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Borderline vs. unknown: comparing three-valued representations of imperfect information

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A B S T R A C T

In this paper we compare the expressive power of elementary representation formats for vague, incomplete or conflicting information. These include Boolean valuation pairs introduced by Lawry and González-Rodríguez, orthopairs of sets of variables, Boolean possibility and necessity measures, three-valued valuations, supervaluations. We make explicit their connections with strong Kleene logic and with Belnap logic of conflicting information. The formal similarities between 3-valued approaches to vagueness and formalisms that handle incomplete information often lead to a confusion between degrees of truth and degrees of uncertainty. Yet there are important differences that appear at the interpretive level: while truth-functional logics of vagueness are accepted by a part of the scientific community (even if questioned by supervaluationists), the truth-functionality assumption of three-valued calculi for handling incomplete information looks questionable, compared to the non-truth-functional approaches based on Boolean possibility–necessity pairs. This paper aims to clarify the similarities and differences between the two situations. We also study to what extent operations for comparing and merging information items in the form of orthopairs can be expressed by means of operations on valuation pairs, three-valued valuations and underlying possibility distributions.

Keywords:

Kleene logic
Partial models
Orthopairs
Vagueness
Incomplete information
Belnap logic
Supervaluations

1. Introduction

Three-valued logics have been used for different purposes, depending on the meaning of the third truth-value. Among them, Kleene logic [29] is typically assumed to deal with incomplete knowledge, with the third truth-value interpreted as *unknown*. Another possibility is to interpret the additional truth-value as *borderline*, as a means of representing borderline cases for vague predicates [28]. It is then tempting to use Kleene logic as a simple logic of non-Boolean predicates as done recently by Lawry and González-Rodríguez [32]. They introduced a new formalism to handle three-valued vague predicates by means of upper and lower Boolean valuations. Basic three-valued connectives of conjunction, disjunction and negation can be expressed by composing Boolean valuation pairs. Moreover, they showed how to recover the connectives of Belnap 4-valued logic of conflict in terms of Boolean valuation pairs. They also use another equivalent representation consisting of pairs of subsets of atomic variables that are disjoint when they represent Kleene valuations. Lawry and Tang [33] interpret

the supervaluation approach to vagueness in terms of general non-truth functional valuation pairs respecting Boolean tautologies.

However, three-valued valuations also encode Boolean partial models, like for instance in partial logic [6]. Orthopairs of variable sets then represent the sets of Boolean variables that are known to be true and of those that are known to be false. The third truth-value then refers to the unknown. The natural generalisation of partial models consists of epistemic sets, understood as non-empty subsets of interpretations of a Boolean language, representing the information possessed by an agent, which can be viewed as all-or-nothing possibility distributions [45]. Such possibility distributions generalise to formulas in the form of possibility and necessity functions [42,18]. Possibility theory, even in its all-or-nothing form, is not truth-functional, which explains apparent anomalies in Kleene logic for handling incomplete information [19]. More recently Lawry and Dubois [31] proposed operations for merging valuation pairs and study their expression in terms of three-valued truth-tables, as well as pairs of subsets of variables.

In this paper, we focus on the use of orthopairs of subsets of variables for handling vague or incomplete information. We compare the expressive power of three-valued valuations, Boolean valuation pairs, and orthopairs. The aim is to expose the differences between these three representation tools. In contrast with these truth-functional representation tools, we consider non-truth functional ones, such as possibility–necessity pairs and supervaluations, respectively used for modelling incomplete information and vagueness, and show their formal similarities, as well as their differences with Boolean valuation pairs. Besides, we show that orthopairs are more difficult to handle than three-valued valuations and Boolean valuation pairs when it comes to evaluating logical expressions in Kleene logic, even if the three notions are in one-to-one correspondence.

We then study informational orderings and combination rules for the merging of pieces of information that can be expressed in terms of orthopairs. Again, some discrepancies are highlighted in their expressive power compared to other representation formats, especially possibility distributions.

This paper is structured as follows. The next section presents three different ways of encoding three-valued valuations. Section 3 presents two uses of these representations, for modelling borderline cases of vague predicates, and for incomplete Boolean information. Section 4 casts them in the setting of possibility theory. The strong formal similarity between supervaluation pairs and possibility–necessity pairs is also highlighted. Section 5 extends these representations so as to account for conflict in accordance with Belnap logic. We then obtain generalised valuation pairs that can be inconsistent, and general pairs of subsets of variables that can overlap (on variables for which knowledge is inconsistent). The problem of importing the recursive definitions of consistent pairs of valuations for logical expressions into the framework of orthopairs of variables is discussed in detail in Section 6. Section 7 considers various notions of ordering relations between three-valued valuations and various combination rules. It studies whether they can be encoded in the three formats proposed in this paper. Finally in Section 8, we check whether these combination operations are expressible in terms of the fusion of general subsets of interpretations.

