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► **To cite this version:**

Jean-Bernard Lasserre. A MAX-CUT FORMULATION OF 0/1 PROGRAMS. 2015. hal-01154698v2

HAL Id: hal-01154698

<https://hal.science/hal-01154698v2>

Preprint submitted on 24 Aug 2015 (v2), last revised 22 Dec 2015 (v3)

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A MAX-CUT FORMULATION OF 0/1 PROGRAMS

JEAN B. LASSERRE

ABSTRACT. We consider the linear or quadratic 0/1 program

$$\mathbf{P} : f^* = \min\{\mathbf{c}^T \mathbf{x} + \mathbf{x}^T \mathbf{F} \mathbf{x} : \mathbf{A} \mathbf{x} = \mathbf{b}; \mathbf{x} \in \{0, 1\}^n\},$$

for some vectors $\mathbf{c} \in \mathbb{R}^n$, $\mathbf{b} \in \mathbb{Z}^m$, some matrix $\mathbf{A} \in \mathbb{Z}^{m \times n}$ and some real symmetric matrix $\mathbf{F} \in \mathbb{R}^{n \times n}$. We show that \mathbf{P} can be formulated as a MAX-CUT problem whose quadratic form criterion is explicit from the data of \mathbf{P} . In particular, to \mathbf{P} one may associate a graph whose connectivity is related to the connectivity of the matrix \mathbf{F} and $\mathbf{A}^T \mathbf{A}$, and \mathbf{P} reduces to finding a maximum (weighted) cut in such a graph. Hence the whole arsenal of approximation techniques for MAX-CUT can be applied. We also briefly mention some extensions. On a sample of 0/1 knapsack and k -cluster problems (with up to 100 variables) we compare the lower bound of the resulting semidefinite (or Shor) relaxation with that of the standard LP-relaxation, the first semidefinite relaxation of the Lasserre hierarchy associated with \mathbf{P} and the semidefinite relaxation of the copositive formulation of \mathbf{P} .

1. INTRODUCTION

Consider the linear or quadratic 0/1 program \mathbf{P} defined by:

$$(1.1) \quad \mathbf{P} : f^* = \min_{\mathbf{x}} \{ \mathbf{c}^T \mathbf{x} + \mathbf{x}^T \mathbf{F} \mathbf{x} : \mathbf{A} \mathbf{x} = \mathbf{b}; \mathbf{x} \in \{0, 1\}^n \}$$

for some cost vector $\mathbf{c} \in \mathbb{R}^n$, some matrix $\mathbf{A} \in \mathbb{Z}^{m \times n}$, some vector $\mathbf{b} \in \mathbb{Z}^m$, and some real symmetric matrix $\mathbf{F} \in \mathbb{R}^{n \times n}$. If $\mathbf{F} = 0$ then \mathbf{P} is a 0/1 linear program and a quadratic 0/1 program otherwise. Obtaining good quality lower bounds on f^* is highly desirable since the efficiency of Branch & Bound algorithms to solve large scale problems \mathbf{P} heavily depends on the quality of bounds of this form computed at nodes of the search tree.

To obtain lower bounds for 0/1 programs (1.1) one may solve a relaxation of \mathbf{P} where the integrality constraints $\mathbf{x} \in \{0, 1\}^n$ are replaced with the box constraints $\mathbf{x} \in [0, 1]^n$. If $\mathbf{F} = 0$ the resulting relaxation is linear whereas if \mathbf{F} is positive definite it is a (convex) quadratic program. If \mathbf{F} is not positive semidefinite then one may also solve a convex quadratic program but now with an appropriate convex quadratic underestimator $\mathbf{x}^T \tilde{\mathbf{F}} \mathbf{x}$ of $\mathbf{x}^T \mathbf{F} \mathbf{x}$ on $[0, 1]^n$. An alternative is to consider an equivalent formulation of \mathbf{P}

Date: August 24, 2015.

Research partially supported by a grant from the MONGE program of the *Fédération Mathématique Jacques Hadamard* (FMJH, Paris) and an ERC-ADG Grant (TAMING 666981) of the European Research Council (ERC).

as a copositive conic program as advocated by Burer [3] and compute a sequence of lower bounds by solving an appropriate hierarchy of LP- or SDP-relaxations associated with the copositive cone (or its dual). For more details on the latter approach the interested reader is referred to e.g. De Klerk and Pasechnik [4], Dür [5], Bomze [1], and Bomze and de Klerk [2].

Contribution. The purpose of this note is to show that solving \mathbf{P} is equivalent to minimizing a quadratic form in $n+1$ variables on the hypercube $\{-1, 1\}^{n+1}$ (and the quadratic form is explicit from the data of \mathbf{P}). In other words \mathbf{P} can be viewed as an explicit instance of the MAX-CUT problem. This idea had been already briefly mentioned in Rendl [15] in the context of partitioning an ordering problems. Hence the MAX-CUT problem which at first glance seems to be a very specific combinatorial optimization problem, in fact can be considered as a canonical model of linear and quadratic 0/1 programs. In particular, to each linear or quadratic 0/1 program (1.1) one may associate a graph $G = (V, E)$ with $n+1$ nodes and $(i, j) \in E$ whenever a product $x_i x_j$ has a nonzero coefficient in some quadratic form built upon the data $\mathbf{c}, \mathbf{b}, \mathbf{F}$ and \mathbf{A} of (1.1). (Among other things, the sparsity of G is related to the sparsity of the matrices \mathbf{F} and $\mathbf{A}^T \mathbf{A}$.) Then solving (1.1) reduces to finding a maximum (weighted) cut of G .

Therefore the whole specialized arsenal of approximation techniques for MAX-CUT can be applied. In particular one may obtain a lower bound f_1^* on f^* by solving the standard (Shor) SDP-relaxation associated with the resulting MAX-CUT problem while solving higher levels of the associated Lasserre-SOS hierarchy [7, 8] would provide a monotone nondecreasing sequence of improved lower bounds $f_1^* \leq f_d^* \leq f^*$, $d = 2, \dots$, but of course at a higher computational cost. Alternatively one may also apply the Handelman hierarchy of LP-relaxations as described and analyzed in Laurent and Sun [11]. For more details on recent developments on computational approaches to MAX-CUT the interested reader is referred to Palagi et al. [14].

In addition one may also obtain performance guarantees *à la Nesterov* [13] in the form

$$f_1^* \leq f^* \leq \frac{2}{\pi} f_1^* + \left(1 - \frac{2}{\pi}\right) h_1^*,$$

(where h_1^* is the optimal value of a similar SDP but with a max-criterion instead of a min-criterion) or their improvements by Marshall [12].

Finally the same methodology also works for general 0/1 optimization problems with feasible set as in (1.1) and polynomial criterion $f \in \mathbb{R}[\mathbf{x}]$ of degree $d > 2$, except that now the problem reduces to minimizing a new polynomial criterion $\tilde{f}(\mathbf{x})$ on the hypercube $\{-1, 1\}^n$ (and not a MAX-CUT problem any more as the degree of \tilde{f} is larger than 2).

Numerical experiments. If $\mathbf{F} = 0$ (i.e. when \mathbf{P} is a linear 0/1 program) the lower bound f_1^* can be better than the standard LP-relaxation which consists in replacing the integrality constraints $\mathbf{x} \in \{0, 1\}^n$ with the box

$[0, 1]^n$, as shown on a (limited) sample of 0/1-knapsack-type examples. On such examples f_1^* also dominates the one obtained from the first relaxation of the copositive formulation (where the dual cone \mathcal{C}^* of completely positive matrices is replaced with $\mathcal{S}^+ \cap \mathcal{N} \supset \mathcal{C}^*$) in about 55% of cases and the maximum relative difference is bounded by 0.55% in all cases.

In fact, and still on the same sample of linear and quadratic 0/1 knapsack examples, one also observes that the resulting lower bound f_1^* is almost always better than the lower bound obtained by solving the first SDP-relaxation of the Lasserre-SOS hierarchy applied to the initial formulation (1.1) of the problem (which is also an SDP of same size). This is good news since typically the SOS-hierarchy is known to produce good lower bounds for general polynomial optimization problems (discrete or not) even at the first level of the hierarchy. Even more, the first level SDP-relaxation has the celebrated Goemans & Williamson performance guarantee ($\approx 87\%$) when the matrix \mathbf{Q} (associated with the quadratic form) has nonnegative entries and a performance guarantee $\approx 64\%$ when $\mathbf{Q} \succeq 0$. (However note that the matrix \mathbf{Q} associated with our MAX-CUT problem equivalent to the initial 0/1 program (1.1) does not have all its entries nonnegative.) This explains why in the linear 0/1 knapsack examples the lower bound f_1^* is almost always better than the one obtained with the standard LP-relaxation and why for quadratic 0/1 knapsack problems (1.1), f_1^* is also likely to be better than the lower bound obtained by relaxing $\{0, 1\}^n$ to $[0, 1]^n$, replacing \mathbf{F} with a convex quadratic underestimator of \mathbf{F} on $[0, 1]^n$, and solving the resulting convex quadratic program.

