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Hausdorff Dimension and Derivatives of Typical Nondecreasing Continuous Functions

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Abstract

We endow the space Cr of nondecreasing functions on the unit interval $[0, 1]$ with the uniform metric and consider its subspace Ccr of continuous nondecreasing functions. Then, we mainly prove:

1) In the sense of Baire categories, for most $f \in \text{Ccr}$, and also for most $f \in \text{Cr}$, we have: (a) the Stieljes measure mes_f of f is carried by a set of Hausdorff dimension zero, (b) More precisely, f has zero left and right Diny lower derivatives everywhere outside a set of Hausdorff dimension zero, (c) For any $0 \leq \alpha \leq \infty$, the set of all $t \in [0, 1]$ at which α is the left and right Diny upper derivative of f , is of Hausdorff dimension 1.

2) For most $f \in \text{Ccr}$, we have: (a) the set of all t at which f has an infinite Diny derivative contains a Cantor set in any nonempty open subset of $[0, 1]$, (b) the same is true for the set of t at which f has positive and finite Diny lower derivative.

3) If A is any countable subset of $[0, 1]$, then for most $f \in \text{Cr}$ (a) mes_f is atomic, and f is (left and right) discontinuous at each $t \in A$, (b) f has a zero lower derivative at each of its continuity point.

Some other properties are proved which often mean that the fact that the measure mes_f is carried by a set of null Hausdorff dimension does not imply similar properties for the derivative of f .

We give direct and elementary proofs of all these properties, except (1c)(when $0 < \alpha < \infty$) which is the more tricky one, and except (2b) and (3b) for the proof of which we need a geometric approach, using closely results and methods of [9].

Finally, we also explain how the property 1_c can be adapted to the case of typical functions with bounded variations.

Keywords: Typical monotone function, Diny derivatives, upper Hausdorff dimension of a measure, convex function, curvature, cut locus.

Mathematical Subject Classifications (2010): 26A27, 26A48, 28A78; 26A30, 26A51.

Thanks are due to Zoltán Buczolicz who gave me indications for the main result as mentioned in Remark 5 p. 6.

1 Introduction

The aim of this section is to introduce the definitions and to present our main results.

We denote by $[a, b]$, $]a, b[$, $[a, b[$, and $]a, b]$ the real intervals, respectively closed, open, left-closed-right-open and left-open-right-closed.

DEFINITION 1 (Hausdorff measures) *By a dimension function, we mean a map $h : [0, +\infty[\rightarrow \mathbb{R}$ continuous, nondecreasing, with $h(0) = 0$ and $t > 0 \Rightarrow h(t) > 0$. Let A be a subset of a metric space (E, δ) , μ a measure on E and h a dimension function.*

$\text{diam } A$ denotes the diameter $\sup\{\delta(a, b) \mid a \in A \text{ and } b \in B\}$ of the set A .

$\mathcal{H}_\varepsilon^h(A) = \inf\{\sum h(\text{diam } A_n) \mid (A_n) \text{ is a countable covering of } A \text{ by sets satisfying } \text{diam } A_n \leq \varepsilon\}$, for $\varepsilon > 0$.

$\mathcal{H}^h(A) = \sup_{\varepsilon > 0} \mathcal{H}_\varepsilon^h(A)$ is the Hausdorff measure of A with respect to the dimension function h .

When $h(t) = t^s$, we also write

$$\mathcal{H}^s \text{ and } \mathcal{H}_\varepsilon^s$$

instead of \mathcal{H}^h and $\mathcal{H}_\varepsilon^h$. Here \mathcal{H}^s is the s -dimensional measure.

The Hausdorff dimension $\dim_{\mathbb{H}}(A)$ of A is defined by the fact that $\mathcal{H}^s(A) = 0$ if $s > \dim_{\mathbb{H}}(A)$ and that $\mathcal{H}^s(A) = \infty$ if $0 < s < \dim_{\mathbb{H}}(A)$.

A carries μ if $\mu(E \setminus A) = 0$. For example the support $\text{Supp } \mu$ of μ is the smallest closed subset of E carrying μ . Only atomic measures are carried by a minimal set.

The upper Hausdorff dimension $\dim_{\mathbb{H}}^s \mu$ of μ is the infimum of the Hausdorff dimensions of the subsets of E carrying μ (see [3] or [2], chapter 10).

Using dimension functions, we can precise the information given by the Hausdorff dimension of a set, as in Propositions 1 p. 8, ?? p. ?? and 2 p. 14 below.

Let X be a topological space and $A \subset X$.

The subset A is a G_{δ} of X if it is the intersection of a countable family of open subsets of X . The subset A is a F_{σ} of X if it is the union of a countable family of closed subsets of X . The subset A is *meager* if it is included in the union of a countable family of closed subsets of X of empty interiors.

A property P is said to be “generic” in X (in the sense of Baire categories), or shared by “most elements” of X , or by “typical” elements of X , if the exceptional set of all the $x \in X$ not satisfying P is meager. We will consider more specifically the generic properties of two topological spaces.

DEFINITION 2 (spaces of nondecreasing functions)

Cr denotes the space of all nondecreasing functions endowed with the uniform metric

$$\|g - f\| = \|g - f\|_{[0,1]} \text{ where } \|g - f\|_I = \sup_{t \in I} |g(t) - f(t)|.$$

Cr is complete, but not separable. We observe that $\{f \in \text{Cr} \mid f([0, 1] \text{ is finite})\}$ is everywhere dense in Cr .

Ccr denotes the separable subspace of Cr , of all $f \in \text{Cr}$ which are continuous on $[0, 1]$.

Remark. We could consider functions defined on any interval I of \mathbb{R} , instead of $[0, 1]$, and then use the topology of the uniform convergence on every compact subset of I . All our results in this paper can be adapted with almost the same proof to such a case.

We will consider the usual (sometimes left, right, or lower, or upper) Diny derivatives of a function f at a real number t :

$$\begin{aligned}
 f'(t) &= \lim_{s \rightarrow 0} \frac{f(t+s) - f(t)}{s} \\
 f'_s(t) &= \limsup_{s \rightarrow 0} \frac{f(t+s) - f(t)}{s} & f'_i(t) &= \liminf_{s \rightarrow 0} \frac{f(t+s) - f(t)}{s} \\
 f'_r(t) &= \lim_{s \rightarrow 0^+} \frac{f(t+s) - f(t)}{s} & f_l(t) &= \lim_{s \rightarrow 0^-} \frac{f(t+s) - f(t)}{s} \\
 f'_{r,s}(t) &= \limsup_{s \rightarrow 0^+} \frac{f(t+s) - f(t)}{s} & f'_{r,i}(t) &= \liminf_{s \rightarrow 0^+} \frac{f(t+s) - f(t)}{s} \\
 f'_{l,s}(t) &= \limsup_{s \rightarrow 0^-} \frac{f(t+s) - f(t)}{s} & f'_{l,i}(t) &= \liminf_{s \rightarrow 0^-} \frac{f(t+s) - f(t)}{s}.
 \end{aligned}$$

$f(t+)$ and $f(t-)$ will denote the right and left limit of f at t and we also set $f(0-) = f(0)$ and $f(1+) = f(1)$ when $f \in \text{Cr}$.

We will use brackets to denote some sets of points associated to f : for example

$$[f'_{r,s} = 2]$$

denotes the set of points where the right upper Diny derivative of the function f equals 2.