2. Three-valued valuations

Let \mathcal{A} be a finite set of (propositional) variables. We denote by $\mathbf{3}$ the set $\{0, \frac{1}{2}, 1\}$, equipped with total order $0 < \frac{1}{2} < 1$, loosely understood as a set of truth-values with 0 and 1 respectively referring to false and true.

Definition 2.1. A *three-valued valuation* is a mapping $\tau : \mathcal{A} \rightarrow \mathbf{3}$.

We denote by $\mathbf{3}^{\mathcal{A}}$ the set of three-valued valuations. A particular subset of $\mathbf{3}^{\mathcal{A}}$ consists of Boolean valuations (or interpretations), i.e. mappings $w : \mathcal{A} \rightarrow \{0, 1\}$; we denote by $\Omega = \{0, 1\}^{\mathcal{A}}$ the set of all such valuations.

In this section we introduce two alternative representations of three-valued valuations together with their extensions to logical expressions in Kleene logic: orthopairs and ordered pairs of Boolean valuations.

2.1. Representations of three-valued valuations

Definition 2.2. By an *orthopair*, we mean a pair (P, N) of disjoint subsets of variables: $P, N \subseteq \mathcal{A}$ and $P \cap N = \emptyset$.

A three-valued valuation $\tau : \mathcal{A} \rightarrow \mathbf{3}$ induces an orthopair as follows: $a \in P$ if $\tau(a) = 1$, $a \in N$ if $\tau(a) = 0$. So P and N stand for positive and negative literals, respectively. Conversely, given an orthopair (P, N) we can define the following three-valued function:

$$\tau(a) = \begin{cases} 0 & a \in N \\ 1 & a \in P \\ \frac{1}{2} & \text{otherwise} \end{cases}$$

There is indeed a bijection between orthopairs and three-valued valuations.

Lawry and González-Rodríguez [32] propose yet another representation of ternary valuations τ by means of consistent Boolean valuation pairs:

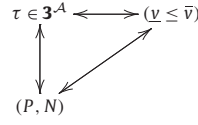


Fig. 1. Bijection among three-valued functions τ , orthopairs (P, N) and pairs of Boolean functions $\underline{v} \leq \bar{v}$.

Table 1
Kleene strong conjunction \sqcap .

\sqcap	0	$\frac{1}{2}$	1
0	0	0	0
$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$
1	0	$\frac{1}{2}$	1

Definition 2.3. A consistent Boolean valuation pair (BVP) is a pair $\bar{v} = (\underline{v}, \bar{v}) \in \Omega^2$ of Boolean valuations on $\{0, 1\}$, such that $\underline{v} \leq \bar{v}$ holds pointwisely. \underline{v} is called a *lower Boolean valuation*, and \bar{v} is an *upper Boolean valuation*.

A three-valued valuation induces a consistent BVP as follows. For variables $a \in \mathcal{A}$:

- if $\tau(a) = 1$ then $\underline{v}(a) = \bar{v}(a) = 1$;
- if $\tau(a) = 0$ then $\underline{v}(a) = \bar{v}(a) = 0$;
- if $\tau(a) = \frac{1}{2}$ then $\underline{v}(a) = 0, \bar{v}(a) = 1$.

Clearly the property $\underline{v} \leq \bar{v}$ holds and expresses consistency of the BVP. There is also a one-to-one correspondence between consistent BVP's such that $\underline{v} \leq \bar{v}$ and orthopairs (P, N) , defined by $P = \{a \in \mathcal{A} : \underline{v}(a) = \bar{v}(a) = 1\}$, $N = \{a \in \mathcal{A} : \underline{v}(a) = \bar{v}(a) = 0\}$, and $\underline{v}(a) = 0, \bar{v}(a) = 1$ if $a \notin P \cup N$. The condition $P \cap N = \emptyset$ is equivalently expressed by $\underline{v} \leq \bar{v}$ using BVPs. Obviously, $\tau(a) = \frac{\underline{v}(a) + \bar{v}(a)}{2}$.

So, in the case where $\underline{v} \leq \bar{v}$ there are bijections between three-valued functions, orthopairs and consistent BVP's as illustrated by Fig. 1.