Finally we have also considered the MAX-CUT formulation to the k -cluster problem $\rho = \min\{\mathbf{x}^T \mathbf{A} \mathbf{x} : \mathbf{e}^T \mathbf{x} = k; \mathbf{x} \in \{0, 1\}^n\}$ where $k \leq n$ is a fixed integer, \mathbf{A} is some real symmetric matrix, for $n = 70$ and $n = 100$ variables and with various percentages of zero entries in the matrix \mathbf{A} . In all examples for different values of k (and where the entries of \mathbf{A} were randomly generated following the uniform distribution on $[0, 1]$), the optimal value of the Shor relaxation was almost indistinguishable of the first semidefinite relaxation of the Lasserre-SOS hierarchy applied to the initial formulation.

Note that if $\mathbf{U}^T \Lambda \mathbf{U}$ is the Schur decomposition of \mathbf{A} then $\rho = -k\theta + \min\{\mathbf{x}^T \tilde{\mathbf{A}} \mathbf{x} : \mathbf{e}^T \mathbf{x} = k; \mathbf{x} \in \{0, 1\}^n\}$ (where $\theta = 1 + \lambda_{\min}(\mathbf{A})$) and $\tilde{\mathbf{A}} = \mathbf{U}^T (\Lambda + \theta \mathbf{I}) \mathbf{U} \succeq 0$ so that $\mathbf{x} \mapsto \mathbf{x}^T \tilde{\mathbf{A}} \mathbf{x}$ is a convex quadratic form. Therefore we have also compared the Shor relaxation with the quadratic convex relaxation $-k\theta + \min\{\mathbf{x}^T \tilde{\mathbf{A}} \mathbf{x} : \mathbf{e}^T \mathbf{x} = k; \mathbf{x} \in [0, 1]^n\}$. In all cases the Shor relaxation provides better lower bounds (and depending on k , sometimes really significantly better). Finally we have also compared the Shor relaxations associated with the MAX-CUT and copositive formulations. Again the respective optimal values are very close (less than 1% relative difference and in a few cases the former is significantly better).

2. MAIN RESULT

Denote by \mathbb{Z} the set of integer numbers and $\mathbb{N} \subset \mathbb{Z}$ the set of natural numbers. Let \mathbf{P} be the 0/1 program defined in (1.1) with $\mathbf{F}^T = \mathbf{F} \in \mathbb{R}^{n \times n}$, $\mathbf{A} \in \mathbb{Z}^{m \times n}$, $\mathbf{c} \in \mathbb{R}^n$ and $\mathbf{b} \in \mathbb{Z}^m$. Let $|\mathbf{c}| := (|c_i|) \in \mathbb{R}_+^n$. With $\mathbf{e} \in \mathbb{Z}^n$ being the vector of all ones, notice first that \mathbf{P} has an equivalent formulation on the hypercube $\{-1, 1\}^n$, by the change of variables $\tilde{\mathbf{x}} := 2\mathbf{x} - \mathbf{e}$. Indeed, \mathbf{A} , \mathbf{b} , \mathbf{c} and \mathbf{F} now become $\mathbf{A}/2$, $\mathbf{b} - \mathbf{A}\mathbf{e}/2$, $(\mathbf{c} + \mathbf{e}^T\mathbf{F})/2$ and $\mathbf{F}/4$ respectively.

Therefore from now on we consider the discrete program:

$$(2.1) \quad \mathbf{P} : \quad f^* = \min_{\mathbf{x} \in \{-1, 1\}^n} \{ \mathbf{c}^T \mathbf{x} + \mathbf{x}^T \mathbf{F} \mathbf{x} : \mathbf{A} \mathbf{x} = \mathbf{b} \},$$

on the hypercube $\{-1, 1\}^n$, with $\mathbf{A} \in \mathbb{Z}^{m \times n}$, $\mathbf{c} \in \mathbb{R}^n$, $\mathbf{b} \in \mathbb{Z}^m$, and $\mathbf{F}^T = \mathbf{F} \in \mathbb{R}^{n \times n}$. With \mathbf{c} and \mathbf{F} , let us associate the scalars:

$$\begin{aligned} r_{\mathbf{c}, \mathbf{F}}^1 &= \min \{ \mathbf{c}^T \mathbf{x} + \langle \mathbf{X}, \mathbf{F} \rangle : \begin{bmatrix} 1 & \mathbf{x}^T \\ \mathbf{x} & \mathbf{X} \end{bmatrix} \succeq 0; X_{ii} = 1, i = 1, \dots, n \} \\ r_{\mathbf{c}, \mathbf{F}}^2 &= \max \{ \mathbf{c}^T \mathbf{x} + \langle \mathbf{X}, \mathbf{F} \rangle : \begin{bmatrix} 1 & \mathbf{x}^T \\ \mathbf{x} & \mathbf{X} \end{bmatrix} \succeq 0; X_{ii} = 1, i = 1, \dots, n \} \end{aligned}$$

(with $\mathbf{X}^T = \mathbf{X}$) and let

$$(2.2) \quad \rho(\mathbf{c}, \mathbf{F}) := \max_{i=1,2} |r_{\mathbf{c}, \mathbf{F}}^i|.$$

It is straightforward to verify that

$$(2.3) \quad \rho(\mathbf{c}, \mathbf{F}) \geq \max \{ |\mathbf{c}^T \mathbf{x} + \mathbf{x}^T \mathbf{F} \mathbf{x}| : \mathbf{x} \in \{-1, 1\}^n \},$$

and $\rho(\mathbf{c}, \mathbf{F}) = |\mathbf{c}|$ if $\mathbf{F} = 0$. Moreover each scalar $r_{\mathbf{c}, \mathbf{F}}^i$ can be computed by solving an SDP which is the Shor relaxation (or first level of the Lasserre-SOS hierarchy [7, 8]) associated with the problems $\min (\max) \{ \mathbf{c}^T \mathbf{x} + \mathbf{x}^T \mathbf{F} \mathbf{x} : \mathbf{x} \in \{-1, 1\}^n \}$.

2.1. A MAX-CUT formulation of \mathbf{P} .

Lemma 2.1. *Let \mathbf{P} be as (2.1) and let $\rho(\mathbf{c}, \mathbf{F})$ be as in (2.2). Then f^* is the optimal value of the quadratic minimization problem:*

$$(2.4) \quad \min_{\mathbf{x} \in \{-1, 1\}^n} \mathbf{c}^T \mathbf{x} + \mathbf{x}^T \mathbf{F} \mathbf{x} + (2\rho(\mathbf{c}, \mathbf{F}) + 1) \cdot \|\mathbf{A} \mathbf{x} - \mathbf{b}\|^2.$$

Proof. Let $\Delta := \{ \mathbf{x} \in \{-1, 1\}^n : \mathbf{A} \mathbf{x} = \mathbf{b} \}$ be the feasible set of \mathbf{P} defined in (2.1), and let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be the function

$$(2.5) \quad x \mapsto f(\mathbf{x}) := \mathbf{c}^T \mathbf{x} + \mathbf{x}^T \mathbf{F} \mathbf{x} + (2\rho(\mathbf{c}, \mathbf{F}) + 1) \cdot \|\mathbf{A} \mathbf{x} - \mathbf{b}\|^2.$$

On $\{-1, 1\}^n$ one has $\max \{ \mathbf{c}^T \mathbf{x} + \mathbf{x}^T \mathbf{F} \mathbf{x} : \mathbf{x} \in \{-1, 1\}^n \} \leq \rho(\mathbf{c}, \mathbf{F})$, and

$$\|\mathbf{A} \mathbf{x} - \mathbf{b}\|^2 \geq 1, \quad \forall \mathbf{x} \in \{-1, 1\}^n \setminus \Delta,$$

because $\mathbf{A} \in \mathbb{Z}^{m \times n}$ and $\mathbf{b} \in \mathbb{Z}^m$. Therefore,

$$(2.6) \quad f(\mathbf{x}) \begin{cases} = \mathbf{c}^T \mathbf{x} + \mathbf{x}^T \mathbf{F} \mathbf{x} & \text{on } \Delta \\ \geq \mathbf{c}^T \mathbf{x} + \mathbf{x}^T \mathbf{F} \mathbf{x} + 2\rho(\mathbf{c}, \mathbf{F}) + 1 > \rho(\mathbf{c}, \mathbf{F}) & \text{on } \{-1, 1\}^n \setminus \Delta. \end{cases}$$

From this and $\mathbf{c}^T \mathbf{x} + \mathbf{x}^T \mathbf{F} \mathbf{x} \leq \rho(\mathbf{c}, \mathbf{F})$ on Δ , the result follows. \square

Next, let $Q : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ be the homogenization of the quadratic polynomial f , i.e., the quadratic form $Q(\mathbf{x}, x_0) := x_0^2 f(\frac{\mathbf{x}}{x_0})$, or in explicitly form:

$$(2.7) \quad Q(\mathbf{x}, x_0) = x_0 \mathbf{c}^T \mathbf{x} + \mathbf{x}^T \mathbf{F} \mathbf{x} + (2\rho(\mathbf{c}, \mathbf{F}) + 1) \cdot \|\mathbf{A}\mathbf{x} - x_0 \mathbf{b}\|^2.$$

Observe that $Q(\mathbf{x}, 1) = f(\mathbf{x})$.

