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From damage to fracture, a modelization based on moving discontinuities and layers

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Abstract. A damage model is proposed based on a continuous transition from undamaged to damaged material. The evolution of damage is associated with a moving layer of finite thickness l_c , then initiation and propagation of damage can be described in the same constitutive law. The driving force associated to the layer motion is a generalized release rate of energy. Using a normality rule based on this force the solution of the rate boundary value problem of propagation and displacement satisfies a variational inequality. Applications of the model are analyzed.

Introduction

Several approaches are proposed to damage modelization in order to describe the initiation of defect and the gradual loss of local stiffness. For instance to avoid spurious mode of localization, constitutive laws based on second gradient of damage ([2]) or phase field ([5, 6]) have been recently proposed. Many others regularizations of damage constitutive laws have been also tempted to control the evolution of damage ([9, 8]).

For brittle material, the evolution of the interface Γ_o separating the undamaged material $d = 0$ to the damaged one $d = 1$ have been studied in a framework based on an energetical description ([4, 11]). In total brittle damage, the damaged zone can not support any further tension after some critical prescribed value in stress, strain, or free energy. With this property, at each point of the moving interface the dissipation defines the driving force $G(s)$ associated to the motion of the interface. When the velocity of propagation ($a(s)$) is governed by a normality rule based on this driving force, many variational formulations of the rate boundary value problems have been established ([4, 10, 11]). The rate boundary value problem for brittle material possesses generally many solutions when the propagation law is governed by a generalized Griffith's law:

$$a(s) \geq 0, \quad G(s) \leq G_c, \quad (G(s) - G_c)a(s) = 0.$$

This model has no characteristic length. Moreover the introduction of a surface energy along the interface plays a role on the uniqueness of the velocity $a(s)$. In this case the front is more stable [12] because the new driving force depends now on the mean curvature κ_m of the moving front:

$$G_\beta(s) = G(s) - \kappa_m(s)\beta \leq G_c,$$

Using this framework, the propagation of an existing interface between a undamaged and a damaged zone can be studied. The presence of surface energy density β stabilized the propagation of the interface. This interface has no thickness and mechanical quantities present discontinuities [12].

Unless initiation of defects can be analysed as an equilibrium bifurcation based on evolution of infinitesimal defect and imperfection analysis [13], this approach does not unify initiation and propagation of defects using the same constitutive behaviour. In presence of a surface energy density β the analysis implies that the critical loading for initiation of defect becomes infinite in the case of spherical infinitesimal defect. Description of moving interface and layer is also useful to study variations of domain. Such description with thin or thick layers permits to describe complex process during contact wear [14].

Here, a new approach is proposed based on the propagation of a moving layer inside which the damage is a continuous function of the position. The evolution of damage is then associated to the motion of a layer of finite thickness using of level-set approach [7]. The initial material and the damaged material are separated by a surface Γ_o . This boundary is a moving interface, defined as an equipotential or a level-set. Through the interface the properties of the material change in a continuous manner.

The model of damage

We consider a body Ω under tension \underline{T}^d over $\partial\Omega_T$ and prescribed displacement \underline{u}^d on the complementary part of the boundary $\partial\Omega_u$. Under this loading, the body is deformed and a displacement field \underline{u} describes the motion of all material points of the body. The material of the body has an elastic behaviour with moduli evolving with damage. The free energy of the body $w(\boldsymbol{\varepsilon}, d)$ is a function of the strain $\boldsymbol{\varepsilon} = \frac{1}{2}(\nabla\underline{u}^T + \nabla\underline{u})$ and of a scalar damage variable $d, 0 \leq d \leq 1$. The state equations are defined classically as:

$$\boldsymbol{\sigma} = \frac{\partial w}{\partial \boldsymbol{\varepsilon}}, Y = -\frac{\partial w}{\partial d}, \quad \mathbb{C}(d) = \frac{\partial^2 W^2}{\partial \boldsymbol{\varepsilon} \partial \boldsymbol{\varepsilon}}$$

