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# A MAX-CUT FORMULATION OF 0/1 PROGRAMS

JEAN B. LASSERRE

ABSTRACT. We consider the linear or quadratic 0/1 program

$$\mathbf{P} : f^* = \min\{\mathbf{c}^T \mathbf{x} + \mathbf{x}^T \mathbf{F} \mathbf{x} : \mathbf{A} \mathbf{x} = \mathbf{b}; \mathbf{x} \in \{0, 1\}^n\},$$

for some vectors  $\mathbf{c} \in \mathbb{R}^n$ ,  $\mathbf{b} \in \mathbb{Z}^m$ , some matrix  $\mathbf{A} \in \mathbb{Z}^{m \times n}$  and some real symmetric matrix  $\mathbf{F} \in \mathbb{R}^{n \times n}$ . We show that  $\mathbf{P}$  can be formulated as a MAX-CUT problem whose quadratic form criterion is explicit from the data of  $\mathbf{P}$ . In particular, to  $\mathbf{P}$  one may associate a graph whose connectivity is related to the connectivity of the matrix  $\mathbf{F}$  and  $\mathbf{A}^T \mathbf{A}$ , and  $\mathbf{P}$  reduces to finding a maximum (weighted) cut in such a graph. Hence the whole arsenal of approximation techniques for MAX-CUT can be applied. We also briefly mention some extensions. On a sample of 0/1 knapsack and  $k$ -cluster problems (with up to 100 variables) we compare the lower bound of the resulting semidefinite (or Shor) relaxation with that of the standard LP-relaxation, the first semidefinite relaxation of the Lasserre hierarchy associated with  $\mathbf{P}$  and the semidefinite relaxation of the copositive formulation of  $\mathbf{P}$ .

## 1. INTRODUCTION

Consider the linear or quadratic 0/1 program  $\mathbf{P}$  defined by:

$$(1.1) \quad \mathbf{P} : f^* = \min_{\mathbf{x}} \{ \mathbf{c}^T \mathbf{x} + \mathbf{x}^T \mathbf{F} \mathbf{x} : \mathbf{A} \mathbf{x} = \mathbf{b}; \mathbf{x} \in \{0, 1\}^n \}$$

for some cost vector  $\mathbf{c} \in \mathbb{R}^n$ , some matrix  $\mathbf{A} \in \mathbb{Z}^{m \times n}$ , some vector  $\mathbf{b} \in \mathbb{Z}^m$ , and some real symmetric matrix  $\mathbf{F} \in \mathbb{R}^{n \times n}$ . If  $\mathbf{F} = 0$  then  $\mathbf{P}$  is a 0/1 linear program and a quadratic 0/1 program otherwise. Obtaining good quality lower bounds on  $f^*$  is highly desirable since the efficiency of Branch & Bound algorithms to solve large scale problems  $\mathbf{P}$  heavily depends on the quality of bounds of this form computed at nodes of the search tree.

To obtain lower bounds for 0/1 programs (1.1) one may solve a relaxation of  $\mathbf{P}$  where the integrality constraints  $\mathbf{x} \in \{0, 1\}^n$  are replaced with the box constraints  $\mathbf{x} \in [0, 1]^n$ . If  $\mathbf{F} = 0$  the resulting relaxation is linear whereas if  $\mathbf{F}$  is positive definite it is a (convex) quadratic program. If  $\mathbf{F}$  is not positive semidefinite then one may also solve a convex quadratic program but now with an appropriate convex quadratic underestimator  $\mathbf{x}^T \tilde{\mathbf{F}} \mathbf{x}$  of  $\mathbf{x}^T \mathbf{F} \mathbf{x}$  on  $[0, 1]^n$ . An alternative is to consider an equivalent formulation of  $\mathbf{P}$

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as a copositive conic program as advocated by Burer [3] and compute a sequence of lower bounds by solving an appropriate hierarchy of LP- or SDP-relaxations associated with the copositive cone (or its dual). For more details on the latter approach the interested reader is referred to e.g. De Klerk and Pasechnik [4], Dür [5], Bomze [1], and Bomze and de Klerk [2].

**Contribution.** The purpose of this note is to show that solving  $\mathbf{P}$  is equivalent to minimizing a quadratic form in  $n+1$  variables on the hypercube  $\{-1, 1\}^{n+1}$  (and the quadratic form is explicit from the data of  $\mathbf{P}$ ). In other words  $\mathbf{P}$  can be viewed as an explicit instance of the MAX-CUT problem. This idea had been already briefly mentioned in Rendl [15] in the context of partitioning an ordering problems. Hence the MAX-CUT problem which at first glance seems to be a very specific combinatorial optimization problem, in fact can be considered as a canonical model of linear and quadratic 0/1 programs. In particular, to each linear or quadratic 0/1 program (1.1) one may associate a graph  $G = (V, E)$  with  $n+1$  nodes and  $(i, j) \in E$  whenever a product  $x_i x_j$  has a nonzero coefficient in some quadratic form built upon the data  $\mathbf{c}, \mathbf{b}, \mathbf{F}$  and  $\mathbf{A}$  of (1.1). (Among other things, the sparsity of  $G$  is related to the sparsity of the matrices  $\mathbf{F}$  and  $\mathbf{A}^T \mathbf{A}$ .) Then solving (1.1) reduces to finding a maximum (weighted) cut of  $G$ .

Therefore the whole specialized arsenal of approximation techniques for MAX-CUT can be applied. In particular one may obtain a lower bound  $f_1^*$  on  $f^*$  by solving the standard (Shor) SDP-relaxation associated with the resulting MAX-CUT problem while solving higher levels of the associated Lasserre-SOS hierarchy [7, 8] would provide a monotone nondecreasing sequence of improved lower bounds  $f_1^* \leq f_d^* \leq f^*$ ,  $d = 2, \dots$ , but of course at a higher computational cost. Alternatively one may also apply the Handelman hierarchy of LP-relaxations as described and analyzed in Laurent and Sun [11]. For more details on recent developments on computational approaches to MAX-CUT the interested reader is referred to Palagi et al. [14].

In addition one may also obtain performance guarantees *à la Nesterov* [13] in the form

$$f_1^* \leq f^* \leq \frac{2}{\pi} f_1^* + \left(1 - \frac{2}{\pi}\right) h_1^*,$$

(where  $h_1^*$  is the optimal value of a similar SDP but with a max-criterion instead of a min-criterion) or their improvements by Marshall [12].

Finally the same methodology also works for general 0/1 optimization problems with feasible set as in (1.1) and polynomial criterion  $f \in \mathbb{R}[\mathbf{x}]$  of degree  $d > 2$ , except that now the problem reduces to minimizing a new polynomial criterion  $\tilde{f}(\mathbf{x})$  on the hypercube  $\{-1, 1\}^n$  (and not a MAX-CUT problem any more as the degree of  $\tilde{f}$  is larger than 2).

**Numerical experiments.** If  $\mathbf{F} = 0$  (i.e. when  $\mathbf{P}$  is a linear 0/1 program) the lower bound  $f_1^*$  can be better than the standard LP-relaxation which consists in replacing the integrality constraints  $\mathbf{x} \in \{0, 1\}^n$  with the box

$[0, 1]^n$ , as shown on a (limited) sample of 0/1-knapsack-type examples. On such examples  $f_1^*$  also dominates the one obtained from the first relaxation of the copositive formulation (where the dual cone  $\mathcal{C}^*$  of completely positive matrices is replaced with  $\mathcal{S}^+ \cap \mathcal{N} \supset \mathcal{C}^*$ ) in about 55% of cases and the maximum relative difference is bounded by 0.55% in all cases.

In fact, and still on the same sample of linear and quadratic 0/1 knapsack examples, one also observes that the resulting lower bound  $f_1^*$  is almost always better than the lower bound obtained by solving the first SDP-relaxation of the Lasserre-SOS hierarchy applied to the initial formulation (1.1) of the problem (which is also an SDP of same size). This is good news since typically the SOS-hierarchy is known to produce good lower bounds for general polynomial optimization problems (discrete or not) even at the first level of the hierarchy. Even more, the first level SDP-relaxation has the celebrated Goemans & Williamson performance guarantee ( $\approx 87\%$ ) when the matrix  $\mathbf{Q}$  (associated with the quadratic form) has nonnegative entries and a performance guarantee  $\approx 64\%$  when  $\mathbf{Q} \succeq 0$ . (However note that the matrix  $\mathbf{Q}$  associated with our MAX-CUT problem equivalent to the initial 0/1 program (1.1) does not have all its entries nonnegative.) This explains why in the linear 0/1 knapsack examples the lower bound  $f_1^*$  is almost always better than the one obtained with the standard LP-relaxation and why for quadratic 0/1 knapsack problems (1.1),  $f_1^*$  is also likely to be better than the lower bound obtained by relaxing  $\{0, 1\}^n$  to  $[0, 1]^n$ , replacing  $\mathbf{F}$  with a convex quadratic underestimator of  $\mathbf{F}$  on  $[0, 1]^n$ , and solving the resulting convex quadratic program.