2.2. Kleene logic expressions

In this paper we focus on the simplest three-valued logic known as Kleene logic [29]. Kleene logic uses two connectives: an idempotent conjunction \sqcap and the involutive negation $'$. Connectives are defined truth-functionally, as follows (see also Table 1):

Definition 2.4. Negation and conjunction are respectively defined by:

- $0' = 1, 1' = 0, \frac{1}{2}' = \frac{1}{2}$, which reads $\tau(\phi') = 1 - \tau(\phi)$.
- $\tau(\phi \sqcap \psi) = \min(\tau(\phi), \tau(\psi))$.

Based on these connectives, we can construct a language \mathcal{L}_K . The language \mathcal{L}_K is recursively generated by: $a \in \mathcal{L}_K, \phi' \in \mathcal{L}_K, \phi \sqcap \psi \in \mathcal{L}_K$. The corresponding disjunction is $\phi \sqcup \psi$ that stands for $(\phi' \sqcap \psi')'$ such that $\tau(\phi \sqcup \psi) = \max(\tau(\phi), \tau(\psi)) = \tau((\phi' \sqcap \psi')')$.¹

This language defines a class of three-valued functions whose expressions are the same as those of Boolean logic, but for the possibility of simplifying terms of the form $a \sqcup a'$ and $a \sqcap a'$. Remarkably, there is no tautology in Kleene logic, that is, no constant function $f : \mathbf{3}^{\mathcal{A}} \rightarrow \mathbf{3}$ taking the value $\{1\}$ that can be expressed in the Kleene logic language. In fact any proposition in Kleene logic takes value $\frac{1}{2}$ if all its variables take value $\frac{1}{2}$.

Upper and lower Boolean valuations can be extended to formulas ϕ built from variables and Kleene connectives $', \sqcap, \sqcup$:

$$\underline{v}(\phi') = 1 - \bar{v}(\phi); \tag{1}$$

$$\bar{v}(\phi \sqcap \psi) = \min(\bar{v}(\phi), \bar{v}(\psi)); \quad \underline{v}(\phi \sqcap \psi) = \min(\underline{v}(\phi), \underline{v}(\psi)); \tag{2}$$

$$\bar{v}(\phi \sqcup \psi) = \max(\bar{v}(\phi), \bar{v}(\psi)); \quad \underline{v}(\phi \sqcup \psi) = \max(\underline{v}(\phi), \underline{v}(\psi)). \tag{3}$$

¹ We shall occasionally abuse notation by identifying truth functions with the corresponding connectives in the language.

Table 2
Borderline vs. unknown.

Name	Tall	Single
John	1	1
Paul	Borderline	1
George	0	Unknown
Ringo	0	0

It can be checked that if $\tau(\phi)$ is computed by means of Kleene truth tables and the pair (\underline{v}, \bar{v}) is computed by the above identities then,

- if $\tau(\phi) = 1$ then $\underline{v}(\phi) = \bar{v}(\phi) = 1$;
- if $\tau(\phi) = 0$ then $\underline{v}(\phi) = \bar{v}(\phi) = 0$;
- if $\tau(\phi) = \frac{1}{2}$ then $\underline{v}(\phi) = 0, \bar{v}(\phi) = 1$.

Furthermore, it can be shown by induction that the three-valued valuation τ can be defined as an arithmetic mean of \underline{v}, \bar{v} for all expressions in Kleene logic:

Proposition 2.1. For any formula ϕ formed using the Kleene connectives it holds that $\tau(\phi) = \frac{\underline{v}(\phi) + \bar{v}(\phi)}{2}$.

Proof. It is easy to check that for atomic propositions $\tau(a) = \frac{\underline{v}(a) + \bar{v}(a)}{2}$. By induction, we can prove by enumerating all the possible situations, that

- $\tau(\phi') = 1 - \tau(\phi) = 1 - \frac{\underline{v}(\phi) + \bar{v}(\phi)}{2} = \frac{\underline{v}(\phi') + \bar{v}(\phi')}{2}$,
- $\tau(\phi \sqcap \psi) = \min\left(\frac{\underline{v}(\phi) + \bar{v}(\phi)}{2}, \frac{\underline{v}(\psi) + \bar{v}(\psi)}{2}\right) = \frac{\min(\underline{v}(\phi), \underline{v}(\psi)) + \min(\bar{v}(\phi), \bar{v}(\psi))}{2}$. \square

The question of whether the evaluation of Kleene expressions can be obtained by combining orthopairs is much less obvious and will be addressed in Section 6. However, we first point out two areas where Kleene logic has been used.