DEFINITION 3 For $f \in \text{Cr}$ we define:

- mes_f the Stieljes measure of f , thus $0 \leq t < 1 \Rightarrow f(t+) = \text{mes}_f[0, t]$.
- $\text{disc}_t(f) = \text{mes}_f(\{t\})$ ($= f(t+) - f(t-)$ when $0 < t < 1$),
- $\text{Disc } f = \{t \in [0, 1] \mid \text{disc}_t(f) > 0\}$.

Thus mes_f is atomic if it is carried by $\text{Disc } f$.

We now recall some previous results.

Young 1911 [11]. Every $f \in \text{Cr}$ is differentiable at almost every $t \in [0, 1]$ (Lebesgues 1904 for the case where f is continuous). Other similar properties are reviewed by D.L. Renfro in [7].

Zamfirescu 1981 [12] For most $f \in \text{Ccr}$ (and also for most $f \in \text{Cr}$) and for almost every $t \in [0, 1]$, $f'(t) = 0$ and we have:

- $\forall t \in]0, 1], f'_{l,i}(t) = 0$ or $f'_{l,s}(t) = +\infty$
- $\forall t \in [0, 1[, f'_{r,i}(t) = 0$ or $f'_{r,s}(t) = +\infty$

For the second point, one observes that $\{f \in \text{Ccr} \mid \exists 0 \leq t \leq 1 - 1/n \mid \forall s \in [t, t + 1/n], (s - t)/n \leq f(s) - f(t) \leq n(s - t)\}$ is a nowhere dense closed subset of Ccr . The first assertion follows then by Young's result above.

Buczolich and Nagi 1999 [1] For most $f \in \text{Ccr}$ and for all $0 \leq \alpha \leq 1$, $\dim_H E_{f,\alpha} = 1$ where $E_{f,\alpha}$ is the set of the $t \in [0, 1]$ where α is the **Höder exponent** of f , that is the larger real number such that the function $s \mapsto |f(s) - f(t)|/|s - t|^\alpha$ is bounded. Moreover mes_f is carried by $E_{f,0}$ and thus $\dim_H^s \text{mes}_f = 0$.

the support Supp mes_f of the measure mes_f is the smallest closed set carrying mes_f , but it is well known that, when mes_f is not atomic, there is no canonical $A \subset [0, 1]$ carrying mes_f and of Hausdorff dimension $\dim_H^s \text{mes}_f$. However one could expect some link between $\dim_H^s \text{mes}_f$ and $\dim_H[f'_s > 0]$, for instance. This is (typically!) not the case, because we may have $\dim_H^s \text{mes}_f = 0$, and $\dim_H[f'_s > 0] = 1$ by our following result.

THEOREM 1 1. For most $f \in \text{Ccr}$, and also for most $f \in \text{Cr}$, we have:

- (a) $\dim_H^s \text{mes}_f = 0$,
- (b) $\dim_H[f'_{l,i} > 0] \cup [f'_{r,i} > 0] = 0$,
- (c) $\forall 0 \leq \alpha \leq \infty, 1 = \dim_H[f'_{l,s} = f'_{r,s} = \alpha]$.

2. For most $f \in \text{Ccr}$, we have:

- (a) $[f' = \infty]$ contains a Cantor set in any nonempty open subset of $[0, 1]$,
- (b) $[0 < f'_i < \infty]$ contains a Cantor set in any nonempty open subset of $[0, 1]$.

3. If A is a countable subset of $[0, 1]$, then for most $f \in \text{Cr}$

- (a) mes_f is carried by $\text{Disc } f \supset A$, and $f(t-) < f(t) < f(t+)$ for all $t \in \text{Disc } f$,
- (b) $\text{Disc } f = [f' = \infty]$ and $[0, 1] \setminus \text{Disc } f = [f'_i = 0]$.

Remarks.

1. 1a was already proved for Ccr in [1], as mentioned above (our proof is shorter and will give a slightly more precise statement concerning the size of set $[f'_{l,i} > 0] \cup [f'_{r,i} > 0]$ carrying mes_f , see Prop.1 p. 8). 1b and 1c implies that $1 = \dim_{\mathbb{H}}[f'_{l,s} = f'_{r,s} = \alpha]$ and $0 = [f'_{l,i} = f'_{r,i}]$ and hence the Hausdorff dimension of the set of all points where f is not differentiable is one (see also the remark 2 p.10) and the remark 3 p.11 after Proposition 1 about the size of $[f'_{l,i} = f'_{r,i} = 0] \cap [f'_{l,s} = f'_{r,s} = \infty]$.
2. (1b) implies (1a). Concerning most $f \in \text{Cr}$, (3a) implies (1a) and (3b) implies (1b).
3. Let $\lambda \in [0, 1]$ and $\text{Cr}_\lambda = \{f \in \text{Cr} \mid \forall t \in \text{Disc } f, f(t) = (1 - \lambda)f(t-) + \lambda f(t+)\}$. Thus Cr_λ is a closed subspace of Cr and Cr_1 is the set of right continuous $f \in \text{Cr}$.
Then Theorem 1 remains true if we delete “ $f(t-) < f(t) < f(t+)$ for all $t \in \text{Disc } f$ ” in 3a, if we add in 3 that $\lambda = 0 \Rightarrow 1 \notin A$ and that $\lambda = 1 \Rightarrow 0 \notin A$, and if we substitute everywhere Cr for Cr_λ ; with the same proof (in which we make the same substitution).
4. Note the paradox in the easy result 3a: For every $f \in \text{Cr}$, $\text{Disc } f$ is countable; but for each $t \in [0, 1]$, for most $f \in \text{Cr}$ we have $t \in \text{Disc } f$.
5. We consider 1c, when $0 < \alpha < \infty$, as our **main result**.

The case $\alpha = 0$ is clear: the set has even full Lebesgue measure.

The case $\alpha = \infty$ is also simpler and more precise: for all \mathcal{H}^1 -null compact set $A \subset [0, 1]$ and for most $f \in \text{Cr}$, we have $A \subset [f'_s = \infty]$. For the case $f \in \text{Ccr}$ and the symmetric derivative, it is also a consequence of the mentioned result of [1].

1c obviously implies that $1 = \dim_{\mathbb{H}}[0 < f'_s < \infty]$. This was our main result in a previous version of this work. The given proof used closely [9], which involves geometric tools, and were more intricate than the proof that we will give of 1c (see the beginning

of section 3 for more details).

This proof of 1c has been suggested to us by Zoltán Buczolic who advised us to adapt some of the proofs of [1] to get the result (it could also apply to the case $\alpha = \infty$).

6. Lemma 1 below gives a way, for the explicit construction of strictly increasing $f \in \text{Cr}$ such that mes_f is atomic and such that

$$f'_i = 0 \text{ on } [0, 1] \text{ and } f'_{r,i} = 0 \text{ on } [0, 1[$$

(but we must have $f'_{r,s} = f'_{l,s} = +\infty$ on a dense G_δ of $[0, 1]$ for such an f). Lemma 2 describes the construction of $f \in \text{Cr}$ with mes_f atomic and with large a set of points at which $f' = 1$. Lemmata 1 and 2 will not be used for the study of typical functions.