where $\boldsymbol{\sigma}$ is the Cauchy stress. The elastic moduli \mathbb{C} are known functions of d . The mechanism of dissipation is only due to damage and the dissipation of the whole body is reduced to

$$D_m = \int_{\Omega} Y \dot{d} \, d\Omega \geq 0.$$

When damage is established the whole body is decomposed in three parts, the undamaged body Ω_1 , the transition zone Ω_c (where $0 < d < 1$) and the damaged material Ω_2 where ($d = 1$). On the boundary $\partial\Omega_c$ the free energy is continuous, there is no discontinuities of the stress vector and the moduli of elasticity are continuous. Then when the layer Ω_c is moving, there exists no dissipation along the boundary of the layer. A more detail discussion is given in ([7]).

The level-set $\phi = 0$ gives the position of Γ_o the part of boundary Ω_c where $d = 0$. We assume that the damage d is a continuous explicit function $d(\phi)$ of the distance ϕ to the surface Γ_o , [7]. In the domain where the gradient of the level-set is continuous, the damage satisfies

$$d = 0, \phi \leq 0 \quad ; \quad d'(\phi) \geq 0, 0 \leq \phi < l_c \quad ; \quad d(\phi) = 1, \phi \geq l_c.$$

The surface $d(X, t) = d_o$ is also a level-set. This representation of damage is illustrated Fig.1. The minimum length separating the level-set $d = 0$ to the level-set $d = 1$ is l_c . To simplify the expressions,

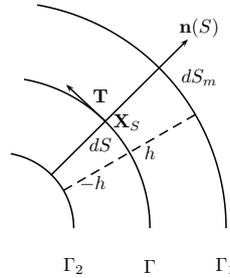


Fig. 1: The local frame

the study is made for plane motion. With this definition, the dissipation is obtained as

$$D_m = \int_{\Gamma} \int_o^l Y d'(\phi) (1 - \kappa\phi) \dot{\phi} d\phi dS.$$

where κ is the local curvature of the curve Γ_o . The evolution of the level-set z is given by the evolution of the moving surface $\phi(X, t) = z$ then

$$\dot{\phi} - a(s)\nabla\phi \cdot \underline{N} = 0,$$

where the velocity a is the normal speed of the iso- ϕ and $\underline{N} = \nabla\phi/|\nabla\phi|$ is the normal vector to the surface $\phi = z$. The driving force G associated to the velocity a is given according to the dissipation

$$D_m = \int_{\Gamma} G(s)a(s) dS, \text{ where } G(s) = \int_o^l Y \nabla d \cdot \underline{N} \det(I - \kappa\phi) d\phi.$$

The velocity a is determined with respect to a constitutive law based on the driving force $G(s)$. The integration of a local normality law based on Y

$$\dot{d} \geq 0, \quad Y \leq Y_c, \quad (Y - Y_c)\dot{d} = 0$$

suggests that the velocity a satisfies the generalized Griffith's law for $(l(s, t) \leq l_c)$

$$a(s) \geq 0, \quad G(s) \leq \bar{G}(s) = \int_o^l Y_c \nabla d \cdot \underline{N} (1 - \kappa\phi) d\phi, \quad (G(s) - \bar{G}(s))a(s) = 0,$$

which is an averaged yielding function on the layer. This definition ensures the positivity of the entropy production.

On the motion of a layer

The curve Γ_o is the interface separating the undamaged material to the damaged one. A point M_t of Γ_o is referred by its curvilinear coordinates s , its position is $\underline{X}_o(S)$. The local frame is defined by the tangential vector $\underline{T} = \frac{d\underline{X}}{dS}$ and the normal vector \underline{N} . They satisfy the Fresnet relations

$$\frac{d\underline{T}}{dS} = \kappa \underline{N}, \quad \frac{d\underline{N}}{dS} = -\kappa \underline{T}$$

where κ is the curvature of the curve Γ_o . A point M of the layer has coordinates S, z in the frame $\underline{T}, \underline{N}$, $\underline{X} = \underline{X}_o + z\underline{N}$ then the local frame at point \underline{X} is defined by