3. Three-valued logic: vagueness vs. incomplete information

There are two main interpretations of Kleene logic: it provides a representation of borderline cases in the modelling of vagueness, or it may account for reasoning under incomplete knowledge.

3.1. Kleene logic as an elementary framework for vague predicates

The existence of borderline cases when evaluating the truth status of propositions is supposed to be one of the main characteristics of vagueness in natural language [28]. In the most elementary representation of vagueness we can assume variables to be three-valued, with the third truth value $\frac{1}{2}$ being interpreted as *borderline*. In this approach, each three-valued valuation is a complete model, that is, whereby each variable is assigned a truth value (false, true or borderline). This is the approach of Lawry and González-Rodríguez [32], who cast this elementary idealised model of vague propositions in Kleene three-valued logic [29]. While $\frac{1}{2}$ means *borderline*, the truth-values 1 and 0 respectively mean *clearly true* and *clearly false*. In other words, the value $\frac{1}{2}$ indicates a situation where a proposition is neither clearly true nor clearly false. It is easily seen that the lower valuation \underline{v} evaluates whether a proposition is clearly true or not, while the upper valuation \bar{v} evaluates whether a proposition is at least borderline true or not. In this view, an orthopair (P, N) is interpreted as follows: P is the set of variables that are *clearly true* in the situation under concern, N the set of *clearly false* ones, and the remainder are considered to be *borderline*.

The corresponding three-valued set-theory is a very elementary variant of fuzzy set theory [44] where sets possess central elements and peripheral ones as first studied, in the scope of linguistics, by Gentilhomme [25]. For instance, in the scale [0, 250] cm, men heights from 1.80 cm up are considered typical of *tall* heights, hence central, while heights between 1.70 to 1.80 cm correspond to borderline cases of *tall*. So, the predicate *tall* may be better described as a three-valued one than as a Boolean notion, as some persons may be judged neither *clearly tall* nor *clearly not tall* (as opposed, e.g. to other clearcut predicates such as *single*, see e.g., Table 2). But these persons may as well be judged at the same time *peripherally tall* and *peripherally not tall*, (the weak forms of opposite predicates overlap) even if not clearly so.

This elementary modelling of borderline cases is discussed in the book of Shapiro [36, page 64]. The three-valued setting reflects the open texture of vague predicates leading to propositions that can go either way. He elaborates a supervaluationist approach on top of the three-valued setting: the borderline truth-value is then understood as reflecting the fact that the

truth status of the statement $tall(x)$ is likely to move from true to false and back in conversational scores. For instance, in Table 2, Paul may alternatively be declared *tall* or *not tall*, because his height lies between 1.6 and 1.8 meters. Of course, this is just a coarse idealisation of the vagueness phenomenon, in fact the most elementary one might think of, since the borderline interval [1.6, 1.8] is questionably overprecise.

The clear-cut boundaries between the borderline area and the truth and falsity ones may be found counterintuitive. However, it is maybe worth noting that Sainsbury [35] argues against the introduction of higher order vagueness in terms of borderline borderlines, etc. He says that even if you do such a thing the important question is which cases are definitely true and which are definitely false. And, following Shapiro [36], one may consider such precise boundaries as artefacts of the model, whose aim is solely to expose the presence of borderline cases in the extension of vague predicates. They do not represent something meaningful for the understanding of natural languages. Besides, even if the truth-functionality of Kleene logic of *borderline* has a well-founded algebraic setting (the so-called Kleene algebras), the adequacy of this assumption is widely questioned by supervaluationists, among others [22]. We will make no attempt here to engage in the debate regarding higher-order vagueness and the limitations of the three-valued logic of *borderline* to account for the representation of vagueness. Instead, our purpose is only to use a very basic representation of non-Boolean predicates (as advocated by Lawry and colleagues) and contrast it with the logic of *unknown*.

3.2. Kleene logic and incomplete information

The original intuition in [29] regarding the interpretation of the third truth-value $\frac{1}{2}$ corresponds to the idea of *unknown* instead of *borderline*. In this section, we assume that the third truth-value $\frac{1}{2}$ means *unknown*, and refers to the ignorance about the actual Boolean truth values of classical binary propositional variables. The truth-tables of Kleene logic are then sometimes used to propagate incomplete knowledge.

In many applications, such as relational databases [11], or logic programming [23], we assume that an agent expresses knowledge only about elementary Boolean variables. In the database area, the value $\frac{1}{2}$ is called a *null* value. For instance, in Table 2, the variable pertaining to the predicate *single* is Boolean, but it is not known whether George is single or not. However, the predicate *single* is in no way vague. This example illustrates the difference between *borderline* and *unknown*.