LEMMA 1 *Let $(a_n)_{n \geq 1}$ be a sequence of pairwise distincts elements of $]0, 1[$ and $A = \{a_n \mid n \geq 1\}$. Let $(p_n)_{n \geq 1}$ be some non increasing sequence of positive numbers satisfying (1): $\limsup p_{n+1}/p_n < 1$ and (2): $p_n / \min\{|a_n - a_k| \mid 0 \leq k < n\} \rightarrow 0$. Let $f \in \text{Cr}$ such that $A = \text{Disc } f$ carries mes_f and such that $\text{mes}_f(\{a_n\}) = p_n$. Thus for $t \in [0, 1]$ we must have*

$$f(t+) - f(0) = \sum_{n \geq 1, a_n \leq t} p_n.$$

Then we also have for all $t \in [0, 1]$: (a) : $f(t) = f(t+) \Rightarrow f'_{r,i}(t) = 0$ and (b) : $f(t) = f(t-) \Rightarrow f'_{l,i}(t) = 0$.

Remark. The function f of Lemma 1 is strictly increasing when the sequence (a_n) is dense in $[0, 1]$ and we can ask it to be right continuous.

Proof of Lemma 1. Let be $t \in [0, 1[$ such that $f(t) = f(t+)$. We define inductively a decreasing sequence (b_n) in $]t, 1]$ converging to t , by the conditions $b_0 = 1$ and, for $n \geq 1$, b_n is the first of the real numbers a_k which belongs to $]t, b_{n-1}[$. We also define $q_n = \text{mes}_f(\{b_n\})$. From (2) we have $q_n/(b_n - b_{n-1}) \rightarrow 0$, hence from (1) we also have $(f(b_n) - f(b_{n-1}))/(b_n - b_{n-1}) \rightarrow 0$, hence $(f(b_n) - f(t))/(b_n - t) \rightarrow 0$ and thus $f'_{r,i}(t) = 0$. This proves (a), the proof of (b) is quite similar.

LEMMA 2 *Let K be a \mathcal{H}^1 -null compact subset of $]0, 1[$, then there is $f \in \text{Cr}$ such that $\text{Disc}_f = A$ carries mes_f , $A \cap K = \emptyset$, $f' = 1$ on K and $\text{Supp } f = K \cup A$.*

Proof of Lemma 2. We can choose a discrete and closed subset A of $[0, 1] \setminus K$ such that for all $n \in \mathbb{N}^*$ and $t \in [0, 1]$, $\text{dist}(t, K) \leq 1/n \Rightarrow \text{dist}(t, A) \leq 1/n^2$ (we need $K \cup A$ to be “porous” at no point of K). Then one can take for the wanted f , the only right continuous $f \in \text{Cr}$ such that $f(t) = t$ if $t \in K \cup A$ and $f'(t) = 0$ if $t \notin K \cup A$.

Questions. We do not know if for most $f \in \text{Ccr}$, there exist points $t \in]0, 1[$ at which $f'_{l,i}(t) \in \mathbb{R}_+^*$ and $f'_{r,i}(t) \in \mathbb{R}_+^*$.

We do not know if for most $f \in \text{Cr}$, we have $[0, 1] \setminus \text{Disc } f = [f'_{l,i} = f'_{r,i} = 0]$ (it is compatible with 3a by Lemma 1).

Concerning Lemma 2, we actually do not know what can be the size, for instance the Hausdorff dimension, of the set $[f' = \infty]$ for a function $f \in \text{Cr}$ with mes_f atomic. In a previous version of this paper, we made a mistake in “proving” that this Hausdorff dimension can be equal to 1.

2 Direct proofs concerning Theorem 1

We will give here the proofs of the statements of Theorem 1 which do not use geometric arguments, excepted 1c when $0 < \alpha < \infty$.

Proof of 3a. Let $A \subset \{a_n, n \in \mathbb{N}\}$. Then we associate to every $n \in \mathbb{N}^*$ an everywhere dense open subset Ω_n of Cr by defining Ω_n as the set of all $f \in \text{Cr}$ such that there exists a finite subset B of $\text{Disc } f$, containing $\{a_0, \dots, a_n\}$, such that $f(b) - f(a) - \sum_{t \in B} \text{disc}_t(f) < 1/n$ and such that $\forall t \in B, f(t-) < f(t) < f(t+)$. Now most $f \in \text{Cr}$ belong to $\bigcap_{n \geq 1} \Omega_n$, and thus satisfy 3a.

Proof of 1a, 1b and “1c when $\alpha = \infty$ ” . 1a and 1b result of 1 of the following proposition, taking for instance $h(t) = -1/\ln t$ when $t > 0$ is small enough, while 1c with $\alpha = \infty$ results of 2, choosing A of Hausdorff dimension 1.

PROPOSITION 1 *Let h be a dimension function and $A \subset [0, 1]$. Then for most $f \in \text{Ccr}$:*

1. mes_f is carried by some \mathcal{H}^h -null subset B of $[0, 1]$ containing the set $[f'_{l,i} > 0] \cup [f'_{r,i} > 0]$.

2. If A is included in an F_σ \mathcal{H}^1 -null subset of $[0, 1]$, then $f'_{l,s} = \infty$ on $]0, 1] \cap A$ and $f'_{r,s} = \infty$ on $]0, 1[\cap A$.

3. If A is meager in $[0, 1]$, then $f'_{l,i} = f'_{r,i} = 0$ on A ,

Moreover 1 and 2 are also true or for most $f \in \text{Cr}$.

We observe that $[f'_{l,i} > 0] \cup [f'_{r,i} > 0] = [0, 1] \setminus [f'_{l,i} = f'_{r,i} = 0]$. Moreover, the third part of Theorem 1 shows that 3 does not hold for most $f \in \text{Cr}$.

Proof of Proposition 1. We only give the proofs concerning most $f \in \text{Ccr}$, because these about most $f \in \text{Cr}$ are almost identical.

1. For $n \in \mathbb{N}^*$, we define an everywhere dense open subset of Ccr by $\Omega_n = \{f \in \text{Ccr} \mid \exists 0 = t_0 < t_1 < \dots < t_{N+1} = 1 \text{ and } r > 0 \text{ such that } Nh(4r) \leq 2^{-n}, \text{mes}_f \left([0, 1] \setminus \bigcup_{1 \leq k \leq N} [t_k - r, t_k + r] \right) < r/n \text{ and for every } 0 \leq k \leq N: 4r < t_{k+1} - t_k < 1/n\}$.

Let $f \in \bigcap_{n \geq 1} \Omega_n$, we chose thus for each $n \in \mathbb{N}^*$ a sequence $(t_{k,n})_{0 \leq k \leq 1+N_n}$ and an $r_n > 0$ corresponding to $f \in \Omega_n$. We set then

$$A_n = \bigcup_{1 \leq k \leq N_n} [t_{k,n} - 2r_n, t_{k,n} + 2r_n], B_n = \bigcup_{m \geq n} A_m \text{ and } B = \{0, 1\} \cup \bigcap_{n \geq 1} B_n.$$

For all $n \geq 1$ we have $\mathcal{H}_{1/n}^h(A_n) \leq 2^{-n}$ and thus $\mathcal{H}_{1/n}^h(B) \leq \mathcal{H}_{1/n}^h(B_n) \leq 2^{-n+1}$, so $\mathcal{H}^h(B) = 0$.

