the mapping:

$$(x_1, x_2) \rightarrow (\zeta_1, \zeta_2), \quad \zeta_i = w_{,i}(x_1, x_2). \quad (3-1)$$

The displacement appears as a potential ($dw = \sum_{i=1}^2 \xi_i dx_i$). This mapping is invertible provided the Jacobian $H = w_{,11}w_{,22} - (w_{,12})^2$ does not vanish. The Legendre transformation U of w with respect to ξ_i is given by:

$$U(\xi_1, \xi_2) = \sum_{i=1}^2 x_i w_{,i}(x_1, x_2) - w(x_1, x_2) = \xi_i x_i - w. \quad (3-2)$$

By differentiating U with respect to ξ , the inverse mapping is obtained:

$$x_i = \frac{\partial U}{\partial \xi_i}, \quad \bar{w} = \xi \cdot \nabla U - U. \quad (3-3)$$

In the hodograph plane (ξ_1, ξ_2), the polar coordinates are used ($\xi_1 = R \cos \Theta, \xi_2 = R \sin \Theta$) and the physical coordinates (x_1, x_2) satisfy:

$$x_1 = \cos \Theta \frac{\partial U}{\partial R} - \frac{\sin \Theta}{R} \frac{\partial U}{\partial \Theta}, \quad x_2 = \sin \Theta \frac{\partial U}{\partial R} + \frac{\cos \Theta}{R} \frac{\partial U}{\partial \Theta}. \quad (3-4)$$

The stress field takes the form

$$\tau_1 = \sigma_{13} = \mu(R)R \cos \Theta, \quad \tau_2 = \sigma_{23} = \mu(R)R \sin \Theta \quad (3-5)$$

and the norm $\tau(R) = \sqrt{\tau_1^2 + \tau_2^2}$ satisfies the constitutive law $\tau(R) = \mu(R)R$.

For a given constitutive law, the Eq. (2-4) is rewritten in the hodograph plane (R, Θ) using the mapping (3-4) and the potential U . The notation ($'$) indicates a derivation with respect to R . The Eq. (2-4) becomes

$$\frac{\partial}{\partial R} \left(\tau(R) \frac{\partial U}{\partial R} \right) + \frac{\tau(R)'}{R} \frac{\partial^2 U}{\partial \Theta^2} = 0. \quad (3-6)$$

This equation has the form proposed by Knowles and Sternberg (1980), Rice (1967). The Eq. (3-6) is homogeneous of degree one in U . The displacement w associated to U satisfies

$$w = R^2 \frac{\partial}{\partial R} \left(\frac{U}{R} \right). \quad (3-7)$$

For the class of constitutive laws defined previously, the solution of the problem of equilibrium is obtained by integration of the Eq. (3-6) on each domain ($R, 0 \leq \Theta \leq \pi$) of the hodograph lane, with respectively ($0 < R < R_o$), ($R_o < R < R_u$), ($R_u < R < R_m$). To each domain in the hodograph plane corresponds a domain in the physical space x_1, x_2 .

When the stress-strain curve is continuous, along the boundaries ($R = R_o$ or $R = R_u$), the potential, the stress, the displacement are continuous, this case arrives when the differential equation are elliptic. When the stress-strain curve is discontinuous or when softening occurs, the intersection of the domains are not reduced to the images of the curves ($R = R_o$ or $R = R_u$). A solution is obtained by ensuring the continuity of the displacement \mathbf{w} and of the stress vector $\boldsymbol{\sigma} \cdot \mathbf{n}$ along a curve (shock curve) across which discontinuities of the displacement gradient can exist. The condition of uniqueness is not discussed in this article, but it is relevant from the property that the

domain of hyperbolicity is embedded by the domain of ellipticity. This will be discuss in another publication.

4. Peculiar solutions of equilibrium Eq. (3-6)

Consider a power stress-strain curve: $\tau(R) = \hat{\tau}R^\alpha$, the set of solutions of (3-6) contains the particular functions

$$U_\alpha = R \cos \Theta \int_R^{R_o} \frac{dt}{\tau(t)t^{2+\alpha}}, \quad V_\alpha = R \left(\frac{2\alpha}{\alpha+1} \log R \cos \Theta - \left(\Theta - \frac{\pi}{2} \right) \sin \Theta \right), \\ T_1 = R \cos \Theta, \quad T_2 = R \sin \Theta,$$

for which the mapping (R, Θ) \rightarrow (x_1, x_2) can be determined. For example, the function T_1 (or T_2) gives a shift of one unity along \mathbf{e}_1 (resp. \mathbf{e}_2) in the physical coordinates. The function V_α permits a shift along x_2 of π when Θ varies from 0 to 2π at fixed R .

For $\Theta = 0$ or $\Theta = \pi$, $\sigma_{32} = \tau_2 = 0$, this is the boundary condition satisfied for a crack or on the lip of the quasi-crack. In this case, the potential U must satisfy:

$$x_2(R, \Theta) = \frac{1}{R} \frac{\partial U}{\partial \Theta} = C(R, \Theta), \\ \forall R, \quad \Theta = 0, \quad \text{and} \quad \Theta = \pi, \quad (4-1)$$

where $C(R, 0)$ and $C(R, \pi)$ are uniform, this condition is satisfied by U_α and V_α .