An orthopair (P, N) then does not have the same meaning as in the previous Subsection 3.1. Here, P is the set of atomic propositions *known to be true* (in the usual sense), and N is the set of atomic propositions *known to be false*. Clearly, $\mathcal{A} \setminus (P \cup N)$ represents variables on which the agent has no knowledge. Note that this is formally the same orthopair as in the previous section, and it can again be encoded as a three-valued valuation. However, the intended meanings of orthopairs are quite different: in the vagueness situation, an orthopair represents *complete* knowledge about a *non-Boolean* description, while here it represents incomplete knowledge about a Boolean proposition in a standard propositional language \mathcal{L} equipped with usual conjunction, disjunction and negation connectives respectively denoted by \wedge, \vee, \neg .

In this Boolean language, the orthopair (P, N) corresponds to a conjunction of literals in \mathcal{A} , i.e., $\bigwedge_{a \in P} a \wedge \bigwedge_{a \in N} \neg a$. At the semantic level, (P, N) is sometimes called a partial interpretation or *partial model*. Usually in partial logic [6], the same satisfiability definitions as for Boolean valuations are applied to partial models, which can be questioned [13].

Let the set of Boolean models of a formula ϕ (that is, the set of Boolean valuations that make ϕ true) be denoted by $E = [\phi]$. An orthopair (P, N) viewed as a partial model corresponds to a non-empty set $E_{(P, N)} = [\bigwedge_{a \in P} a \wedge \bigwedge_{a \in N} \neg a]$ of Boolean valuations, namely all Boolean interpretations w that complete the partial model. A Boolean valuation w then corresponds to an orthopair (P, P^c) , where P^c denotes the complement of P , that partitions the set of propositional variables; or, equivalently, w is the only model of a maximal conjunction of literals $\bigwedge_{a \in P_w} a \wedge \bigwedge_{a \notin P_w} \neg a$. Boolean valuations that complete an orthopair (P, N) can be represented by (P_w, P_w^c) where $P \subseteq P_w$ and $N \cap P_w = \emptyset$. It is often the case (e.g. in logic programming) that a Boolean valuation w is simply represented by the subset $P_w = \{a \in \mathcal{A} : w(a) = 1\} = P$ of positive literals it satisfies as per the above expression. This exposes the one-to-one correspondence between Ω and $2^{\mathcal{A}}$.

Now, let us suppose that an agent expresses knowledge by means of positive literals, $P \subseteq \mathcal{A}$. We can have two attitudes with respect to the other variables in $\mathcal{A} \setminus P$. First, the open world assumption. In this case nothing is assumed about $\mathcal{A} \setminus P$. Then the corresponding orthopair is (P, \emptyset) . In contrast, under the closed world assumption, it is supposed that what is not said to be true is false, thus $N = \mathcal{A} \setminus P$ contains the negative facts. This corresponds to selecting just a single valuation (P, P^c) among all the possible ones in $E_{(P, \emptyset)}$.

Now let us turn to the use of upper and lower Boolean valuations in this context. The lower valuation $\underline{v}(a) = 1$ means that variable a is known to be true, the upper valuation $\bar{v}(a) = 0$ means that a is known to be false, while the case where $\underline{v}(a) = 0$ and $\bar{v}(a) = 1$ corresponds to the case where it is unknown whether a is true or false. Very naturally, the pair $(\underline{v}(a), \bar{v}(a))$ represents the set of possible (Boolean) truth-values of a , according to the available state of information. The truth-tables of Kleene logic are then precisely the set-valued extensions of the Boolean truth-tables [34,15], namely:

$$\begin{aligned} (\underline{v}(a \wedge b), \bar{v}(a \wedge b)) &= \{w(a \wedge b) : \underline{v} \leq w \leq \bar{v}\} \\ &= (\min(\underline{v}(a), \underline{v}(b)), \min(\bar{v}(a), \bar{v}(b))) \\ (\underline{v}(a \vee b), \bar{v}(a \vee b)) &= \{w(a \vee b) : \underline{v} \leq w \leq \bar{v}\} \\ &= (\max(\underline{v}(a), \underline{v}(b)), \max(\bar{v}(a), \bar{v}(b))) \end{aligned}$$

$$(\underline{v}(\neg a), \bar{v}(\neg a)) = \{w(\neg a) : \underline{v} \leq w \leq \bar{v}\} = (1 - \bar{v}(a), 1 - \underline{v}(a))$$