$$d\underline{X} = dS \underline{\tau} + \nu$$

it depends on the position inside the layer,

$$\underline{\tau} = (1 - \kappa z)\underline{T}, \quad \nu = \underline{N}$$

The level-set $\phi(\underline{X}_o, t) = 0$ is the curve Γ_o , during the motion defined by the normal velocity $a(S)$ of Γ_o , the local frame $\underline{T}, \underline{N}$ is changing. We know that $\underline{N} = \frac{\partial\phi}{\partial\underline{X}}/|\frac{\partial\phi}{\partial\underline{X}}|$, and

$$\frac{\partial \phi}{\partial \underline{X}} \cdot \underline{\phi} + \frac{\partial \phi}{\partial t} = 0,$$

this defines $\underline{\phi} = a(s)\underline{N}$, The same is true for all level set $\phi(\underline{X}, t) = z = \phi(\underline{X}_o + z\underline{N}, t)$.

Actual geometry and convected geometry. The actual position \underline{x}_o of a point of Γ_o satisfies the equations of motion $\underline{x}_o = \underline{X}_o + \underline{\phi}(S)dt$, the evolution of the local frame is then deduced. Introducing the derivative following the motion of the surface Γ_o by

$$D_\phi G = \lim_{dt \rightarrow 0} \frac{G_{t+dt} - G_t}{dt}.$$

we obtain

$$D_\phi \underline{T} = \frac{d\phi}{dS} \cdot \underline{N} \underline{N}, \quad D_\phi \underline{N} = -\frac{d\phi}{dS} \cdot \underline{N} \underline{T}, \quad D_\phi \kappa = \underline{N} \cdot \frac{d^2 \phi}{dS^2} - 2\kappa \frac{d\phi}{dS} \cdot \underline{T}.$$

At a point $\underline{X} = \underline{X}_o + z\underline{N}$ of the layer the value of mechanical quantity $f f(\underline{X}, t)$ is following the motion of the layer. At time $t + dt$, the actual position is $\underline{x} = \underline{x}_o + z \underline{N}_{t+dt}$ and the evolution is given by

$$D_\phi \underline{X} = \lim_{dt \rightarrow 0} \frac{(\underline{x} - \underline{X})}{dt} = a(s)\underline{N} - z \frac{da}{dS} \underline{T}$$

At point \underline{X} the variation of any mechanical quantities following the motion of the layer is then

$$D_\phi f = \lim_{\eta \rightarrow 0} \frac{f(\underline{X} + \eta D_a \underline{X}, t + \eta) - f(\underline{X}, t)}{\eta} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial \underline{X}} \cdot D_\phi \underline{X}$$

For the function $\phi(\underline{X}, t) = z$, we have $\nabla \phi = \underline{N}$, $\Delta \phi = -\frac{\Gamma}{1 - z\Gamma}$ and $D_a dS = -\kappa a dS$.

Variations of averaged quantity on the layer. To study the evolution of the driving force we must studied the evolution of quantities such as

$$F = \int_o^l f(1 - z\kappa) dz \, dS \quad D_\phi F = \int_o^l D_\phi f (1 - \kappa z) dz - \int_o^l z f D_\phi \kappa dz$$

On the rate boundary value problem

At time t , the position of equilibrium is defined by the displacement \underline{u} , the position of the layer is known and the solution is inside the domain of reversibility, then the driving force satisfies:

$$F(S, t) = \int_o^l Y(\underline{\varepsilon}(\underline{u}), d) d'(\phi)(1 - \kappa z) dz - Y_c \int_o^l d'(\phi) dz \leq 0.$$

The quantity F is defined on the moving curve Γ_o . For an increasing of the loading, the evolution of the position of Γ_o is governed by the normality rule

$$a(S) \geq 0, \quad F(S, t) \leq 0, \quad a(S)F(S, t) = 0$$

It is obvious that $a(S)$ is positive if $F = 0$. At this state F satisfies $\dot{F} \leq 0$. The derivation of the consistency condition $aF = 0$ implies that $a(S) > 0$ if $D_\phi F = 0$. Then the set of admissible field $a(S)$ satisfies

$$\int_{\Gamma} (a(s) - a^*(s)) D_{\phi} F \, ds \leq 0$$

this is a variational inequation to solve on the set of admissible fields ($a^*(s) \geq 0$) along $\Gamma_o^+ = \{S \in \Gamma_o / F(S, t) = 0\}$.