4.1. Anti-plane shear and mode III loading

In linear elasticity, the differential equation reduces to Laplace's equation and for a straight crack ($-\infty \leq x_1 \leq 0, x_2 = 0$) the solution of equilibrium is well known. For polar coordinates (r, θ) centre at the crack tip, the displacement is obtained as

$$w \rightarrow \frac{2K}{\mu_o} \sqrt{\frac{r}{2\pi}} \sin \frac{\theta}{2}, \quad r \rightarrow \infty \quad (4-2)$$

For a linear elastic solid $R_o = \infty$ and $\mu(R) = \mu_o$, the solution is then

$$U_1(R, \Theta) = \frac{A}{2\mu_o R} \cos \Theta, \quad w = -\frac{A}{\mu_o R} \cos \Theta \quad (4-3)$$

The potential U_1 satisfies the boundary condition along the crack lips $\Theta = 0$ and $\Theta = \pi$, the lips are stress free. The comparison with (4-2) gives the relations

$$\Theta = \frac{1}{2}(\pi + \theta), \quad A = \frac{K^2}{\pi \mu_o}. \quad (4-4)$$

This defines the loading to apply far ahead of the crack in the following sections.

5. The quasi-crack solution in mode III

The mode III displacement field is applied at infinity. The local behaviour is non-linear and the amount of shear is limited by a critical value R_m , as described by the proposed constitutive law.

Consider a brittle material with ($\mu_1 = \mu_o, R_u = R_m$) the solution in the hodograph plane is build by the combination of the proposed potential U, V as:

$$0 \leq R \leq R_o, \quad \bar{U}_o(R, \Theta) = A_o U_1(R, \Theta) - B_o V_1(R, \Theta) + C_o T_1(R, \Theta)$$

$$R_o \leq R \leq R_m, \quad \bar{U}_1(R, \Theta) = A_1 U_x(R, \Theta) - B_1 V_x(R, \Theta) + C_1 T_1(R, \Theta)$$

The constants (A_o, A_1, B_o) and (B_1, C_o, C_1) are determined by:

- asymptotic behaviour when $R \rightarrow 0$,
- traction-free conditions along the boundary of the damaged zone,
- continuity of the displacement w everywhere,
- continuity of the stress vector along curves where \bar{U} and ∇w are discontinuous.

For sake of simplicity, we introduce dimensionless constants

$$a = \frac{A_o}{\tau_o R_o}, \quad a_1 = \frac{A_1}{a \tau_o R_o}, \quad b_o = \frac{B_o}{a}, \quad b_1 = \frac{B_1}{a}, \quad \rho = \frac{R}{R_o},$$

$$\rho_m = \frac{R_m}{R_o}, \quad X_i = \frac{x_i}{a}, \quad W = \frac{w}{a}$$

5.1. Determination of the constants

The displacement is given on each domain by

$$\begin{cases} 0 \leq \rho \leq 1, & W_o(\rho, \Theta) = -\left(\frac{1}{\rho} + b_o \rho\right) \cos \Theta \\ 1 \leq \rho \leq \rho_m, & W_1(\rho, \Theta) = -\left(\frac{a_1}{\rho^\alpha} + \frac{2b_1 \alpha}{\alpha+1} \rho\right) \cos \Theta \end{cases} \quad (5-1)$$

The displacement must satisfied the matching condition at ∞ , so we obtain

$$a = \frac{K^2}{\pi \tau_o^2} \quad (5-2)$$

The boundary along the damaged zone must be traction-free. For that, the image of the hodograph plane in the physical plane must be determined. The constants C_i are chosen such that at point $(\rho = 1, \Theta = \frac{\pi}{2})$: $X_1(1^-, \frac{\pi}{2}) = X_1(1^+, \frac{\pi}{2})$. And the physical coordinates are given in each domain by

- $0 \leq \rho \leq 1$:

$$X_1^o = -\frac{1}{2} - \frac{1}{2\rho^2} \cos(2\Theta) - b_o \log \rho$$

$$X_2^o = -\frac{1}{2\rho^2} \sin(2\Theta) + b_o \left(\Theta - \frac{\pi}{2}\right)$$

- $1 \leq \rho \leq \rho_m$:

$$X_1^1 = -\left(\frac{a_1}{\rho^{\alpha+1}} - \beta b_1\right) \frac{\cos 2\Theta}{2} + \left(\frac{a_1}{\rho^{\alpha+1}} + b_1\right) \frac{\beta}{2} - \frac{2b_1 \alpha}{\alpha+1} \log \rho - \frac{a_1}{\alpha+1}$$

$$X_2^1 = -\left(\frac{a_1}{\rho^{\alpha+1}} - \beta b_1\right) \frac{\sin 2\Theta}{2} + b_1 \left(\Theta - \frac{\pi}{2}\right)$$

where $\beta = (1 - \alpha)/(1 + \alpha)$.

