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Yavar Kian. Unique determination of a time-dependent potential for wave equations from partial data. 2015. hal-01154831v1

HAL Id: hal-01154831

<https://hal.science/hal-01154831v1>

Preprint submitted on 24 May 2015 (v1), last revised 18 Jun 2015 (v2)

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UNIQUE DETERMINATION OF A TIME-DEPENDENT POTENTIAL FOR WAVE EQUATIONS FROM PARTIAL DATA

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ABSTRACT. We consider the inverse problem of determining a time-dependent coefficient of order zero q , appearing in a wave equation $\partial_t^2 u - \Delta u + q(t, x)u = 0$ in $Q = (0, T) \times \Omega$ with Ω a C^2 bounded domain of \mathbb{R}^n , $n \geq 2$, from partial observations of the solutions on ∂Q . Using suitable geometric optics solutions and Carleman estimates, we prove global unique determination of a coefficient $q \in L^\infty(Q)$ from these observations.

Keywords: Inverse problems, wave equation, scalar time-dependent potential, Carleman estimates, partial data.

Mathematics subject classification 2010 : 35R30, 35L05.

1. INTRODUCTION

1.1. Statement of the problem. We fix Ω a C^2 bounded domain of \mathbb{R}^n , $n \geq 2$, and we set $\Sigma = (0, T) \times \partial\Omega$, $Q = (0, T) \times \Omega$ with $0 < T < \infty$. We consider the wave equation

$$\partial_t^2 u - \Delta_x u + q(t, x)u = 0, \quad (t, x) \in Q, \quad (1.1)$$

where the potential $q \in L^\infty(Q)$ is assumed to be real valued. We study the inverse problem of determining q from observations of the solutions of (1.1) on ∂Q .

It is well known that the data

$$\mathcal{A}_q = \{(u|_\Sigma, \partial_\nu u|_\Sigma) : u \in L^2(Q), \square u + qu = 0, u|_{t=0} = \partial_t u|_{t=0} = 0\} \quad (1.2)$$

determines uniquely a time-independent potential $q \in L^\infty(\Omega)$ (e.g. [24]). Here ν denotes the outward unit normal vector to Ω and from now on \square denotes the differential operator $\partial_t^2 - \Delta$. In fact, it has been proved that even partial knowledge of \mathcal{A}_q is enough for the determination of time-independent potential q (e.g. [7]). In contrast to time-independent potentials, due to domain of dependence arguments, there is no hope to recover general time dependent potential q from the data \mathcal{A}_q on the sets

$$D_0 = \{(t, x) \in Q : 0 < t < \text{Diam}(\Omega)/2, \text{dist}(x, \partial\Omega) < t\},$$

$$D_T = \{(t, x) \in Q : T - \text{Diam}(\Omega)/2 < t < T, \text{dist}(x, \partial\Omega) < T - t\}.$$

Indeed, assume that $\Omega = \{x \in \mathbb{R}^n : |x| < R\}$, $T > R > 0$. Now let u solves

$$\square u = 0, \quad u|_\Sigma = f, \quad u|_{t=0} = \partial_t u|_{t=0} = 0.$$

Since $u|_{t=0} = \partial_t u|_{t=0} = 0$, the finite speed of propagation implies that $u|_{D_0} = 0$. Therefore, for any $q \in C_0^\infty(D_0)$, we have $qu = 0$ and u solves

$$\square u + qu = 0, \quad u|_\Sigma = f, \quad u|_{t=0} = \partial_t u|_{t=0} = 0.$$

This last result means that for any $q \in C_0^\infty(D_0)$ we have $\mathcal{A}_q = \mathcal{A}_0$ where \mathcal{A}_0 stands for \mathcal{A}_q when $q = 0$.

Facing this obstruction to uniqueness, it appears that four different approaches have been considered so far to solve this problem:

- 1) Considering the equation (1.1) for any time $t \in \mathbb{R}$ instead of $0 < t < T$ (e.g. [25], [26]).
- 2) Recovering a time-dependent potential q on a subset of Q from the data \mathcal{A}_q (e.g. [23]).
- 3) Recovering a time-dependent potential q from the extended data C_q (e.g. [11]) given by

$$C_q = \{(u|_\Sigma, u|_{t=0}, \partial_t u|_{t=0}, \partial_\nu u|_\Sigma, u|_{t=T}, \partial_t u|_{t=T}) : u \in L^2(Q), (\partial_t^2 - \Delta + q)u = 0\}.$$

- 4) Recovering time-dependent coefficients that are analytic with respect to the t variable (e.g. [8]).

Therefore, it seems that the only unique global determinations of time-dependent potentials q proved so far at finite time involve strong smoothness assumptions such as analyticity with respect to the t variable or the important set of data C_q . The goal of this paper is to prove unique global determination of general time-dependent potentials q with partial knowledge of the set C_q .

1.2. Physical and mathematical interest. In contrast to time-independent potentials, the consideration of time dependent coefficients corresponds to more realistic models since it takes into account the evolution in time of the perturbation. Practically, our inverse problem can be stated as the determination of physical properties such as the time evolving density of an inhomogeneous medium by probing it with disturbances generated on the boundary and at initial time. The data is the response of the medium to these disturbances, measured on the boundary and at the end of the experiment, and the purpose is to recover the function which measures the property of the medium.

Note also that the determination of time-dependent potentials can also be an important tool for the more difficult problem of determining a non-linear term appearing in a nonlinear wave equation from observations of the solution in ∂Q . Indeed, in [13] Isakov applied such results for the determination of a semilinear term appearing in a semilinear parabolic equation from observations of the solutions in ∂Q .

1.3. Existing papers. In recent years the problem of recovering coefficients for hyperbolic equations from boundary measurements has attracted many attention. Many authors have considered this problem with an observation given by the set \mathcal{A}_q (see (1.2)). In [24], the authors proved that the data \mathcal{A}_q determines uniquely a time-independent potential q and in [12] Isakov considered the determination of a coefficient of order zero and a damping coefficient. These results are concerned with measurements on the whole boundary. The uniqueness by local boundary observations has been considered in [7]. For sake of completeness we also mention that the stability issue related to this problem has been treated by [14, 2, 16, 21, 28, 29]. Note that [16] extended the results of [24] to determine a large class of time-independent coefficients of order zero in an unbounded cylindrical domain. It has been proved that only measurements on a bounded subset of the infinite domain are required for the determination of some class of coefficients including periodic coefficients and compactly supported coefficients.

All the above mentioned results are concerned only with time-independent coefficients. Several authors considered the problem of determining time-dependent coefficients for hyperbolic equations. In [27], Stefanov proved unique determination of a time-dependent potential for the wave equation from the knowledge of scattering data. The result of [27] is equivalent to the consideration of the problem with boundary measurements. In [25], Ramm and Sjöstrand considered the problem of determining the time-dependent coefficient q from the data $(u|_{\mathbb{R} \times \partial\Omega}, \partial_\nu u|_{\mathbb{R} \times \partial\Omega})$ of forward solutions of (1.1) on the infinite time-space cylindrical domain $\mathbb{R}_t \times \Omega$ instead of Q ($t \in \mathbb{R}$ instead of $0 < t < T < \infty$). In [23], Rakesh and Ramm considered the same problem at finite time on Q , with $T > \text{Diam}(\Omega)$, and they proved a uniqueness result for the determination of q restricted to the subset S of Q , made of lines with angle 45° with the t -axis and which meet the planes $t = 0$ and $t = T$ outside \overline{Q} , from the data \mathcal{A}_q . In Theorem 4.2 of [11], Isakov established a result of uniqueness for a time-dependent potential on the whole domain Q from observations of the solution on ∂Q . Applying a result of unique continuation borrowed from [30], Eskin [8] proved that the \mathcal{A}_q uniquely determines time-dependent coefficients that are analytic with respect to the time variable t . In some recent work, [26] extended the

result of [25] to more general coefficients and proved a result of stability for compactly supported coefficients provided T is sufficiently large. Moreover, [31] proved stability in the recovery of X-ray transforms of time-dependent potentials on a Riemannian manifold.

We also mention that [4], [5], [6] and [9] consider the problem of determining a time-dependent coefficient for parabolic and Schrödinger equations and derive stability estimate for these problems.

1.4. Main result. In order to state our main result, we first introduce some intermediate tools and notations. For all $\omega \in \mathbb{S}^{n-1} = \{y \in \mathbb{R}^n : |y| = 1\}$ we introduce the ω -illuminated face

$$\partial\Omega_{-, \omega} = \{x \in \partial\Omega : \nu(x) \cdot \omega \leq 0\}$$

and the ω -shadowed face

$$\partial\Omega_{+, \omega} = \{x \in \partial\Omega : \nu(x) \cdot \omega \geq 0\}$$

of $\partial\Omega$. We associate to $\partial\Omega_{\pm, \omega}$ the part of the lateral boundary Σ given by $\Sigma_{\pm, \omega} = (0, T) \times \partial\Omega_{\pm, \omega}$. From now on we fix $\omega_0 \in \mathbb{S}^{n-1}$ and we consider $F = (0, T) \times F'$ (resp $G = (0, T) \times G'$) with F' (resp G') a closed neighborhood of $\partial\Omega_{+, \omega_0}$ (resp $\partial\Omega_{-, \omega_0}$) in $\partial\Omega$.

The main purpose of this paper is to prove the unique global determination of the time-dependent potential q from the data

$$C'_q = \{(u|_{\Sigma}, \partial_t u|_{t=0}, \partial_\nu u|_G, u|_{t=T}) : u \in L^2(Q), \square u + qu = 0, u|_{t=0} = 0, \text{supp } u|_{\Sigma} \subset F\}.$$

See also Section 2 for a rigorous definition of this set. Our main result can be stated as follows.

Theorem 1. *Let $q_1, q_2 \in L^\infty(Q)$. Assume that*

$$C'_{q_1} = C'_{q_2}. \quad (1.3)$$

Then $q_1 = q_2$.