Finally we have also considered the MAX-CUT formulation to the  $k$ -cluster problem  $\rho = \min\{\mathbf{x}^T \mathbf{A} \mathbf{x} : \mathbf{e}^T \mathbf{x} = k; \mathbf{x} \in \{0, 1\}^n\}$  where  $k \leq n$  is a fixed integer,  $\mathbf{A}$  is some real symmetric matrix, for  $n = 70$  and  $n = 100$  variables and with various percentages of zero entries in the matrix  $\mathbf{A}$ . In all examples for different values of  $k$  (and where the entries of  $\mathbf{A}$  were randomly generated following the uniform distribution on  $[0, 1]$ ), the optimal value of the Shor relaxation was almost indistinguishable of the first semidefinite relaxation of the Lasserre-SOS hierarchy applied to the initial formulation.

Note that if  $\mathbf{U}^T \Lambda \mathbf{U}$  is the Schur decomposition of  $\mathbf{A}$  then  $\rho = -k\theta + \min\{\mathbf{x}^T \tilde{\mathbf{A}} \mathbf{x} : \mathbf{e}^T \mathbf{x} = k; \mathbf{x} \in \{0, 1\}^n\}$  (where  $\theta = 1 + \lambda_{\min}(\mathbf{A})$ ) and  $\tilde{\mathbf{A}} = \mathbf{U}^T (\Lambda + \theta \mathbf{I}) \mathbf{U} \succeq 0$  so that  $\mathbf{x} \mapsto \mathbf{x}^T \tilde{\mathbf{A}} \mathbf{x}$  is a convex quadratic form. Therefore we have also compared the Shor relaxation with the quadratic convex relaxation  $-k\theta + \min\{\mathbf{x}^T \tilde{\mathbf{A}} \mathbf{x} : \mathbf{e}^T \mathbf{x} = k; \mathbf{x} \in [0, 1]^n\}$ . In all cases the Shor relaxation provides better lower bounds (and depending on  $k$ , sometimes really significantly better). Finally we have also compared the Shor relaxations associated with the MAX-CUT and copositive formulations. Again the respective optimal values are very close (less than 1% relative difference and in a few cases the former is significantly better).

## 2. MAIN RESULT

Denote by  $\mathbb{Z}$  the set of integer numbers and  $\mathbb{N} \subset \mathbb{Z}$  the set of natural numbers. Let  $\mathbf{P}$  be the 0/1 program defined in (1.1) with  $\mathbf{F}^T = \mathbf{F} \in \mathbb{R}^{n \times n}$ ,  $\mathbf{A} \in \mathbb{Z}^{m \times n}$ ,  $\mathbf{c} \in \mathbb{R}^n$  and  $\mathbf{b} \in \mathbb{Z}^m$ . Let  $|\mathbf{c}| := (|c_i|) \in \mathbb{R}_+^n$ . With  $\mathbf{e} \in \mathbb{Z}^n$  being the vector of all ones, notice first that  $\mathbf{P}$  has an equivalent formulation on the hypercube  $\{-1, 1\}^n$ , by the change of variables  $\tilde{\mathbf{x}} := 2\mathbf{x} - \mathbf{e}$ . Indeed,  $\mathbf{A}$ ,  $\mathbf{b}$ ,  $\mathbf{c}$  and  $\mathbf{F}$  now become  $\mathbf{A}/2$ ,  $\mathbf{b} - \mathbf{A}\mathbf{e}/2$ ,  $(\mathbf{c} + \mathbf{e}^T\mathbf{F})/2$  and  $\mathbf{F}/4$  respectively.

Therefore from now on we consider the discrete program:

$$(2.1) \quad \mathbf{P} : \quad f^* = \min_{\mathbf{x} \in \{-1, 1\}^n} \{ \mathbf{c}^T \mathbf{x} + \mathbf{x}^T \mathbf{F} \mathbf{x} : \mathbf{A} \mathbf{x} = \mathbf{b} \},$$

on the hypercube  $\{-1, 1\}^n$ , with  $\mathbf{A} \in \mathbb{Z}^{m \times n}$ ,  $\mathbf{c} \in \mathbb{R}^n$ ,  $\mathbf{b} \in \mathbb{Z}^m$ , and  $\mathbf{F}^T = \mathbf{F} \in \mathbb{R}^{n \times n}$ . With  $\mathbf{c}$  and  $\mathbf{F}$ , let us associate the scalars:

$$\begin{aligned} r_{\mathbf{c}, \mathbf{F}}^1 &= \min \{ \mathbf{c}^T \mathbf{x} + \langle \mathbf{X}, \mathbf{F} \rangle : \begin{bmatrix} 1 & \mathbf{x}^T \\ \mathbf{x} & \mathbf{X} \end{bmatrix} \succeq 0; X_{ii} = 1, i = 1, \dots, n \} \\ r_{\mathbf{c}, \mathbf{F}}^2 &= \max \{ \mathbf{c}^T \mathbf{x} + \langle \mathbf{X}, \mathbf{F} \rangle : \begin{bmatrix} 1 & \mathbf{x}^T \\ \mathbf{x} & \mathbf{X} \end{bmatrix} \succeq 0; X_{ii} = 1, i = 1, \dots, n \} \end{aligned}$$

(with  $\mathbf{X}^T = \mathbf{X}$ ) and let

$$(2.2) \quad \rho(\mathbf{c}, \mathbf{F}) := \max_{i=1,2} |r_{\mathbf{c}, \mathbf{F}}^i|.$$

It is straightforward to verify that

$$(2.3) \quad \rho(\mathbf{c}, \mathbf{F}) \geq \max \{ |\mathbf{c}^T \mathbf{x} + \mathbf{x}^T \mathbf{F} \mathbf{x}| : \mathbf{x} \in \{-1, 1\}^n \},$$

and  $\rho(\mathbf{c}, \mathbf{F}) = |\mathbf{c}|$  if  $\mathbf{F} = 0$ . Moreover each scalar  $r_{\mathbf{c}, \mathbf{F}}^i$  can be computed by solving an SDP which is the Shor relaxation (or first level of the Lasserre-SOS hierarchy [7, 8]) associated with the problems  $\min (\max) \{ \mathbf{c}^T \mathbf{x} + \mathbf{x}^T \mathbf{F} \mathbf{x} : \mathbf{x} \in \{-1, 1\}^n \}$ .

### 2.1. A MAX-CUT formulation of $\mathbf{P}$ .

**Lemma 2.1.** *Let  $\mathbf{P}$  be as (2.1) and let  $\rho(\mathbf{c}, \mathbf{F})$  be as in (2.2). Then  $f^*$  is the optimal value of the quadratic minimization problem:*

$$(2.4) \quad \min_{\mathbf{x} \in \{-1, 1\}^n} \mathbf{c}^T \mathbf{x} + \mathbf{x}^T \mathbf{F} \mathbf{x} + (2\rho(\mathbf{c}, \mathbf{F}) + 1) \cdot \|\mathbf{A} \mathbf{x} - \mathbf{b}\|^2.$$

*Proof.* Let  $\Delta := \{ \mathbf{x} \in \{-1, 1\}^n : \mathbf{A} \mathbf{x} = \mathbf{b} \}$  be the feasible set of  $\mathbf{P}$  defined in (2.1), and let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be the function

$$(2.5) \quad x \mapsto f(\mathbf{x}) := \mathbf{c}^T \mathbf{x} + \mathbf{x}^T \mathbf{F} \mathbf{x} + (2\rho(\mathbf{c}, \mathbf{F}) + 1) \cdot \|\mathbf{A} \mathbf{x} - \mathbf{b}\|^2.$$

On  $\{-1, 1\}^n$  one has  $\max \{ \mathbf{c}^T \mathbf{x} + \mathbf{x}^T \mathbf{F} \mathbf{x} : \mathbf{x} \in \{-1, 1\}^n \} \leq \rho(\mathbf{c}, \mathbf{F})$ , and

$$\|\mathbf{A} \mathbf{x} - \mathbf{b}\|^2 \geq 1, \quad \forall \mathbf{x} \in \{-1, 1\}^n \setminus \Delta,$$

because  $\mathbf{A} \in \mathbb{Z}^{m \times n}$  and  $\mathbf{b} \in \mathbb{Z}^m$ . Therefore,

$$(2.6) \quad f(\mathbf{x}) \begin{cases} = \mathbf{c}^T \mathbf{x} + \mathbf{x}^T \mathbf{F} \mathbf{x} & \text{on } \Delta \\ \geq \mathbf{c}^T \mathbf{x} + \mathbf{x}^T \mathbf{F} \mathbf{x} + 2\rho(\mathbf{c}, \mathbf{F}) + 1 > \rho(\mathbf{c}, \mathbf{F}) & \text{on } \{-1, 1\}^n \setminus \Delta. \end{cases}$$

From this and  $\mathbf{c}^T \mathbf{x} + \mathbf{x}^T \mathbf{F} \mathbf{x} \leq \rho(\mathbf{c}, \mathbf{F})$  on  $\Delta$ , the result follows.  $\square$

Next, let  $Q : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$  be the homogenization of the quadratic polynomial  $f$ , i.e., the quadratic form  $Q(\mathbf{x}, x_0) := x_0^2 f(\frac{\mathbf{x}}{x_0})$ , or in explicitly form:

$$(2.7) \quad Q(\mathbf{x}, x_0) = x_0 \mathbf{c}^T \mathbf{x} + \mathbf{x}^T \mathbf{F} \mathbf{x} + (2\rho(\mathbf{c}, \mathbf{F}) + 1) \cdot \|\mathbf{A}\mathbf{x} - x_0 \mathbf{b}\|^2.$$

Observe that  $Q(\mathbf{x}, 1) = f(\mathbf{x})$ .