For fixed ρ : $\rho = \rho_a$, the curve $(X_1(\rho_a, \Theta), X_2(\rho_a, \Theta))$ is a cycloid.

For the values $\Theta_l = 0$ or $\Theta_l = \pi$, the curves $(X_2^i(\rho, \Theta_l) = \pm B_i \pi/2, \forall \rho)$ are two lines parallel to \mathbf{e}_1 where $\tau_2 = 0$.

In the hodograph plane, the traction-free boundary condition is given by the equation

$$\tau_1 dX_2 - \tau_2 dX_1 = 0. \quad (5-3)$$

These conditions must be satisfied along the curves $\rho = \rho_m$ or $\Theta = \Theta_l$. It is obvious that the traction-free boundary condition is satisfied for $\Theta_l = \frac{\pi}{2} \pm \frac{\pi}{2}$ for all ρ . The relation along the cycloid where $\rho = \rho_m$ gives the property

$$\frac{2b_1}{\alpha+1} - \frac{a_1}{\rho_m^{\alpha+1}} = 0. \quad (5-4)$$

The thickness H of the damaged zone is $H = b_1 a \pi$. For $\rho_m \rightarrow \infty$, the quantity b_1 vanishes, the quasi-crack tends to the classical crack, the thickness H vanishes, the cycloids are circles as presented in Knowles and Sternberg (1980)

The determination of the solution depends on the mechanical property, especially on the non linearity of the constitutive behaviour defined by α .

5.2. Solutions for $\alpha \geq 0$

The balance Eq. (2-4) are elliptic. This is the case described study by Neuber (1968). If $\rho_2 \geq \rho_1$ the volume bounded by the cycloid $(X(\rho_2, \Theta), Y(\rho_2, \Theta))$ contains the volume bounded by the cycloid $(X(\rho_1, \Theta), Y(\rho_1, \Theta))$.

For $\rho = 1$ the stress τ is continuous. The cycloid $(X_1^1(1, \Theta), X_2^1(1, \Theta))$ and the cycloid $(X_1^2(1, \Theta), X_2^2(1, \Theta))$ are identical when $W_o(1, \Theta) = W_1(1, \Theta)$ under the conditions:

$$b_1 = b_o, \quad \text{and} \quad 1 = a_1 - \beta b. \quad (5-5)$$

All the constants are then determined.

In addition, the thickness of the damaged zone with respect to the external loading is determined. Inserting the value of a_1 in (5-4):

$$H = ab\pi = \frac{K^2}{\tau_o^2} b, \quad (5-6)$$

$$1 = \frac{2b}{\alpha+1} \left(\rho_m^{\alpha+1} + \frac{\alpha-1}{2} \right). \quad (5-7)$$

The last equation is related to the value of strain energy along the cycloid $\rho = \rho_m$ given by

$$\mathcal{E}(R_m) = \int_0^{R_m} \tau(t) dt = \frac{1}{\mu_o} \tau_o^2 \left(\rho_m^{\alpha+1} + \frac{\alpha-1}{2} \right). \quad (5-8)$$

Then the constant b is related to the flux of energy along the cycloid $\rho = \rho_m$.

$$D = H\mathcal{E}(R_m) = ab\pi\mathcal{E}(R_m) = \frac{K^2}{2\mu_o}. \quad (5-9)$$

This can be interpreted as the damage dissipation for a steady state propagation of the quasi-crack and the equivalent crack defined by the same loading at infinity (Stolz, 2010).

The equations of equilibrium are elliptic and there is no discontinuities of the displacement gradient except for $\rho = \rho_m$.

For $\rho_m = 1$, the results obtained by Bui (1980) are recovered: $b = 1$, that is the case of elastic-brittle material. The thickness of the damaged zone satisfies the relation

$$H = \pi a = \frac{K^2}{\tau_0^2} \quad (5-10)$$

An estimation of the thickness for steel gives H of the order of 100 μm .

When ρ_m tends to ∞ , the quasi-crack tends to be a crack because the thickness vanishes and the solution of Rice (1967) is recovered.

5.3. Solution for $\alpha \leq 0$ and $R_m = \infty$

Now the equilibrium equations are no longer elliptic, there is softening for $\rho > 1$. We must construct a continuous displacement field.

For $-1 < \alpha < 0$ the solution is given by ($\beta = (1 - \alpha)/(1 + \alpha)$).

- **Ellipticity domain:** For $0 \leq \rho_e \leq 1$, the Eq. (2-4) are elliptic

$$X_1^e = r \cos \theta + \frac{a_1}{1 + \alpha} = \frac{\beta a_1}{2} - \frac{1}{2\rho_e^2} \cos(2\Theta_e) = X_c a_1 + r_1 \cos \theta_1,$$

$$X_2^e = r \sin \theta = -\frac{1}{2\rho_e^2} \sin(2\Theta_e) = r_1 \sin \theta_1,$$

$$W_e = -\frac{1}{\rho_e} \cos \Theta_e.$$

In the physical plane, the ellipticity domain is the exterior of the disc Γ_1 , see Fig. 2.