Let us observe that our uniqueness result is stated for bounded potentials with, roughly speaking, half of the data C'_q considered in [11, Theorem 4.2] which seems to be the only unique global determination of general time-dependent coefficients for the wave equation in the mathematical literature. More precisely, we consider $u \in L^2(Q)$ solutions of $(\partial_t^2 - \Delta + q)u = 0$, on Q , with initial condition $u|_{t=0} = 0$ and Dirichlet boundary condition $u|_{\Sigma}$ supported on F (which, roughly speaking, corresponds to half of the boundary). Moreover, we exclude the data $\partial_t u|_{t=T}$ and we consider the Neumann data $\partial_\nu u$ only on G (which, roughly speaking, corresponds to the other half of the boundary). We also mention that in contrast to [8], we do not use result of unique continuation where the analyticity of the coefficients with respect to t are required. To our best knowledge condition (1.3) is the weakest condition that guaranty global uniqueness of general time dependent potentials. Moreover, taking into account the obstruction to uniqueness given by domain of dependence arguments (see Subsection 1.1), the restriction to solutions u of (1.1) satisfying $u|_{t=0} = 0$ and the restriction of observations at $t = T$ to $u|_{t=T}$ seems close to the best condition that we can expect on the initial and final data for the determination of time-dependent potentials.

The main tools in our analysis are suitable geometric optics (GO in short) solutions and Carleman estimates. More precisely, following an approach used for elliptic equations (e.g. [3], [15], [22]) and for determination of time-independent potentials by [2], we construct two kind of geometric optics solutions growing exponentially: solutions lying in $H^1(Q)$ without condition on ∂Q (see Section 3) and GO solutions associated to (1.1) that vanish on parts of ∂Q (see Section 5). With this solutions and a Carleman estimate with linear weight (see Section 4), we prove Theorem 1.

1.5. Outline. This paper is organized as follows. In Section 2 we treat the direct problem. We give a suitable definition of the set of data C'_q and we define the associated boundary operator. In Section 3, using some results of [4] and [10], we build suitable GO solutions associated to (1.1) without condition on ∂Q . In Section 4, we establish a Carleman estimate for the wave equation with linear weight. In Section 5, we use the Carleman estimate introduced in Section 4 to build GO solutions associated to (1.1) that vanish on parts of ∂Q . More precisely, we build GO u which are solutions of (1.1) with $u|_{t=0} = 0$ and $\text{supp } u|_{\Sigma} \subset F$.

In Section 6 we combine all the results of the previous sections in order to prove Theorem 1. We prove also some auxiliary results in the appendix.

Acknowledgements. The author would like to thank Mourad Bellassoued, Mourad Choulli and Eric Soccorsi for their remarks and suggestions.

2. FUNCTIONAL SPACES

The goal of this section is to give a suitable definition to the set of data C'_q for any $q \in L^\infty(Q)$ real valued. We first introduce the space

$$J = \{u \in L^2(Q) : (\partial_t^2 - \Delta)u = 0\}$$

and topologize it as a closed subset of $L^2(Q)$. We work with the space

$$H_\square(Q) = \{u \in L^2(Q) : \square u = (\partial_t^2 - \Delta)u \in L^2(Q)\},$$

with the norm

$$\|u\|_{H_\square(Q)}^2 = \|u\|_{L^2(Q)}^2 + \|(\partial_t^2 - \Delta)u\|_{L^2(Q)}^2.$$

Repeating some arguments of Theorem 6.4 in chapter 2 of [19] we prove in the appendix (see Theorem 4) that $H_\square(Q)$ embedded continuously into the closure of $C^\infty(\overline{Q})$ in the space

$$K_\square(Q) = \{u \in H^{-1}(0, T; L^2(\Omega)) : \square u = (\partial_t^2 - \Delta)u \in L^2(Q)\}$$

topologized by the norm

$$\|u\|_{K_\square(Q)}^2 = \|u\|_{H^{-1}(0, T; L^2(\Omega))}^2 + \|(\partial_t^2 - \Delta)u\|_{L^2(Q)}^2.$$

Then, following Theorem 6.5 in chapter 2 of [19], we prove in the appendix that the maps

$$\tau_0 w = (w|_\Sigma, w|_{t=0}, \partial_t w|_{t=0}), \quad \tau_1 w = (\partial_\nu w|_\Sigma, w|_{t=T}, \partial_t w|_{t=T}), \quad w \in C^\infty(\overline{Q}),$$

can be extended continuously to $\tau_0 : H_\square(Q) \rightarrow H^{-3}(0, T; H^{-\frac{1}{2}}(\partial\Omega)) \times H^{-2}(\Omega) \times H^{-4}(\Omega)$, $\tau_1 : H_\square(Q) \rightarrow H^{-3}(0, T; H^{-\frac{3}{2}}(\partial\Omega)) \times H^{-2}(\Omega) \times H^{-4}(\Omega)$ (see Proposition 4). Here for all $w \in C^\infty(\overline{Q})$ we set

$$\tau_0 w = (\tau_{0,1}w, \tau_{0,2}w, \tau_{0,3}w), \quad \tau_1 w = (\tau_{1,1}w, \tau_{1,2}w, \tau_{1,3}w),$$

where

$$\tau_{0,1}w = w|_\Sigma, \quad \tau_{0,2}w = w|_{t=0}, \quad \tau_{0,3}w = \partial_t w|_{t=0}, \quad \tau_{1,1}w = \partial_\nu w|_\Sigma, \quad \tau_{1,2}w = w|_{t=T}, \quad \tau_{1,3}w = \partial_t w|_{t=T}.$$

Therefore, we can introduce

$$\mathcal{H}(\partial Q) = \{\tau_0 u : u \in H_\square(Q)\} \subset H^{-3}(0, T; H^{-\frac{1}{2}}(\partial\Omega)) \times H^{-2}(\Omega) \times H^{-4}(\Omega).$$

Following [3] and [22], in order to define an appropriate topology on $\mathcal{H}(\partial Q)$ we consider the restriction of τ_0 to the space J .

Proposition 1. *The restriction of τ_0 to J is one to one and onto.*

Proof. Let $u, v \in J$ with $\tau_0 u = \tau_0 v$. Then $w = u - v$ solves

$$\begin{cases} \partial_t^2 w - \Delta w &= 0, & (t, x) \in Q, \\ w|_{t=0} = \partial_t w|_{t=0} &= 0, \\ w|_\Sigma &= 0 \end{cases}$$

and the uniqueness of solutions of this initial boundary value problem implies that $w = 0$. Thus, the restriction of τ_0 to J is one to one. Now let $(g, v_0, v_1) \in \mathcal{H}(\partial Q)$. There exists $F \in H_\square(Q)$ such that $\tau_0 F = (g, v_0, v_1)$. Consider the initial boundary value problem

$$\begin{cases} \partial_t^2 v - \Delta v &= -(\partial_t^2 - \Delta)F, & (t, x) \in Q, \\ v|_{t=0} = \partial_t v|_{t=0} &= 0, \\ v|_\Sigma &= 0. \end{cases}$$

Since $-(\partial_t^2 - \Delta)F \in L^2(Q)$, from the theory introduced in Section 8 of Chapter 3 of [19] we deduce that this IBVP admits a unique solution $v \in \mathcal{C}^1([0, T]; L^2(\Omega)) \cap \mathcal{C}([0, T]; H_0^1(\Omega))$. Then, $u = v + F \in L^2(Q)$ satisfies $(\partial_t^2 - \Delta)u = 0$ and $\tau_0 u = \tau_0 v + \tau_0 F = (g, v_0, v_1)$. Thus τ_0 is onto. \square

From now on, we set \mathcal{P}_0 the inverse of $\tau_0 : J \rightarrow \mathcal{H}(\partial Q)$ and define the norm of $\mathcal{H}(\partial Q)$ by

$$\|(g, v_0, v_1)\|_{\mathcal{H}(\partial Q)} = \|\mathcal{P}_0(g, v_0, v_1)\|_{L^2(Q)}.$$

Now let us consider the initial boundary value problem (IBVP in short)

$$\begin{cases} \partial_t^2 u - \Delta u + q(t, x)u = 0, & \text{in } Q, \\ u(0, \cdot) = v_0, \quad \partial_t u(0, \cdot) = v_1, & \text{in } \Omega, \\ u = g, & \text{on } \Sigma. \end{cases} \quad (2.1)$$

We are now in position to state existence and uniqueness of solutions for the IBVP (2.1) for $(g, v_0, v_1) \in \mathcal{H}(\partial Q)$.

Proposition 2. *Let $(g, v_0, v_1) \in \mathcal{H}(\partial Q)$ and $q \in L^\infty(Q)$. Then the IBVP (2.1) admits a unique weak solution $u \in L^2(Q)$ satisfying*

$$\|u\|_{L^2(Q)} \leq C \|(g, v_0, v_1)\|_{\mathcal{H}(\partial Q)} \quad (2.2)$$

and the boundary operator $B_q : (g, v_0, v_1) \mapsto \tau_1 u$ is a bounded operator from $\mathcal{H}(\partial Q)$ to $H^{-3}(0, T; H^{-\frac{3}{2}}(\partial\Omega)) \times H^{-2}(\Omega) \times H^{-4}(\Omega)$.