**Theorem 2.2.** *Let  $f^* = \min\{\mathbf{c}^T \mathbf{x} + \mathbf{x}^T \mathbf{F} \mathbf{x} : \mathbf{A}\mathbf{x} = \mathbf{b}; \mathbf{x} \in \{-1, 1\}^n\}$  and let  $Q$  be the quadratic form in (2.7). Then*

$$(2.8) \quad f^* = \min_{(\mathbf{x}, x_0) \in \{-1, 1\}^{n+1}} Q(\mathbf{x}, x_0),$$

that is,  $f^*$  is the optimal value of the MAX-CUT problem associated with the quadratic form  $Q$ .

*Proof.* Let  $f$  be as in (2.5). By definition of  $Q$ ,

$$(2.9) \quad \min_{\mathbf{x} \in \{-1, 1\}^n} f(\mathbf{x}) = \min_{(\mathbf{x}, x_0) \in \{-1, 1\}^{n+1}} \{Q(\mathbf{x}, x_0) : x_0 = 1\}.$$

On the other hand, let  $(\mathbf{x}^*, x_0^*) \in \{-1, 1\}^{n+1}$  be a global minimizer of  $\min\{Q(\mathbf{x}, x_0) : (\mathbf{x}, x_0) \in \{-1, 1\}^{n+1}\}$ . Then by homogeneity of  $Q$ ,  $(-\mathbf{x}^*, -x_0^*)$  is also a global minimizer and so one may decide arbitrarily to fix  $x_0 = 1$ . That is,

$$\min_{(\mathbf{x}, x_0) \in \{-1, 1\}^{n+1}} Q(\mathbf{x}, x_0) = \min_{(\mathbf{x}, x_0) \in \{-1, 1\}^{n+1}} \{Q(\mathbf{x}, x_0) : x_0 = 1\},$$

which combined with (2.9) yields the desired result.  $\square$

Next, write  $Q(\mathbf{x}, x_0) = (\mathbf{x}, x_0) \mathbf{Q} (\mathbf{x}, x_0)^T$  for an appropriate real symmetric matrix  $\mathbf{Q} \in \mathbb{R}^{(n+1) \times (n+1)}$ , and introduce the semidefinite programs

$$(2.10) \quad \min_{\mathbf{X}} \{\langle \mathbf{Q}, \mathbf{X} \rangle : \mathbf{X} \succeq 0; X_{ii} = 1, \quad i = 1, \dots, n+1\}$$

with optimal value denoted by  $\min \mathbf{Q}_+$ , and

$$(2.11) \quad \max_{\mathbf{X}} \{\langle \mathbf{Q}, \mathbf{X} \rangle : \mathbf{X} \succeq 0; X_{ii} = 1, \quad i = 1, \dots, n+1\}$$

with optimal value denoted by  $\max \mathbf{Q}^+$ .

**Proposition 2.3.** *Let  $\mathbf{P}$  be the problem defined in (2.1) with optimal value  $f^*$  and let  $\rho(\mathbf{c}, \mathbf{F})$  be as in (2.2). Then:*

$$(2.12) \quad \min \mathbf{Q}_+ \leq f^* \leq \frac{2}{\pi} \min \mathbf{Q}_+ + (1 - \frac{2}{\pi}) \max \mathbf{Q}^+$$

where  $\mathbf{Q}$  is the real symmetric matrix associated with the quadratic form (2.7) and  $\min \mathbf{Q}_+$  (resp.  $\max \mathbf{Q}^+$ ) is the optimal value of the semidefinite program (2.10) (resp. (2.11)).

Moreover, if  $\min \mathbf{Q}_+ > \rho(\mathbf{c}, \mathbf{F})$  then  $\mathbf{P}$  has no feasible solution (i.e.,  $f^* = +\infty$ ).

*Proof.* The bounds in (2.12) are from Nesterov [13]. In addition, one may also use the bounds provided in Marshall [12] which sometimes improve those in (2.12). Next, let  $\Delta := \{\mathbf{x} \in \{-1, 1\}^n : \mathbf{A}\mathbf{x} = \mathbf{b}\}$  and assume that  $\Delta \neq \emptyset$ . Then by (2.3) and (2.6),  $\min \mathbf{Q}_+ \leq f^* \leq \rho(\mathbf{c}, \mathbf{F})$ . Therefore  $\Delta = \emptyset$  if  $\min \mathbf{Q}_+ > \rho(\mathbf{c}, \mathbf{F})$ .  $\square$

The quality of the upper bound in (2.12) depends strongly on the magnitude of the “penalty coefficient”  $2\rho(\mathbf{c}, \mathbf{F}) + 1$  in the definition of the function  $f$  in (2.5). Whether or not the penalty parameter  $\rho(\mathbf{c}, \mathbf{F})$  could be taken smaller (at least in some cases) has not been investigated and is beyond the scope of this paper. However for a practical use of relaxations what matters most is the quality of the *lower* bound  $\min \mathbf{Q}_+$  which in principle is very good for MAX-CUT problems (even if  $\mathbf{Q} \not\geq 0$  or  $\mathbf{Q} \not\leq 0$ ). For instance in a Branch & Bound algorithm the lower bound  $\min \mathbf{Q}_+$  has an important impact in the pruning of nodes in the search tree.

**2.2. Sparsity.** Hence to each 0/1 program (1.1) one may associate a graph  $G = (V, E)$  with  $n + 1$  nodes and an arc  $(i, j) \in E$  connects the nodes  $i, j \in V$  if and only if the coefficient  $\mathbf{Q}_{ij}$  of the quadratic form  $Q(\mathbf{x}, x_0)$  does not vanish. *Sparsity* properties of  $G$  are of primary interest, e.g. for computational reasons. From the definition of the matrix  $\mathbf{Q}$ , this sparsity is in turn related to sparsity of the matrix  $\mathbf{F} + (2\rho_{\mathbf{c}, \mathbf{F}} + 1) \cdot \mathbf{A}^T \mathbf{A}$ , hence of sparsity of  $\mathbf{F}$  and  $\mathbf{A}^T \mathbf{A}$ .

In particular, two nodes  $i, j$  are not connected if  $\mathbf{F}_{ij} = 0$  and  $\mathbf{A}_{ki} \mathbf{A}_{kj} = 0$  for all  $k = 1, \dots, m$ , that is, in any of the constraints  $\mathbf{A}\mathbf{x} = \mathbf{b}$ , at most one of the two variables  $(x_i, x_j)$  appears (a structural condition). A weaker condition (non-structural as it depends on values of entries of  $\mathbf{A}$ ) is that  $\sum_k \mathbf{A}_{ki} \mathbf{A}_{kj} = 0$ .

**2.3. Extension to inequalities.** Let  $f(\mathbf{x}) := \mathbf{c}^T \mathbf{x} + \mathbf{x}^T \mathbf{F} \mathbf{x}$  for some  $\mathbf{c} \in \mathbb{R}^n$  and some  $\mathbf{F}^T = \mathbf{F} \in \mathbb{R}^{n \times n}$ , and consider the problem:

$$(2.13) \quad \mathbf{P} : \quad f^* = \min_{\mathbf{x}} \{ f(\mathbf{x}) : \mathbf{A} \mathbf{x} \leq \mathbf{b}; \quad \mathbf{x} \in \{0, 1\}^n \},$$

for some cost vector  $\mathbf{c} \in \mathbb{Z}^n$ , some matrix  $\mathbf{A} \in \mathbb{Z}^{m \times n}$ , and some vector  $\mathbf{b} \in \mathbb{Z}^m$ . We may and will replace (2.13) with the equivalent pure integer program:

$$\mathbf{P}' : \quad f^* = \min_{\mathbf{x}, \mathbf{y}} \{ f(\mathbf{x}) : \mathbf{A} \mathbf{x} + \mathbf{y} = \mathbf{b}; \quad \mathbf{x} \in \{0, 1\}^n; \mathbf{y} \in \mathbb{N}^m \}.$$

Next, as  $\mathbf{x} \in \{0, 1\}^n$  we can bound each integer variable  $y_j$  by  $M_j := b_j - \min\{\mathbf{A}_j \mathbf{x} : \mathbf{x} \in \{0, 1\}^n\}$ ,  $j = 1, \dots, m$ , where  $\mathbf{A}_j$  denotes the  $j$ -th row vector of the matrix  $\mathbf{A}$ ; and in fact  $M_j = b_j - \sum_i \min[0, \mathbf{A}_{ji}]$ ,  $j = 1, \dots, m$ . Then we may use the standard decomposition of  $y_j$  into a weighted sum of boolean variables :

$$y_j = \sum_{k=0}^{s_j} 2^k z_{jk}, \quad z_{jk} \in \{0, 1\}^n, \quad j = 1, \dots, m,$$

(where  $s_j := \lceil \log(M_j) \rceil$ ) and replace (2.13) with the equivalent 0/1 program:

$$f^* = \min_{\mathbf{x}, \mathbf{z}} \{ f(\mathbf{x}) : \mathbf{A}_j^T \mathbf{x} + \sum_{k=0}^{s_j} 2^k z_{jk} = b_j, j \leq m; (\mathbf{x}, \mathbf{z}) \in \{0, 1\}^{n+s} \},$$

(where  $s := \sum_j (1 + s_j)$ ) which is of the form (1.1).