For all $n \geq m \geq 1$ we have $\text{mes}_f([0, 1] \setminus B_m) \leq \text{mes}_f([0, 1] \setminus A_n) \leq r_n/n \leq 1/n$, thus $\text{mes}_f([0, 1] \setminus B_m) = 0$, thus $\text{mes}_f([0, 1] \setminus B) = \lim_{m \rightarrow \infty} \text{mes}_f([0, 1] \setminus B_m) = 0$.

Let be $t \in [0, 1] \setminus B$ and $m \in \mathbb{N}^*$, we can choose $n \geq m$ such that $t \in [0, 1] \setminus A_n$. We have then $t + r_n < 1 \Rightarrow f(t + r_n) - f(t) < r_n/n$ and $t - r_n > 0 \Rightarrow f(t) - f(t - r_n) < r_n/n$, moreover $r_n < 1/4n < 1/m$, thus $f'_{l,i}(t) = f'_{r,i}(t) = 0$.

2. We can suppose that A is compact. We define now an everywhere dense open subset of Ccr by $\Omega_n = \{f \in \text{Ccr} \mid \exists 0 = t_0 < t_1 < \dots < t_{N+1} = 1 \text{ and, for each } 0 \leq k \leq N, 0 < r_k < 1/n \text{ such that } k < N \Rightarrow t_k + 2r_k < 1 \text{ and } f(t_k + 2r_k) - f(t_k + r_k) > nr_k, k > 0 \Rightarrow t_k - 2r_k > 0 \text{ and } f(t_k - r_k) - f(t_k - 2r_k) > nr_k, \text{ and also such that } A \subset \bigcup_{k=0}^N]t_k - r, t_k + r_k[\}$.

But $f \in \bigcap_{n \geq 1} \Omega_n \Rightarrow f'_{l,i} = f'_{r,i} = \infty$ on A .

3. It is the same proof as for 2), using now Ω_n as the set of all $f \in \text{Ccr}$ for which there exists an open neighborhood ω of A with a finite number of connected components, all of them with lengths less than $1/n$, and such that $\text{mes}_f(\omega) < \text{dist}(A, [0, 1] \setminus \omega)/n$.

Second proof of 1 of Proposition 1 for most $f \in \text{Ccr}$. it results of 3, indeed one can choose an \mathcal{H}^h -null G_δ -subset G of $[0, 1]$ containing $\mathbb{Q} \cap [0, 1]$ and apply 3 to $A = [0, 1] \setminus B$. We conclude because for every $f \in \text{Cr}$ we have $\text{mes}_f([f'_{l,i} = f'_{r,i} = 0]) = 0$.

Remarks.

1. In 2 of Proposition 1, A is a meager and \mathcal{H}^1 -null subset of $[0, 1]$.

However there exists a nowhere dense \mathcal{H}^h -null subset A' of $[0, 1]$ which cannot be included in the union of a countable family of \mathcal{H}^1 -null compact subsets (choose a Cantor subspace K of $[0, 1]$ such that every nonempty open subset of K has a positive \mathcal{H}^1 -measure, and then a \mathcal{H}^h -null dense G_δ subset A' of K).

One get thus a “first category Lebesgue null set (and even of null Hausdorff dimension) A' that is not σ -Jordan null” See [8] for more details and a historical account of the subject. *Jordan-null* means of \mathcal{H}^1 -null adherence, or equivalently, for every $\varepsilon > 0$, included in the union of a finite family of intervals with the sum of lengths $< \varepsilon$.

2. I don't know if for every \mathcal{H}^1 -null subset A of $[0, 1]$, most $f \in \text{Ccr}$ (or most $f \in \text{Cr}$) satisfy $f'_s = +\infty$ on A . I don't know if for every \mathcal{H}^1 -null subset A of $[0, 1]$, most $f \in \text{Ccr}$ satisfy $f'_i = 0$ on A .

However, Gyorgy Petruska [5] has announced that an analytic subset of $[0, 1]$ is included in an \mathcal{H}^1 -Null F_σ subset if and only if for most $f \in \text{Cr}$, f is differentiable at no point of A . G. Petruska mentions there, as an essential feature from [4], that when an analytic¹ subset of $[0, 1]$ is not included in an \mathcal{H}^1 -Null F_σ subset, then it is residual in a compact subset K such every nonempty open subset of K have positive

¹J. Mycieszki and R. Laver have communicated to G. Petruska the existence of an \mathcal{H}^1 -null subset A of $[0, 1]$, not included in any \mathcal{H}^1 -Null F_σ subset, and not residual in any perfect set.

\mathcal{H}^1 -measure. This fact is also used in a previous similar result of D. Preiss and J. Tišer [6] about typical Lipschitz functions.

The following result, analogous to 2 and 3 of Prop. 1, is also proved in [6] (in the proof of the theorem, p.224-225): Let A be any \mathcal{H}^1 -Null F_σ subset of $[0, 1]$, then most 1-Lipschitz maps f on $[0, 1]$ satisfy $f'_s = 1$ and $f'_i = -1$ on A .

3. For any $\lambda \in [0, +\infty]$ and $f \in \text{Ccr}$, $[f'_s \geq \lambda]$, $[f'_i \leq \lambda]$ and $[f'_{l,i} = f'_{r,i} = 0] \cap [f'_{l,s} = f'_{r,s} = \infty]$ are G_δ subsets of $[0, 1] \setminus \text{Disc } f$, hence of $[0, 1]$. Moreover, by 1 and 2 for a suitable A , they are also everywhere dense if f is typical in Ccr or in Cr , and more precisely for every nonempty open subset ω of $[0, 1]$, each of them contains a subset of ω of Hausdorff dimension 1.
4. Let us observe the following paradox: For most $f \in \text{Cr}$, $f'(t) = 0$ at \mathcal{H}^1 -almost all $t \in [0, 1]$; however for every $t \in [0, 1]$, most $f \in \text{Cr}$ satisfy $f'_s(t) = +\infty$ (taking $A = \{t\}$). Idem for most $f \in \text{Ccr}$.

LEMMA 3 *Let $R > 0$, $a < b$ and $f : [a, b] \rightarrow \mathbb{R}$ continuous and such that $E = \{t \in [a, b] \mid f'_{r,i} > R\}$ is countable. Then $f(b) - f(a) \leq R(b - a)$.*

Proof of Lemma 3. We choose $g : [a, b] \rightarrow \mathbb{R}$ nondecreasing and discontinuous at each $t \in E$. We have to check for every $\varepsilon > 0$ that $b \in E_\varepsilon = \{t \in [a, b] \mid f(t) - f(a) \leq (R + \varepsilon)(t - a) + \varepsilon(g(t) - g(a))\}$. But if it was not true, we should get a contradiction at $T = \sup E_\varepsilon$.

LEMMA 4 *Let $f \in \text{Ccr}$ such that mes_f is carried by an \mathcal{H}^1 -null set $A \subset [0, 1]$. Then for mes_f -almost every $t \in A$ we have $f'(t) = +\infty$.*

(It is also true for mes_f -almost every $t \in [0, 1]$!)

Proof of Lemma 4. It will be enough to prove that for mes_f -almost every $t \in A$ we have $f'_r(t) = +\infty$. If that last statement were not true, we could choose $R \in \mathbb{R}$ and K a compact subset of $[0, 1]$ such that $\text{mes}_f(K) > 0$ and $f'_{r,i} \leq R$ on K . Let $g : [0, 1] \rightarrow \mathbb{R}$ be defined by $g(t) = \text{mes}_f(K \cap [0, t])$. If $K \subset L$ and if L is the union of a finite family of closed intervals, by Lemma 3 applied to g we have $\text{mes}_f(K) = \text{mes}_g(K) \leq R\mathcal{H}^1(L)$, thus $\text{mes}_f(K) \leq R\mathcal{H}^1(K) = 0$, which is absurd and ends the proof.