Proof. We split u into two terms $u = v + \mathcal{P}_0(g, v_0, v_1)$ where v solves

$$\begin{cases} \partial_t^2 v - \Delta v + qv = -q\mathcal{P}_0(g, v_0, v_1), & (t, x) \in Q, \\ v|_{t=0} = \partial_t v|_{t=0} = 0, \\ v|_\Sigma = 0. \end{cases} \quad (2.3)$$

Since $\mathcal{P}_0(g, v_0, v_1) \in L^2(Q)$, the IBVP (2.3) admits a unique solution $v \in \mathcal{C}^1([0, T]; L^2(\Omega)) \cap \mathcal{C}([0, T]; H_0^1(\Omega))$ (e.g. Section 8 of Chapter 3 of [19]) satisfying

$$\|v\|_{\mathcal{C}^1([0, T]; L^2(\Omega))} + \|v\|_{\mathcal{C}([0, T]; H_0^1(\Omega))} \leq C \|-q\mathcal{P}_0(g, v_0, v_1)\|_{L^2(Q)} \leq C \|q\|_{L^\infty(Q)} \|\mathcal{P}_0(g, v_0, v_1)\|_{L^2(Q)}. \quad (2.4)$$

Therefore, $u = v + \mathcal{P}_0(g, v_0, v_1)$ is the unique solution of (2.1) and estimate (2.4) implies (2.2). Now let us show the last part of the proposition. For this purpose fix $(g, v_0, v_1) \in \mathcal{H}(\partial Q)$ and consider u the solution of (2.1). Note first that $u \in L^2(Q)$ and $(\partial_t^2 - \Delta)u = -qu \in L^2(Q)$. Thus, $u \in H_\square(Q)$ and $\tau_1 u \in H^{-3}(0, T; H^{-\frac{3}{2}}(\partial\Omega)) \times H^{-2}(\Omega) \times H^{-4}(\Omega)$ with

$$\|\tau_1 u\|^2 \leq C^2 \|u\|_{H_\square(Q)}^2 = C^2 (\|u\|_{L^2(Q)}^2 + \|qu\|_{L^2(Q)}^2) \leq C^2 (1 + \|q\|_{L^\infty(Q)}^2) \|u\|_{L^2(Q)}^2.$$

Combining this with (2.2) we deduce that B_q is a bounded operator from $\mathcal{H}(\partial Q)$ to $H^{-3}(0, T; H^{-\frac{3}{2}}(\partial\Omega)) \times H^{-2}(\Omega) \times H^{-4}(\Omega)$. \square

From the definition of the boundary operator B_q one can deduce the set C'_q which corresponds (in some suitable sense) to a subspace of the graph of B_q .

3. GEOMETRIC OPTICS SOLUTIONS WITHOUT BOUNDARY CONDITIONS

The goal of this section is to build geometric optics solutions $u \in H^1(Q)$ associated to the equation

$$\partial_t^2 u - \Delta u + q(t, x)u = 0 \quad \text{on } Q. \quad (3.1)$$

More precisely, for $\lambda > 1$, $\omega \in \mathbb{S}^{n-1} = \{y \in \mathbb{R}^n : |y| = 1\}$ and $\xi \in \mathbb{R}^{1+n}$ satisfying $\xi \cdot (1, -\omega) = 0$, we consider solutions of the form

$$u(t, x) = e^{-\lambda(t+x\cdot\omega)} (e^{-i\xi \cdot (t, x)} + w(t, x)), \quad (t, x) \in Q. \quad (3.2)$$

Here w is the remainder term in the asymptotic expansion of u with respect to λ and we have

$$\|w\|_{L^2(Q)} \leq \frac{C}{\lambda}$$

with $C > 0$ independent of λ . For this purpose, for all $s \in \mathbb{R}$ and all $\omega \in \mathbb{S}^{n-1}$, we consider the operators $P_{s,\omega}$ defined by $P_{s,\omega} = e^{-s(t+x \cdot \omega)} \square e^{s(t+x \cdot \omega)}$. One can check that

$$P_{s,\omega} = p_{s,\omega}(D_t, D_x) = \square + 2s(\partial_t - \omega \cdot \nabla_x)$$

with $D_t = -i\partial_t$, $D_x = -i\nabla_x$ and $p_{s,\omega}(\mu, \eta) = -\mu^2 + |\eta|^2 + 2si(\mu - \omega \cdot \eta)$, $\mu \in \mathbb{R}$, $\eta \in \mathbb{R}^n$. Applying some results of [4] and [10] about solutions of PDEs with constant coefficients we obtain the following.

Lemma 1. *For every $\lambda > 1$ and $\omega \in \mathbb{S}^{n-1}$ there exists a bounded operator $E_{\lambda,\omega} : L^2(Q) \rightarrow L^2(Q)$ such that:*

$$P_{-\lambda,\omega} E_{\lambda,\omega} f = f, \quad f \in L^2(Q), \quad (3.3)$$

$$\|E_{\lambda,\omega}\|_{\mathcal{B}(L^2(Q))} \leq C\lambda^{-1}, \quad f \in L^2(Q), \quad (3.4)$$

$$E_{\lambda,\omega} \in \mathcal{B}(L^2(Q); H^1(Q)) \quad \text{and} \quad \|E_{\lambda,\omega}\|_{\mathcal{B}(L^2(Q); H^1(Q))} \leq C \quad (3.5)$$

with $C > 0$ depending only on T and Ω .

Proof. In light of [4, Theorem 2.3] (see also [10, Theorem 10.3.7]), there exists a bounded operator $E_{\lambda,\omega} : L^2(Q) \rightarrow L^2(Q)$, defined from a fundamental solution associated to $P_{-\lambda,\omega}$ (see Section 10.3 of [10]), such that (3.3) is fulfilled. In addition, for all differential operator $Q(D_t, D_x)$ such that $\frac{Q(\mu, \eta)}{\tilde{p}_{-\lambda,\omega}(\mu, \eta)}$ is bounded, we have $Q(D_t, D_x)E_{\lambda,\omega} \in \mathcal{B}(L^2(Q))$ and

$$\|Q(D_t, D_x)E_{\lambda,\omega}\|_{\mathcal{B}(L^2(Q))} \leq C \sup_{(\mu, \eta) \in \mathbb{R}^{1+n}} \frac{|Q(\mu, \eta)|}{\tilde{p}_{-\lambda,\omega}(\mu, \eta)}, \quad (3.6)$$

where $\tilde{p}_{-\lambda,\omega}$ is given by

$$\tilde{p}_{-\lambda,\omega}(\mu, \eta) = \left(\sum_{k \in \mathbb{N}} \sum_{\alpha \in \mathbb{N}^n} |\partial_\mu^k \partial_\eta^\alpha p_{-\lambda,\omega}(\mu, \eta)|^2 \right)^{\frac{1}{2}}.$$

Note that $\tilde{p}_{-\lambda,\omega}(\mu, \eta) \geq |\Im \partial_\mu p_{-\lambda,\omega}(\mu, \eta)| = 2\lambda$. Therefore, (3.6) implies

$$\|E_{\lambda,\omega}\|_{\mathcal{B}(L^2(Q))} \leq C \sup_{(\mu, \eta) \in \mathbb{R}^{1+n}} \frac{1}{\tilde{p}_{-\lambda,\omega}(\mu, \eta)} \leq C\lambda^{-1}$$

and (3.4) is fulfilled. In a same way, we have $\tilde{p}_{-\lambda,\omega}(\mu, \eta) \geq |\Re \partial_\mu p_{-\lambda,\omega}(\mu, \eta)| = 2|\mu|$ and $\tilde{p}_{-\lambda,\omega}(\mu, \eta) \geq |\Re \partial_{\eta_i} p_{-\lambda,\omega}(\mu, \eta)| = 2|\eta_i|$, $i = 1, \dots, n$ and $\eta = (\eta_1, \dots, \eta_n)$. Therefore, in view of [4, Theorem 2.3], we have $E_{\lambda,\omega} \in \mathcal{B}(L^2(Q); H^1(Q))$ with

$$\|E_{\lambda,\omega}\|_{\mathcal{B}(L^2(Q); H^1(Q))} \leq C \sup_{(\mu, \eta) \in \mathbb{R}^{1+n}} \frac{|\mu| + |\eta_1| + \dots + |\eta_n|}{\tilde{p}_{-\lambda,\omega}(\mu, \eta)} \leq C(n+1)$$

and (3.5) is proved. \square

Armed with this result, we are now in position to build geometric optics solutions of the form (3.2).

Proposition 3. *Let $q \in L^\infty(Q)$, $\omega \in \mathbb{S}^{n-1}$, $\lambda > 1$. Then, there exists $\lambda_0 > 1$ such that for $\lambda \geq \lambda_0$ the equation (3.1) admits a solution $u \in H^1(Q)$ of the form (3.2) with*

$$\|w\|_{H^k(Q)} \leq C\lambda^{k-1}, \quad k = 0, 1, \quad (3.7)$$

where C and λ_0 depend on Ω , ξ , T , $M \geq \|q\|_{L^\infty(Q)}$.

Proof. We start by recalling that

$$\begin{aligned} \square e^{-\lambda(t+x \cdot \omega)} e^{-i\xi \cdot (t,x)} &= e^{-\lambda(t+x \cdot \omega)} \left(\square e^{-i\xi \cdot (t,x)} + 2i\lambda\xi \cdot (1, -\omega) e^{-i\xi \cdot (t,x)} \right) \\ &= e^{-\lambda(t+x \cdot \omega)} \square e^{-i\xi \cdot (t,x)}, \quad (t, x) \in Q. \end{aligned}$$

Thus, w should be a solution of

$$\partial_t^2 w - \Delta w - 2\lambda(\partial_t - \omega \cdot \nabla_x)w = -\left((\square + q)e^{-i\xi \cdot (t,x)} + qw\right). \quad (3.8)$$

Therefore, according to Lemma 1, we can define w as a solution of the equation

$$w = -E_{\lambda,\omega} \left((\square + q)e^{-i\xi \cdot (t,x)} + qw \right), \quad w \in L^2(Q)$$

with $E_{\lambda,\omega} \in \mathcal{B}(L^2(Q))$ given by Lemma 1. For this purpose, we will use a standard fixed point argument associated to the map

$$\begin{aligned} \mathcal{G} : L^2(Q) &\rightarrow L^2(Q), \\ F &\mapsto -E_{\lambda,\omega} \left[(\square + q)e^{-i\xi \cdot (t,x)} + qF \right]. \end{aligned}$$

Indeed, in view of (3.4), fixing $M_1 > 0$, there exists $\lambda_0 > 1$ such that for $\lambda \geq \lambda_0$ the map \mathcal{G} admits a unique fixed point w in $\{u \in L^2(Q) : \|u\|_{L^2(Q)} \leq M_1\}$. In addition, condition (3.4)-(3.5) imply that $w \in H^1(Q)$ fulfills (3.7). This completes the proof. \square

4. CARLEMAN ESTIMATES

This section is devoted to the proof of Carleman estimates similar to [2] and [3]. More precisely, we consider the following estimates.