- **Hyperbolicity domain:** For $1 \leq \rho$, the Eq. (2-4) are hyperbolic

$$X_1^h = r \cos \theta + \frac{a_1}{1 + \alpha} = -\frac{a_1}{\rho_h^{2+1}} \frac{\cos 2\Theta_h}{2} - \frac{a_1}{\rho_h^{2+1}} \frac{\beta}{2} + \frac{a_1}{1 + \alpha},$$

$$X_2^h = r \sin \theta = -\frac{a_1}{\rho_h^{2+1}} \frac{\sin 2\Theta_h}{2},$$

$$W_h = -\frac{a_1}{\rho_h^2} \cos \Theta_h.$$

The domain of hyperbolicity \mathcal{H} is plotted on Fig. 2.

To ensure the continuity of the displacement, we must explain the displacement in terms of (r, θ) in the two domains.

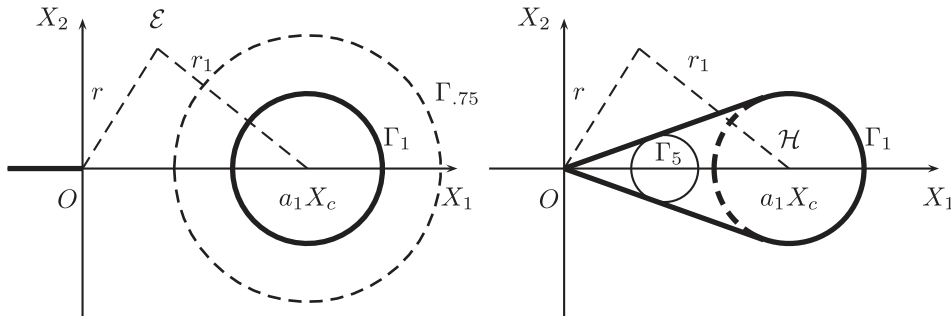


Fig. 2. The ellipticity (left) domain and the hyperbolicity domain (right) for $\alpha = -5$.

The domain of ellipticity $0 \leq \rho \leq 1$ is the exterior of the circle Γ_1 of radius $1/2$ centred in $a_1 X_c$, see Fig. 2. The circle $\Gamma_{.75}$ for $\rho = .75$ belongs to the domain of ellipticity. With respect to the polar coordinates at this point, we have

$$\frac{1}{\rho_e} = \sqrt{2r_1}, \quad 2\Theta_e = \theta_1 + \pi \quad W_e = \sqrt{2r_1} \sin \frac{\theta_1}{2} \quad (5-11)$$

We proceed now for the hyperbolic case, $\rho > 1$. For $\rho_h = \rho_a$, the point (X_1, X_2) is on the circle Γ_h of radius $1/(2\rho_h^{1+\alpha})$. In Fig. 2 the circle Γ_5 for $\rho = 5$ belongs to the domain of hyperbolicity, these points are also in the domain of ellipticity. Upon referring to the polar coordinates (r, θ) and solving the quadratic equation in ρ_h , one is led to

$$\frac{a_1}{2\rho_h^{2+1}} = r \frac{\beta \cos \theta \pm S(\theta)}{\beta^2 - 1}, \quad S(\theta) = \sqrt{1 - \beta^2 + \beta^2 \cos^2(\theta)} \quad (5-12)$$

The value of θ is limited by the condition of existence of $S(\theta)$. After some algebraic manipulations, we get the definition of Θ_h as

$$\cos \Theta_h = -\frac{1}{\sqrt{2}} \frac{(1 + \beta) \sin \theta}{\sqrt{(1 + \beta - \beta \cos^2 \theta + S(\theta) \cos \theta)}} \quad (5-13)$$

$$\sin \Theta_h = \frac{1}{\sqrt{2}} \frac{\cos \theta + S(\theta)}{\sqrt{(1 + \beta - \beta \cos^2 \theta + S(\theta) \cos \theta)}} \quad (5-14)$$

The value of the displacement has an analogous form as HRR field

$$W_h = a_1 f(\theta) \sin \theta \gamma^{\frac{\alpha}{2+1}}, \quad f = K \frac{(\beta \cos \theta + S(\theta))^{\frac{\beta}{2}}}{(\cos \theta + S(\theta))^{\frac{1}{2}}}, \quad (5-15)$$

where

$$\gamma = \frac{r}{a_1}, \quad K = \frac{\sqrt{\beta + 1}}{2^{\beta/2}} \quad (5-16)$$

The continuity of w is ensured along the line $\Theta = \pi/2$.

The domain of hyperbolicity must contain the disk Γ_1 , and the constant a_1 must be determined. The solution is obtained by ensuring the continuity of w on a curve, this curve is inside the intersection of the domains \mathcal{E} and \mathcal{H} .

Along this curve, we must ensure the equality $(X_1^e, X_2^e, W_e) = (X_1^h, X_2^h, W_h)$, this is an implicit equation of a curve as a relation between ρ_e , Θ_e and ρ_h , Θ_h .