Proof of 2a. Now for most $f \in \text{Ccr}$, we know from Proposition 1 that mes_f is carried by an \mathcal{H}^1 -null set and that f is strictly increasing. Thus 2a results from Lemma 4.

3 Geometric proofs concerning Theorem 1

In the following definitions, to any nondecreasing function f , we will associate a primitive, thus a convex function, and then a convex subset of \mathbb{R}^2 . From this, properties of typical f will be, roughly speaking, compared with properties of typical convex sets.

DEFINITION 4 *Let be $f \in \text{Cr}$ and $I \subset \mathbb{R}$, we define:*

- \tilde{f} the primitive of f vanishing at zero.
- $\Gamma_{f,I} = \{(t, \tilde{f}(t)) \mid t \in I\}$ and $\Gamma_f = \Gamma_{f,[0,1]}$.

Now suppose that B is a Euclidean disk in \mathbb{R}^2 whose boundary circle ∂B meets Γ_f in exactly the point $p = (t, \tilde{f}(t))$ for some $t \in]0, 1[$. If $B \supset \Gamma_f$ and is minimal for that, it should be reasonable to think that the radius of B is a kind of radius of lower curvature of Γ_f at p , and thus related to $f'_i(t)$. Because of this we will be concerned with the cut locus relative to the farthest projection to Γ_f .

In a previous version of this work, a similar link between a radius of upper curvature of Γ_f and f'_s were used to investigate $\dim_H[0 < f'_s < \infty]$, using more closely the main proofs of [9], concerned with the cut locus relative to the nearest projection to Γ_f . Let us mention that those links between this work and [9] were the reason why we were interested in the derivatives of typical nondecreasing functions (first in view of solving the geometric problem, secondly in view of applying the geometric results).

3.1 Cut locus relative to the farthest projection

From now, d is an integer ≥ 2 , but in this work we are only concerned with the case $d = 2$. \mathbb{R}^d is endowed with its usual Euclidean norm $\|\cdot\|$.

DEFINITION 5 (farthest projection)

Let $\emptyset \neq F \subset \mathbb{R}^d$, $a \in \mathbb{R}^d$, $r > 0$, and $A \subset \mathbb{R}^d$. We define

$$\text{adist}(a, F) = \sup\{\|a - p\| \mid p \in F\} \in [0, +\infty],$$

$$\text{Aproj}_F(a) = \{p \in F \mid \|a - p\| = \text{adist}(a, F)\},$$

$$\text{Aproj}_F(A) = \bigcup_{b \in A} \text{Aproj}_F(b).$$

When there is only one farthest projection of a to F , we denote it by $\text{aproj}_F(a)$, we have then $\text{Aproj}_F(a) = \{\text{aproj}_F(a)\}$.

Proof of 3b. Let K be a compact subset of \mathbb{R}^2 , we define an everywhere dense open subset Ω_K of Cr by $\Omega_K = \{f \in \text{Cr} \mid \{0, 1\} \subset \text{Disc } f \text{ and } \exists \varepsilon > 0 \mid \forall a \in K \text{ and } b \in \mathbb{R}^2 \text{ with } \|b - a\| < \varepsilon, \text{Aproj}_{\Gamma_f}(b) \subset \text{Aproj}_{\Gamma_f}(a) \subset \Gamma_{f, \text{Disc } f}\}$. Thus for most $f \in \text{Cr}$ we have $\{0, 1\} \subset \text{Disc } f$ and $\text{Aproj}_{\Gamma_f}(\mathbb{R}^2) \subset \Gamma_{f, \text{Disc } f}$. But this implies that $f'_i = 0$ on $[0, 1] \setminus \text{Disc } f$, 3b follows because $f'_i = +\infty$ on $\text{Disc } f$ by 3a.

DEFINITION 6 (cut locus)

When $a, b \in \mathbb{R}^d$ we write $[a, b] = \{(1 - \lambda)a + \lambda b \mid 0 \leq \lambda \leq 1\}$.

Let F be a nonempty compact subset of \mathbb{R}^d . We define

$$\text{a}\mathcal{M} \text{ or } \text{a}\mathcal{M}_F = \{a \in \mathbb{R}^d \mid \text{card } \text{Aproj}_F(a) \geq 2\},$$

$$\text{a}\mathcal{N} \text{ or } \text{a}\mathcal{N}_F = \{a \in \mathbb{R}^d \mid \forall b \in \mathbb{R}^d \setminus \{a\} \text{ and } p \in \text{Aproj}_F(b), b \notin [a, p]\}, \text{ the cut locus relative to the farthest projection to } F.$$

We will need the following easy properties of these sets:

LEMMA 5 *Let F be a nonempty compact subset of \mathbb{R}^d . Then*

1. $\text{a}\mathcal{M}_F \subset \text{a}\mathcal{N}_F$,
2. $\text{a}\mathcal{M}_F$ is an F_σ -subset of \mathbb{R}^d ,
3. $\text{a}\mathcal{N}_F$ is a G_δ -subset of \mathbb{R}^d ,
4. $\text{a}\mathcal{N}_F$ is of empty interior in \mathbb{R}^d .

Proof of Lemma 5. Because the Euclidean norm is strictly convex, for all $a \in \mathbb{R}^d$, $b \in \mathbb{R}^d \setminus \{a\}$ and $p \in \text{Aproj}_F(b)$ we have $b \in [a, p] \Rightarrow p = \text{aproj}_F(a)$, thus $a \notin \text{a}\mathcal{N} \Rightarrow a \notin \text{a}\mathcal{M}$, this proves 1; thus also $b \notin \text{int}_{\mathbb{R}^d} \text{a}\mathcal{N}$, this proves 4. Property 2 results from the fact that for every $\varepsilon > 0$, the set $\{a \in \mathbb{R}^d \mid \text{diam } \text{Aproj}_F(a) \geq \varepsilon\}$ is closed. Finally, property 3 results from the fact that for every $\varepsilon > 0$, the set $\{a \in \mathbb{R}^d \mid \exists b \in \mathbb{R}^d \text{ such that } \|b - a\| \geq \varepsilon, p \in \text{Aproj}_F(b) \text{ and } b \in [a, p]\}$ is closed.

Proof of 2b. It follows from the lemme:

LEMMA 6 *Let $f \in Ccr$ strictly increasing and such that the set $\{t \in [0, 1] \mid f'_i(t) = 0\}$ is everywhere dense in $[0, 1]$. Then $[0 < f'_i < \infty]$ contains a Cantor set in any nonempty open subset of $[0, 1]$*

Proof of Lemma 6. Let $0 < \alpha < \beta < 1$ and $F = \Gamma_f$. Then $\Omega = \{a \in \mathbb{R}^2 \mid \text{Aproj}_F(a) \subset \Gamma_{f,] \alpha, \beta[}\}$ is a nonempty open subset of \mathbb{R}^2 .