Theorem 2.2. *Let $f^* = \min\{\mathbf{c}^T \mathbf{x} + \mathbf{x}^T \mathbf{F} \mathbf{x} : \mathbf{A}\mathbf{x} = \mathbf{b}; \mathbf{x} \in \{-1, 1\}^n\}$ and let Q be the quadratic form in (2.7). Then*

$$(2.8) \quad f^* = \min_{(\mathbf{x}, x_0) \in \{-1, 1\}^{n+1}} Q(\mathbf{x}, x_0),$$

that is, f^* is the optimal value of the MAX-CUT problem associated with the quadratic form Q .

Proof. Let f be as in (2.5). By definition of Q ,

$$(2.9) \quad \min_{\mathbf{x} \in \{-1, 1\}^n} f(\mathbf{x}) = \min_{(\mathbf{x}, x_0) \in \{-1, 1\}^{n+1}} \{Q(\mathbf{x}, x_0) : x_0 = 1\}.$$

On the other hand, let $(\mathbf{x}^*, x_0^*) \in \{-1, 1\}^{n+1}$ be a global minimizer of $\min\{Q(\mathbf{x}, x_0) : (\mathbf{x}, x_0) \in \{-1, 1\}^{n+1}\}$. Then by homogeneity of Q , $(-\mathbf{x}^*, -x_0^*)$ is also a global minimizer and so one may decide arbitrarily to fix $x_0 = 1$. That is,

$$\min_{(\mathbf{x}, x_0) \in \{-1, 1\}^{n+1}} Q(\mathbf{x}, x_0) = \min_{(\mathbf{x}, x_0) \in \{-1, 1\}^{n+1}} \{Q(\mathbf{x}, x_0) : x_0 = 1\},$$

which combined with (2.9) yields the desired result. \square

Next, write $Q(\mathbf{x}, x_0) = (\mathbf{x}, x_0) \mathbf{Q} (\mathbf{x}, x_0)^T$ for an appropriate real symmetric matrix $\mathbf{Q} \in \mathbb{R}^{(n+1) \times (n+1)}$, and introduce the semidefinite programs

$$(2.10) \quad \min_{\mathbf{X}} \{\langle \mathbf{Q}, \mathbf{X} \rangle : \mathbf{X} \succeq 0; X_{ii} = 1, \quad i = 1, \dots, n+1\}$$

with optimal value denoted by $\min \mathbf{Q}_+$, and

$$(2.11) \quad \max_{\mathbf{X}} \{\langle \mathbf{Q}, \mathbf{X} \rangle : \mathbf{X} \succeq 0; X_{ii} = 1, \quad i = 1, \dots, n+1\}$$

with optimal value denoted by $\max \mathbf{Q}^+$.

Proposition 2.3. *Let \mathbf{P} be the problem defined in (2.1) with optimal value f^* and let $\rho(\mathbf{c}, \mathbf{F})$ be as in (2.2). Then:*

$$(2.12) \quad \min \mathbf{Q}_+ \leq f^* \leq \frac{2}{\pi} \min \mathbf{Q}_+ + (1 - \frac{2}{\pi}) \max \mathbf{Q}^+$$

where \mathbf{Q} is the real symmetric matrix associated with the quadratic form (2.7) and $\min \mathbf{Q}_+$ (resp. $\max \mathbf{Q}^+$) is the optimal value of the semidefinite program (2.10) (resp. (2.11)).

Moreover, if $\min \mathbf{Q}_+ > \rho(\mathbf{c}, \mathbf{F})$ then \mathbf{P} has no feasible solution (i.e., $f^* = +\infty$).

Proof. The bounds in (2.12) are from Nesterov [13]. In addition, one may also use the bounds provided in Marshall [12] which sometimes improve those in (2.12). Next, let $\Delta := \{\mathbf{x} \in \{-1, 1\}^n : \mathbf{A}\mathbf{x} = \mathbf{b}\}$ and assume that $\Delta \neq \emptyset$. Then by (2.3) and (2.6), $\min \mathbf{Q}_+ \leq f^* \leq \rho(\mathbf{c}, \mathbf{F})$. Therefore $\Delta = \emptyset$ if $\min \mathbf{Q}_+ > \rho(\mathbf{c}, \mathbf{F})$. \square

The quality of the upper bound in (2.12) depends strongly on the magnitude of the “penalty coefficient” $2\rho(\mathbf{c}, \mathbf{F}) + 1$ in the definition of the function f in (2.5). Whether or not the penalty parameter $\rho(\mathbf{c}, \mathbf{F})$ could be taken smaller (at least in some cases) has not been investigated and is beyond the scope of this paper. However for a practical use of relaxations what matters most is the quality of the *lower* bound $\min \mathbf{Q}_+$ which in principle is very good for MAX-CUT problems (even if $\mathbf{Q} \not\geq 0$ or $\mathbf{Q} \not\leq 0$). For instance in a Branch & Bound algorithm the lower bound $\min \mathbf{Q}_+$ has an important impact in the pruning of nodes in the search tree.

2.2. Sparsity. Hence to each 0/1 program (1.1) one may associate a graph $G = (V, E)$ with $n + 1$ nodes and an arc $(i, j) \in E$ connects the nodes $i, j \in V$ if and only if the coefficient \mathbf{Q}_{ij} of the quadratic form $Q(\mathbf{x}, x_0)$ does not vanish. *Sparsity* properties of G are of primary interest, e.g. for computational reasons. From the definition of the matrix \mathbf{Q} , this sparsity is in turn related to sparsity of the matrix $\mathbf{F} + (2\rho_{\mathbf{c}, \mathbf{F}} + 1) \cdot \mathbf{A}^T \mathbf{A}$, hence of sparsity of \mathbf{F} and $\mathbf{A}^T \mathbf{A}$.

In particular, two nodes i, j are not connected if $\mathbf{F}_{ij} = 0$ and $\mathbf{A}_{ki} \mathbf{A}_{kj} = 0$ for all $k = 1, \dots, m$, that is, in any of the constraints $\mathbf{A}\mathbf{x} = \mathbf{b}$, at most one of the two variables (x_i, x_j) appears (a structural condition). A weaker condition (non-structural as it depends on values of entries of \mathbf{A}) is that $\sum_k \mathbf{A}_{ki} \mathbf{A}_{kj} = 0$.

2.3. Extension to inequalities. Let $f(\mathbf{x}) := \mathbf{c}^T \mathbf{x} + \mathbf{x}^T \mathbf{F} \mathbf{x}$ for some $\mathbf{c} \in \mathbb{R}^n$ and some $\mathbf{F}^T = \mathbf{F} \in \mathbb{R}^{n \times n}$, and consider the problem:

$$(2.13) \quad \mathbf{P} : \quad f^* = \min_{\mathbf{x}} \{ f(\mathbf{x}) : \mathbf{A} \mathbf{x} \leq \mathbf{b}; \quad \mathbf{x} \in \{0, 1\}^n \},$$

for some cost vector $\mathbf{c} \in \mathbb{Z}^n$, some matrix $\mathbf{A} \in \mathbb{Z}^{m \times n}$, and some vector $\mathbf{b} \in \mathbb{Z}^m$. We may and will replace (2.13) with the equivalent pure integer program:

$$\mathbf{P}' : \quad f^* = \min_{\mathbf{x}, \mathbf{y}} \{ f(\mathbf{x}) : \mathbf{A} \mathbf{x} + \mathbf{y} = \mathbf{b}; \quad \mathbf{x} \in \{0, 1\}^n; \mathbf{y} \in \mathbb{N}^m \}.$$

Next, as $\mathbf{x} \in \{0, 1\}^n$ we can bound each integer variable y_j by $M_j := b_j - \min\{\mathbf{A}_j \mathbf{x} : \mathbf{x} \in \{0, 1\}^n\}$, $j = 1, \dots, m$, where \mathbf{A}_j denotes the j -th row vector of the matrix \mathbf{A} ; and in fact $M_j = b_j - \sum_i \min[0, \mathbf{A}_{ji}]$, $j = 1, \dots, m$. Then we may use the standard decomposition of y_j into a weighted sum of boolean variables :

$$y_j = \sum_{k=0}^{s_j} 2^k z_{jk}, \quad z_{jk} \in \{0, 1\}^n, \quad j = 1, \dots, m,$$

(where $s_j := \lceil \log(M_j) \rceil$) and replace (2.13) with the equivalent 0/1 program:

$$f^* = \min_{\mathbf{x}, \mathbf{z}} \{ f(\mathbf{x}) : \mathbf{A}_j^T \mathbf{x} + \sum_{k=0}^{s_j} 2^k z_{jk} = b_j, j \leq m; (\mathbf{x}, \mathbf{z}) \in \{0, 1\}^{n+s} \},$$

(where $s := \sum_j (1 + s_j)$) which is of the form (1.1).