By replacing of θ_1 and r_1 in terms of polar coordinates r , θ , and using the equality $W_e = W_h$, the curve $r_s(\theta)$ satisfies the equation

$$-\gamma^{2\beta} - 2(a_1\gamma \cos \theta - X_c)K^2 f^2 \gamma^{\beta-1} + a_1^2 K^4 f^4 \sin^2 \theta = 0 \quad (5-17)$$

This generalises the results of Knowles and Sternberg (1980) for different α . On this curve the gradient of the displacement has a discontinuity, but the stress vector must be continuous.

As $W_e = W_h$ the normal vector \mathbf{n} to the curve $r_s(\theta)$ in the plane (x_1, x_2) is

$$\mathbf{n} \|\nabla W_e - \nabla W_h\| = \nabla W_e - \nabla W_h, \quad (5-18)$$

then the continuity of the stress vector implies

$$(\mu(R_h)\nabla W_h - \mu(R_e)\nabla W_e) \cdot \mathbf{n} = 0. \quad (5-19)$$

For the considered behaviour we obtain

$$\begin{aligned} \mu_1 \rho_h^{\alpha-1} \|\nabla W_h\|^2 + \mu_o \|\nabla W_e\|^2 \\ - (\mu_1 \rho_e^{\alpha-1} + \mu_o) \rho_e \rho_h \cos(\Theta_e - \Theta_h) = 0. \end{aligned} \quad (5-20)$$

By elimination of (ρ_e, Θ_e) using the relations

$$\begin{aligned} -\frac{\sin(2\Theta_e)}{2\rho_e^2} &= -\frac{a_1}{\rho_h^{\alpha+1}} \frac{\sin(2\Theta_h)}{2}, & -\frac{\cos(\Theta_e)}{\rho_e} \\ &= -\frac{a_1}{\rho_h^\alpha} \cos \Theta_h, & \lambda \rho_h^{\alpha+1} = 1 \end{aligned} \quad (5-21)$$

we obtain finally

$$(\mu_o - \mu_1 a_1) \left(1 - \frac{a_1}{\lambda^\beta}\right) \cos^2 \Theta_h = 0. \quad (5-22)$$

As the stress τ is continuous at point $\rho = 1$, $\mu_1 = \mu_o$ then $a_1 = 1$. The displacement w is then continuous along the circle Γ_1 . The domain of hyperbolicity contains the interior of the disc Γ_1 and is limited by two ray-segments, see Fig. 2.

$$0 < r < \frac{1}{2} \sqrt{\beta^2 - 1}, \quad \theta_x = \pm \arccos \sqrt{\beta^2 - 1} / \beta \quad (5-23)$$

The curve L begins at $(r, \theta) = (0, 0)$ and finishes at $(r_s(\theta_x), \theta_x)$. When $\theta \rightarrow 0^+$, $\theta_1 \rightarrow \pi$, and $0 = X_c - r_1$ then we know the value of $W_x = w_e$ at this point. The displacement w_h along the curve L tends to W_x and we obtain the asymptotic value of $r_s(\theta)$

$$r_s(\theta) = W_x \quad (5-24)$$

and the slope at the origin of the shock line is zero. When the value of $\alpha < 0$ decreases, the length of the domain of hyperbolicity along X_1 increases, see Fig. 3

For a finite ρ_m , the domain of ellipticity must contain the domain of hyperbolicity then $b_1 \geq b_0$. The Eq. (5-4) is conserved. The potential \bar{U} is continuous along the line $\Theta = \frac{\pi}{2}$. A numerical solution is obtained ensuring the continuity of the displacement at the point image of $(\rho = 1, \Theta = 0)$. The two constants b_o, b_1 are evaluated ensuring the existence of a point $(\rho_e \leq 1, \Theta = 0)$ of the domain of elasticity and a point (ρ_m, Θ_h) where the

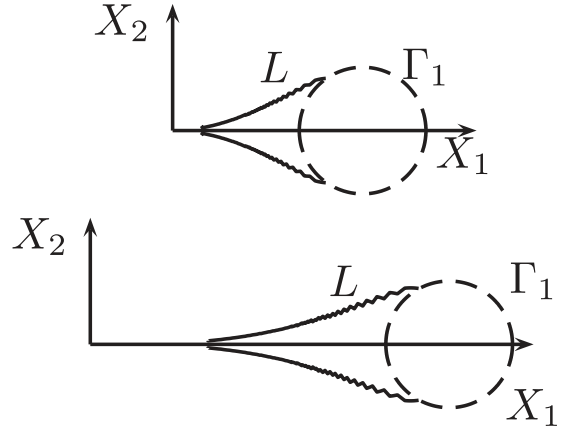


Fig. 3. The shape of the shock curve L , for $\alpha = -0.5$ (left) and for $\alpha = -0.7$ (right) under the same loading and the same scale, Γ_1 is the circle $\rho = 1$.

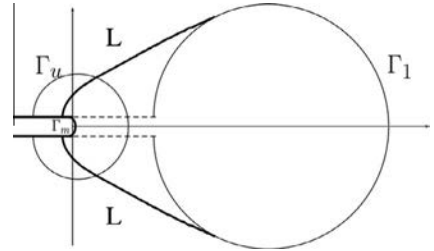


Fig. 4. The shock line for special law $\alpha = -0.5$, $\rho_m = 20$, $\rho_u = 8$.

displacement $W_o(\rho_e, 0)$ is equal to $W_h(\rho_m, \Theta_h)$. The solution is theoretically closed, however the proof of the continuity of the stress vector and of the displacement, along the determined shock curve L , is obtained only numerically. This point request more investigation.