We must now write the consistency condition $D_{\phi} F = 0$, we have the property ($f(\varepsilon, \phi) = Y d'(\phi)$):

$$D_{\phi} f(\varepsilon, \phi) = \frac{\partial f}{\partial \varepsilon} D_{\phi} \varepsilon = \frac{\partial f}{\partial \varepsilon} : (\dot{\varepsilon} + \nabla \varepsilon \cdot D_{\phi} \varepsilon = \dot{f} + a \frac{\partial f}{\partial z} - \frac{z}{1 - z\kappa} \frac{da}{ds} \frac{\partial f}{\partial S}$$

Taking account of $D_{\phi} \phi = 0$ and $\phi(\underline{X}_o + z\underline{N}, t) = z$, the inequation is rewritten as

$$\begin{aligned} 0 &\geq \int_{\Gamma} (a - a^*) \left(\int_l \dot{f}(1 - \kappa z) dz + a \left(\frac{\partial f}{\partial z} (1 - \kappa z) - z f \kappa^2 \right) dz \right) ds \\ &\quad - \int_{\Gamma} (a - a^*) \left(\int_l z f \left(\frac{d^2 a}{ds^2} + \frac{da}{ds} \frac{df}{ds} \right) dz \right) ds \end{aligned}$$

and by integration by parts

$$0 \geq \int_{\Gamma} (a - a^*) \left(\int_l \dot{f}(1 - \kappa z) dz + \int_l a \left(\frac{\partial f}{\partial z} (1 - \kappa z) - z f \kappa^2 \right) dz \right) ds - \int_{\Gamma} \int_l f z dz \frac{da}{ds} \frac{d}{ds} (a - a^*) ds$$

As $\dot{f} = \frac{\partial f}{\partial \varepsilon} \dot{\varepsilon} - a \frac{\partial f}{\partial \phi}$, the inequation becomes

$$0 \geq \int_{\Gamma} (a - a^*) \int_l \frac{\partial f}{\partial \varepsilon} : \dot{\varepsilon} (1 - \kappa z) dz \, ds + \int_{\Gamma} (a(s) - a^*(s)) L a(s) + \int_{\Gamma} \frac{da}{ds} M \frac{d}{ds} (a(s) - a^*(s)) \, ds$$

where L, M are functions of the actual state

$$L = \int_l \left(\frac{\partial f}{\partial \varepsilon} : \frac{\partial \varepsilon}{\partial z} (1 - \kappa z) + z f \kappa^2 \right) dz, \quad M = \int_l f z dz$$

Property of the solution. The solution of the rate boundary value problem satisfies the variational inequation

$$\frac{\partial \mathcal{F}}{\partial \underline{v}} \cdot (\underline{v} - \underline{v}^*) + \frac{\partial \mathcal{F}}{\partial a} (a - a^*) \leq 0$$

among the set of admissible fields \mathcal{K}

$$\mathcal{K} = \{(\underline{v}, a) / \underline{v} = \underline{v}^d \text{ over } \partial\Omega_u, a(S) \geq 0 \text{ along } \Gamma_o^+, a(s) = 0 \text{ otherwise } \}$$

where

$$\mathcal{F}(\underline{v}, a) = \int_{\Omega} \frac{1}{2} \varepsilon(\underline{v}) : C(d) : \varepsilon(\underline{v}) \, d\Omega + \int_{\Gamma} a \int_l \frac{\partial f}{\partial \varepsilon} : \varepsilon(\underline{v}) (1 - z\kappa) dz \, ds + \int_{\Gamma} \frac{1}{2} (a^2 L + M \frac{da^2}{ds}) \, ds$$