We claim that $\tilde{\Omega} \cap \text{a}\mathcal{M}$ is everywhere dense in Ω . Else we could choose $t \in]\alpha, \beta[, p = (t, \tilde{f}(t)), a \in \Omega$ such that $p = \text{aproj}_F(a)$ and $r > 0$ such that $B(a, r) \subset \Omega \setminus \text{a}\mathcal{M}$. If x is a unit vector of \mathbb{R}^2 orthogonal to $p - a$, then $t \mapsto \text{aproj}_F(a + tx)$ defines a continuous map $\varphi :]-\varepsilon, \varepsilon[\rightarrow F$ which is not constant, and then the subset $\varphi(]-\varepsilon, \varepsilon[)$ of $\text{Aproj}_F(\Omega)$ is of nonempty interior in F . But this is impossible because $\text{Aproj}_F(\mathbb{R}^2)$ contains no point $(s, \tilde{f}(s))$ such that $f'_i(s) = 0$, and because these points are everywhere dense in F .

Now by Lemma 5 and Baire's theorem, $\Omega \cap \text{a}\mathcal{N} \setminus \text{a}\mathcal{M}$ is a G_δ -dense subset of Ω , and contains thus a Cantor set K . Then, if we set $\text{pr}_1(x_1, x_2) = x_1$, the image set $\text{pr}_1(\text{aproj}_F(K))$ is a Cantor set included in $[0 < f'_i < \infty] \cap]\alpha, \beta[$ (observe that aproj_F is continuous on $\mathbb{R}^2 \setminus \mathcal{M}_F$ and thus on K), that ends the proof.

4 Proof of 1c when $0 < \alpha < \infty$

The only statement not already proved in Theorem 1 is 1c when $0 < \alpha < \infty$, which results from the following proposition, taking for instance $h(t) = t|\ln t|$, for $t > 0$ small enough. Moreover we get thus that $[0 < f'_s < \infty]$ has no countable covering (A_n) by sets of Hausdorff dimensions < 1 .

PROPOSITION 2 *Let h be some concave² dimension function such that $t/h(t) \rightarrow 0$ in zero. Then for most $f \in Cr$, and also for most $f \in Ccr$, we have for all $\alpha \in]0, +\infty[$:*

$$0 < \mathcal{H}^h[\alpha = f'_{l,s} = f'_{r,s}].$$

Hence we also have

$$1 = \dim_{\text{H}}[\alpha = f'_{l,s} = f'_{r,s}].$$

²Actually the concavity is not needed because one can substitute h with the smallest concave function h_2 which is $\geq h_1(t) = t \inf_{]0, t]} h(t)/t$.

We already know that, for $\alpha = 0$ and $\alpha = \infty$, with more precise result.

Obviously we cannot use the same paradoxal method of proof as for $\alpha = \infty$. We will have to build a large Cantor set $K_{f,\alpha}$ depending on the typical function f and on α , which will be “almost included” in the wanted set, but in a last step we will have to eliminate a small part of it using the following rather easy but **crucial** Lemma:

LEMMA 7 (Lemma 7 of [1])

Let $f \in \text{Cr}$, $0 < \alpha < \infty$, $0 \leq a < b \leq 1$ and E be the union of all $[s, t] \subset [a, b]$ such that $s \neq t$ and $\frac{f(t)-f(s)}{t-s} \geq \alpha$. Then we have

$$\mathcal{H}^1(E) \leq \frac{2}{\alpha} (f(b) - f(a)). \quad (1)$$

Remark. We will prove more precisely that the lemma remains true when we substitute $\frac{f(t)-f(s)}{t-s}$ with $\frac{f(t+)-f(s-)}{t-s}$.

Proof of Lemma 7. For the case $f \in \text{Ccr}$ Buczolic and Nagi, give a short and elegant proof, related to the link between the variation of some $f \in \text{Ccr}$ and the Lebesgue integration of f' . But their proof is easily adaptable to the case $f \in \text{Cr}$. So we just sketch below a more direct proof, longer but perhaps more instructive to get a picture of why this is true:

It is enough to prove for any $\varepsilon > 0$ that $\mathcal{H}^1(E_\varepsilon) \leq \frac{2}{\alpha} (f(b) - f(a))$ where $E_\varepsilon = E_{f,[a,b],\alpha,\varepsilon}$ denotes the union of all $[s, t] \subset [a, b]$ such that $t - s \geq \varepsilon$ and $\frac{f(t+)-f(s-)}{t-s} \geq \alpha$.

When $E_\varepsilon \neq \emptyset$, it admits a smallest element. Now, by a proof similar to the proof of the compactness of $[a, b]$, one checks that each E_ε is included in a (minimal) finite union of such intervals $[s, t]$. Such a covering can be decomposed in two parvise disjoint subfamilies and one get the wanted inequality by considering the subfamily with union having the greatest length.

The building of $K_{f,\alpha}$ will use the following rather standard Lemma (other standard simillary lemma could be used, we do not proceed here as Buczolic and Nagi for instance).

LEMMA 8 Let $h = t^s \theta(t)$ be some dimension function such that $\theta(t)$ is decreasing on $]0, +\infty[$ (thus h is doubling). Let K be the intersection of a decreasing sequence of compact sets (K_n) , each K_n being the finite union of pairwise disjoint intervals of side r_n . $K_0 = [0, 1]$. If C is one of the intervals of K_{n-1} , we suppose that C is divided in N_n intervals C_i with sides $\rho_n = r_{n-1}/N_n$ and $C \cap K_n$ will be the union of the intervals C'_i where C'_i is

the interval of length r_n and with same center as C_i . r_n , and thus K_n is defined by N_n and by the condition of \mathcal{H}^h -mass repartition:

$$h(r_{n-1}) = N_n^d h(r_n). \quad (2)$$

Then K is an h -set, i. e. $0 < \mathcal{H}^h(K) < \infty$.

Thus to define such a K , there only remains to explain how are choosed the integers $N_n \geq 2$ at the step n .

Observe also that the sets $K \cap C$, where C are intervals of K_n , are pairwise isometric and thus of same \mathcal{H}^h -measure, so that, up to a multiplicative constant, \mathcal{H}^h restricted to K is the same as the canonical probability on K associated with the description of K .

If we suppose $h(r_{n-1}) \leq N_n^d h(r_n)$ instead of (2) we then get $0 < \mathcal{H}^h(K)$ by (part of) the same proof. We will not formally use it in the proof of Prop 2 (but think of it!).

Proof of Lemma 8 This lemma is just the first statement of Lemma 4 of [10], when $d = 1$.

Proof of Proposition 2

1) First we give an idea of the scheme of the proof. We will use some countable dense subset D of $]0, +\infty[$ (which actually needs only to be choosed at the beginning of the part 6 of this proof). The needed G_δ will be defined by $G = \bigcap_{m \geq 1} \Omega_m^*$ with $\Omega_m^* = \bigcap_{\alpha \in D} \Omega_{m,\alpha}^*$, $\Omega_{m,\alpha}^* = \bigcup_{n \geq m} \Omega_{n,\alpha}$ where for any $\alpha > 0$, the set $\Omega_{n,\alpha}$ will be open in Cr , $\Omega_{m,\alpha}^*$ dense in Cr and $\text{Cr} \cap \Omega_{m,\alpha}^*$ dense in Ccr . Then each $f \in G$ will satisfy the statement of Proposition 2.