6. A particular constitutive law

Consider now the special material introduced by Abeyaratne (1981) for which the material characteristics are $(\mu_1 = \mu_o, \alpha = -1/2, \mu_2 = \mu_o \left(\frac{\rho_o}{\rho_h}\right)^{3/2}, R_m = \infty)$. For this case the domains of ellipticity and hyperbolicity are more complex, see Fig. 5.

We consider now the same law with a finite R_m . By applying the above results we can conclude that the local solution is decomposed into three parts

- Ellipticity domain $0 \leq \rho \leq 1$

$$\begin{aligned} X_1 &= -\frac{1}{2} - \frac{1}{2\rho^2} \cos(2\Theta) - b \log \rho \\ X_2 &= -\frac{1}{2\rho^2} \sin(2\Theta) + b \left(\Theta - \frac{\pi}{2}\right) \\ W &= -\left(\frac{1}{\rho} + b\rho\right) \cos \Theta \end{aligned}$$

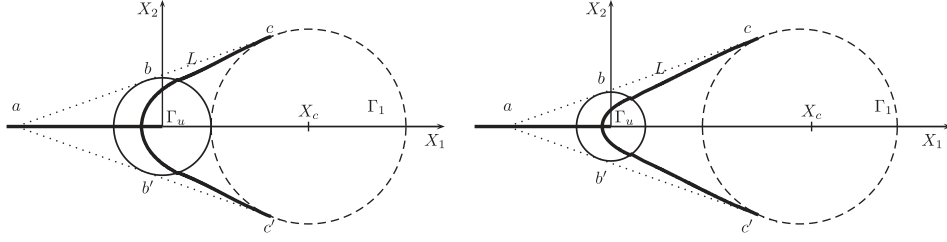


Fig. 5. The domains and the shock curve L for $\alpha = -.5$, $R_m = \infty$, for $\rho_u = 4$ (left) and $\rho_u = 8$ (right).

- Hyperbolicity domain $1 \leq \rho \leq \rho_u$

$$X_1 = -\left(\frac{a_1}{\sqrt{\rho}} - 3b\right) \frac{\cos 2\Theta}{2} + 3\frac{a_1}{2\sqrt{\rho}} + \frac{3b}{2} + 2b \log \rho - 2a$$

$$X_2 = -\left(\frac{a_1}{\sqrt{\rho}} - 3b\right) \frac{\sin 2\Theta}{2} + b\left(\Theta - \frac{\pi}{2}\right)$$

$$W = -(a_1\sqrt{\rho} - 2b\rho) \cos \Theta$$

- Ellipticity domain $\rho_u \leq \rho \leq \rho_m$

$$X_1 = -b \log \rho - \frac{a_2}{2\rho^2} \cos 2\Theta - \frac{a_2}{2\rho_u^2} + 2a_1 \left(\sqrt{\frac{1}{\rho_u}} - 1 \right) + 3b \log \rho_u$$

$$X_2 = -\frac{a_2}{\rho^2} \sin(2\Theta) + b\left(\Theta - \frac{\pi}{2}\right)$$

$$W = -\left(\frac{a_2}{\rho} + b\rho\right) \cos \Theta$$

In this case, the domains are now defined by cycloids. The continuity of w for ρ_o , ρ_u implies

$$1 = a_1 - 3b, \quad \frac{a_2}{\rho_u^2} = \frac{a_1}{\sqrt{\rho_u}} - 3b \quad (6-1)$$

The free stress boundary along the cycloid $\rho = \rho_m$ implies

$$b - \frac{a_2}{\rho_m^2} = 0 \quad (6-2)$$

then the constants are determined

$$b = \frac{1}{\sqrt{\rho_u} \left(3 + \left(\frac{\rho_m}{\rho_u} \right)^2 \right) - 3}, \quad a_1 = b\sqrt{\rho_u} \left(3 + \left(\frac{\rho_m}{\rho_u} \right)^2 \right) \quad a_2 = b\rho_m^2 \quad (6-3)$$

The determination of the shock curve is made numerically and the solution is compared to the solution of Abeyaratne (1981). The geometry is of the same type but the circles are replaced by cycloids.

When ρ_m tends to infinity, b tends to 0, the solution of Abeyaratne (1981) is recovered, $b = 0$, $a_1 = 1$, $a_2 = \rho_u\sqrt{\rho_u}$ and the cycloids are transformed in circles. In Fig. 5, the curves $(X_1(\rho, \Theta), X_2(\rho, \Theta))$ at fixed ρ are circles Γ_ρ . The domain of ellipticity ($0 < \rho < 1$) is the exterior of the disc Γ_1 , the domain of ellipticity $\rho_u < \rho$ is the interior of disc Γ_u . The domain of hyperbolicity is limited by the two

ray-segment (bc) , $(b'c')$ contains the disc Γ_1 and is limited by the circle Γ_u .