The proof is easy to obtain. Studying the properties of \mathcal{F} gives the conditions on the stability and uniqueness of the velocity a in the form proposed in [10, 12] When a is given, the solution

$\underline{v}(\underline{v}^d, \underline{\dot{T}}^d, a(S))$ is a classical problem of elasticity with a particular distribution of volumic forces due to $a(S)$ at each point of the layer:

$$\dot{\sigma} = \mathbb{C}(d) : \varepsilon(\underline{v}) + \frac{\partial \mathbb{C}}{\partial d} : \varepsilon(\underline{u}) \nabla d \cdot \underline{N} a(S)$$

The solution \underline{v} depends on the field $a(S)$, defining $\mathcal{W}(a, \underline{v}^d, \underline{\dot{T}}^d) = \mathcal{F}(\underline{v}(\underline{v}^d, \underline{\dot{T}}^d, a), a)$, we have the following properties

The solution a is unique if $\forall \beta$

$$\int_{\Gamma} \int_{\Gamma} \frac{\partial^2 \mathcal{W}}{\partial a \partial a} \beta(S) \beta(S') \, ds \, ds' \geq 0$$

The solution is stable if $\forall \beta \geq 0$ over Γ_o^+

$$\int_{\Gamma} \int_{\Gamma} \frac{\partial^2 \mathcal{W}}{\partial a \partial a} \beta(S) \beta(S') \, ds \, ds' \geq 0$$

Typical examples and applications

Sharp interfaces - brittle damage. When the thickness l_c tends to 0, the functional \mathcal{F} tends to the primal functional obtained with sharp interface as proposed in [10]. In this case the fields are discontinuous along the curve Γ_o and the driving force is the jump of Eshelby tensor through this curve.

A model of bar with a moving layer. The free energy w for uniaxial response is

$$w(\varepsilon, d) = \frac{1}{2} E(d) \varepsilon^2, \quad Y = -\frac{\partial w}{\partial d},$$

where d varies from 0 to 1, the Young modulus $E(d)$ is a continuous function of d , then there is no discontinuity at $d = 0$. For comparison with a sharp interface we consider the matching conditions $E(0) = E_1$ and $E(1) = E_2$. On the layer the value of the damage parameter is given by $(\phi(X, t) = \Gamma(t) - X)$. The damage parameter d is an increasing function of the distance ϕ to the boundary Γ separating the sound material to the damaged one. The function $d(\phi)$ is a given continuous function D of ϕ

Initially $\Gamma(t) = 0$ and the propagation of the layer begins at the origin of the bar, so the thickness $\Gamma(t) = l(t)$ is smaller than l_c . The thickness increases to l_c and after this step of initialization, the thickness is kept constant.

For the given constitutive laws, the dissipation is local and only due to damage $d_m = Y \dot{d}$. From the integration over the layer we get the total dissipation due to damage inside the bar:

$$D_m = \int_o^l Y \dot{d} \, d\phi.$$

Assuming that d is a continuous function of ϕ , then the young modulus E becomes a continuous function of ϕ . The prime denotes the derivative with respect to ϕ , $d'(\phi) = \frac{dd}{d\phi}$ and $E'(\phi) = \frac{dE}{d\phi}$. To

define the local force Y we need the derivative $\frac{\partial E}{\partial d} \frac{\partial E}{\partial d} = \frac{\partial E}{\partial \phi} \frac{\partial \phi}{\partial d}$. The fact that this derivative must be finite implies properties on $E'(\phi)$, $d'(\phi)$. Moreover, the local force Y is

$$Y = -\frac{E'(\phi)}{2d'(\phi)}\varepsilon^2 = -\frac{E'(\phi)}{2E^2d'(\phi)}\Sigma^2.$$

As the velocity ϕ satisfies $\dot{\phi} = a(s)$, the total dissipation is finally expressed as

$$D_m = \frac{a}{2} \int_0^l \Sigma^2 \left(-\frac{E'(\phi)}{E^2(\phi)}\right) d\phi = G(l, \Sigma)a, \quad G(l, \Sigma) = \frac{1}{2} \Sigma^2 \left(\frac{1}{E(l/l_c)} - \frac{1}{E(0)}\right).$$

When $l = l_c$, we recover the expression obtained for a sharp interface.