**2.4. Extension to polynomial programs.** Let  $f \in \mathbb{R}[\mathbf{x}]$  be a polynomial of even degree  $d > 2$ , written as

$$\mathbf{x} \mapsto f(\mathbf{x}) = \sum_{\alpha \in \mathbb{N}^n} f_\alpha \mathbf{x}^\alpha,$$

for some vector of coefficients  $\mathbf{f} = (f_\alpha) \in \mathbb{R}^{s(d)}$  (with  $s(d) = \binom{n+d}{n}$ ). Given a sequence  $\mathbf{y} = (y_\alpha)$ ,  $\alpha \in \mathbb{N}^n$ , define the Riesz functional  $L_{\mathbf{y}} : \mathbb{R}[\mathbf{x}] \rightarrow \mathbb{R}$  by:

$$f \mapsto L_{\mathbf{y}}(f) = \sum_{\alpha \in \mathbb{N}^n} f_\alpha y_\alpha, \quad f \in \mathbb{R}[\mathbf{x}].$$

Consider the polynomial program:

$$(2.14) \quad f^* = \min \{ f(\mathbf{x}) : \mathbf{A}\mathbf{x} = \mathbf{b}; \quad \mathbf{x} \in \{-1, 1\}^n \},$$

on the hyper cube  $\{-1, 1\}^n$ . Let  $d' := d/2$ ,  $\mathbf{x} \mapsto g_j(\mathbf{x}) := 1 - x_j^2$ ,  $j = 1, \dots, n$ , and with  $f$  let us associate the scalars:

$$\begin{aligned} r_f^1 &= \min \{ L_{\mathbf{y}}(f) : \mathbf{M}_{d'}(\mathbf{y}) \succeq 0; \mathbf{M}_{d'-1}(g_j \mathbf{y}) = 0, j = 1, \dots, n \} \\ r_f^1 &= \max \{ L_{\mathbf{y}}(f) : \mathbf{M}_{d'}(\mathbf{y}) \succeq 0; \mathbf{M}_{d'-1}(g_j \mathbf{y}) = 0, j = 1, \dots, n \} \end{aligned}$$

where  $\mathbf{M}_{d'}(\mathbf{y})$  (resp.  $\mathbf{M}_{d'-1}(g_j \mathbf{y})$ ) is the moment matrix (resp. localizing matrix) of order  $d'$  associated with the real sequence  $\mathbf{y} = (y_\alpha)$ ,  $\alpha \in \mathbb{N}^n$ , (resp. with the sequence  $\mathbf{y}$  and the polynomial  $g_j$ ). It turns out that  $r_f^1$  (resp.  $r_f^2$ ) is the optimal value of the first SDP-relaxation of the Lasserre-SOS hierarchy associated with the optimization problem  $\min$  (resp.  $\max$ )  $\{ f(\mathbf{x}) : \mathbf{x} \in \{-1, 1\}^n \}$  and so  $r_f^1 \leq \min \{ f(\mathbf{x}) : \mathbf{x} \in \{-1, 1\}^n \}$  whereas  $r_f^2 \geq \max \{ f(\mathbf{x}) : \mathbf{x} \in \{-1, 1\}^n \}$ . For more details, see e.g. [9, 10].

Next, if we define

$$(2.15) \quad \rho_f := \max_{i=1,2} |r_f^i|,$$

then it is straightforward to verify that  $\rho_f \geq \max \{ |f(\mathbf{x})| : \mathbf{x} \in \{-1, 1\}^n \}$ . Then we have the following analogue of Lemma 2.1:

**Lemma 2.4.** *Let  $f^*$  be as (2.14) and let  $\rho_f$  be as in (2.15). Then  $f^*$  is the optimal value of the polynomial minimization problem:*

$$(2.16) \quad \min_{\mathbf{x} \in \{-1, 1\}^n} f(\mathbf{x}) + (2\rho_f + 1) \cdot \|\mathbf{A}\mathbf{x} - \mathbf{b}\|^2.$$

The proof being almost a verbatim copy of that of Lemma 2.1, is omitted.

As for the quadratic case and with same arguments, one may also show that if  $d$  is even, the polynomial optimization problem (2.16) is equivalent



to minimizing the homogeneous polynomial  $\tilde{f}$  of degree  $d$  on the hypercube  $\{-1, 1\}^{n+1}$ , where

$$(\mathbf{x}, x_0) \mapsto \tilde{f}(\mathbf{x}) := x_0^d f(\mathbf{x}/x_0) + (2r_f + 1) x_0^{d-2} \cdot \|\mathbf{A} \mathbf{x} - x_0 \mathbf{b}\|^2.$$

But since it is not a MAX-CUT problem, to obtain a lower bound on  $f^*$  one may just as well consider solving the first level of the Lasserre-SOS hierarchy associated with (2.16) or even directly with (2.14). The advantage of using the formulation (2.16) is that one always minimizes on the hypercube  $\{-1, 1\}^n$  instead of minimizing on the subset  $\{-1, 1\}^n \cap \{\mathbf{x} : \mathbf{A} \mathbf{x} = \mathbf{b}\}$  of the hypercube which is problem dependent.

### 3. SOME NUMERICAL EXPERIMENTS

All semidefinite programs were solved by using the GloptiPoly software dedicated to solving the Generalized Problem of Moments and which implements the Lasserre-SOS hierarchy of semidefinite relaxations. For more details the interested reader is referred to Henrion et al. [6]. Typically, for a MAX-CUT problem of size  $n = 100$ , solving the Shor relaxation (with semidefinite solver MOSEK<sup>1</sup>) takes approximately 42s on a Macbook-pro lap-top with Intel Core i5 processor at 2.6 GHz.

**3.1. Some 0/1-knapsack examples.** To test the efficiency of the MAX-CUT formulation we have first considered 0/1-knapsack problems of small size (up to  $n = 15$  variables) and compared the first semidefinite relaxation (or Shor-relaxation) of the MAX-CUT formulation (2.4) with some other relaxations, and in particular with:

- The LP-relaxation of (2.1) (when  $\mathbf{F} = 0$ ) which consists of replacing the constraint  $\mathbf{x} \in \{-1, 1\}^n$  with  $\mathbf{x} \in [-1, 1]^n$ , and
- The first semidefinite relaxation of (2.1) in the Lasserre-SOS hierarchy.

**Example 3.1.** To evaluate the quality of the lower bound obtained with the MAX-CUT formulation consider the following simple linear knapsack-type examples:

$$(3.1) \quad \min \{ \mathbf{c}^T \mathbf{x} : \mathbf{a}^T \mathbf{x} = b; \mathbf{x} \in \{-1, 1\}^n \},$$

on  $\{-1, 1\}^n$ , with 4 and 10 variables. For  $n = 4$ ,  $\mathbf{c} = (13, 11, 7, 3)$  and  $\mathbf{a} = (3, 7, 11, 13)$ , while for  $n = 10$ ,  $\mathbf{c} = (37, 31, 29, 23, 19, 17, 13, 11, 7, 3)$ , and  $\mathbf{a} = (3, 7, 11, 13, 17, 19, 23, 29, 31, 37)$  and for  $n = 15$

$$\mathbf{a} = (3, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 51, 53)$$

and  $\mathbf{c} = (53, 51, 47, 43, 41, 37, 31, 29, 23, 19, 17, 13, 11, 7, 3)$ .

The right-hand-side  $b$  is taken into  $[-|\mathbf{a}|, |\mathbf{a}|] \cap \mathbb{Z}$ . Figure 1 displays the difference  $\min \mathbf{Q}_+ - \min \text{LP}$  where the lower bound  $\min \text{LP}$  is obtained by relaxing the integrality constraints  $\mathbf{x} \in \{-1, 1\}^n$  to the box constraint

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<sup>1</sup>MOSEK is a high performance semidefinite solver which can be called in the GloptiPoly software [6] for solving the resulting SDP-relaxations; more details on MOSEK can be found at <https://www.mosek.com/>

$\mathbf{x} \in [-1, 1]^n$  and solving the resulting LP. On the  $x$ -axis one reads  $b + |\mathbf{a}|$ . As expected the lower bound  $\min \mathbf{Q}_+$  is much better than  $\min \text{LP}$ . In fact the cases where the LP-bound is slightly better is for right-hand-side  $b$  such that the relaxation provides the optimal value  $f^*$ .

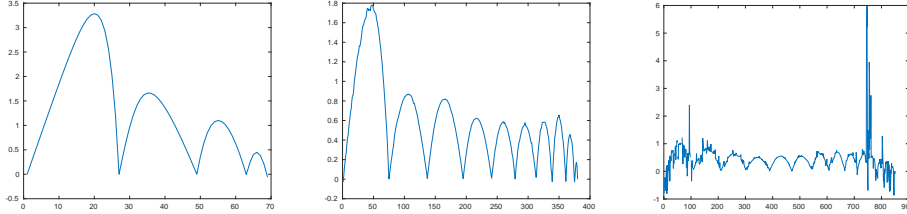


FIGURE 1. Difference  $\min \mathbf{Q}_+ - \min \text{LP}$  with  $n=(4,10,15)$ ; One reads  $b + |\mathbf{a}|$  on the  $x$ -axis

Moreover Figure 2 displays the difference  $\min \mathbf{Q}_+ - \min \hat{\mathbf{Q}}_+$  where  $\min \hat{\mathbf{Q}}_+$  is the optimal value of the first SDP-relaxation of the Lasserre-SOS hierarchy applied to the initial formulation (3.1) of the knapsack problem where one has even included the redundant constraints  $x_i (\mathbf{a}^T \mathbf{x} - b) = 0, i = 1, \dots, n$ . One observes that in most cases the lower bound  $\min \mathbf{Q}_+$  is slightly better than  $\min \hat{\mathbf{Q}}_+$ .