Theorem 2. *Let $q \in L^\infty(Q)$ and $u \in \mathcal{C}^2(\overline{Q})$. If u satisfies the condition*

$$u|_\Sigma = 0, \quad u|_{t=0} = \partial_t u|_{t=0} = 0 \quad (4.1)$$

then there exists $\lambda_1 > 0$ depending only on Ω , T and $M \geq \|q\|_{L^\infty(Q)}$ such that the estimate

$$\begin{aligned} &\lambda \int_\Omega e^{-2\lambda(T+\omega \cdot x)} |\partial_t u|_{t=T}|^2 dx + \lambda \int_{\Sigma_{+, \omega}} e^{-2\lambda(t+\omega \cdot x)} |\partial_\nu u|^2 |\omega \cdot \nu(x)| d\sigma(x) dt + \lambda^2 \int_Q e^{-2\lambda(t+\omega \cdot x)} |u|^2 dx dt \\ &\leq C \left(\int_Q e^{-2\lambda(t+\omega \cdot x)} |(\partial_t^2 - \Delta + q)u|^2 dx dt + \lambda^3 \int_\Omega e^{-2\lambda(T+\omega \cdot x)} |u|_{t=T}|^2 dx + \lambda \int_\Omega e^{-2\lambda(T+\omega \cdot x)} |\nabla_x u|_{t=T}|^2 dx \right) \\ &\quad + C\lambda \int_{\Sigma_{-, \omega}} e^{-2\lambda(t+\omega \cdot x)} |\partial_\nu u|^2 |\omega \cdot \nu(x)| d\sigma(x) dt \end{aligned} \quad (4.2)$$

holds true for $\lambda \geq \lambda_1 > 1$ with C and λ_1 depending only on Ω , T and $M \geq \|q\|_{L^\infty(Q)}$. If u satisfies the condition

$$u|_\Sigma = 0, \quad u|_{t=T} = \partial_t u|_{t=T} = 0 \quad (4.3)$$

then the estimate

$$\begin{aligned} &\lambda \int_\Omega e^{2\lambda\omega \cdot x} |\partial_t u|_{t=0}|^2 dx + \lambda \int_{\Sigma_{-, \omega}} e^{2\lambda(t+\omega \cdot x)} |\partial_\nu u|^2 |\omega \cdot \nu(x)| d\sigma(x) dt + \lambda^2 \int_Q e^{2\lambda(t+\omega \cdot x)} |u|^2 dx dt \\ &\leq C \left(\int_Q e^{2\lambda(t+\omega \cdot x)} |(\partial_t^2 - \Delta + q)u|^2 dx dt + \lambda^3 \int_\Omega e^{2\lambda\omega \cdot x} |u|_{t=0}|^2 dx + \lambda \int_\Omega e^{2\lambda\omega \cdot x} |\nabla_x u|_{t=0}|^2 dx \right) \\ &\quad + C\lambda \int_{\Sigma_{+, \omega}} e^{2\lambda(t+\omega \cdot x)} |\partial_\nu u|^2 |\omega \cdot \nu(x)| d\sigma(x) dt \end{aligned} \quad (4.4)$$

holds true for $\lambda \geq \lambda_1 > 0$.

In order to prove these estimates, we fix $u \in \mathcal{C}^2(\overline{Q})$ satisfying (4.1) (resp (4.3)) and we set $v = e^{-\lambda(t+\omega \cdot x)} u$ (resp $v = e^{\lambda(t+\omega \cdot x)} u$) in such a way that

$$\int_Q e^{-\lambda(t+\omega \cdot x)} \square u dx dt = \int_Q P_{\lambda,\omega} v dt dx, \quad \left(\text{resp } \int_Q e^{\lambda(t+\omega \cdot x)} \square u dx dt = \int_Q P_{-\lambda,\omega} v dt dx \right). \quad (4.5)$$

In order to prove these Carleman estimates we consider first the estimate associated to the weighted operators $P_{\pm\lambda,\omega}$ introduced in the previous section.

Lemma 2. *Let $v \in \mathcal{C}^2(\overline{Q})$ and $\lambda > 1$. If v satisfies the condition*

$$v|_{\Sigma} = 0, \quad v|_{t=0} = \partial_t v|_{t=0} = 0 \quad (4.6)$$

then the estimate

$$\begin{aligned} & \lambda \int_{\Omega} |\partial_t v|_{t=T}|^2 dx + 2\lambda \int_{\Sigma_{+, \omega}} |\partial_{\nu} v|^2 \omega \cdot \nu(x) d\sigma(x) dt + c\lambda^2 \int_Q |v|^2 dx dt \\ & \leq \int_Q |P_{\lambda, \omega} v|^2 dx dt + 62\lambda \int_{\Omega} |\nabla_x v|_{t=T}|^2 dx + 2\lambda \int_{\Sigma_{-, \omega}} |\partial_{\nu} v|^2 |\omega \cdot \nu(x)| d\sigma(x) dt \end{aligned} \quad (4.7)$$

holds true for $c > 0$ depending only on Ω and T . If v satisfies the condition

$$v|_{\Sigma} = 0, \quad v|_{t=T} = \partial_t v|_{t=T} = 0 \quad (4.8)$$

then the estimate

$$\begin{aligned} & \lambda \int_{\Omega} |\partial_t v|_{t=0}|^2 dx + 2\lambda \int_{\Sigma_{-, \omega}} |\partial_{\nu} v|^2 |\omega \cdot \nu(x)| d\sigma(x) dt + c\lambda^2 \int_Q |v|^2 dx dt \\ & \leq \int_Q |P_{-\lambda, \omega} v|^2 dx dt + 62\lambda \int_{\Omega} |\nabla_x v|_{t=0}|^2 dx + 2\lambda \int_{\Sigma_{+, \omega}} |\partial_{\nu} v|^2 \omega \cdot \nu(x) d\sigma(x) dt \end{aligned} \quad (4.9)$$

holds true.

Proof. We start with (4.7). For this purpose we fix $v \in \mathcal{C}^2(\overline{Q})$ satisfying (4.6) and we consider

$$I_{\lambda, \omega} = \int_Q |P_{\lambda, \omega} v|^2 dx dt.$$

Without loss of generality we assume that v is real valued. Repeating some arguments of [2] (see the formula 2 line before (2.4) in page 1225 of [2] and formula (2.5) in page 1226 of [2]) we obtain the following

$$\begin{aligned} I_{\lambda, \omega} & \geq \int_Q |\square v|^2 dx dt + c\lambda^2 \int_Q |v|^2 dx dt + 2\lambda \int_{\Sigma} |\partial_{\nu} v|^2 \omega \cdot \nu(x) d\sigma(x) dt \\ & + 2\lambda \int_{\Omega} |\partial_t v|_{t=T}|^2 dx + 2\lambda \int_{\Omega} |\nabla_x v|_{t=T}|^2 dx - 4\lambda \int_{\Omega} (\partial_t v|_{t=T})(\omega \cdot \nabla_x v|_{t=T}) dx. \end{aligned}$$

On the other hand, an application of the Cauchy-Schwarz inequality yields

$$4\lambda \left| \int_{\Omega} (\partial_t v|_{t=T})(\omega \cdot \nabla_x v|_{t=T}) dx \right| \leq \frac{\lambda}{2} \int_{\Omega} |\partial_t v|_{t=T}|^2 dx + 64\lambda \int_{\Omega} |\nabla_x v|_{t=T}|^2 dx$$

and we deduce that

$$\begin{aligned} & I_{\lambda, \omega} + 62\lambda \int_{\Omega} |\nabla_x v|_{t=T}|^2 dx \\ & \geq \int_Q |\square v|^2 dx dt + c\lambda^2 \int_Q |v|^2 dx dt + 2\lambda \int_{\Sigma} |\partial_{\nu} v|^2 \omega \cdot \nu(x) d\sigma(x) dt + \lambda \int_{\Omega} |\partial_t v|_{t=T}|^2 dx. \end{aligned}$$

From this last estimate we deduce easily (4.7). Now let us consider (4.9). For this purpose note that for v satisfying (4.8), w defined by $w(t, x) = v(T - t, x)$ satisfies (4.6). Thus, applying (4.7) to w with ω replaced by $-\omega$ we obtain (4.9). \square

In light of Lemma 2, we are now in position to prove Theorem 2.