For sharp interfaces, the strain ε and the moduli of elasticity are discontinuous. The total energy and the dissipation are given by

$$W = \frac{1}{2} \left(\frac{x}{E_1} + \frac{1-x}{E_2}\right) \Sigma^2 \quad D_m = -\frac{\partial W}{\partial x} \dot{x} = \frac{1}{2} \left(\frac{1}{E_2} - \frac{1}{E_1}\right) \Sigma^2$$

So when the layer is established, the dissipation described by a sharp interface and a moving layer is the same.

If ϕ vanishes the limit value $Y(0^+, \Sigma)$ is

$$Y(0^+, \Sigma) = -\frac{1}{2} \frac{\Sigma^2}{E_1^2} \lim_{\phi \rightarrow 0} \frac{E'(\phi)}{d'(\phi)} \leq -\frac{1}{2} \frac{\Sigma_o^2}{E_1^2} \lim_{\phi \rightarrow 0} \frac{E'(\phi)}{d'(\phi)} = Y_c$$

When we adopt the local normality rule, the value of Y must be smaller than Y_c . This defines the critical value Σ_o for initiation of damage in a point of the bar. It can be noticed that the critical value Σ_o depends of the damage law and is generally different from Σ_c . We assume that the dissipation of the system is the same when the layer moves with the limit thickness l_c this gives a relation between the value Y_c and G_c .

$$D_m = G_c a = Y_c d(l_c) a$$

On complete study on such a system with different constitutive laws are studied in ([15]).

Case of a cylinder. We consider a cylinder with external radius R_e in plane strain. The elasticity is assumed isotropic with a constant and homogenous shear modulus μ . The damage governs only the bulk modulus K . In this case the Lamé coefficient $\lambda = K + 2\mu/3$ is a function of d . Under radial loading $u(R_e) = ER_e$ the solution is $\underline{u} = u(R)\underline{e}_r$ and satisfies

$$R^2 u(R) = A \int_0^R \frac{r dr}{L(r)} = A I(R), \quad L = \lambda + 2\mu$$

The constant A is determined by the boundary condition $R_e E = A I(R_e)$. Initially the body is homogeneous with characteristics $\lambda_1 = \lambda_o$ and $A = (\lambda_o + 2\mu)E$. Behind a critical value E_c the damage initiates and λ is no more uniform, we assume for example $d(R) = \phi/l_c$, $\phi = \gamma - R$

$$(a) \frac{1}{L(\phi)} = \frac{1-d}{L_1} + \frac{d}{L_2}, \text{ or } (b) L(\phi) = (1-d) L_1 + d L_2$$

where γ is the position of the level-set Γ_o . Then we obtain in case (a)

$$K(\gamma) = I(R_e) = \frac{R_e^2}{2L_1} + \frac{\gamma^3}{6l_c} \frac{1}{L_2 - L_1}, \quad W = 2\pi R_e^2 \left(\frac{R_e}{K(\gamma)} - 2\mu\right)$$

and $G = \frac{\partial W}{\partial \gamma} = \pi A^2 \frac{\gamma^2}{l_c} \left(\frac{1}{L_2} - \frac{1}{L_1}\right)$, For case (b), K is a logarithm function, depending strongly on the curvature $1/\gamma$. This shows the influence of the curvature on the answer of the system.

Numerical examples. Numerical simulation based on different algorithms have been proposed to solve more complex geometry not only in 2D but also in 3D [7].

Conclusion

We have proposed a variational formulation for the description of damage with a continuous transition to rupture on a layer of finite thickness. This provides a complete analysis of stability and uniqueness of the damage propagation. The model proposes also a complete framework from initiation of damage to propagation of rupture.

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