This is encouraging since the Lasserre-SOS hierarchy [7, 8] is known to produce good lower bounds in general, and especially at the first level of the hierarchy for MAX-CUT problems whose matrix  $\mathbf{Q}$  of the associated quadratic form has certain properties, e.g.,  $Q_{ij} \geq 0$  for all  $i, j$  or  $\mathbf{Q} \succeq 0$  (in the maximizing case); see e.g. Marshall [12].

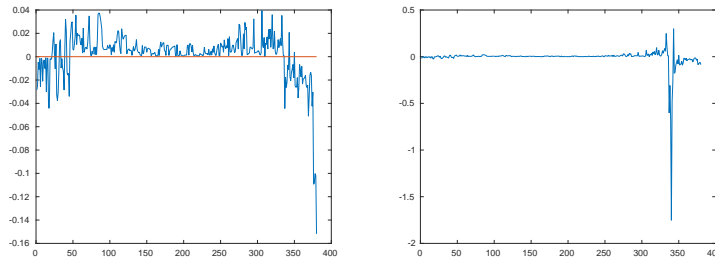


FIGURE 2. Difference  $\min \mathbf{Q}_+ - \min \hat{\mathbf{Q}}_+$  ( $n=10$ ) (left) and relative difference  $100 * (\min \mathbf{Q}_+ - \min \hat{\mathbf{Q}}_+) / \min \hat{\mathbf{Q}}_+$ ; One reads  $b + |\mathbf{a}|$  on the  $x$ -axis

**Example 3.2.** In a second sample of linear knapsack problems (3.1) with  $n = 10, 15$ , we have chosen the same vector  $\mathbf{a}$  as in Example 3.1 but now with a cost criterion of the form  $c(i) = a(i) + s\eta, i = 1, \dots, n$ , where  $\eta$  is a

random variable uniformly distributed on  $[0, 1]$ , and  $s = 20, 10$  is a weighting factor. The reason is that knapsack problems with ratios  $c(i)/a(i) \approx 1$  for all  $i$ , can be difficult to solve. As before the right-hand-side  $b$  is taken into  $[-|\mathbf{a}|, |\mathbf{a}|] \cap \mathbb{Z}$ . Figure 3 displays the results obtained for  $s = 20$ , and  $n = 10, 15$ . Figure 4 displays the same example for another sample with

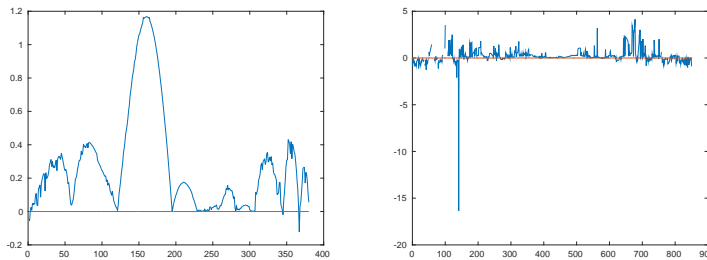


FIGURE 3. Difference  $\min \mathbf{Q}_+ - \min \text{LP}$ ,  $\mathbf{c} = \mathbf{a} + 20 * \eta$  ( $n=10,15$ );  $b + |\mathbf{a}|$  on the  $x$ -axis

cost  $\mathbf{c}$  and  $s = 10$ .

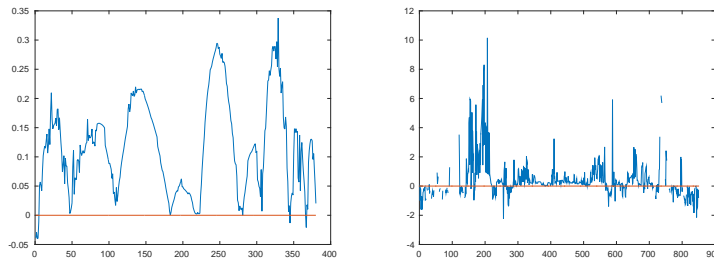


FIGURE 4. Difference  $\min \mathbf{Q}_+ - \min \text{LP}$ ,  $\mathbf{c} = \mathbf{a} + 10 * \eta$  ( $n=10,15$ );  $b + |\mathbf{a}|$  on the  $x$ -axis

Finally, as for Example 3.1, Figure 5 displays the difference  $\min \mathbf{Q}_+ - \min \hat{\mathbf{Q}}_+$  (where  $\min \hat{\mathbf{Q}}_+$  is the optimal value of the first SDP-relaxation of the Lasserre-SOS hierarchy applied to the initial formulation (3.1) of the knapsack problem where one has even included the redundant constraints  $x_i(\mathbf{a}^T \mathbf{x} - b) = 0, i = 1, \dots, n$ ). Again one observes that in most cases the lower bound  $\min \mathbf{Q}_+$  is slightly better than  $\min \hat{\mathbf{Q}}_+$ .

**Example 3.3.** We next consider the same knapsack problems (3.1) as in Example 3.2 but now with quadratic criterion  $\mathbf{c}^T \mathbf{x} + \mathbf{x}^T \mathbf{F} \mathbf{x}$ , again with a cost criterion of the form  $c(i) = a(i) + s\eta, i = 1, \dots, n$ , where  $\eta$  is a random variable uniformly distributed in  $[0, 1]$  and  $s$  is some weighting factor. The real symmetric matrix  $\mathbf{F}$  is also randomly generated and is not positive

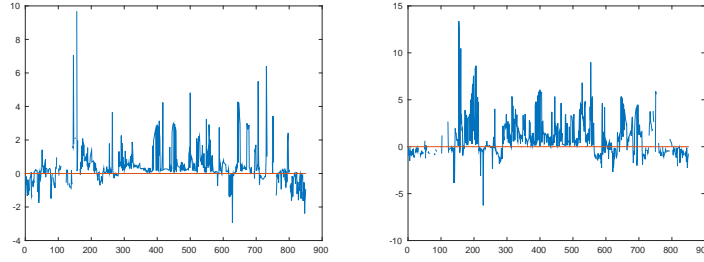


FIGURE 5. Example 3.2: Difference  $\min \mathbf{Q}_+ - \min \hat{\mathbf{Q}}_+$ , ( $n=15$ )  $\mathbf{c} = \mathbf{a} + 20\eta$  (left) and  $\mathbf{c} = \mathbf{a} + 10\eta; b + |\mathbf{a}|$  on the  $x$ -axis

definite in general. Again in Figures 6 and 7 one observes that the lower bound  $\min \mathbf{Q}_+$  is almost always better than the optimal value  $\min \hat{\mathbf{Q}}_+$  of the first level of the Lasserre-SOS hierarchy applied to the original formulation (3.1) of the problem.

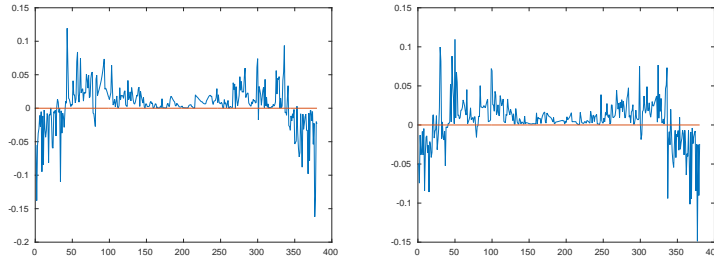


FIGURE 6. Example 3.3: Difference  $\min \mathbf{Q}_+ - \min \hat{\mathbf{Q}}_+$ , ( $n=10$ )  $\mathbf{c} = \mathbf{a} + 20\eta$  (left) and  $\mathbf{c} = \mathbf{a} + 10\eta; b + |\mathbf{a}|$  on the  $x$ -axis

**3.2. Larger size problems: The  $k$ -cluster problem.** As generating data  $(\mathbf{c}, \mathbf{a}, b)$  for difficult 0/1-knapsack problems of larger size is not obvious, we next consider the so-called  $k$ -cluster problem in  $n = 70$  and  $n = 80$  variables:

$$(3.2) \quad \min \{ \mathbf{x}^T \mathbf{A} \mathbf{x} : \mathbf{e}^T \mathbf{x} = k; \mathbf{x} \in \{0, 1\}^n \},$$

where  $k \in \mathbb{N}$  is a fixed parameter,  $\mathbf{e} = (1, 1, \dots, 1) \in \mathbb{R}^n$ ,  $\mathbf{A} \in \mathbb{R}^{n \times n}$  with  $\mathbf{A} = \mathbf{A}^T$  and  $\mathbf{A}_{ii} = 0$  for all  $i = 1, \dots, n$ . Indeed the constraint  $\mathbf{e}^T \mathbf{x} = k$  of the  $k$ -cluster problem is structural and does not depend on a vector  $\mathbf{a}$  as in knapsack examples. After the change of variables  $\mathbf{x} \rightarrow (\mathbf{x} + \mathbf{e})/2$  the  $k$ -cluster problem reads:

$$(3.3) \quad \mathbf{P} : \quad \min \{ 2\mathbf{e}^T \mathbf{A} \mathbf{x} + \mathbf{x}^T \mathbf{A} \mathbf{x} : \mathbf{e}^T \mathbf{x} = 2k - n; \mathbf{x} \in \{-1, 1\}^n \}.$$