However, if used to evaluate the knowledge about the truth or falsity of any Boolean formula, the recursive application of these definitions becomes counterintuitive in the incomplete information setting. Indeed, using Kleene logic (or the recursive formulas on the upper and lower Boolean valuations as in the previous section), one finds that for instance, $(\underline{v}(a \wedge \neg a), \bar{v}(a \wedge \neg a)) = (0, 1)$, if $\tau(a) = \frac{1}{2}$. However, under the incomplete information setting, since $w(a)$ can only be true or false, the truth-value $w(a \wedge \neg a)$ is necessarily 0 (known to be false). And this can be retrieved directly by applying the set-valued calculus to all formulas ϕ , namely, by computing

$$V(\phi) = \{w(\phi) : \underline{v}(a) \leq w(a) \leq \bar{v}(a), \forall a \in \mathcal{A}\},$$

rather than using the truth-tables for conjunction, and negation recursively. For instance, $\forall \underline{v} \leq \bar{v}, V(a \wedge \neg a) = \{0\}$ (and not $\{1, 0\}$ provided by Kleene truth-tables) since $\forall w, w(a \wedge \neg a) = 0$. It suggests that the lack of tautology in Kleene logic can be considered surprising in the scope of incomplete Boolean information: if variables are Boolean, classical tautologies should be preserved, even if the truth-value of some variables is unknown. For instance, from Table 2 we do not know whether George is single, but it is always true that he is “single or not single”. The explanation of this apparent anomaly is clarified in Section 4.3.

So, while there is a one-to-one correspondence between partial models representing incomplete information and three-valued valuations (relying on the bijection between $\mathbf{3}$ and the set $\{\{0\}, \{1\}, \{0, 1\}\}$ of non-empty subsets of the unit interval), Kleene truth-tables do not correctly model the incomplete information setting for evaluating the epistemic status of complex Boolean formulae. To overcome the limited expressiveness of orthopairs when modelling incomplete information, one needs a more general view of epistemic states as can be provided in terms of $\{0, 1\}$ -valued possibility distributions over interpretations [42,18].

4. Non-truth-functional frameworks for incomplete or vague information

The aim of this section is to discuss formal settings that are slightly more expressive than Kleene logic for addressing the problem of incomplete information in the classical logic setting and the question of vagueness. They share the feature of dropping the truth-functionality assumption, and turn out to be formally very close.

We first explain the limitation of orthopairs in the representation of epistemic states, viewing them as $\{0, 1\}$ -valued possibility distributions over the set of classical interpretations. Then we recall $\{0, 1\}$ -valued possibility and necessity measures which we compare to upper and lower Boolean valuations. We show their close connection with modalities in epistemic logic. Finally, we show that supervaluations, usually applied to the modelling of vagueness, are formally very close to possibility theory.

4.1. The limited expressiveness of orthopairs for incomplete information

In the following, a non-empty subset of Boolean valuations $E \subseteq \Omega$ is called an *epistemic set*. It represents an agent’s epistemic state, that is, a state of information according to which all that is known is that the real world is properly described by one and only one of the valuations in E . The sets $E_{(P,N)}$ of Boolean valuations compatible with a partial model (P, N) are special cases of *epistemic sets*.

We can equivalently represent an epistemic set E by a *possibility distribution* [45]:

Definition 4.1. A Boolean possibility distribution is a mapping $\pi : \Omega \rightarrow \{0, 1\}$ such that $\pi(w) = 1$ for some $w \in \Omega$.

The possibility distribution π_E attached to E is simply its characteristic function:

$$\pi_E(w) = \begin{cases} 1 & \text{if } w \in E \text{ (it means possible)} \\ 0 & \text{otherwise (impossible)} \end{cases} \quad (4)$$

Then, the possibility distribution $\pi_{(P,N)}$ of the epistemic set $E_{(P,N)}$ associated with an orthopair (P, N) , i.e. the set of models of $\phi = \bigwedge_{a \in P} a \wedge \bigwedge_{a \in N} \neg a$, can be factorised as:

$$\pi_{(P,N)}(w) = \min\left(\min_{a \in P} \pi_a(w), \min_{a \in N} 1 - \pi_a(w)\right) \quad (5)$$

with convention $\min_{\emptyset} = 0$ and

$$\pi_a(w) = \begin{cases} 1 & \text{if } w \models a \\ 0 & \text{otherwise} \end{cases}$$

The epistemic set $E_{(P,N)}$ is a set of Boolean valuations that takes the form of a Cartesian product of subsets of $\{a, \neg a\}$, $a \in \mathcal{A}$. We refer to this as *hyper-rectangle*, by analogy with the Cartesian product of intervals in the real line \mathbb{R}^n . Thus, partial models

Table 3
Basic epistemic sets representable by orthopairs.