Proof of Theorem 2. Let us first consider the case $q = 0$. Note that for u satisfying (4.1), $v = e^{-\lambda(t+\omega \cdot x)} u$ satisfies (4.6). Moreover, we have (4.5) and (4.1) implies $\partial_{\nu} v|_{\Sigma} = e^{-\lambda(t+\omega \cdot x)} \partial_{\nu} u|_{\Sigma}$. Finally, using the fact that

$$\partial_t u = \partial_t (e^{\lambda(t+\omega \cdot x)} v) = \lambda u + e^{\lambda(t+\omega \cdot x)} \partial_t v, \quad \nabla_x v = e^{-\lambda(t+\omega \cdot x)} (\nabla_x u - \lambda u \omega),$$

we obtain

$$\begin{aligned} & \int_{\Omega} e^{-\lambda(T+\omega \cdot x)} |\partial_t u|_{t=T}|^2 dx \leq 2 \int_{\Omega} |\partial_t v|_{t=T}|^2 dx + 2\lambda^2 \int_{\Omega} e^{-\lambda(T+\omega \cdot x)} |u|_{t=T}|^2 dx, \\ & \int_{\Omega} |\nabla v|_{t=T}|^2 dx \leq 2\lambda^2 \int_{\Omega} e^{-\lambda(T+\omega \cdot x)} |u|_{t=T}|^2 dx + 2 \int_{\Omega} e^{-\lambda(T+\omega \cdot x)} |\nabla u|_{t=T}|^2 dx. \end{aligned}$$

Thus, applying the Carleman estimate (4.7) to v , we deduce (4.2). For $q \neq 0$, we have

$$|\partial_t^2 u - \Delta u|^2 = |\partial_t^2 u - \Delta u + qu - qu|^2 \leq 2 |(\partial_t^2 - \Delta + q)u|^2 + 2 \|q\|_{L^\infty(Q)}^2 |u|^2$$

and hence if we choose $\lambda_1 > 2C \|q\|_{L^\infty(Q)}^2$, replacing C by

$$C_1 = \frac{C\lambda_1^2}{\lambda_1^2 - 2C \|q\|_{L^\infty(Q)}^2},$$

we deduce (4.2) from the same estimate when $q = 0$. Using similar arguments, we prove (4.4). \square

Remark 1. Note that, by density, estimate (4.2) can be extended to function $u \in C^1([0, T]; L^2(\Omega)) \cap C([0, T]; H^1(\Omega))$ satisfying (4.6), $(\partial_t^2 - \Delta)u \in L^2(Q)$ and $\partial_\nu u \in L^2(\Sigma)$.

5. GEOMETRIC OPTICS SOLUTIONS VANISHING ON PARTS OF THE BOUNDARY

In this section we fix $q \in L^\infty(Q)$. The goal of this section is to use the Carleman estimate (4.4) in order to build solutions $u \in H_\square(Q)$ to

$$\begin{cases} (\partial_t^2 - \Delta + q(t, x))u = 0 & \text{in } Q, \\ u|_{t=0} = 0, \\ u = 0, & \text{on some neighborhood of } \Sigma \setminus F, \end{cases} \quad (5.1)$$

of the form

$$u(t, x) = e^{\lambda(t+\omega \cdot x)} (1 + z(t, x)), \quad (t, x) \in Q, \quad (5.2)$$

where $z \in e^{-\lambda(t+\omega \cdot x)} H_\square(Q)$ fulfills: $z(0, x) = -1$, $x \in \Omega$, $z = -1$ on some neighborhood of $\Sigma \setminus F$ and

$$\|z\|_{L^2(Q)} \leq C\lambda^{-\frac{1}{2}}. \quad (5.3)$$

The main result of this section can be stated as follows.

Theorem 3. Let $q \in L^\infty(Q)$. For all $\lambda \geq \lambda_1$, with λ_1 the constant of Theorem 2. Then, there exists a solution $u \in H_\square(Q)$ of (5.1) of the form (5.2) with z satisfying (5.3).

In order to prove existence of such solutions of (5.1) we need some preliminary tools and an intermediate result.

5.1. Weighted spaces. In this subsection we give the definition of some weighted spaces. We set $s \in \mathbb{R}$ and denote by γ the function defined on $\partial\Omega$ by

$$\gamma(x) = |\omega \cdot \nu(x)|, \quad x \in \partial\Omega.$$

We introduce the spaces $L_s(Q)$, $L_s(\Omega)$, and for all non negative measurable function h on $\partial\Omega$ the spaces $L_{s,h,\pm}$ defined respectively by

$$L_s(Q) = e^{s(t+\omega \cdot x)} L^2(Q), \quad L_s(\Omega) = e^{s\omega \cdot x} L^2(\Omega), \quad L_{s,h,\pm} = \{f : e^{s(t+\omega \cdot x)} h(x) f \in L^2(\Sigma_{\pm, \omega})\}$$

with the associated norm

$$\begin{aligned} \|u\|_s &= \left(\int_Q e^{2s(t+\omega \cdot x)} |u|^2 dx dt \right)^{\frac{1}{2}}, \quad u \in L_s(Q), \\ \|u\|_{s,0} &= \left(\int_\Omega e^{2s\omega \cdot x} |u|^2 dx \right)^{\frac{1}{2}}, \quad u \in L_s(\Omega), \\ \|u\|_{s,h,\pm} &= \left(\int_{\Sigma_{\pm, \omega}} e^{2s(t+\omega \cdot x)} h(x) |u|^2 d\sigma(x) dt \right)^{\frac{1}{2}}, \quad u, v \in L_{s,h,\pm}. \end{aligned}$$

5.2. Intermediate result. We set the space

$$\mathcal{D} = \{v \in \mathcal{C}^2(\overline{Q}) : v|_{\Sigma} = 0, v|_{t=T} = \partial_t v|_{t=T} = v|_{t=0} = 0\}$$

and, in view of Theorem 2, applying the Carleman estimate (4.4) to any $f \in \mathcal{D}$ we obtain

$$\lambda \|f\|_{\lambda} + \lambda^{\frac{1}{2}} \|\partial_t f|_{t=0}\|_{\lambda,0} + \lambda^{\frac{1}{2}} \|\partial_{\nu} f\|_{\lambda,\gamma,-} \leq C(\|(\partial_t^2 - \Delta + q)f\|_{\lambda} + \|\partial_{\nu} f\|_{\lambda,\lambda\gamma,+}), \quad \lambda \geq \lambda_1. \quad (5.4)$$

We introduce also the space

$$\mathcal{M} = \{((\partial_t^2 - \Delta + q)v, \partial_{\nu} v|_{\Sigma_{+,\omega}}) : v \in \mathcal{D}\}$$

and think of \mathcal{M} as a subspace of $L_{\lambda}(Q) \times L_{\lambda,\lambda\gamma,+}$. We consider the following intermediate result.

Lemma 3. *Given $\lambda \geq \lambda_1$, with λ_1 the constant of Theorem 2, and*

$$v \in L_{-\lambda}(Q), \quad v_- \in L_{-\lambda,\gamma^{-1},-}, \quad v_0 \in L_{-\lambda}(\Omega),$$

there exists $u \in L_{-\lambda}(Q)$ such that:

$$1) (\partial_t^2 - \Delta + q)u = v,$$

$$2) u|_{\Sigma_{-,\omega}} = v_-, \quad u|_{t=0} = v_0,$$

$$3) \|u\|_{-\lambda} \leq C \left(\lambda^{-1} \|v\|_{-\lambda} + \lambda^{-\frac{1}{2}} \|v_-\|_{-\lambda,\gamma^{-1},-} + \lambda^{-\frac{1}{2}} \|v_0\|_{-\lambda,0} \right) \text{ with } C \text{ depending on } \Omega, T,$$

$$M \geq \|q\|_{L^{\infty}(Q)}.$$

Proof. In view of (5.4), we can define the linear function S on \mathcal{M} by

$$S[(\square + q)f, \partial_{\nu} f|_{\Sigma_{+,\omega}}] = \langle f, v \rangle_{L^2(Q)} - \langle \partial_{\nu} f, v_- \rangle_{L^2(\Sigma_{-,\omega})} + \langle \partial_t f|_{t=0}, v_0 \rangle_{L^2(\Omega)}, \quad f \in \mathcal{D}.$$

Then, using (5.4), for all $f \in \mathcal{D}$, we obtain

$$\begin{aligned} & |S[(\square + q)f, \partial_{\nu} f|_{\Sigma_{+,\omega}}]| \\ & \leq \|f\|_{\lambda} \|v\|_{-\lambda} + \|\partial_{\nu} f\|_{\lambda,\gamma,-} \|v_-\|_{-\lambda,\gamma^{-1},-} + \|\partial_t f|_{t=0}\|_{\lambda,0} \|v_0\|_{-\lambda,0} \\ & \leq \lambda^{-1} \|v\|_{-\lambda} (\lambda \|f\|_{\lambda}) + \lambda^{-\frac{1}{2}} \|v_-\|_{-\lambda,\gamma^{-1},-} \left(\lambda^{\frac{1}{2}} \|\partial_{\nu} f\|_{\lambda,\gamma,-} \right) + \lambda^{-\frac{1}{2}} \|v_0\|_{-\lambda,0} \left(\lambda^{\frac{1}{2}} \|\partial_t f|_{t=0}\|_{\lambda,0} \right) \\ & \leq C \left(\lambda^{-1} \|v\|_{-\lambda} + \lambda^{-\frac{1}{2}} \|v_-\|_{-\lambda,\gamma^{-1},-} + \lambda^{-\frac{1}{2}} \|v_0\|_{-\lambda,0} \right) \left(\|(\square + q)f\|_{\lambda} + \|\partial_{\nu} f\|_{\lambda,\lambda\gamma,+} \right) \\ & \leq 2C \left(\lambda^{-1} \|v\|_{-\lambda} + \lambda^{-\frac{1}{2}} \|v_-\|_{-\lambda,\gamma^{-1},-} + \lambda^{-\frac{1}{2}} \|v_0\|_{-\lambda,0} \right) \|((\square + q)f, \partial_{\nu} f|_{\Sigma_{+,\omega}})\|_{L_{\lambda}(Q) \times L_{\lambda,\lambda\gamma,+}} \end{aligned}$$

with C the constant of (5.4). Applying the Hahn Banach theorem we deduce that S can be extended to a continuous linear form, also denoted by S , on $L_{\lambda}(Q) \times L_{\lambda,\lambda\gamma,+}$ satisfying

$$\|S\| \leq C \left(\lambda^{-1} \|v\|_{-\lambda} + \lambda^{-\frac{1}{2}} \|v_-\|_{-\lambda,\gamma^{-1},-} + \lambda^{-\frac{1}{2}} \|v_0\|_{-\lambda,0} \right). \quad (5.5)$$