2.4. Extension to polynomial programs. Let $f \in \mathbb{R}[\mathbf{x}]$ be a polynomial of even degree $d > 2$, written as

$$\mathbf{x} \mapsto f(\mathbf{x}) = \sum_{\alpha \in \mathbb{N}^n} f_\alpha \mathbf{x}^\alpha,$$

for some vector of coefficients $\mathbf{f} = (f_\alpha) \in \mathbb{R}^{s(d)}$ (with $s(d) = \binom{n+d}{n}$). Given a sequence $\mathbf{y} = (y_\alpha)$, $\alpha \in \mathbb{N}^n$, define the Riesz functional $L_{\mathbf{y}} : \mathbb{R}[\mathbf{x}] \rightarrow \mathbb{R}$ by:

$$f \mapsto L_{\mathbf{y}}(f) = \sum_{\alpha \in \mathbb{N}^n} f_\alpha y_\alpha, \quad f \in \mathbb{R}[\mathbf{x}].$$

Consider the polynomial program:

$$(2.14) \quad f^* = \min \{ f(\mathbf{x}) : \mathbf{A}\mathbf{x} = \mathbf{b}; \quad \mathbf{x} \in \{-1, 1\}^n \},$$

on the hyper cube $\{-1, 1\}^n$. Let $d' := d/2$, $\mathbf{x} \mapsto g_j(\mathbf{x}) := 1 - x_j^2$, $j = 1, \dots, n$, and with f let us associate the scalars:

$$\begin{aligned} r_f^1 &= \min \{ L_{\mathbf{y}}(f) : \mathbf{M}_{d'}(\mathbf{y}) \succeq 0; \mathbf{M}_{d'-1}(g_j \mathbf{y}) = 0, j = 1, \dots, n \} \\ r_f^1 &= \max \{ L_{\mathbf{y}}(f) : \mathbf{M}_{d'}(\mathbf{y}) \succeq 0; \mathbf{M}_{d'-1}(g_j \mathbf{y}) = 0, j = 1, \dots, n \} \end{aligned}$$

where $\mathbf{M}_{d'}(\mathbf{y})$ (resp. $\mathbf{M}_{d'-1}(g_j \mathbf{y})$) is the moment matrix (resp. localizing matrix) of order d' associated with the real sequence $\mathbf{y} = (y_\alpha)$, $\alpha \in \mathbb{N}^n$, (resp. with the sequence \mathbf{y} and the polynomial g_j). It turns out that r_f^1 (resp. r_f^2) is the optimal value of the first SDP-relaxation of the Lasserre-SOS hierarchy associated with the optimization problem \min (resp. \max) $\{ f(\mathbf{x}) : \mathbf{x} \in \{-1, 1\}^n \}$ and so $r_f^1 \leq \min \{ f(\mathbf{x}) : \mathbf{x} \in \{-1, 1\}^n \}$ whereas $r_f^2 \geq \max \{ f(\mathbf{x}) : \mathbf{x} \in \{-1, 1\}^n \}$. For more details, see e.g. [9, 10].

Next, if we define

$$(2.15) \quad \rho_f := \max_{i=1,2} |r_f^i|,$$

then it is straightforward to verify that $\rho_f \geq \max \{ |f(\mathbf{x})| : \mathbf{x} \in \{-1, 1\}^n \}$. Then we have the following analogue of Lemma 2.1:

Lemma 2.4. *Let f^* be as (2.14) and let ρ_f be as in (2.15). Then f^* is the optimal value of the polynomial minimization problem:*

$$(2.16) \quad \min_{\mathbf{x} \in \{-1, 1\}^n} f(\mathbf{x}) + (2\rho_f + 1) \cdot \|\mathbf{A}\mathbf{x} - \mathbf{b}\|^2.$$

The proof being almost a verbatim copy of that of Lemma 2.1, is omitted.

As for the quadratic case and with same arguments, one may also show that if d is even, the polynomial optimization problem (2.16) is equivalent

to minimizing the homogeneous polynomial \tilde{f} of degree d on the hypercube $\{-1, 1\}^{n+1}$, where

$$(\mathbf{x}, x_0) \mapsto \tilde{f}(\mathbf{x}) := x_0^d f(\mathbf{x}/x_0) + (2r_f + 1) x_0^{d-2} \cdot \|\mathbf{A} \mathbf{x} - x_0 \mathbf{b}\|^2.$$

But since it is not a MAX-CUT problem, to obtain a lower bound on f^* one may just as well consider solving the first level of the Lasserre-SOS hierarchy associated with (2.16) or even directly with (2.14). The advantage of using the formulation (2.16) is that one always minimizes on the hypercube $\{-1, 1\}^n$ instead of minimizing on the subset $\{-1, 1\}^n \cap \{\mathbf{x} : \mathbf{A} \mathbf{x} = \mathbf{b}\}$ of the hypercube which is problem dependent.

3. SOME NUMERICAL EXPERIMENTS

All semidefinite programs were solved by using the GloptiPoly software dedicated to solving the Generalized Problem of Moments and which implements the Lasserre-SOS hierarchy of semidefinite relaxations. For more details the interested reader is referred to Henrion et al. [6]. Typically, for a MAX-CUT problem of size $n = 100$, solving the Shor relaxation (with semidefinite solver MOSEK¹) takes approximately 42s on a Macbook-pro lap-top with Intel Core i5 processor at 2.6 GHz.

3.1. Some 0/1-knapsack examples. To test the efficiency of the MAX-CUT formulation we have first considered 0/1-knapsack problems of small size (up to $n = 15$ variables) and compared the first semidefinite relaxation (or Shor-relaxation) of the MAX-CUT formulation (2.4) with some other relaxations, and in particular with:

- The LP-relaxation of (2.1) (when $\mathbf{F} = 0$) which consists of replacing the constraint $\mathbf{x} \in \{-1, 1\}^n$ with $\mathbf{x} \in [-1, 1]^n$, and
- The first semidefinite relaxation of (2.1) in the Lasserre-SOS hierarchy.

Example 3.1. To evaluate the quality of the lower bound obtained with the MAX-CUT formulation consider the following simple linear knapsack-type examples:

$$(3.1) \quad \min \{ \mathbf{c}^T \mathbf{x} : \mathbf{a}^T \mathbf{x} = b; \mathbf{x} \in \{-1, 1\}^n \},$$

on $\{-1, 1\}^n$, with 4 and 10 variables. For $n = 4$, $\mathbf{c} = (13, 11, 7, 3)$ and $\mathbf{a} = (3, 7, 11, 13)$, while for $n = 10$, $\mathbf{c} = (37, 31, 29, 23, 19, 17, 13, 11, 7, 3)$, and $\mathbf{a} = (3, 7, 11, 13, 17, 19, 23, 29, 31, 37)$ and for $n = 15$

$$\mathbf{a} = (3, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 51, 53)$$

and $\mathbf{c} = (53, 51, 47, 43, 41, 37, 31, 29, 23, 19, 17, 13, 11, 7, 3)$.

The right-hand-side b is taken into $[-|\mathbf{a}|, |\mathbf{a}|] \cap \mathbb{Z}$. Figure 1 displays the difference $\min \mathbf{Q}_+ - \min \text{LP}$ where the lower bound $\min \text{LP}$ is obtained by relaxing the integrality constraints $\mathbf{x} \in \{-1, 1\}^n$ to the box constraint

¹MOSEK is a high performance semidefinite solver which can be called in the GloptiPoly software [6] for solving the resulting SDP-relaxations; more details on MOSEK can be found at <https://www.mosek.com/>

$\mathbf{x} \in [-1, 1]^n$ and solving the resulting LP. On the x -axis one reads $b + |\mathbf{a}|$. As expected the lower bound $\min \mathbf{Q}_+$ is much better than $\min \text{LP}$. In fact the cases where the LP-bound is slightly better is for right-hand-side b such that the relaxation provides the optimal value f^* .