3	Orthopairs	Valuation pairs	Subsets of Boolean valuations
0	(\emptyset, \mathcal{A})	$\vec{0} = (\vec{0}, \vec{0})$	$\{w_0\} = [\wedge_{a \in \mathcal{A}} \neg a]$
$\frac{1}{2}$	(\emptyset, \emptyset)	$(\vec{0}, \vec{1})$	Ω
1	(\mathcal{A}, \emptyset)	$\vec{1} = (\vec{1}, \vec{1})$	$\{w_1\} = [\wedge_{a \in \mathcal{A}} a]$
undef.	undef.	undef.	\emptyset

are viewed as hyper-rectangles in the space $\Omega = \{0, 1\}^n$ if there are n variables. In the case of the special orthopair (P, P^c) , the possibility distribution $\pi_{(P, P^c)}$ takes value 1 only for the valuation characterised by P .

Table 3 shows basic epistemic sets that are modelled by orthopairs. Note that the use of orthopairs highlights the interpretation with all positive literals and also the one with all negative literals, which play no specific role in the more general possibility theory setting. Moreover, there is no way of representing the empty set of interpretations (the contradiction) with orthopairs or consistent valuations.

However we can generate a rectangular upper approximation of any non-empty subset $E \subseteq \Omega$ of Boolean interpretations by means of a single partial model $RC(E) = (P_E, N_E)$ as follows:

- $a \in P_E$ iff $w(a) = 1, \forall w \in E$,
- $a \in N_E$ iff $w(a) = 0, \forall w \in E$.

The map $E \mapsto (P_E, N_E)$ defines an equivalence relation on possibility distributions over Ω and $E_{(P, N)} = \cup\{E : (P_E, N_E) = (P, N)\}$. $RC(E)$ can be called the *rectangular closure* of E . Note that the rectangular closure of E may even lose all the information contained in E . For example, if E is a disjunction of interpretations having different projections on each atomic space, such as any diagonal of the hyper-rectangle Ω , then $RC(E) = \Omega$.

In general, a non-empty epistemic set $E \subseteq \Omega$ can be represented by a collection of orthopairs, which can be viewed as a disjunction of partial models. To see this, consider a Boolean formula ϕ whose set of models is exactly E . Then, convert ϕ to disjunctive normal form (as a disjunction of conjunctions of literals). Each such conjunction can be represented by an orthopair.

Remark 1. We can always represent E by a set of mutually exclusive subsets of interpretations, each representing partial models. This is because it is always possible to convert a disjunction of conjunctions into an equivalent disjunction of mutually exclusive conjuncts, for instance using specific normal forms (such as binary decision diagrams [7]).

So, the use of Kleene logic to handle incomplete knowledge is subject to severe limitations in terms of expressiveness, since only a small subfamily of epistemic sets can be captured by orthopairs. Using orthopairs to represent incomplete knowledge presupposes that information can only have a bearing on variables, independently of each other. While in logical representations, variables are supposed to be logically independent, the use of orthopairs carries this assumption over to epistemic independence (independence between pieces of knowledge pertaining to variables). Replacing possibility distributions by probability distributions, it is similar to assuming stochastic independence between variables (which is a strong assumption).

4.2. Possibility–necessity pairs

If the available information takes the form of an epistemic set E , we can attach to any proposition its possibility and necessity degrees $\mathcal{N}(\phi)$ and $\Pi(\phi)$ [19].

Definition 4.2. Necessity measure \mathcal{N} and possibility measure Π are respectively defined as

$$\Pi(\phi) = 1 \quad \text{if and only if} \quad \exists w \in E, w \models \phi \text{ and } 0 \text{ otherwise;} \quad (6)$$

$$\mathcal{N}(\phi) = 1 \quad \text{if and only if} \quad \forall w \in E, w \models \phi \text{ and } 0 \text{ otherwise.} \quad (7)$$

$\mathcal{N}(\phi) = 1$ means that ϕ is certainly true, and $\Pi(\phi) = 1$ that ϕ is possibly true, in the corresponding epistemic state. So, if $\mathcal{N}(\phi) = 0$ and $\Pi(\phi) = 1$, it means that the truth of ϕ is unknown in the epistemic state [19]. The set-function $\Pi(\phi)$ (resp. \mathcal{N}) is an extreme case of numerical possibility measure [45] (resp. necessity measure [18]).