Actually $f \in G_\alpha := \bigcap_{n \geq 1} \Omega_{n,\alpha}^*$ will be enough to check the statement of Prop. 2 relative to that α , but $f \in G$ will permit to consider all $\alpha > 0$, and not only those belonging to D .

Given some $f \in G$ and $\alpha > 0$, we will build some standard \mathcal{H}^h -large Cantor set $K_{f,\alpha}$. We will then use Lemma 7 to find a \mathcal{H}^h -non negligible subset $K'_{f,\alpha} \subset [\alpha = f'_{l,s} = f'_{r,s}]$, by eliminating some intervals at each step of the building of $K_{f,\alpha}$. Lemma 8 will actually only be used to check that $\mathcal{H}^{h_0}(K'_{f,\alpha}) > 0$ for some dimension function $h_0 \leq h$, and thus to check that $\mathcal{H}^h(K'_{f,\alpha}) > 0$. Indeed, because $F_{f,\alpha}$ will be an h_0 -set, it will be possible to give sense, to the ‘‘proportion’’ of $K'_{f,\alpha}$ in $K_{f,\alpha}$, and to use it.

2) It will be convenient to use triadic numbers to have an exact compatibility between the various scales of our setting, and also because we will choose

at each step central intervals (thus any odd basis would suit instead of 3). We will first define an increasing sequence (P_n) of integers, related with h . Then we will define for each $0 < \alpha < \infty$ a sequence $(\varphi_{n,\alpha})$ of continuous nondecreasing functions with 3^{-P_n} -periodic derivative, and finally the open sets $\Omega_{n,\alpha}$.

3) Because h is concave, we observe that the function $\theta_0(t) = h(t)/t$ is non increasing on $]0, +\infty[$, and has infinite limit in zero.

We define $P_0 = 0$. We suppose now $n \geq 1$. We choose $P_n > 2n + P_{n-1}$ large enough so that $3^{P_n - n + 1 - P_{n-1}} h(3^{-n - P_n}) \geq h(3^{-n + 1 - P_{n-1}})$, which just means that $\theta(3^{-n - P_n}) \geq 3^n \theta(3^{-n + 1 - P_{n-1}})$.

4) We now explain how behaves the derivative $\varphi'_{n,\alpha}$ on the interval $J_{n,k} = 3^{-P_n}[k - 1, k]$. It is null except on two intervals of length $n3^{-n - P_n}$ and wich are just one before and one after the interval $J'_{n,k}$ which as same center as $J_{n,k}$ and which has lengtth $3^{-n - P_n}$. In those two intervals $\varphi'_{n,\alpha} = \alpha$.

Thus $\varphi_{n,\alpha}$ has some difference quotients closed to α and involving points of the centered interval $J'_{n,k}$: First indeed, all difference quotient of $\varphi_{n,k}$ are $\leq \alpha$. Secondly, for any $s \in J'_{n,k}$, one can find two $t \in J_{n,k}$, one greater than s , with $3^{-n - P_n} \leq |t - s| \leq 3^{-1 - P_n}$ and for which $\frac{\varphi_{n,\alpha}(t) - \varphi_{n,\alpha}(s)}{t - s} \geq n\alpha/(n + 1)$.

5) Now, for a ρ_n small enough³, $\Omega_{n,\alpha}$ will be the set of those $f \in Cr$ such that for some set $A_{f,n,\alpha} \subset \{1, \dots, 3^{N_n}\}$

$$k \in A_{f,n,\alpha} \Rightarrow \|f - f(k3^{-P_n}) - \varphi_{n,\alpha}\|_{J_{n,k}} < \rho_n \quad (3)$$

with exceptional set $B_{f,n,\alpha} = \{1, \dots, 3^{N_n}\} \setminus A_{f,n,\alpha}$ satisfying

$$n \geq 2 \Rightarrow \text{card } B_{f,n,\alpha} < 2^{-3-n} 3^{P_n - P_{n-1} - n + 1} \quad (4)$$

We ask that:

$$4n\rho_n < 2^{-3-n} 3^{-n-1-P_n} \quad (5)$$

Moreover, we also ask ρ_n to be small enough so that for any $f \in \Omega_{\alpha,n}$ and $k \in A_{f,n,\alpha}$, the function f must satisfy the following slightly relaxed difference quotient condition: For all $s \in J'_{n,k}$ and $t \in J_{n,k}$ with $3^{-n - P_n} \leq |t - s|$, we must have $\frac{f(t) - f(s)}{t - s} \leq (n + 1)\alpha/n$, moreover for each such s , one can choose two such numbers t , one greater than s , one lesser than s , and for which $\frac{f(t) - f(s)}{t - s} \geq n\alpha/(n + 2)$.

³it happens that the choice of ρ_n does not really depend on α , but the contrary would not have been a problem

6) We have already given all the conditions useful for defining G . $\Omega_{n,\alpha}$ is obviously open. The density of $\Omega_{m,\alpha}^*$ follows from the fact that $\|\varphi_{n,\alpha}\|_{[0,1]} \rightarrow 0$ and from the fact that the right member of the majoration (4) of the cardinal of exceptional set tends to infinity. Indeed this allows to dispose almost arbitrary the growth of the functions $f \in \Omega_{m,\alpha}^*$, which are constrained to make most of their growth in the “exceptional intervals” $I_{n,k}$ for $k \in B_{n,f}$ (we recall that step functions are dense in Cr).

7) Let $f \in G$ and $0 < \alpha < \infty$. We choose a sequence (α_n) in D converging to α . We then choose a strictly increasing sequence (Q_n) of integers such that $Q_0 = 0$ and $n \geq 1 \Rightarrow f \in \Omega_{Q_n,\alpha_n}$.

Following Lemma7, we then define an h_0 -set $K = K_{f,\alpha}$, taking

$$N_n = 3^{P_{Q_n} - Q_{n-1} - P_{Q_{n-1}}}$$

and asking the intervals of the step n in the building of K to have length $r_n = 3^{-Q_n - P_{Q_n}}$. The involved dimension function h_0 must then satisfy $h_0(r_{n-1}) = N_n h_0(r_n)$ which defines the values $h_0(r_n)$ if we add that $h_0(1) = h(1)$. We complete the definition of h_0 by asking h_0 to be affine on each interval $[r_n, r_{n-1}]$, for $n \geq 1$, and that $h_0 = h(1)$ on $[1, +\infty[$.

The sequence $(h_0(r_n))$ is decreasing of null limit and thus h is a dimension function. We also have $h_0(r_n) \leq h(r_n)$, hence $h \leq h_0$ because h is concave. We must also check that $\theta_0(t) = h_0(t)/t$ is nondecreasing, but this follows from $\theta_0(r_n) = 3^{Q_n} \theta_0(r_{n-1})$. Thus Lemma 8 can be applied to (K, h_0) instead of (K, h) .