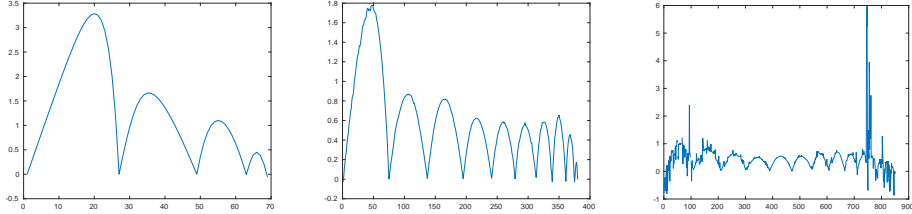


FIGURE 1. Difference $\min \mathbf{Q}_+ - \min \text{LP}$ with $n=(4,10,15)$; One reads $b + |\mathbf{a}|$ on the x -axis

Moreover Figure 2 displays the difference $\min \mathbf{Q}_+ - \min \hat{\mathbf{Q}}_+$ where $\min \hat{\mathbf{Q}}_+$ is the optimal value of the first SDP-relaxation of the Lasserre-SOS hierarchy applied to the initial formulation (3.1) of the knapsack problem where one has even included the redundant constraints $x_i (\mathbf{a}^T \mathbf{x} - b) = 0, i = 1, \dots, n$. One observes that in most cases the lower bound $\min \mathbf{Q}_+$ is slightly better than $\min \hat{\mathbf{Q}}_+$.

This is encouraging since the Lasserre-SOS hierarchy [7, 8] is known to produce good lower bounds in general, and especially at the first level of the hierarchy for MAX-CUT problems whose matrix \mathbf{Q} of the associated quadratic form has certain properties, e.g., $Q_{ij} \geq 0$ for all i, j or $\mathbf{Q} \succeq 0$ (in the maximizing case); see e.g. Marshall [12].

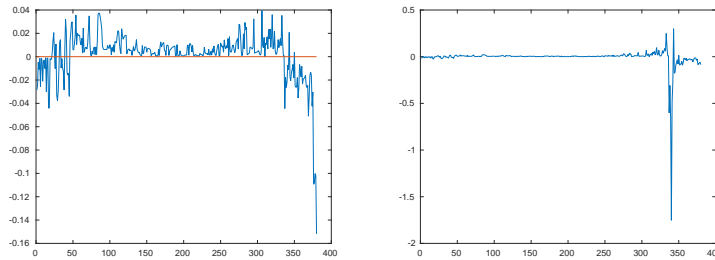


FIGURE 2. Difference $\min \mathbf{Q}_+ - \min \hat{\mathbf{Q}}_+$ ($n=10$) (left) and relative difference $100 * (\min \mathbf{Q}_+ - \min \hat{\mathbf{Q}}_+) / \min \hat{\mathbf{Q}}_+$; One reads $b + |\mathbf{a}|$ on the x -axis

Example 3.2. In a second sample of linear knapsack problems (3.1) with $n = 10, 15$, we have chosen the same vector \mathbf{a} as in Example 3.1 but now with a cost criterion of the form $c(i) = a(i) + s\eta, i = 1, \dots, n$, where η is a

random variable uniformly distributed on $[0, 1]$, and $s = 20, 10$ is a weighting factor. The reason is that knapsack problems with ratios $c(i)/a(i) \approx 1$ for all i , can be difficult to solve. As before the right-hand-side b is taken into $[-|\mathbf{a}|, |\mathbf{a}|] \cap \mathbb{Z}$. Figure 3 displays the results obtained for $s = 20$, and $n = 10, 15$. Figure 4 displays the same example for another sample with

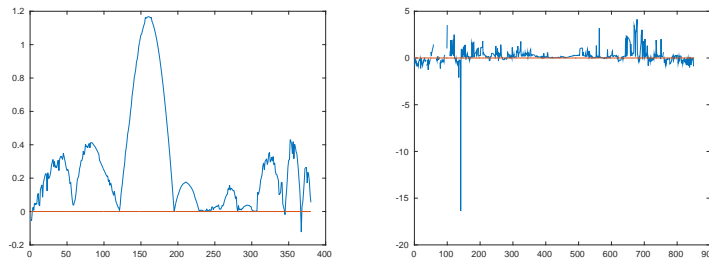


FIGURE 3. Difference $\min \mathbf{Q}_+ - \min \text{LP}$, $\mathbf{c} = \mathbf{a} + 20 * \eta$ ($n=10,15$); $b + |\mathbf{a}|$ on the x -axis

cost \mathbf{c} and $s = 10$.

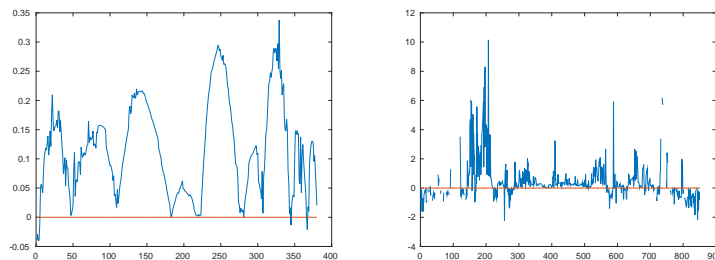


FIGURE 4. Difference $\min \mathbf{Q}_+ - \min \text{LP}$, $\mathbf{c} = \mathbf{a} + 10 * \eta$ ($n=10,15$); $b + |\mathbf{a}|$ on the x -axis

Finally, as for Example 3.1, Figure 5 displays the difference $\min \mathbf{Q}_+ - \min \hat{\mathbf{Q}}_+$ (where $\min \hat{\mathbf{Q}}_+$ is the optimal value of the first SDP-relaxation of the Lasserre-SOS hierarchy applied to the initial formulation (3.1) of the knapsack problem where one has even included the redundant constraints $x_i(\mathbf{a}^T \mathbf{x} - b) = 0, i = 1, \dots, n$). Again one observes that in most cases the lower bound $\min \mathbf{Q}_+$ is slightly better than $\min \hat{\mathbf{Q}}_+$.

Example 3.3. We next consider the same knapsack problems (3.1) as in Example 3.2 but now with quadratic criterion $\mathbf{c}^T \mathbf{x} + \mathbf{x}^T \mathbf{F} \mathbf{x}$, again with a cost criterion of the form $c(i) = a(i) + s\eta, i = 1, \dots, n$, where η is a random variable uniformly distributed in $[0, 1]$ and s is some weighting factor. The real symmetric matrix \mathbf{F} is also randomly generated and is not positive

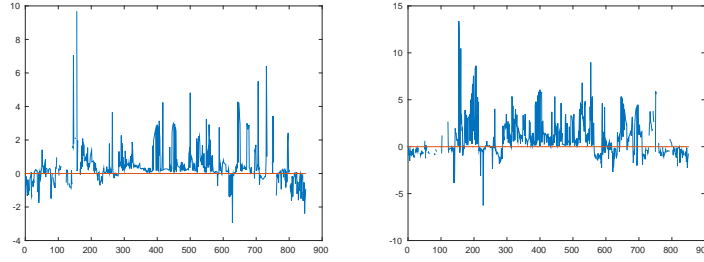


FIGURE 5. Example 3.2: Difference $\min \mathbf{Q}_+ - \min \hat{\mathbf{Q}}_+$, ($n=15$) $\mathbf{c} = \mathbf{a} + 20\eta$ (left) and $\mathbf{c} = \mathbf{a} + 10\eta; b + |\mathbf{a}|$ on the x -axis

definite in general. Again in Figures 6 and 7 one observes that the lower bound $\min \mathbf{Q}_+$ is almost always better than the optimal value $\min \hat{\mathbf{Q}}_+$ of the first level of the Lasserre-SOS hierarchy applied to the original formulation (3.1) of the problem.