Thus, there exists

$$(u, u_+) \in L_{-\lambda}(Q) \times L_{-\lambda,(\lambda\gamma)^{-1},+}$$

such that for all $f \in \mathcal{D}$ we have

$$S[(\square + q)f, \partial_{\nu} f|_{\Sigma_{+,\omega}}] = \langle (\square + q)f, u \rangle_{L^2(Q)} - \langle \partial_{\nu} f, u_+ \rangle_{L^2(\Sigma_{+,\omega})}.$$

Therefore, for all $f \in \mathcal{D}$ we have

$$\begin{aligned} & \langle (\square + q)f, u \rangle_{L^2(Q)} - \langle \partial_{\nu} f, u_+ \rangle_{L^2(\Sigma_{+,\omega})} \\ & = \langle f, v \rangle_{L^2(Q)} - \langle \partial_{\nu} f, v_- \rangle_{L^2(\Sigma_{-,\omega})} + \langle \partial_t f|_{t=0}, v_0 \rangle_{L^2(\Omega)}. \end{aligned} \quad (5.6)$$

Note first that, since $L_{\pm\lambda}(Q)$ embedded continuously into $L^2(Q)$, we have $u \in L^2(Q)$. Therefore, taking $f \in \mathcal{C}_0^{\infty}(Q)$ shows 1). For condition 2), using the fact that $L_{\pm\lambda}(Q)$ embedded continuously into $L^2(Q)$ we deduce that $u \in H_{\square}(Q)$ and define the trace $u|_{\Sigma}$ and $u|_{t=0}$. Thus, allowing $f \in \mathcal{D}$ to be arbitrary shows that $u|_{\Sigma_{-,\omega}} = v_-$, $u|_{t=0} = v_0$ and $u|_{\Sigma_{+,\omega}} = -u_+$. Finally, condition 3) follows from the fact that

$$\|u\|_{-\lambda} \leq \|S\| \leq C \left(\lambda^{-1} \|v\|_{-\lambda} + \lambda^{-\frac{1}{2}} \|v_-\|_{-\lambda,\gamma^{-1},-} + \lambda^{-\frac{1}{2}} \|v_0\|_{-\lambda,0} \right).$$

□

Armed with this lemma we are now in position to prove Theorem 3.

5.3. Proof of Theorem 3. Note first that z must satisfy

$$\begin{cases} z \in L^2(Q) \\ (\partial_t^2 - \Delta + q)(e^{\lambda(t+\omega \cdot x)} z) = -qe^{\lambda(t+\omega \cdot x)} & \text{in } Q \\ z(0, x) = -1, & x \in \Omega, \\ z = -1 & \text{on some neighborhood of } \Sigma \setminus F. \end{cases} \quad (5.7)$$

Let \tilde{F}' be an open neighborhood of $\partial\Omega_{+, \omega_0}$ in $\partial\Omega$ whose closure (with respect to $\partial\Omega$) is contained into interior of F' (with respect to $\partial\Omega$) and let $\psi \in C_0^\infty(\mathbb{R}^n)$ be such that $\text{supp}\psi \cap \partial\Omega \subset \partial\Omega \setminus \tilde{F}'$ and $\psi = 1$ on a neighborhood of $\partial\Omega \setminus F'$. Choose $v_-(t, x) = -e^{\lambda(t+\omega \cdot x)}\psi(x)$, $(t, x) \in \Sigma_{-, \omega}$. Since $v_-(t, x) = 0$ for $(t, x) \in (0, T) \times \tilde{F}'$ we have $v_- \in L_{-\lambda, \gamma^{-1}, -}$. Fix also $v(t, x) = -qe^{\lambda(t+\omega \cdot x)}$ and $v_0(x) = -e^{\lambda\omega \cdot x}$, $(t, x) \in Q$. From Lemma 3, we deduce that there exists $w \in H_\square(Q)$ such that

$$\begin{cases} (\partial_t^2 - \Delta + q)w = v(t, x) = -qe^{\lambda(t+\omega \cdot x)} & \text{in } Q, \\ w(0, x) = v_0(x) = -e^{\lambda\omega \cdot x}, & x \in \Omega, \\ w(t, x) = v_-(t, x) = -e^{\lambda(t+\omega \cdot x)}\psi(x), & (t, x) \in \Sigma_{-, \omega}. \end{cases}$$

Then, for $z = e^{-\lambda(t+\omega \cdot x)}w$ condition (5.7) will be fulfilled. Moreover, condition 3) of Lemma 3 implies

$$\begin{aligned} \|z\|_{L^2(Q)} &= \|w\|_{-\lambda} \leq C \left(\lambda^{-1} \|v\|_{-\lambda} + \lambda^{-\frac{1}{2}} \|v_-\|_{-\lambda, \gamma^{-1}, -} + \lambda^{-\frac{1}{2}} \|v_0\|_{-\lambda, 0} \right) \\ &\leq \left(\lambda^{-1} \|q\|_{L^2(Q)} + \lambda^{-\frac{1}{2}} \|\chi\gamma^{-1}\|_{L^2(\Sigma_{-, \omega})} + \lambda^{-\frac{1}{2}} \|1\|_{L^2(\Omega)} \right) \leq C\lambda^{-\frac{1}{2}} \end{aligned}$$

with C depending only on Ω , T and $\|q\|_{L^\infty(Q)}$. Therefore, estimate (5.3) holds. Using the fact that $e^{\lambda(t+\omega \cdot x)}z = w \in H_\square(Q)$, we deduce that u defined by (5.2) is lying in $H_\square(Q)$ and is a solution of (5.1) with $\text{supp}\tau_{0,1}u \subset F$, $\tau_{0,2}u = 0$ (recall that for $v \in C^\infty(\overline{Q})$, $\tau_{0,1}v = v|_\Sigma$, $\tau_{0,2}v = v|_{t=0}$).

6. UNIQUENESS RESULT

This section is devoted to the proof of Theorem 1. From now on we set $q = q_2 - q_1$ on Q and we assume that $q = 0$ on $\mathbb{R}^{1+n} \setminus Q$. Moreover, for all $y \in \mathbb{S}^{n-1} = \{y \in \mathbb{R}^n : |y| = 1\}$ and all $r > 0$, we set

$$\partial\Omega_{+, r, y} = \{x \in \partial\Omega : \nu(x) \cdot y > r\}, \quad \partial\Omega_{-, r, y} = \{x \in \partial\Omega : \nu(x) \cdot y \leq r\}$$

and $\Sigma_{\pm, r, y} = (0, T) \times \partial\Omega_{\pm, r, y}$. Here and in the remaining of this text we always assume, without mentioning it, that y and r are chosen in such way that $\partial\Omega_{\pm, r, \pm y}$ contain a non-empty relatively open subset of $\partial\Omega$. Without lost of generality we can assume that there exists $0 < \varepsilon < 1$ such that for all $\omega \in \{y \in \mathbb{S}^{n-1} : |y - \omega_0| \leq \varepsilon\}$ we have $\partial\Omega_{-, \varepsilon, y} \subset G'$.

Let $\lambda > \max(\lambda_1, \lambda_0)$ and fix $\omega \in \{y \in \mathbb{S}^{n-1} : |y - \omega_0| \leq \varepsilon\}$. According to Proposition 3, we can introduce

$$u_1(t, x) = e^{-\lambda(t+\omega \cdot x)} \left(e^{-i\xi \cdot (t, x)} + w(t, x) \right), \quad (t, x) \in Q,$$

where $u_1 \in H^1(Q)$ satisfies $\partial_t^2 u_1 - \Delta u_1 + q_1 u_1 = 0$, $\xi \cdot (1, -\omega) = 0$ and w satisfies (3.7). Moreover, in view of Theorem 3, we consider $u_2 \in H_\square(Q)$ solution of (5.1) with $q = q_2$ of the form

$$u_2(t, x) = e^{\lambda(t+\omega \cdot x)} (1 + z(t, x)), \quad (t, x) \in Q$$

with z satisfying (5.3), such that $\text{supp}\tau_{0,1}u_2 \subset F$ and $\tau_{0,2}u_2 = 0$ (we recall that $\tau_{0,j}$, $j = 1, 2$, are the extensions on $H_\square(Q)$ of the operators defined by $\tau_{0,1}v = v|_\Sigma$ and $\tau_{0,2}v = v|_{t=0}$, $v \in C^\infty(\overline{Q})$). Let w_1 solves

$$\begin{cases} \partial_t^2 w_1 - \Delta w_1 + q_1 w_1 = 0 & \text{in } Q, \\ \tau_0 w_1 = \tau_0 u_2, \end{cases} \quad (6.1)$$

Then, $u = w_1 - u_2$ solves

$$\begin{cases} \partial_t^2 - \Delta u + q_1 u = (q_2 - q_1)u_2 & \text{in } Q, \\ u(0, x) = \partial_t u(0, x) = 0 & \text{on } \Omega, \\ u = 0 & \text{on } \Sigma \end{cases} \quad (6.2)$$

and since $(q_2 - q_1)u_2 \in L^2(Q)$, in view of Theorem A.2 in [1] (see also Theorem 2.1 in [18] for $q = 0$), we deduce that $u \in \mathcal{C}^1([0, T]; L^2(\Omega)) \cap \mathcal{C}([0, T]; H_0^1(\Omega)) \cap H_\square(Q) \subset H^1(Q) \cap H_\square(Q)$ with $\partial_\nu u \in L^2(\Sigma)$. Using the fact that $u_1 \in H^1(Q) \cap H_\square(Q)$, we deduce that $(\partial_t u_1, -\nabla_x u_1) \in H_{\text{div}}(Q) = \{F \in L^2(Q; \mathbb{C}^{n+1}) : \text{div}_{(t,x)} F \in L^2(Q)\}$. Therefore, in view of [17, Lemma 2.2], we can apply the Green formula to get

$$\int_Q u(\square u_1) dt dx = - \int_Q (\partial_t u \partial_t u_1 - \nabla_x u \cdot \nabla_x u_1) dt dx + \langle (\partial_t u_1, -\nabla_x u_1) \cdot \mathbf{n}, u \rangle_{H^{-\frac{1}{2}}(\partial Q), H^{\frac{1}{2}}(\partial Q)}$$

with \mathbf{n} the outward unit normal vector to Q . In the same way, we find

$$\int_Q u_1(\square u) dt dx = - \int_Q (\partial_t u \partial_t u_1 - \nabla_x u \cdot \nabla_x u_1) dt dx + \langle (\partial_t u, -\nabla_x u) \cdot \mathbf{n}, u_1 \rangle_{H^{-\frac{1}{2}}(\partial Q), H^{\frac{1}{2}}(\partial Q)}$$