8) We then define a closed subset $K' = K'_{f,\alpha} = K \setminus \bigcup_{n \geq 1} \bigcup_{J \in \mathcal{J}_n} J$, where \mathcal{J}_n denotes the set of intervals J of one of the two following kinds:

$$J = J'_{Q_n,k} \text{ where } k \in B_{f,Q_n,\alpha_n}$$

$$J = J'_{Q_n,k} \subset J'_{Q_{n-1},l} \text{ where } n \geq 2, k \in A_{f,Q_n}, l \in A_{f,Q_{n-1},\alpha_{n-1}} \text{ such that}$$

there exists $s \in J$ and $t \in J'_{Q_{n-1},l}$ satisfying $|s - t| \geq 3^{-1 - P_{Q_n}}$ and

$$\frac{f(t) - f(s)}{t - s} \geq 1/Q_{n-1}.$$

With in mind the observation following Lemma 8, we can claim that K' is also an h_0 -set because we have $\mathcal{H}^{h_0}(K') \leq (\frac{1}{4} + \frac{1}{4})\mathcal{H}^{h_0}(K)$ as a consequence of the inequalities (4) (with Q_n instead of n) and (5) (with Q_{n-1} instead of n and applying Lemma 7 to f in $J'_{n-1,k}$ with $\alpha = 1/Q_{n-1}$).

$K' \subset [\alpha \leq f'_{l,s}] \cap [\alpha \leq f'_{r,s}]$ follows from the second condition (in step 5 of this proof) when choosing the number ρ_n small enough. So if $s \in K'$, it

remains to explain why we also have $f'_s(s) \leq \alpha$. But this follows from the same condition we just mentioned, and also from the second kind of intervals occurring in the definition of K' . Indeed, if $t \in [0, 1]$ we have

$$\frac{f(t) - f(s)}{t - s} \leq (Q_n + 1)\alpha_n/Q_n \text{ when } 3^{-Q_n - P_{Q_n}} \leq |t - s| \leq 3^{-1 - P_{Q_n}} \text{ with } n \geq 1$$

$$\frac{f(t) - f(s)}{t - s} \leq 1/Q_{n-1} \text{ when } 3^{-1 - P_{Q_n}} \leq |t - s| \leq 3^{-Q_{n-1} - P_{Q_{n-1}}} \text{ with } n \geq 2.$$

5 Typical functions with bounded variation

Here we adapt the main result above to the case where the topology of bounded variation is involved. The proof of the results obtained will be an easy adaptation of the proof above.

Let Vb denote the space of all functions $f : [0, 1] \rightarrow \mathbb{R}$ endowed with the norm $\|f\|_v = |f(0)| + \text{var}_{[0,1]} f \geq \|f\|_{[0,1]}$. It contains Cr and Ccr , and we also consider the subsets Cvb , Avb , Avb^+ of $f \in \text{Vb}$ which are respectively continuous, with $[f \neq 0]$ countable, ≥ 0 with $[f \neq 0]$ countable (here the letter A stands for f atomic and not for $\text{mes } f$ atomic).

Vb , Cvb , Avb and Avb^+ are always endowed with $\|\cdot\|_v$ while we precise $(\text{Cr}, \|\cdot\|_v)$ and $(\text{Ccr}, \|\cdot\|_v)$ when we will consider Cr and Ccr as topological subspaces of Vb .

Let us recall that $(\text{Vb}, \|\cdot\|_v)$ and $(\text{Cvb}, \|\cdot\|_v)$ are Banach spaces, but not separables. $(\text{Cr}, \|\cdot\|_v)$ and $(\text{Ccr}, \|\cdot\|_v)$ are also complete and not separables.

To check the non separability of $(\text{Ccr}, \|\cdot\|_v)$, and hence of $(\text{Cvb}, \|\cdot\|_v)$, one may associate to each Cantor subset K of $[0, 1]$ some $f_K \in \text{Ccr}$ such that $\text{Supp } \text{mes}_{f_K} = K$, $f_K(0) = 0$ and $f_K(1) = 1$. Using the fact that such a K is homeomorphic to $\{0, 1\}^{\mathbb{N}}$, thus to $\{0, 1\}^{\mathbb{Z}}$, thus to K^2 , one can choose some set \mathcal{K} of Cantor subsets of $[0, 1]$ with $\text{card } \mathcal{K} = \text{card } \mathbb{R}$ and such that for all distincts elements A, B of \mathcal{K} we have $A \cap B = \emptyset$ and thus $\|f_A - f_B\|_v = 2$.

The crucial lemma 7 can be adapted to functions of bounded variations:

LEMMA 9 *Let $f \in \text{Vb}$, $0 < \alpha < \infty$, $0 \leq a < b \leq 1$ and E be the union of all $[s, t] \subset [a, b]$ such that $s \neq t$ and $\left| \frac{f(t) - f(s)}{t - s} \right| \geq \alpha$. Then we have*

$$\mathcal{H}^1(E) \leq \frac{2}{\alpha} \text{var}_{[a,b]} f. \quad (6)$$

Proof of Lemma 9 Results from Lemma 7 because of the Jordan decomposition $f = g - h$ with some (unique up to additive constant) $g, h \in \text{Cr}$ also satisfying $\text{var } f = \text{var } g + \text{var } h$, and because we have then $E_f \subset E_g \cup E_h$.

THEOREM 2 *Let h be some concave dimension function such that $t/h(t) \rightarrow 0$ in zero. Then for most f in resp Vb , Cvb , Avb , and for all $\alpha > 0$, we have*

$$0 < \mathcal{H}^h[\alpha = f'_{l,s} = f'_{r,s}] \text{ and } 0 < \mathcal{H}^h[-\alpha = f'_{l,i} = f'_{r,i}].$$

Hence we also have

$$1 = \dim_{\text{H}}[\alpha = f'_{l,s} = f'_{r,s}] = \dim_{\text{H}}[-\alpha = f'_{l,i} = f'_{r,i}].$$

$0 < \mathcal{H}^h[\alpha = f'_{l,s} = f'_{r,s}]$ is also true for most f in resp $(\text{Cr}, \|\cdot\|_{\text{vb}})$, $(\text{Ccr}, \|\cdot\|_{\text{vb}})$. Finally, $0 < \mathcal{H}^h[\alpha = -f'_{l,i} = f'_{r,i}]$ is also true for most $f \text{ Avb}^+$.

Proof of Th 2. For the cases Vb , Cvb , $(\text{Cr}, \|\cdot\|_{\text{vb}})$, $(\text{Ccr}, \|\cdot\|_{\text{vb}})$, it is almost the same as that of Theorem 2, using Lemma 9, in stead of Lemma 7. For example, $\|\cdot\|_{\text{vb}}$ restricted to $J'_{n,k}$ must be used instead of the uniform norm when defining $\Omega_{n,\alpha}$.

For the cases Avb and Avb^+ , one must also change the definition of $\varphi_{n,\alpha}$ which will be now periodic and atomic. More precisely $\varphi_{n,\alpha}$ will be now null except in exactly two points of each interval $I'_{n,k}$:

$$\begin{aligned} \varphi_{n,\alpha}((k - 1/2)3^{-P_n} + (n + \frac{1}{2})3^{-n-P_n}) &= n3^{-n-P_n}\alpha \\ \varphi_{n,\alpha}((k - 1/2)3^{-P_n} - (n + \frac{1}{2})3^{-n-P_n}) &= -n3^{-n-P_n}\alpha \text{ for the case Avb and} \\ &= n3^{-n-P_n}\alpha \text{ for the case Avb}^+. \end{aligned}$$

Remark. Actually the last statement of Th 2 is also true for most f of respectively Avb , Vb , Cvb (with an easy change of φ_n in the proof). For given $\alpha, \beta \geq 0$, one could also adapt the proof of the main resul to get that most f in Cr or in Ccr satisfy

$$1 = \dim_{\text{H}}[f'_{l,s} = \alpha \text{ and } f'_{r,s} = \beta]$$

and similar improvement of Th 2.

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