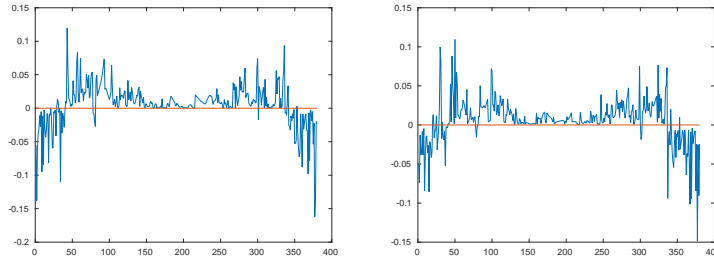


FIGURE 6. Example 3.3: Difference $\min \mathbf{Q}_+ - \min \hat{\mathbf{Q}}_+$, ($n=10$) $\mathbf{c} = \mathbf{a} + 20\eta$ (left) and $\mathbf{c} = \mathbf{a} + 10\eta; b + |\mathbf{a}|$ on the x -axis

3.2. Larger size problems: The k -cluster problem. As generating data $(\mathbf{c}, \mathbf{a}, b)$ for difficult 0/1-knapsack problems of larger size is not obvious, we next consider the so-called k -cluster problem in $n = 70$ and $n = 80$ variables:

$$(3.2) \quad \min \{ \mathbf{x}^T \mathbf{A} \mathbf{x} : \mathbf{e}^T \mathbf{x} = k; \mathbf{x} \in \{0, 1\}^n \},$$

where $k \in \mathbb{N}$ is a fixed parameter, $\mathbf{e} = (1, 1, \dots, 1) \in \mathbb{R}^n$, $\mathbf{A} \in \mathbb{R}^{n \times n}$ with $\mathbf{A} = \mathbf{A}^T$ and $\mathbf{A}_{ii} = 0$ for all $i = 1, \dots, n$. Indeed the constraint $\mathbf{e}^T \mathbf{x} = k$ of the k -cluster problem is structural and does not depend on a vector \mathbf{a} as in knapsack examples. After the change of variables $\mathbf{x} \rightarrow (\mathbf{x} + \mathbf{e})/2$ the k -cluster problem reads:

$$(3.3) \quad \mathbf{P} : \quad \min \{ 2\mathbf{e}^T \mathbf{A} \mathbf{x} + \mathbf{x}^T \mathbf{A} \mathbf{x} : \mathbf{e}^T \mathbf{x} = 2k - n; \mathbf{x} \in \{-1, 1\}^n \}.$$

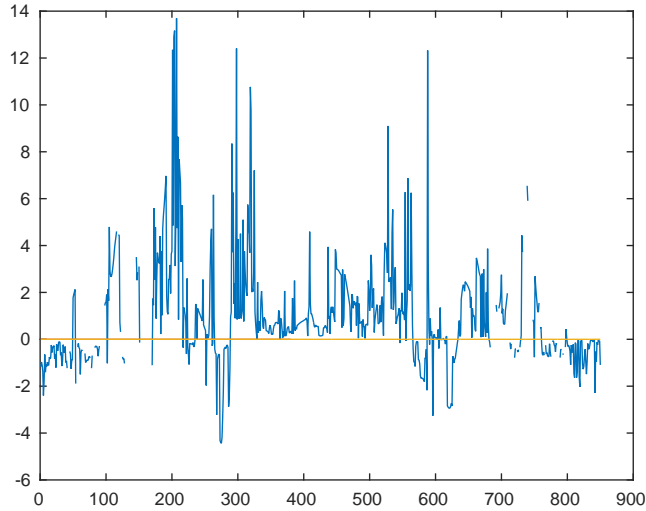


FIGURE 7. Example 3.3: Difference $\min \mathbf{Q}_+ - \min \hat{\mathbf{Q}}_+$, ($n=15$) $\mathbf{c} = \mathbf{a} + 20\eta$ (left) and $\mathbf{c} = \mathbf{a} + 10\eta$; $b + |\mathbf{a}|$ on the x -axis

We have compared the optimal value f_{maxcut}^* for the Shor relaxation of the MAX-CUT formulation (2.4) of (3.3) with the optimal value f_{Shor}^* for the first semidefinite relaxation of (3.3) in the Lasserre-SOS hierarchy. As can be seen from results displayed in Table 1 for $n = 70$ and in Table 2 for $n = 80$, both values are almost identical since the relative difference is no more than 0.014% and 0.06% respectively.

	f_{maxcut}^*	f_{Shor}^*	$100 * (f_{maxcut}^* - f_{Shor}^*) / f_{Shor}^* $
$k = 1$	-2456.391	-2456.269	-0.005%
$k = 6$	-2442.986	-2442.992	0.0002%
$k = 11$	-2319.395	-2319.716	0.014%
$k = 16$	-2089.292	-2090.238	0.045%
$k = 21$	-1756.995	-1756.974	-0.001%
$k = 26$	-1320.461	-1320.437	-0.001%
$k = 31$	-779.733	-779.692	-0.005%

TABLE 1. k -cluster with $n = 70$; f_{maxcut}^* , f_{Shor}^* and the relative difference $100 * (f_{maxcut}^* - f_{Shor}^*) / |f_{Shor}^*|$

Detecting infeasibility. Of course when $k > n$ the k -cluster problem (3.3) has no feasible solution to \mathbf{P} . In view of the special structure of the k -cluster problem it is straightforward to check that the first semidefinite relaxation

	f_{maxcut}^*	f_{Shor}^*	$100 * (f_{Shor}^* - f_{maxcut}^*) / f_{Shor}^* $
$k = 10$	-3122.913	-3124.925	0.06%
$k = 20$	-2613.463	-2613.748	0.01%
$k = 30$	-1684.377	-1684.419	0.002%
	f_{maxcut}^*	f_{Shor}^*	$100 * (f_{maxcut}^* - f_{Shor}^*) / f_{Shor}^* $
$k = 70$	6373.92	6373.65	0.004%
$k = 60$	3675.92	3674.98	0.025%
$k = 50$	1445.74	1445.67	0.005%

TABLE 2. k -cluster with $n = 80$; f_{maxcut}^* , f_{Shor}^* and the relative difference $100 * (f_{maxcut}^* - f_{Shor}^*) / |f_{Shor}^*|$ (with two different \mathbf{F})

of (3.3) (or equivalently of (3.2)) in the Lasserre-SOS hierarchy is infeasible, hence certifying that $f^* = +\infty$. This is because in the first semidefinite relaxation of (3.2), the constraints are:

$$\mathbf{e}^T \mathbf{x} = k; \quad \left[\begin{array}{c|c} 1 & \mathbf{x}^T \\ \hline - & - \\ \mathbf{x} & \mathbf{X} \end{array} \right] \succeq 0, \quad \mathbf{X}_{ii} = \mathbf{x}_i, \quad i = 1, \dots, n.$$

The semidefinite constraint implies $0 \leq \mathbf{x} \leq 1$, and therefore one cannot have $\mathbf{e}^T \mathbf{x} = k$ if $k > n$.

On the other hand the Shor relaxation of (3.3) (and so the Shor relaxation of the MAX-CUT formulation of (3.3)) has always a finite value, but if $k > n$ then the optimal value is larger than $\rho(\mathbf{c}, \mathbf{F})$, which by Proposition 2.3 also certifies that $f^* = +\infty$. Indeed in the Shor relaxation of (3.3) one has the constraint

$$(3.4) \quad \left[\begin{array}{c|c} 1 & \mathbf{x}^T \\ \hline - & - \\ \mathbf{x} & \mathbf{X} \end{array} \right] \succeq 0, \quad \mathbf{X}_{ii} = 1, \quad \forall i = 1, \dots, n$$

and with $b = 2k - n$ the criterion reads

$$2\mathbf{e}^T \mathbf{x} + \mathbf{x}^T \mathbf{A} \mathbf{x} + (2\rho(\mathbf{c}, \mathbf{F}) + 1) \underbrace{\begin{bmatrix} -b \\ \mathbf{e} \end{bmatrix}^T \begin{bmatrix} 1 & | & \mathbf{x}^T \\ \hline - & - & - \\ \mathbf{x} & | & \mathbf{X} \end{bmatrix} \begin{bmatrix} -b \\ \mathbf{e} \end{bmatrix}}_{(b - \mathbf{e}^T \mathbf{x})^2 + \mathbf{e}^T (\mathbf{X} - \mathbf{x} \mathbf{x}^T) \mathbf{e}}.$$

Therefore as $\mathbf{X} \succeq \mathbf{x} \mathbf{x}^T$, $(b - \mathbf{e}^T \mathbf{x})^2 + \mathbf{e}^T (\mathbf{X} - \mathbf{x} \mathbf{x}^T) \mathbf{e} \geq (b - \mathbf{e}^T \mathbf{x})^2$, and with so if $k > n$ then $b = 2k - n > n$ which in turn implies $(b - \mathbf{e}^T \mathbf{x}) > 1$ whenever (3.4) holds (as $-1 \leq \mathbf{x} \leq 1$). Next, from the definition of $\rho(\mathbf{c}, \mathbf{F})$ in (2.2) we have seen that $|2\mathbf{e}^T \mathbf{x} + \mathbf{x}^T \mathbf{A} \mathbf{x}| \leq \rho(\mathbf{c}, \mathbf{F})$ whenever (3.4) holds. Hence

$$2\mathbf{e}^T \mathbf{x} + \mathbf{x}^T \mathbf{A} \mathbf{x} + (2\rho(\mathbf{c}, \mathbf{F}) + 1) ((b - \mathbf{e}^T \mathbf{x})^2 + \mathbf{e}^T (\mathbf{X} - \mathbf{x} \mathbf{x}^T) \mathbf{e}) > \rho(\mathbf{c}, \mathbf{F}),$$

whenever (3.4) holds. This is confirmed in the results displayed in Table 3 where again \mathbf{A} was randomly generated and $k = n + i$, $i = 1, 4, 7$ so that the k -cluster problem is infeasible.