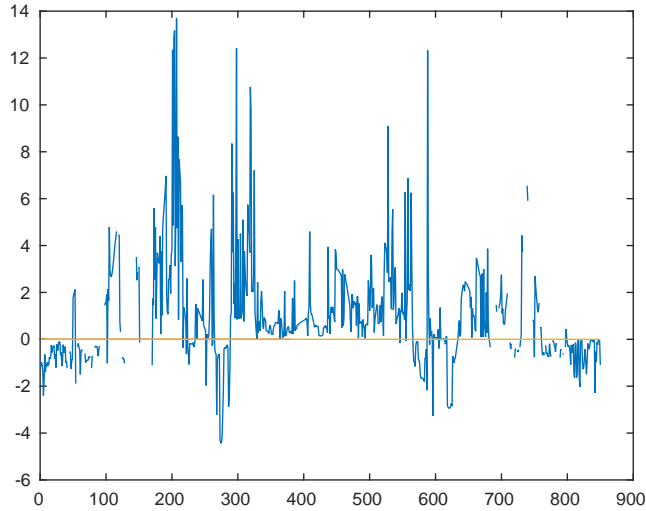


FIGURE 7. Example 3.3: Difference  $\min \mathbf{Q}_+ - \min \hat{\mathbf{Q}}_+$ , ( $n=15$ )  $\mathbf{c} = \mathbf{a} + 20\eta$  (left) and  $\mathbf{c} = \mathbf{a} + 10\eta$ ;  $b + |\mathbf{a}|$  on the  $x$ -axis

We have compared the optimal value  $f_{maxcut}^*$  for the Shor relaxation of the MAX-CUT formulation (2.4) of (3.3) with the optimal value  $f_{Shor}^*$  for the first semidefinite relaxation of (3.3) in the Lasserre-SOS hierarchy. As can be seen from results displayed in Table 1 for  $n = 70$  and in Table 2 for  $n = 80$ , both values are almost identical since the relative difference is no more than 0.014% and 0.06% respectively.

	$f_{maxcut}^*$	$f_{Shor}^*$	$100 * (f_{maxcut}^* - f_{Shor}^*) /  f_{Shor}^* $
$k = 1$	-2456.391	-2456.269	-0.005%
$k = 6$	-2442.986	-2442.992	0.0002%
$k = 11$	-2319.395	-2319.716	0.014%
$k = 16$	-2089.292	-2090.238	0.045%
$k = 21$	-1756.995	-1756.974	-0.001%
$k = 26$	-1320.461	-1320.437	-0.001%
$k = 31$	-779.733	-779.692	-0.005%

TABLE 1.  $k$ -cluster with  $n = 70$ ;  $f_{maxcut}^*$ ,  $f_{Shor}^*$  and the relative difference  $100 * (f_{maxcut}^* - f_{Shor}^*) / |f_{Shor}^*|$

**Detecting infeasibility.** Of course when  $k > n$  the  $k$ -cluster problem (3.3) has no feasible solution to  $\mathbf{P}$ . In view of the special structure of the  $k$ -cluster problem it is straightforward to check that the first semidefinite relaxation

	$f_{maxcut}^*$	$f_{Shor}^*$	$100 * (f_{Shor}^* - f_{maxcut}^*) /  f_{Shor}^* $
$k = 10$	-3122.913	-3124.925	0.06%
$k = 20$	-2613.463	-2613.748	0.01%
$k = 30$	-1684.377	-1684.419	0.002%
	$f_{maxcut}^*$	$f_{Shor}^*$	$100 * (f_{maxcut}^* - f_{Shor}^*) /  f_{Shor}^* $
$k = 70$	6373.92	6373.65	0.004%
$k = 60$	3675.92	3674.98	0.025%
$k = 50$	1445.74	1445.67	0.005%

TABLE 2.  $k$ -cluster with  $n = 80$ ;  $f_{maxcut}^*$ ,  $f_{Shor}^*$  and the relative difference  $100 * (f_{maxcut}^* - f_{Shor}^*) / |f_{Shor}^*|$  (with two different  $\mathbf{F}$ )

of (3.3) (or equivalently of (3.2)) in the Lasserre-SOS hierarchy is infeasible, hence certifying that  $f^* = +\infty$ . This is because in the first semidefinite relaxation of (3.2), the constraints are:

$$\mathbf{e}^T \mathbf{x} = k; \quad \left[ \begin{array}{c|c} 1 & \mathbf{x}^T \\ \hline - & - \\ \mathbf{x} & \mathbf{X} \end{array} \right] \succeq 0, \quad \mathbf{X}_{ii} = \mathbf{x}_i, \quad i = 1, \dots, n.$$

The semidefinite constraint implies  $0 \leq \mathbf{x} \leq 1$ , and therefore one cannot have  $\mathbf{e}^T \mathbf{x} = k$  if  $k > n$ .

On the other hand the Shor relaxation of (3.3) (and so the Shor relaxation of the MAX-CUT formulation of (3.3)) has always a finite value, but if  $k > n$  then the optimal value is larger than  $\rho(\mathbf{c}, \mathbf{F})$ , which by Proposition 2.3 also certifies that  $f^* = +\infty$ . Indeed in the Shor relaxation of (3.3) one has the constraint

$$(3.4) \quad \left[ \begin{array}{c|c} 1 & \mathbf{x}^T \\ \hline - & - \\ \mathbf{x} & \mathbf{X} \end{array} \right] \succeq 0, \quad \mathbf{X}_{ii} = 1, \quad \forall i = 1, \dots, n$$

and with  $b = 2k - n$  the criterion reads

$$2\mathbf{e}^T \mathbf{x} + \mathbf{x}^T \mathbf{A} \mathbf{x} + (2\rho(\mathbf{c}, \mathbf{F}) + 1) \underbrace{\begin{bmatrix} -b \\ \mathbf{e} \end{bmatrix}^T \begin{bmatrix} 1 & | & \mathbf{x}^T \\ \hline - & - & - \\ \mathbf{x} & | & \mathbf{X} \end{bmatrix} \begin{bmatrix} -b \\ \mathbf{e} \end{bmatrix}}_{(b - \mathbf{e}^T \mathbf{x})^2 + \mathbf{e}^T (\mathbf{X} - \mathbf{x} \mathbf{x}^T) \mathbf{e}}.$$

Therefore as  $\mathbf{X} \succeq \mathbf{x} \mathbf{x}^T$ ,  $(b - \mathbf{e}^T \mathbf{x})^2 + \mathbf{e}^T (\mathbf{X} - \mathbf{x} \mathbf{x}^T) \mathbf{e} \geq (b - \mathbf{e}^T \mathbf{x})^2$ , and with so if  $k > n$  then  $b = 2k - n > n$  which in turn implies  $(b - \mathbf{e}^T \mathbf{x}) > 1$  whenever (3.4) holds (as  $-1 \leq \mathbf{x} \leq 1$ ). Next, from the definition of  $\rho(\mathbf{c}, \mathbf{F})$  in (2.2) we have seen that  $|2\mathbf{e}^T \mathbf{x} + \mathbf{x}^T \mathbf{A} \mathbf{x}| \leq \rho(\mathbf{c}, \mathbf{F})$  whenever (3.4) holds. Hence

$$2\mathbf{e}^T \mathbf{x} + \mathbf{x}^T \mathbf{A} \mathbf{x} + (2\rho(\mathbf{c}, \mathbf{F}) + 1) ((b - \mathbf{e}^T \mathbf{x})^2 + \mathbf{e}^T (\mathbf{X} - \mathbf{x} \mathbf{x}^T) \mathbf{e}) > \rho(\mathbf{c}, \mathbf{F}),$$

whenever (3.4) holds. This is confirmed in the results displayed in Table 3 where again  $\mathbf{A}$  was randomly generated and  $k = n + i$ ,  $i = 1, 4, 7$  so that the  $k$ -cluster problem is infeasible.

n=70	$f_{maxcut}^*$	$f_{Shor}^*$
$k = n + 1$	22129.64 > $\rho(\mathbf{c}, \mathbf{F})$	infeas
$k = n + 4$	243431.21 > $\rho(\mathbf{c}, \mathbf{F})$	infeas
$k = n + 7$	730294.15 > $\rho(\mathbf{c}, \mathbf{F})$	infeas
n=80	$f_{maxcut}^*$	$f_{Shor}^*$
$k = n + 1$	28563.26 > $\rho(\mathbf{c}, \mathbf{F})$	infeas

TABLE 3. Infeasible  $k$ -cluster problems with  $n = 70, 80$

**Comparing with a quadratic convex relaxation.** Decompose  $\mathbf{A}$  as  $\mathbf{U}^T \Lambda \mathbf{U}$  with  $\mathbf{U}^T \mathbf{U} = \mathbf{I}$  (the identity matrix) and  $\Lambda$  is the diagonal matrix of eigenvalues of  $\mathbf{A}$ . Let  $\theta := |\lambda_{min}(\mathbf{A})| + 1$  where  $\lambda_{min}(\mathbf{A})$  is the smallest eigenvalue of  $\mathbf{A}$ . Replace  $\mathbf{A}$  with  $\tilde{\mathbf{A}} := \mathbf{U}^T (\Lambda + \theta \mathbf{I}) \mathbf{U}$  so that the resulting the quadratic form  $\mathbf{x} \mapsto \mathbf{x}^T \tilde{\mathbf{A}} \mathbf{x}$  is convex. Then observe that

$$\min \{ 2\mathbf{e}^T \mathbf{A} \mathbf{x} + \mathbf{x}^T \tilde{\mathbf{A}} \mathbf{x} : \mathbf{e}^T \mathbf{x} = 2k - n; \mathbf{x} \in \{-1, 1\}^n \} = f^* + n\theta$$

because  $\mathbf{x}^T \tilde{\mathbf{A}} \mathbf{x} = \mathbf{x}^T \mathbf{A} \mathbf{x} + \theta \|\mathbf{x}\|^2$  and  $x_i^2 = 1$  for all  $i = 1, \dots, n$ . Therefore one may compare the optimal value  $f_{maxcut}^*$  of the Shor relaxation of (3.3) with  $\tilde{\psi} = \psi - n\theta$ , where

$$\psi = \min \{ 2\mathbf{e}^T \mathbf{A} \mathbf{x} + \mathbf{x}^T \tilde{\mathbf{A}} \mathbf{x} : \mathbf{e}^T \mathbf{x} = 2k - n; x_i^2 \leq 1, i = 1, \dots, n \},$$

which is a convex quadratic program. Results displayed in Table 4 show that the Shor relaxation is always better (and sometimes significantly better) than the quadratic convex relaxation.