From these two formulas we deduce that

$$\begin{aligned} \int_Q (q_2 - q_1) u_2 u_1 dt dx &= \int_Q u_1(\square u + q_1 u) dt dx - \int_Q u(\square u_1 + q_1 u_1) dt dx \\ &= \langle (\partial_t u, -\nabla_x u) \cdot \mathbf{n}, u_1 \rangle_{H^{-\frac{1}{2}}(\partial Q), H^{\frac{1}{2}}(\partial Q)} - \langle (\partial_t u_1, -\nabla_x u_1) \cdot \mathbf{n}, u \rangle_{H^{-\frac{1}{2}}(\partial Q), H^{\frac{1}{2}}(\partial Q)}. \end{aligned}$$

On the other hand we have $u|_{t=0} = \partial_t u|_{t=0} = u|_\Sigma = 0$ and condition (1.3) implies that $u|_{t=T} = \partial_\nu u|_G = 0$. Combining this with the fact that $u \in \mathcal{C}^1([0, T]; L^2(\Omega))$ and $\partial_\nu u \in L^2(\Sigma)$, we obtain

$$\int_Q q u_2 u_1 dt dx = - \int_{\Sigma \setminus G} \partial_\nu u u_1 d\sigma(x) dt + \int_\Omega \partial_t u(T, x) u_1(T, x) dx. \quad (6.3)$$

Applying the Cauchy-Schwarz inequality to the first expression on the right hand side of this formula, we get

$$\begin{aligned} \left| \int_{\Sigma \setminus G} \partial_\nu u u_1 d\sigma(x) dt \right| &\leq \int_{\Sigma_{+, \varepsilon, \omega}} \left| \partial_\nu u e^{-\lambda(t+\omega \cdot x)} (1 + w) \right| dt d\sigma(x) \\ &\leq C \left(\int_{\Sigma_{+, \varepsilon, \omega}} \left| e^{-\lambda(t+\omega \cdot x)} \partial_\nu u \right|^2 d\sigma(x) dt \right)^{\frac{1}{2}} \end{aligned}$$

for some C independent of λ . Here we have used both (3.7) and the fact that $(\Sigma \setminus G) \subset \Sigma_{+, \varepsilon, \omega}$. In the same way, we have

$$\begin{aligned} \left| \int_\Omega \partial_t u(T, x) u_1(T, x) dx \right| &\leq \int_\Omega \left| \partial_t u(T, x) e^{-\lambda(T+\omega \cdot x)} (1 + w(T, x)) \right| dx \\ &\leq C \left(\int_\Omega \left| e^{-\lambda(T+\omega \cdot x)} \partial_t u(T, x) \right|^2 d\sigma(x) dt \right)^{\frac{1}{2}} \end{aligned}$$

Combining these estimates with the Carleman estimate (4.2), the fact that $u|_{t=T} = \partial_\nu u|_{\Sigma_-, \omega} = 0$ and the fact that $\partial\Omega_{+, \varepsilon, \omega} \subset \partial\Omega_{+, \omega}$, we find

$$\begin{aligned}
& \left| \int_Q (q_2 - q_1) u_2 u_1 dt dx \right|^2 \\
& \leq 2C \left(\int_{\Sigma_{+, \varepsilon, \omega}} |e^{-\lambda(t+\omega \cdot x)} \partial_\nu u|^2 d\sigma(x) dt + \int_\Omega |e^{-\lambda(T+\omega \cdot x)} \partial_t u(T, x)|^2 dx \right) \\
& \leq 2\varepsilon^{-1} C \left(\int_{\Sigma_{+, \omega}} |e^{-\lambda(t+\omega \cdot x)} \partial_\nu u|^2 \omega \cdot \nu(x) d\sigma(x) dt + \int_\Omega |e^{-\lambda(T+\omega \cdot x)} \partial_t u(T, x)|^2 dx \right) \\
& \leq \frac{\varepsilon^{-1} C}{\lambda} \left(\int_Q |e^{-\lambda(t+\omega \cdot x)} (\partial_t^2 - \Delta + q_1) u|^2 dx dt \right) \\
& \leq \frac{\varepsilon^{-1} C}{\lambda} \left(\int_Q |e^{-\lambda(t+\omega \cdot x)} q u_2|^2 dx dt \right) = \frac{\varepsilon^{-1} C}{\lambda} \left(\int_Q |q|^2 (1 + |z|)^2 dx dt \right).
\end{aligned}$$

Here $C > 0$ stands for some generic constant independent of λ . It follows that

$$\limsup_{\lambda \rightarrow +\infty} \int_Q q u_2 u_1 dt dx = 0 \quad (6.4)$$

On the other hand, we have

$$\int_Q q u_1 u_2 dx dt = \int_{\mathbb{R}^{1+n}} q(t, x) e^{-i\xi \cdot (t, x)} dx dt + \int_Q Z(t, x) dx dt$$

with $Z(t, x) = q(t, x)(z(t, x)e^{-i\xi \cdot (t, x)} + w(t, x) + z(t, x)w(t, x))$. Then, in view of (3.7) and (5.3), an application of the Cauchy-Schwarz inequality yields

$$\left| \int_Q Z(t, x) dx dt \right| \leq C \lambda^{-\frac{1}{2}}$$

with C independent of λ . Combining this with (6.4), we deduce that for all $\omega \in \{y \in \mathbb{S}^{n-1} : |y - \omega_0| \leq \varepsilon\}$ and all $\xi \in \mathbb{R}^{1+n}$ orthogonal to $(1, -\omega)$, we have

$$\int_{\mathbb{R}^{1+n}} q(t, x) e^{-i\xi \cdot (t, x)} dx dt = 0.$$

Thus, since $q \in L^\infty(Q)$ is compactly supported, we deduce that $q = 0$ and $q_1 = q_2$.

APPENDIX

In this appendix we prove that the space $\mathcal{C}^\infty(\overline{Q})$ is dense in $H_\square(Q)$ in some appropriate sense and we show that the maps τ_0 and τ_1 can be extended continuously on these spaces. Without lost of generality we consider only these spaces for real valued functions. The results of this section are well known, nevertheless we prove them for sake of completeness.

Density result in $H_\square(Q)$. Let us first recall the definition of $K_\square(Q)$:

$$K_\square(Q) = \{u \in H^{-1}(0, T; L^2(\Omega)) : \square u = (\partial_t^2 - \Delta)u \in L^2(Q)\}$$

with the norm

$$\|u\|_{K_\square(Q)}^2 = \|u\|_{H^{-1}(0, T; L^2(\Omega))}^2 + \|\square u\|_{L^2(Q)}^2.$$

The goal of this subsection is to prove the following.

Theorem 4. $H_\square(Q)$ embedded continuously into the closure of $\mathcal{C}^\infty(\overline{Q})$ with respect to $K_\square(Q)$.

Proof. Let N be a continuous linear form on $K_\square(Q)$ satisfying

$$Nf = 0, \quad f \in C^\infty(\overline{Q}). \quad (6.5)$$

In order to show the required density result we will prove that this condition implies that $N|_{H_\square(Q)} = 0$.

By considering the application $u \mapsto (u, \square u)$ we can identify $K_\square(Q)$ to a subspace of $H^{-1}(0, T; L^2(\Omega)) \times L^2(Q)$. Then, applying the Hahn Banach theorem we deduce that N can be extended to a continuous linear form on $H^{-1}(0, T; L^2(\Omega)) \times L^2(Q)$. Therefore, there exist $h_1 \in H_0^1(0, T; L^2(\Omega))$, $h_2 \in L^2(Q)$ such that

$$N(u) = \langle u, h_1 \rangle_{H^{-1}(0, T; L^2(\Omega)), H_0^1(0, T; L^2(\Omega))} + \langle \square u, h_2 \rangle_{L^2(Q)}, \quad u \in K_\square(Q).$$

Now let $\mathcal{O} \subset \mathbb{R}^n$ be a bounded C^∞ domain such that $\overline{\Omega} \subset \mathcal{O}$ and fix $Q_\varepsilon = (-\varepsilon, T + \varepsilon) \times \mathcal{O}$ with $\varepsilon > 0$. Let \tilde{h}_j be the extension of h_j on \mathbb{R}^{1+n} by 0 outside of Q for $j = 1, 2$. In view of (6.5) we have

$$\langle f, \tilde{h}_1 \rangle_{L^2(Q_\varepsilon)} + \langle (\partial_t^2 - \Delta)f, \tilde{h}_2 \rangle_{L^2(Q_\varepsilon)} = N(f|_Q) = 0, \quad f \in C_0^\infty(Q_\varepsilon).$$

Thus, in the sense of distribution we have

$$\square \tilde{h}_2 = -\tilde{h}_1 \quad \text{on } Q_\varepsilon.$$

Moreover, since $\tilde{h}_2 = 0$ on $\mathbb{R}^{1+n} \setminus \overline{Q} \supset \partial Q_\varepsilon$, we deduce that \tilde{h}_2 solves

$$\begin{cases} \partial_t^2 \tilde{h}_2 - \Delta \tilde{h}_2 = -\tilde{h}_1 & \text{in } Q_\varepsilon, \\ \tilde{h}_2(-\varepsilon, x) = \partial_t \tilde{h}_2(-\varepsilon, x) = 0, & x \in \mathcal{O}, \\ \tilde{h}_2(t, x) = 0, & (t, x) \in (-\varepsilon, T + \varepsilon) \times \partial \mathcal{O}. \end{cases}$$