n=70	f_{maxcut}^*	f_{Shor}^*
$k = n + 1$	22129.64 > $\rho(\mathbf{c}, \mathbf{F})$	infeas
$k = n + 4$	243431.21 > $\rho(\mathbf{c}, \mathbf{F})$	infeas
$k = n + 7$	730294.15 > $\rho(\mathbf{c}, \mathbf{F})$	infeas
n=80	f_{maxcut}^*	f_{Shor}^*
$k = n + 1$	28563.26 > $\rho(\mathbf{c}, \mathbf{F})$	infeas

TABLE 3. Infeasible k -cluster problems with $n = 70, 80$

Comparing with a quadratic convex relaxation. Decompose \mathbf{A} as $\mathbf{U}^T \Lambda \mathbf{U}$ with $\mathbf{U}^T \mathbf{U} = \mathbf{I}$ (the identity matrix) and Λ is the diagonal matrix of eigenvalues of \mathbf{A} . Let $\theta := |\lambda_{min}(\mathbf{A})| + 1$ where $\lambda_{min}(\mathbf{A})$ is the smallest eigenvalue of \mathbf{A} . Replace \mathbf{A} with $\tilde{\mathbf{A}} := \mathbf{U}^T (\Lambda + \theta \mathbf{I}) \mathbf{U}$ so that the resulting the quadratic form $\mathbf{x} \mapsto \mathbf{x}^T \tilde{\mathbf{A}} \mathbf{x}$ is convex. Then observe that

$$\min \{ 2\mathbf{e}^T \mathbf{A} \mathbf{x} + \mathbf{x}^T \tilde{\mathbf{A}} \mathbf{x} : \mathbf{e}^T \mathbf{x} = 2k - n; \mathbf{x} \in \{-1, 1\}^n \} = f^* + n\theta$$

because $\mathbf{x}^T \tilde{\mathbf{A}} \mathbf{x} = \mathbf{x}^T \mathbf{A} \mathbf{x} + \theta \|\mathbf{x}\|^2$ and $x_i^2 = 1$ for all $i = 1, \dots, n$. Therefore one may compare the optimal value f_{maxcut}^* of the Shor relaxation of (3.3) with $\tilde{\psi} = \psi - n\theta$, where

$$\psi = \min \{ 2\mathbf{e}^T \mathbf{A} \mathbf{x} + \mathbf{x}^T \tilde{\mathbf{A}} \mathbf{x} : \mathbf{e}^T \mathbf{x} = 2k - n; x_i^2 \leq 1, i = 1, \dots, n \},$$

which is a convex quadratic program. Results displayed in Table 4 show that the Shor relaxation is always better (and sometimes significantly better) than the quadratic convex relaxation.

Next we do the same comparison for a k -cluster problem where some sparsity is introduced in the cost matrix \mathbf{A} . Namely once \mathbf{A} has been generated then for each non-diagonal entry \mathbf{A}_{ij} we decide $\mathbf{A}_{ij} = \mathbf{A}_{ji} := 0$ with probability 0.4, 0.7 and 0.8. Results displayed in Table 5 for $n = 70$ and in Table 6 for $n = 100$ show that again the Shor relaxation is always better (and some times significantly better) than the quadratic convex relaxation. The sparser is \mathbf{A} the more efficient is the Shor relaxation compared with the quadratic convex relaxation.

n=70	k=50	k=45	k=40	k=35	k=30	k=25
f_{maxcut}^*	2324.68	1350.74	488.83	-261.78	-900.55	-1429.66
$\tilde{\psi}$	2260.00	1278.08	410.48	-343.95	-985.57	-1514.77
$\frac{100(f_{maxcut}^* - \tilde{\psi})}{\tilde{\psi}}$	2.86%	5.68%	19.08%	23.89%	8.62%	5.61%

TABLE 4. k -cluster: Shor relaxation (f_{maxcut}^*) vs convex quadratic relaxation ($\tilde{\psi} = \psi - n\theta$) with $n = 70$

n=70	k=50	k=45	k=40	k=35	k=30	k=25
$\mathbf{A}_{ij} = 0$ with probability 0.4						
f_{maxcut}^*	1188.41	569.19	32.38	-422.36	-794.41	-1084.71
$\tilde{\psi}$	1129.49	505.55	-32.54	-485.26	-855.35	-1149.94
$\frac{100(f_{maxcut}^* - \tilde{\psi})}{\tilde{\psi}}$	5.2%	12.5%	199.5%	12.9%	7.1%	5.6%
$\mathbf{A}_{ij} = 0$ with probability 0.7						
f_{maxcut}^*	427.01	106.85	-156.52	-362.12	-519.68	-637.74
$\tilde{\psi}$	376.80	60.31	-203.72	-418.46	-588.56	-718.30
$\frac{100(f_{maxcut}^* - \tilde{\psi})}{\tilde{\psi}}$	13.3%	77.1%	23.1%	13.4%	11.7%	11.2%
$\mathbf{A}_{ij} = 0$ with probability 0.8						
f_{maxcut}^*	232.09	26.12	-143.13	-279.53	-384.83	-463.19
$\tilde{\psi}$	172.80	-43.89	-223.92	-370.54	-485.54	-569.36
$\frac{100(f_{maxcut}^* - \tilde{\psi})}{\tilde{\psi}}$	34.3%	159.5%	36.0%	24.5%	20.7%	18.6%

TABLE 5. k -cluster: Shor relaxation (f_{maxcut}^*) vs convex quadratic relaxation ($\tilde{\psi} = \psi - n\theta$) with $n = 70$. $\mathbf{A}_{ij} = 0$ with probability $p = 0.4, 0.7, 0.8$.

3.3. Comparing with the copositive formulation. As already mentioned in the introduction, the 0/1 program (1.1) also has a copositive formulation. Namely, let $e_i = (\delta_{i=j}) \in \mathbb{R}^n$, $i = 1, \dots, n$, and $\mathbf{e} = (1, \dots, 1) \in \mathbb{R}^n$. Following Burer [3, p. 481–482], introduce n additional variables $\mathbf{z} = (z_1, \dots, z_n)$ and the n additional equality constraints $x_i + z_i = 1$, $i = 1, \dots, n$, with $\mathbf{z} \geq 0$ (which are necessary to obtain an equivalent formulation). So let $\tilde{\mathbf{x}}^T = (\mathbf{x}^T, \mathbf{z}^T) \in \mathbb{R}^{2n}$ and with $\mathbf{I} \in \mathbb{R}^{n \times n}$ being the identity matrix, introduce the real matrices

$$\tilde{\mathbf{F}} := \begin{bmatrix} \mathbf{F} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \mathbf{S} := \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{I} & \mathbf{I} \end{bmatrix}$$

and the real vectors $\tilde{\mathbf{c}}^T := (\mathbf{c}^T, \mathbf{0}) \in \mathbb{R}^{2n}$ and $\tilde{\mathbf{b}}^T = (\mathbf{b}^T, \mathbf{e}^T) \in \mathbb{R}^{m+n}$. Let \mathbf{S}_i denote the i -th row vector of \mathbf{S} , $i = 1, \dots, 2n$. Then the copositive

n=100	k=70	k=60	k=50	k=40	k=20
$\mathbf{A}_{ij} = 0$ with probability 0.8					
f_{maxcut}^*	513.32	-60.83	-450.91	-781.84	-1084.98
$\tilde{\psi}$	419.07	-147.77	-572.27	-894.72	-1188.59
$\frac{100(f_{maxcut}^* - \tilde{\psi})}{\tilde{\psi}}$	22.4%	58.8%	21.2%	12.6%	8.7%
$\mathbf{A}_{ij} = 0$ with probability 0.6					
f_{maxcut}^*	1278.01	256.43	-627.72	-1248.39	-1988.94
$\tilde{\psi}$	1192.99	155.44	-723.34	-1372.94	-2095.66
$\frac{100(f_{maxcut}^* - \tilde{\psi})}{\tilde{\psi}}$	7.1%	64.9%	13.2%	9.0%	5.0%