Next we do the same comparison for a  $k$ -cluster problem where some sparsity is introduced in the cost matrix  $\mathbf{A}$ . Namely once  $\mathbf{A}$  has been generated then for each non-diagonal entry  $\mathbf{A}_{ij}$  we decide  $\mathbf{A}_{ij} = \mathbf{A}_{ji} := 0$  with probability 0.4, 0.7 and 0.8. Results displayed in Table 5 for  $n = 70$  and in Table 6 for  $n = 100$  show that again the Shor relaxation is always better (and some times significantly better) than the quadratic convex relaxation. The sparser is  $\mathbf{A}$  the more efficient is the Shor relaxation compared with the quadratic convex relaxation.

n=70	k=50	k=45	k=40	k=35	k=30	k=25
$f_{maxcut}^*$	2324.68	1350.74	488.83	-261.78	-900.55	-1429.66
$\tilde{\psi}$	2260.00	1278.08	410.48	-343.95	-985.57	-1514.77
$\frac{100(f_{maxcut}^* - \tilde{\psi})}{\tilde{\psi}}$	2.86%	5.68%	19.08%	23.89%	8.62%	5.61%

TABLE 4.  $k$ -cluster: Shor relaxation ( $f_{maxcut}^*$ ) vs convex quadratic relaxation ( $\tilde{\psi} = \psi - n\theta$ ) with  $n = 70$

n=70	k=50	k=45	k=40	k=35	k=30	k=25
$\mathbf{A}_{ij} = 0$ with probability 0.4						
$f_{maxcut}^*$	1188.41	569.19	32.38	-422.36	-794.41	-1084.71
$\tilde{\psi}$	1129.49	505.55	-32.54	-485.26	-855.35	-1149.94
$\frac{100(f_{maxcut}^* - \tilde{\psi})}{\tilde{\psi}}$	5.2%	12.5%	199.5%	12.9%	7.1%	5.6%
$\mathbf{A}_{ij} = 0$ with probability 0.7						
$f_{maxcut}^*$	427.01	106.85	-156.52	-362.12	-519.68	-637.74
$\tilde{\psi}$	376.80	60.31	-203.72	-418.46	-588.56	-718.30
$\frac{100(f_{maxcut}^* - \tilde{\psi})}{\tilde{\psi}}$	13.3%	77.1%	23.1%	13.4%	11.7%	11.2%
$\mathbf{A}_{ij} = 0$ with probability 0.8						
$f_{maxcut}^*$	232.09	26.12	-143.13	-279.53	-384.83	-463.19
$\tilde{\psi}$	172.80	-43.89	-223.92	-370.54	-485.54	-569.36
$\frac{100(f_{maxcut}^* - \tilde{\psi})}{\tilde{\psi}}$	34.3%	159.5%	36.0%	24.5%	20.7%	18.6%

TABLE 5.  $k$ -cluster: Shor relaxation ( $f_{maxcut}^*$ ) vs convex quadratic relaxation ( $\tilde{\psi} = \psi - n\theta$ ) with  $n = 70$ .  $\mathbf{A}_{ij} = 0$  with probability  $p = 0.4, 0.7, 0.8$ .

**3.3. Comparing with the copositive formulation.** As already mentioned in the introduction, the 0/1 program (1.1) also has a copositive formulation. Namely, let  $e_i = (\delta_{i=j}) \in \mathbb{R}^n$ ,  $i = 1, \dots, n$ , and  $\mathbf{e} = (1, \dots, 1) \in \mathbb{R}^n$ . Following Burer [3, p. 481–482], introduce  $n$  additional variables  $\mathbf{z} = (z_1, \dots, z_n)$  and the  $n$  additional equality constraints  $x_i + z_i = 1$ ,  $i = 1, \dots, n$ , with  $\mathbf{z} \geq 0$  (which are necessary to obtain an equivalent formulation). So let  $\tilde{\mathbf{x}}^T = (\mathbf{x}^T, \mathbf{z}^T) \in \mathbb{R}^{2n}$  and with  $\mathbf{I} \in \mathbb{R}^{n \times n}$  being the identity matrix, introduce the real matrices

$$\tilde{\mathbf{F}} := \begin{bmatrix} \mathbf{F} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \mathbf{S} := \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{I} & \mathbf{I} \end{bmatrix}$$

and the real vectors  $\tilde{\mathbf{c}}^T := (\mathbf{c}^T, \mathbf{0}) \in \mathbb{R}^{2n}$  and  $\tilde{\mathbf{b}}^T = (\mathbf{b}^T, \mathbf{e}^T) \in \mathbb{R}^{m+n}$ . Let  $\mathbf{S}_i$  denote the  $i$ -th row vector of  $\mathbf{S}$ ,  $i = 1, \dots, 2n$ . Then the copositive



n=100	k=70	k=60	k=50	k=40	k=20
$\mathbf{A}_{ij} = 0$ with probability 0.8					
$f_{maxcut}^*$	513.32	-60.83	-450.91	-781.84	-1084.98
$\tilde{\psi}$	419.07	-147.77	-572.27	-894.72	-1188.59
$\frac{100(f_{maxcut}^* - \tilde{\psi})}{\tilde{\psi}}$	22.4%	58.8%	21.2%	12.6%	8.7%
$\mathbf{A}_{ij} = 0$ with probability 0.6					
$f_{maxcut}^*$	1278.01	256.43	-627.72	-1248.39	-1988.94
$\tilde{\psi}$	1192.99	155.44	-723.34	-1372.94	-2095.66
$\frac{100(f_{maxcut}^* - \tilde{\psi})}{\tilde{\psi}}$	7.1%	64.9%	13.2%	9.0%	5.0%

TABLE 6.  $k$ -cluster: Shor relaxation ( $f_{maxcut}^*$ ) vs convex quadratic relaxation ( $\tilde{\psi} = \psi - n\theta$ ) with  $n = 100$ .  $\mathbf{A}_{ij} = 0$  with probability  $p = 0.8, 0.6$ .

formulation of (1.1) reads:

$$\begin{aligned}
(3.5) \quad f^* = \min \quad & \tilde{\mathbf{c}}^T \tilde{\mathbf{x}} + \langle \tilde{\mathbf{F}}, \mathbf{X} \rangle \\
\text{s.t.} \quad & \mathbf{S}_i \tilde{\mathbf{x}} = \tilde{\mathbf{b}}_i, \quad i = 1, \dots, m+n \\
& \mathbf{S}_i \mathbf{X} \mathbf{S}_i^T = \tilde{\mathbf{b}}_i^2, \quad i = 1, \dots, m+n \\
& \mathbf{X}_{ii} = \tilde{\mathbf{x}}_i, \quad i = 1, \dots, 2n \\
& \begin{bmatrix} 1 & \tilde{\mathbf{x}}^T \\ \tilde{\mathbf{x}} & \mathbf{X} \end{bmatrix} \in \mathcal{C}_{2n+1}^*,
\end{aligned}$$

where  $\mathcal{C}_{2n+1}$  is the convex cone of  $(2n+1) \times (2n+1)$  copositive matrices and  $\mathcal{C}_{2n+1}^*$  is its dual, i.e., the convex cone of *completely positive matrices*.