The curve L does not emanate from the crack tip. In Fig. 5 the position of the crack tip is the origin. The slope of the ray-segment does not depend on ρ_u .

7. The particular case $\alpha = 0$

For $\alpha = 0$, the stress $\tau = \tau_1$ is constant, the potential U and the associated displacements, the polar coordinates are

- Ellipticity domain: $0 \leq \rho \leq 1$

$$U_e = \rho \cos \Theta \int_\rho^1 \frac{dt}{t^3}$$

$$X_1^e = -\frac{1}{2} - \frac{1}{2\rho^2} \cos 2\Theta = r \cos \theta - a = r_1 \cos \theta_1$$

$$X_2^e = -\frac{1}{2} \sin 2\Theta = r \sin \theta = r_1 \sin \theta$$

$$W_e = -\frac{1}{\rho} \cos \Theta$$

- Parabolic domain $1 \leq \rho$

$$U_p = a\rho \cos \Theta \int_\rho^1 \frac{dt}{t^2} = a\left(\frac{1}{\rho} - 1\right) \rho \cos \Theta$$

$$X_1^p = \frac{a}{2\rho} (1 - \cos \Theta) - a = r \cos \theta$$

$$X_2^p = -\frac{a}{2\rho} \sin \Theta = r \sin \theta$$

$$W_p = -a \cos \Theta$$

Assuming that $W_e(1, \Theta) = W_p(1, \Theta)$ for all Θ , the constant a must be equal to 1. Consider now the constitutive law, in the elliptic domain, $\tau = \tau_o$ and when $\rho \geq 1$, the stress is τ_1 for the same Θ , such a continuous solution is obtained only if $\tau_1 = \tau_o$. The solution is that of Rice (1966).

When $\tau_o \neq \tau_1$, we must determine a curve along which the displacement is continuous simultaneously with the stress vector. Then we must solve the system: find the two families of points $(\rho_e \leq 1, \Theta_e)$, $(\rho_p \geq 1, \Theta_p)$ such that

$$W_e(\rho_e, \Theta_e) = W_p(\rho_p, \Theta_p), \quad X_1^e(\rho_e, \Theta_e) = X_1^p(\rho_p, \Theta_p), \quad X_2^e(\rho_e, \Theta_e) = X_2^p(\rho_p, \Theta_p).$$

This system determines a curve implicitly. Along this curve the normal vector is defined by

$$\mathbf{n} \parallel \nabla W_e - \nabla W_p \parallel = \nabla W_e - \nabla W_p \quad (7-1)$$

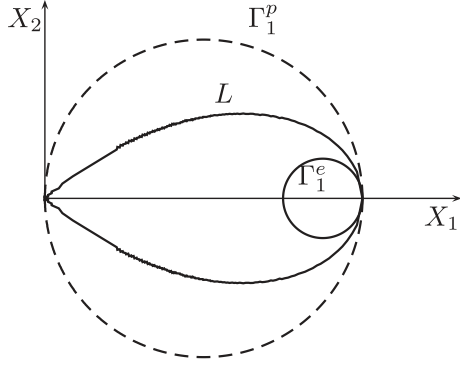


Fig. 6. The shock curve for $\alpha = 0$, $R_m = \infty$, $\rho_u = \rho_o = 1$, $a = 4$.

and the continuity of the stress vector $(\sigma_p - \sigma_e) \cdot \mathbf{n} = 0$ determines the value of the constant a_1 .

As $\tau_1^p = \tau_1 \cos \Theta_p$, $\tau_2^p = \tau_1 \sin \Theta_p$ and $\tau_1^e = \tau_o \rho_e \cos \Theta_e$, $\tau_2^e = \tau_o \rho_e \sin \Theta_e$, the conditions of continuity is written as

$$\tau_1 \rho_p + \tau_o \rho_e^2 - (\tau_1 \rho_e + \tau_o \rho_e \rho_p) \cos(\Theta_p - \Theta_e) = 0 \quad (7-2)$$

From the definition of the curve we have

$$\cos(\Theta_p - \Theta_e) = a \cos^2 \Theta_p + \frac{\sin^2 \Theta_p}{\rho_p}, \quad \frac{1}{\rho_e^2} = a^2 \cos^2 \Theta_p + \frac{\sin^2 \Theta_p}{\rho_p^2} \quad (7-3)$$

then reporting these values in the expression of the stress-vector continuity, we obtain

$$\cos^2 \Theta_p (1 - \rho_p a) (a \tau_1 - \tau_o) = 0 \quad (7-4)$$

then this determines the value of a : $a = \tau_o / \tau_1$ for which the stress vector is continuous.

The solution is continuous only at point $\rho = 1$, $\Theta = \pi/2$ of the circle Γ_e . For this case the general form of the potential is recovered

$$U = \rho \cos \Theta \int_{\rho}^{\infty} \frac{dt}{\tau(t)t^2}. \quad (7-5)$$

When ρ tends to infinity, X_1 tends to $X_o = -a$, the parabolic domain is inside the circle of radius $a/2$ centred at point $-a/2$. For increasing ρ , the position of point at fixed ρ is a circle tangent at point $(X_o, 0)$. The polar coordinates are referred to this point.