TABLE 6. k -cluster: Shor relaxation (f_{maxcut}^*) vs convex quadratic relaxation ($\tilde{\psi} = \psi - n\theta$) with $n = 100$. $\mathbf{A}_{ij} = 0$ with probability $p = 0.8, 0.6$.

formulation of (1.1) reads:

$$\begin{aligned}
(3.5) \quad f^* = \min \quad & \tilde{\mathbf{c}}^T \tilde{\mathbf{x}} + \langle \tilde{\mathbf{F}}, \mathbf{X} \rangle \\
\text{s.t.} \quad & \mathbf{S}_i \tilde{\mathbf{x}} = \tilde{\mathbf{b}}_i, \quad i = 1, \dots, m+n \\
& \mathbf{S}_i \mathbf{X} \mathbf{S}_i^T = \tilde{\mathbf{b}}_i^2, \quad i = 1, \dots, m+n \\
& \mathbf{X}_{ii} = \tilde{\mathbf{x}}_i, \quad i = 1, \dots, 2n \\
& \begin{bmatrix} 1 & \tilde{\mathbf{x}}^T \\ \tilde{\mathbf{x}} & \mathbf{X} \end{bmatrix} \in \mathcal{C}_{2n+1}^*,
\end{aligned}$$

where \mathcal{C}_{2n+1} is the convex cone of $(2n+1) \times (2n+1)$ copositive matrices and \mathcal{C}_{2n+1}^* is its dual, i.e., the convex cone of *completely positive matrices*.

The hard constraint being membership in \mathcal{C}_{2n+1}^* , a strategy is to use hierarchies of tractable approximations (of increasing size) of \mathcal{C}_{2n+1}^* , as described in e.g. Dür [5]. In particular a possible choice for the first relaxation in such hierarchies is to replace (3.5) with the semidefinite program:

$$\begin{aligned}
(3.6) \quad f_{copo}^* = \min \quad & \tilde{\mathbf{c}}^T \tilde{\mathbf{x}} + \langle \tilde{\mathbf{F}}, \mathbf{X} \rangle \\
\text{s.t.} \quad & \mathbf{S}_i \tilde{\mathbf{x}} = \tilde{\mathbf{b}}_i, \quad i = 1, \dots, m+n \\
& \mathbf{S}_i \mathbf{X} \mathbf{S}_i^T = \tilde{\mathbf{b}}_i^2, \quad i = 1, \dots, m+n \\
& \mathbf{X}_{ii} = \tilde{\mathbf{x}}_i, \quad i = 1, \dots, 2n \\
& \begin{bmatrix} 1 & \tilde{\mathbf{x}}^T \\ \tilde{\mathbf{x}} & \mathbf{X} \end{bmatrix} \in \mathcal{S}_{2n+1}^+ \cap \mathcal{N}_{2n+1},
\end{aligned}$$

where \mathcal{S}_{2n+1}^+ (resp. \mathcal{N}_{2n+1}) is the convex cone of real symmetric positive semidefinite (resp. entrywise nonnegative) matrices. Then (3.6) is a semidefinite relaxation of (3.5) because $\mathcal{C}_{2n+1}^* \subset \mathcal{S}_{2n+1}^+ \cap \mathcal{N}_{2n+1}$, and so $f_{copo}^* \leq f^*$. In fact if one considers the problem

$$(3.7) \quad \min_{\mathbf{x} \in \{0,1\}^n} \{ \mathbf{c}^T + \mathbf{x}^T \mathbf{F} \mathbf{x} : \mathbf{A} \mathbf{x} = \mathbf{b}; (\mathbf{A}_i^T \mathbf{x})^2 = b_i^2, i = 1, \dots, m \},$$

which is clearly equivalent to (1.1), then the first SDP-relaxation of the Lasserre-SOS hierarchy associated with (3.7) reads:

$$\begin{aligned}
(3.8) \quad & \min \quad \mathbf{c}^T \mathbf{x} + \langle \mathbf{F}, \mathbf{X} \rangle \\
& \text{s.t.} \quad \mathbf{A} \mathbf{x} = \mathbf{b} \\
& \quad \mathbf{A}_i \mathbf{X} \mathbf{A}_i^T = \mathbf{b}_i^2, \quad i = 1, \dots, n \\
& \quad \mathbf{X}_{ii} = x_i, \quad i = 1, \dots, n \\
& \quad \begin{bmatrix} 1 & \mathbf{x}^T \\ \mathbf{x} & \mathbf{X} \end{bmatrix} \succeq 0,
\end{aligned}$$

which is of the same flavor as the semidefinite program (3.6) but with dimension twice as less than (3.6).

Example 3.4. We first compare the first SDP-relaxation of the MAX-CUT formulation with the first SDP-relaxation (3.6) of the copositive formulation for the linear 0/1 knapsack problems (1.1) with $\mathbf{F} = 0$, $n = 10, 15$ and vector \mathbf{a} as in Example 3.2.

We have kept the formulation on the hypercube $\{0, 1\}^n$ rather than on the hypercube $\{-1, 1\}^n$ and so in fact the first SDP relaxation is for problem (2.4) with a quadratic cost function (and not a quadratic form as in the MAX-CUT formulation on $\{-1, 1\}^n$).

In each case $n = 10$ (resp. $n = 15$), we have chosen 19 (resp. 18) values of the right-hand-side $b = 10s$, $s = 1, \dots, 19$ (resp. $b = 20s$, $s = 1, \dots, 18$), and for each problem we have run a sample of 10 problems with cost vector $\mathbf{c} = \mathbf{a} + 10\eta$ where η is a random variable uniformly distributed in $[0, 1]$.

For $n = 10$, the lower bound f_{maxcut}^* from the MAX-CUT formulation dominates the lower bound f_{copo}^* in (3.6), in 111 out of 190 problems ($\approx 58\%$) and the relative difference $100 \cdot |f_{maxcut}^* - f_{copo}^*| / \max[f_{maxcut}^*, f_{copo}^*]$ never exceeds 0.05% over all 190 problems!

For $n = 15$, $f_{maxcut}^* > f_{copo}^*$ in 94 out of 180 problems ($\approx 52\%$) and $100 \cdot |f_{maxcut}^* - f_{copo}^*| / \max[f_{maxcut}^*, f_{copo}^*]$ never exceeds 0.55% in all 180 problems!

Example 3.5. Next we consider the k -cluster problem (3.2) with $n = 50$ variables and where \mathbf{F} is randomly generated with 80% of zero entries in \mathbf{F} . Then we compare the optimal values:

- f_{Shor}^* of the Shor- (or first SDP-relaxation of the Lasserre-SOS hierarchy) associated with (3.2).
- f_{maxcut}^* of the Shor-relaxation of the MAX-CUT formulation (2.4).
- f_{copo}^* of the Shor-relaxation (3.6) associated with the copositive formulation of (3.2).

Results displayed in Table 7 show that f_{maxcut}^* almost identical to the optimal value f_{copo}^* of Shor relaxation associated with the copositive formulation. However notice that the size of the semidefinite constraint in the latter is twice as large as the one in the former.

n=50	f_{Shor}^*	f_{maxcut}^*	f_{copo}^*	$100(f_{maxcut}^* - f_{copo}^*)/ f_{copo}^* $
k=12	-3.3836	-3.3791	-10.8609	68.8%
k= 14	-0.3708	-0.3674	-0.6658	44.8%
k=16	3.3725	3.3733	3.3725	0.02%
k =18	7.8361	7.8421	7.8361	0.07%
k=20	13.2627	13.2769	13.2627	0.10%
k=22	19.8905	19.8926	19.8905	0.01%
k=24	27.6835	27.6837	27.6835	0.0%
k= 26	36.5762	36.6012	36.5762	0.06%
k=28	46.6564	46.6630	46.6564	0.01%
k=30	57.9790	57.9762	57.9790	-0.00%

TABLE 7. k -cluster: Shor-relaxations of MAX-CUT (f_{maxcut}^*) and copositive (f_{copo}^*) formulations with $n = 50$ and 80% sparsity.

4. CONCLUSION

In this paper we have shown that a linear or quadratic 0/1 program \mathbf{P} has an equivalent MAX-CUT formulation and so the whole arsenal of approximation techniques for the latter can be applied. In particular, and as suggested by some preliminary tests on a (limited sample) of 0/1 knapsack and k -cluster examples, it is expected that the lower bound obtained from the Shor relaxation of this MAX-CUT formulation will be in general better than the one obtained from the standard LP-relaxation (for linear 0/1 programs) of the original problem. The situation might be even better for quadratic 0/1 programs when the Shor relaxation of the MAX-CUT formulation is compared with a convex quadratic relaxation.

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