The hard constraint being membership in  $\mathcal{C}_{2n+1}^*$ , a strategy is to use hierarchies of tractable approximations (of increasing size) of  $\mathcal{C}_{2n+1}^*$ , as described in e.g. Dür [5]. In particular a possible choice for the first relaxation in such hierarchies is to replace (3.5) with the semidefinite program:

$$\begin{aligned}
(3.6) \quad f_{copo}^* = \min \quad & \tilde{\mathbf{c}}^T \tilde{\mathbf{x}} + \langle \tilde{\mathbf{F}}, \mathbf{X} \rangle \\
\text{s.t.} \quad & \mathbf{S}_i \tilde{\mathbf{x}} = \tilde{\mathbf{b}}_i, \quad i = 1, \dots, m+n \\
& \mathbf{S}_i \mathbf{X} \mathbf{S}_i^T = \tilde{\mathbf{b}}_i^2, \quad i = 1, \dots, m+n \\
& \mathbf{X}_{ii} = \tilde{\mathbf{x}}_i, \quad i = 1, \dots, 2n \\
& \begin{bmatrix} 1 & \tilde{\mathbf{x}}^T \\ \tilde{\mathbf{x}} & \mathbf{X} \end{bmatrix} \in \mathcal{S}_{2n+1}^+ \cap \mathcal{N}_{2n+1},
\end{aligned}$$

where  $\mathcal{S}_{2n+1}^+$  (resp.  $\mathcal{N}_{2n+1}$ ) is the convex cone of real symmetric positive semidefinite (resp. entrywise nonnegative) matrices. Then (3.6) is a semidefinite relaxation of (3.5) because  $\mathcal{C}_{2n+1}^* \subset \mathcal{S}_{2n+1}^+ \cap \mathcal{N}_{2n+1}$ , and so  $f_{copo}^* \leq f^*$ . In fact if one considers the problem

$$(3.7) \quad \min_{\mathbf{x} \in \{0,1\}^n} \{ \mathbf{c}^T + \mathbf{x}^T \mathbf{F} \mathbf{x} : \mathbf{A} \mathbf{x} = \mathbf{b}; (\mathbf{A}_i^T \mathbf{x})^2 = b_i^2, i = 1, \dots, m \},$$

which is clearly equivalent to (1.1), then the first SDP-relaxation of the Lasserre-SOS hierarchy associated with (3.7) reads:

$$\begin{aligned}
(3.8) \quad & \min \quad \mathbf{c}^T \mathbf{x} + \langle \mathbf{F}, \mathbf{X} \rangle \\
& \text{s.t.} \quad \mathbf{A} \mathbf{x} = \mathbf{b} \\
& \quad \mathbf{A}_i \mathbf{X} \mathbf{A}_i^T = \mathbf{b}_i^2, \quad i = 1, \dots, n \\
& \quad \mathbf{X}_{ii} = x_i, \quad i = 1, \dots, n \\
& \quad \begin{bmatrix} 1 & \mathbf{x}^T \\ \mathbf{x} & \mathbf{X} \end{bmatrix} \succeq 0,
\end{aligned}$$

which is of the same flavor as the semidefinite program (3.6) but with dimension twice as less than (3.6).

**Example 3.4.** We first compare the first SDP-relaxation of the MAX-CUT formulation with the first SDP-relaxation (3.6) of the copositive formulation for the linear 0/1 knapsack problems (1.1) with  $\mathbf{F} = 0$ ,  $n = 10, 15$  and vector  $\mathbf{a}$  as in Example 3.2.

We have kept the formulation on the hypercube  $\{0, 1\}^n$  rather than on the hypercube  $\{-1, 1\}^n$  and so in fact the first SDP relaxation is for problem (2.4) with a quadratic cost function (and not a quadratic form as in the MAX-CUT formulation on  $\{-1, 1\}^n$ ).

In each case  $n = 10$  (resp.  $n = 15$ ), we have chosen 19 (resp. 18) values of the right-hand-side  $b = 10s$ ,  $s = 1, \dots, 19$  (resp.  $b = 20s$ ,  $s = 1, \dots, 18$ ), and for each problem we have run a sample of 10 problems with cost vector  $\mathbf{c} = \mathbf{a} + 10\eta$  where  $\eta$  is a random variable uniformly distributed in  $[0, 1]$ .

For  $n = 10$ , the lower bound  $f_{maxcut}^*$  from the MAX-CUT formulation dominates the lower bound  $f_{copo}^*$  in (3.6), in 111 out of 190 problems ( $\approx 58\%$ ) and the relative difference  $100 \cdot |f_{maxcut}^* - f_{copo}^*| / \max[f_{maxcut}^*, f_{copo}^*]$  never exceeds 0.05% over all 190 problems!

For  $n = 15$ ,  $f_{maxcut}^* > f_{copo}^*$  in 94 out of 180 problems ( $\approx 52\%$ ) and  $100 \cdot |f_{maxcut}^* - f_{copo}^*| / \max[f_{maxcut}^*, f_{copo}^*]$  never exceeds 0.55% in all 180 problems!

**Example 3.5.** Next we consider the  $k$ -cluster problem (3.2) with  $n = 50$  variables and where  $\mathbf{F}$  is randomly generated with 80% of zero entries in  $\mathbf{F}$ . Then we compare the optimal values:

- $f_{Shor}^*$  of the Shor- (or first SDP-relaxation of the Lasserre-SOS hierarchy) associated with (3.2).
- $f_{maxcut}^*$  of the Shor-relaxation of the MAX-CUT formulation (2.4).
- $f_{copo}^*$  of the Shor-relaxation (3.6) associated with the copositive formulation of (3.2).

Results displayed in Table 7 show that  $f_{maxcut}^*$  almost identical to the optimal value  $f_{copo}^*$  of Shor relaxation associated with the copositive formulation. However notice that the size of the semidefinite constraint in the latter is twice as large as the one in the former.

n=50	$f_{Shor}^*$	$f_{maxcut}^*$	$f_{copo}^*$	$100(f_{maxcut}^* - f_{copo}^*)/ f_{copo}^* $
k=12	-3.3836	-3.3791	-10.8609	68.8%
k= 14	-0.3708	-0.3674	-0.6658	44.8%
k=16	3.3725	3.3733	3.3725	0.02%
k =18	7.8361	7.8421	7.8361	0.07%
k=20	13.2627	13.2769	13.2627	0.10%
k=22	19.8905	19.8926	19.8905	0.01%
k=24	27.6835	27.6837	27.6835	0.0%
k= 26	36.5762	36.6012	36.5762	0.06%
k=28	46.6564	46.6630	46.6564	0.01%
k=30	57.9790	57.9762	57.9790	-0.00%

TABLE 7.  $k$ -cluster: Shor-relaxations of MAX-CUT ( $f_{maxcut}^*$ ) and copositive ( $f_{copo}^*$ ) formulations with  $n = 50$  and 80% sparsity.

#### 4. CONCLUSION

In this paper we have shown that a linear or quadratic 0/1 program  $\mathbf{P}$  has an equivalent MAX-CUT formulation and so the whole arsenal of approximation techniques for the latter can be applied. In particular, and as suggested by some preliminary tests on a (limited sample) of 0/1 knapsack and  $k$ -cluster examples, it is expected that the lower bound obtained from the Shor relaxation of this MAX-CUT formulation will be in general better than the one obtained from the standard LP-relaxation (for linear 0/1 programs) of the original problem. The situation might be even better for quadratic 0/1 programs when the Shor relaxation of the MAX-CUT formulation is compared with a convex quadratic relaxation.

#### REFERENCES

- [1] I.M. Bomze. Copositive optimization - recent developments and applications, *European J. Oper. Res.* **216** (2012), 509–520.
- [2] I.M. Bomze, E. de Klerk. Solving standard quadratic optimization problems via linear, semidefinite and copositive programming, *J. Global Optim.* **24** (2002), 163–185.
- [3] S. Burer. Copositive programming, in *Handbook of Semidefinite, Conic and Polynomial Optimization*, M. Anjos and J.B. Lasserre, Eds., Springer, New York, 2012, pp. 201–218.
- [4] E. de Klerk, D.V. Pasechnik. Approximation of the stability number of a graph via copositive programming, *SIAM J. Optim.* **12** (2002), 875–892.
- [5] M. Dür. Copositive Programming - a survey, In *Recent Advances in Optimization and its Applications in Engineering*, M. Diehl, F. Glineur, E. Jarlebring and W. Michiels Eds., Springer, New York, 2010, pp. 3 – 20.
- [6] D. Henrion, J.B. Lasserre, J. Lofberg. GloptiPoly 3: moments, optimization and semidefinite programming, *Optim. Methods & Softwares* **24** (2009), pp. 761–779.
- [7] J.B. Lasserre. Optimisation globale et théorie des moments, *C. R. Acad. Sci. Paris* **331** (2000), Série 1, pp. 929–934.

- [8] J.B. Lasserre. Global optimization with polynomials and the problem of moments, *SIAM J. Optim.* **11** (2001), pp. 796–817.
- [9] J.B. Lasserre. *Moments, Positive Polynomials and Their Applications*, Imperial College Press, London, 2010.
- [10] J.B. Lasserre. *An Introduction to Polynomial and Semi-Algebraic Optimization*, Cambridge University Press, Cambridge, 2015.
- [11] M. Laurent, Zhao Sun. Handelman’s hierarchy for the maximum stable set problem, *J. Global Optim.* **60** (2014), pp. 393–423.
- [12] M. Marshall. Error estimates in the optimization of degree two polynomials on a discrete hypercube, *SIAM J. Optim.* **16** (2005), 297–309.
- [13] Y. Nesterov. Semidefinite relaxation and nonconvex quadratic optimization, *Optim. Methods Softw.* **9** (1998), 141–160.
- [14] L. Palagi, V. Piccialli, F. Rendl, G. Rinaldi, A. Wiegele. Computational approaches to Max-Cut, in *Handbook of Semidefinite, Conic and Polynomial Optimization*, M. Anjos and J.B. Lasserre, Eds., Springer, New York, 2012, pp. 821–847.
- [15] F. Rendl. Semidefinite relaxations for partitioning, assignment and ordering problems, *4OR* **10**, pp. 321–346, 2012.

LAAS-CNRS AND INSTITUTE OF MATHEMATICS, LAAS, 7 AVENUE DU COLONEL ROCHE, BP 54200, 31031 TOULOUSE CÉDEX 4, FRANCE.  
*E-mail address:* `lasserre@laas.fr`