But, since $h_1 \in H_0^1(0, T; L^2(\Omega))$, we have $\tilde{h}_1 \in H_0^1(-\varepsilon, T + \varepsilon; L^2(\mathcal{O}))$ and we deduce from Theorem 2.1 in Chapter 5 of [20] that this IBVP admits a unique solution lying in $H^2(Q_\varepsilon)$. Therefore, $\tilde{h}_2 \in H^2(Q_\varepsilon)$. Combining this with the fact that $\tilde{h}_2 = 0$ on $Q_\varepsilon \setminus Q$, we deduce that $h_2 \in H_0^2(Q)$, with $H_0^2(Q)$ the closure of $C_0^\infty(Q)$ in $H^2(Q)$, and that $\square h_2 = -h_1$ on Q . Thus, for every $u \in H_\square(Q)$ we have

$$\langle \square u, h_2 \rangle_{L^2(Q)} = \langle \square u, h_2 \rangle_{H^{-2}(Q), H_0^2(Q)} = \langle u, \square h_2 \rangle_{L^2(Q)} = -\langle u, h_1 \rangle_{L^2(Q)}.$$

Here we use the fact that $H_\square(Q) \subset L^2(Q)$. Then it follows that

$$N(u) = \langle u, h_1 \rangle_{L^2(Q)} - \langle u, h_1 \rangle_{L^2(Q)} = 0, \quad u \in H_\square(Q).$$

From this last result we deduce that $H_\square(Q)$ is contained into the closure of $C^\infty(\overline{Q})$ with respect to $K_\square(Q)$. Combining this with the fact that $H_\square(Q)$ embedded continuously into $K_\square(Q)$ we deduce the required result. \square

Trace operator in $H_\square(Q)$. In this subsection we extend the trace maps τ_0 and τ_1 into $H_\square(Q)$ by duality in the following way.

Proposition 4. *The maps*

$$\tau_0 w = (\tau_{0,1} w, \tau_{0,2} w, \tau_{0,3} w) = (w|_\Sigma, w|_{t=0}, \partial_t w|_{t=0}), \quad w \in C^\infty(\overline{Q}),$$

$$\tau_1 w = (\tau_{1,1} w, \tau_{1,2} w, \tau_{1,3} w) = (\partial_\nu w|_\Sigma, w|_{t=T}, \partial_t w|_{t=T}), \quad w \in C^\infty(\overline{Q}),$$

can be extended continuously to $\tau_0 : H_\square(Q) \rightarrow H^{-3}(0, T; H^{-\frac{1}{2}}(\partial\Omega)) \times H^{-2}(\Omega) \times H^{-4}(\Omega)$,
 $\tau_1 : H_\square(Q) \rightarrow H^{-3}(0, T; H^{-\frac{3}{2}}(\partial\Omega)) \times H^{-2}(\Omega) \times H^{-4}(\Omega)$.

Proof. It is well known that the trace maps

$$u \mapsto (u|_{\partial\Omega}, \partial_\nu u|_{\partial\Omega})$$

can be extended continuously to a bounded operator from $H^2(\Omega)$ to $H^{\frac{3}{2}}(\partial\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ which is onto. Therefore, there exists a bounded operator $R : H^{\frac{3}{2}}(\partial\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \rightarrow H^2(\Omega)$ such that

$$R[h_1, h_2]|_{\partial\Omega} = h_1, \quad \partial_\nu R[h_1, h_2]|_{\partial\Omega} = h_2, \quad (h_1, h_2) \in H^{\frac{3}{2}}(\partial\Omega) \times H^{\frac{1}{2}}(\partial\Omega).$$

Fix $g \in H_0^3(0, T; H^{\frac{1}{2}}(\partial\Omega))$ and choose $G(t, \cdot) = R(0, g(t, \cdot))$. One can check that $G \in H_0^3(0, T; H^2(\Omega))$ and

$$\|G\|_{H^3(0, T; H^2(\Omega))} \leq \|R\| \|g\|_{H^3(0, T; H^{\frac{1}{2}}(\partial\Omega))}. \quad (6.6)$$

Applying twice the Green formula we obtain

$$\int_{\Sigma} v g d\sigma(x) dt = \int_Q \square v G dx dt - \int_Q v \square G dx dt, \quad v \in \mathcal{C}^\infty(\overline{Q}).$$

But $\square G \in H_0^1(0, T; H^2(\Omega))$, and we have

$$\langle \tau_{0,1} v, g \rangle_{H^{-3}(0, T; H^{-\frac{1}{2}}(\partial\Omega)), H_0^3(0, T; H^{\frac{1}{2}}(\partial\Omega))} = \langle \square v, G \rangle_{L^2(Q)} - \langle v, \square G \rangle_{H^{-1}(0, T; L^2(\Omega)), H_0^1(0, T; L^2(\Omega))}.$$

Then, using (6.6) and the Cauchy Schwarz inequality, for all $v \in \mathcal{C}^\infty(\overline{Q})$, we obtain

$$|\langle \tau_{0,1} v, g \rangle| \leq \|\square v\|_{L^2(Q)} \|G\|_{L^2(Q)} + \|v\|_{H^{-1}(0, T; L^2(\Omega))} \|\square G\|_{H_0^1(0, T; L^2(\Omega))}$$

$$\leq C \|v\|_{K_{\square}(Q)} \|g\|_{H^3(0, T; H^{\frac{1}{2}}(\partial\Omega))}$$

which, combined with the density result of Theorem 4, implies that $\tau_{0,1} : v \mapsto v|_{\Sigma}$ extend continuously to a bounded operator from $H_{\square}(Q)$ to $H^{-3}(0, T; H^{-\frac{1}{2}}(\partial\Omega))$. In a same way we prove that

$$\tau_1^1 v = \partial_{\nu} v|_{\Sigma}, \quad v \in \mathcal{C}^\infty(\overline{Q})$$

extend continuously to a bounded operator from $H_{\square}(Q)$ to $H^{-3}(0, T; H^{-\frac{3}{2}}(\partial\Omega))$.

Now let us consider the operators $\tau_{i,j}$, $i = 0, 1$, $j = 2, 3$. We start with

$$\tau_{0,2} : v \mapsto v|_{t=0}, \quad v \in \mathcal{C}^\infty(\overline{Q}).$$

Let $h \in H_0^2(\Omega)$ and fix $H(t, x) = t\psi(t)h(x)$ with $\psi \in \mathcal{C}_0^\infty(-T, \frac{T}{2})$ satisfying $0 \leq \psi \leq 1$ and $\psi = 1$ on $[-\frac{T}{3}, \frac{T}{3}]$. Then, using the fact that $\psi = 1$ on a neighborhood of $t = 0$, we deduce that

$$H|_{\Sigma} = \partial_{\nu} H|_{\Sigma} = H|_{t=0} = \square H|_{t=0} = \square H|_{t=T} = 0, \quad \partial_t H|_{t=0} = h.$$

Therefore, $\square H \in H_0^1(0, T; L^2(\Omega))$ and repeating the above arguments, for all $v \in \mathcal{C}^\infty(\overline{Q})$, we obtain the representation

$$\langle \tau_{0,2} v, h \rangle_{H^{-2}(\Omega), H_0^2(\Omega)} = \langle v, \square H \rangle_{H^{-1}(0, T; L^2(\Omega)), H_0^1(0, T; L^2(\Omega))} - \langle H, \square v \rangle_{L^2(Q)}.$$

Then, we prove by density that $\tau_{0,2}$ extends continuously to $\tau_{0,2} : H_{\square}(Q) \longrightarrow H^{-2}(\Omega)$.

For

$$\tau_{0,3} : v \mapsto \partial_t v|_{t=0}, \quad v \in \mathcal{C}^\infty(\overline{Q}),$$

let $\varphi \in H_0^4(\Omega)$ and fix

$$\Phi(t, x) = \psi(t)\varphi(x) + \frac{\psi(t)t^2\Delta\varphi(x)}{2}.$$

Then, Φ satisfies

$$\Phi|_{\Sigma} = \partial_{\nu} \Phi|_{\Sigma} = \partial_t \Phi|_{t=0} = 0, \quad \Phi|_{t=0} = \varphi.$$

Moreover, we have $\square \Phi \in H^1(0, T; L^2(\Omega))$ with

$$(\partial_t^2 - \Delta)\Phi|_{t=0} = -\Delta\varphi + \Delta\varphi = 0, \quad (\partial_t^2 - \Delta)\Phi|_{t=T} = 0$$

and it follows that $\square \Phi \in H_0^1(0, T; L^2(\Omega))$. Therefore, repeating the above arguments we obtain the representation

$$\langle \tau_{0,3} v, \varphi \rangle_{H^{-4}(\Omega), H_0^4(\Omega)} = \langle \square v, \Phi \rangle_{L^2(Q)} - \langle v, \square \Phi \rangle_{H^{-1}(0, T; L^2(\Omega)), H_0^1(0, T; L^2(\Omega))}$$

and we deduce that $\tau_{0,3}$ extends continuously to $\tau_{0,3} : H_{\square}(Q) \longrightarrow H^{-4}(\Omega)$. In a same way, one can check that

$$\tau_{1,2} v = v|_{t=T}, \quad \tau_{1,3} v = \partial_t v|_{t=T}, \quad v \in \mathcal{C}^\infty(\overline{Q})$$

extend continuously to $\tau_{1,2} : H_{\square}(Q) \longrightarrow H^{-2}(\Omega)$ and $\tau_{1,3} : H_{\square}(Q) \longrightarrow H^{-4}(\Omega)$. \square

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