Using the preceding results, in the elliptic domain, we have the connection

$$\frac{1}{\rho} = \sqrt{2r_1}, \quad 2\Theta = \theta_1 + \pi, \quad W = \sqrt{2r_1} \sin \frac{\theta_1}{2} \quad (7-6)$$

And in parabolic domain,

$$r = \frac{a}{\rho} \cos \theta, \quad \Theta = \theta + \frac{\pi}{2}. \quad (7-7)$$

Now, we must find a curve, belonging the domain Γ_1^p / Γ_1^e , along which the displacement is continuous. This curve is defined by the equation $w_e(r_s, \theta) = w_p(r_s, \theta)$, so we have

$$r_s^2 - 2a^2 r_s \cos \theta - (a^4 \sin^2 \theta - 2a^3 + a^2) = 0, \quad (7-8)$$

then the equation of the shock curve L is given by:

$$r_s = a^2 \cos \theta + 1 - a^2. \quad (7-9)$$

The shape of L is plotted in Fig. 6

We consider now that ρ_m is finite, the solution is obtained by addition of peculiar functions

- Ellipticity domain (exterior of Γ_e^e)

$$U = \frac{1}{2} \rho \cos \Theta \left(\frac{1}{\rho^2} - 1 \right) - b_1 \rho (\log \rho \cos \Theta - \left(\Theta - \frac{\pi}{2} \right) \sin \Theta)$$

$$X_1 = -\frac{1}{2\rho^2} \cos 2\Theta - \frac{1}{2} b_1 \log \rho, \quad X_2 = -\frac{1}{2\rho^2} \sin 2\Theta + b_1 \left(\Theta - \frac{\pi}{2} \right)$$

$$W = -\left(\frac{1}{\rho} + b_1 \rho \right) \cos \Theta$$

- Parabolic domain (interior of Γ_1^p)

$$U = a \rho \cos \Theta \left(\frac{1}{\rho} - 1 \right) + b_2 \rho \left(\Theta - \frac{\pi}{2} \right) \sin \Theta,$$

$$X_1 = -\left(a \frac{1}{\rho} - b_2 \right) \frac{\cos 2\Theta - 1}{2} - a + b_2,$$

$$X_2 = -\left(a \frac{1}{\rho} - b_2 \right) \frac{\sin 2\Theta}{2} + b_2 \left(\Theta - \frac{\pi}{2} \right)$$

$$W = -a \cos \Theta$$

The domain of ellipticity contains the parabolic domain if $b_2 < b_1$.

Along the cycloid $\rho = \rho_m$, the stress free condition gives

$$b_2 = \frac{a}{2\rho_m} \quad (7-10)$$

The construction of the curve is now made numerically. The solution is plotted in Fig. 7. Further studies is necessary to prove analytically the continuity of the displacement and of the stress vector along the line L .

When $\tau_1 = \tau_o$ and R_m finite, the continuity of the displacement, along $\rho = 1$, and the condition along ρ_m , imply:

$$a = 1 + b, \quad b_1 = b_2 = b, \quad 2b\rho_m = a \quad (7-11)$$

and the solution of Bui (1980) is recovered. The domain where $(\tau = \tau_o)$ is limited by the two cycloids Γ_1 , Γ_m (See Fig. 8).

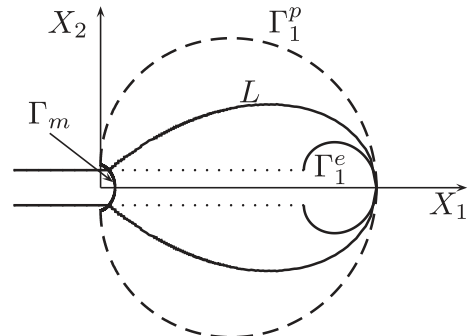


Fig. 7. The shock line for $\alpha = 0$, $\rho_m = 2$, $\rho_u = \rho_o = 1$, $a = 4$.

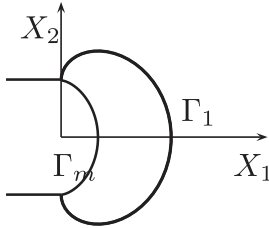


Fig. 8. The shock line for $\alpha = 0$, $\rho_m = 2$, $\rho_u = \rho_o = 1$.

8. Conclusion

We have determined the shape of the damaged zone under anti-plane shear condition for hyperelastic brittle material. The analytical results are a generalisation of preceding results obtained by Bui and Ehrlacher (1980,) for brittle material. The thickness of the damaged zone is determined by the critical strain energy at rupture and the loading.

We have extend the theory for more complex constitutive law and recovered results obtained many years ago. The case of power law and extension on a class of non linear elastic law is discussed with and without brittle damage. With brittle damage one obtains for mode III loading, we have found the geometry of the quasi-crack proposed by Neuber for hardening law, this results is extended to some case of softening especially for a generalisation of the special material introduced by Abeyaratne (1981).

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