We can compute the pair (\mathcal{N}, Π) of functions $\mathcal{L} \rightarrow \{0, 1\}$ induced by the epistemic set representing an orthopair (P, N) as follows:

- $\mathcal{N}(\phi) = 1$ if $\bigwedge_{a \in P} a \wedge \bigwedge_{a \in N} \neg a \models \phi$ and 0 otherwise.
- $\Pi(\phi) = 1$ if $\bigwedge_{a \in P} a \wedge \bigwedge_{a \in N} \neg a \wedge \phi$ is consistent, and 0 otherwise.

There is an obvious similarity between consistent BVPs extended to the whole language and necessity–possibility pairs (\mathcal{N}, Π) , namely the following identities hold:

$$\mathcal{N}(\neg\phi) = 1 - \Pi(\phi) \quad \text{and} \quad \Pi(\neg\phi) = 1 - \mathcal{N}(\phi) \quad (8)$$

$$\mathcal{N}(\theta \wedge \varphi) = \min(\mathcal{N}(\theta), \mathcal{N}(\varphi)) \quad (9)$$

$$\Pi(\theta \vee \varphi) = \max(\Pi(\theta), \Pi(\varphi)) \quad (10)$$

However there is a difference between them: while $\underline{\nu}(\theta \vee \varphi) = \max(\underline{\nu}(\theta), \underline{\nu}(\varphi))$ and $\bar{\nu}(\theta \wedge \varphi) = \min(\bar{\nu}(\theta), \bar{\nu}(\varphi))$, in general, it only holds that

$$\mathcal{N}(\theta \vee \varphi) \geq \max(\mathcal{N}(\theta), \mathcal{N}(\varphi))$$

and

$$\Pi(\theta \wedge \varphi) \leq \min(\Pi(\theta), \Pi(\varphi))$$

In particular, $\Pi(\theta \wedge \neg\theta) = 0$ (non-contradiction law) and $\mathcal{N}(\theta \vee \neg\theta) = 1$ (excluded middle law), thus escaping the anomaly of losing the Boolean tautologies when information is incomplete and propositions are Boolean. Due to these properties, it is clear that:

Proposition 4.1. *If the epistemic set is a partial model encoded by the consistent BVP $(\underline{\nu}, \bar{\nu})$ the associated possibility–necessity pairs are recovered as*

$$\begin{aligned} \Pi(\phi) &= \max_{\underline{\nu} \leq w \leq \bar{\nu}} w(\phi) \\ \mathcal{N}(\phi) &= \min_{\underline{\nu} \leq w \leq \bar{\nu}} w(\phi) \end{aligned}$$

Proof. $\underline{\nu} \leq w \leq \bar{\nu}$ is equivalent to saying that w is a completion of the partial model (P, N) encoded by $(\underline{\nu}, \bar{\nu})$, i.e. $w \in E_{(P, N)}$. \square

It is easy to identify logical expressions ϕ for which extended consistent BVP's $(\bar{\nu}, \underline{\nu})$ and possibility–necessity pairs (Π, \mathcal{N}) differ.

Example 4.1. $a \sqcup b$ and $a \sqcup (a' \sqcap b)$ are not equivalent propositions under extended consistent BVP's. Indeed, consider the orthopair $(P, N) = (\{b\}, \emptyset)$. Since $\bar{\nu}(b) = \underline{\nu}(b) = 1$, it is obvious that $\bar{\nu}(a \sqcup b) = \underline{\nu}(a \sqcup b) = 1$ too. For the other formula, we can proceed as follows:

- $\bar{\nu}(a) = 1$ and $\underline{\nu}(a) = 0$ since $a \notin P \cup N$;
- $\bar{\nu}(a') = 1$ and $\underline{\nu}(a') = 0$ likewise;
- $\bar{\nu}(a' \sqcap b) = \min(\bar{\nu}(a'), \bar{\nu}(b)) = 1$;
- $\underline{\nu}(a' \sqcap b) = \min(\underline{\nu}(a'), \underline{\nu}(b)) = 0$;
- So $\bar{\nu}(a \sqcup (a' \sqcap b)) = \max(\bar{\nu}(a), \bar{\nu}(a' \sqcap b)) = 1$;
- So $\underline{\nu}(a \sqcup (a' \sqcap b)) = \max(\underline{\nu}(a), \underline{\nu}(a' \sqcap b)) = 0$.

Note that this result is more easily checked with Kleene truth-tables, since, with $\tau(a) = \frac{1}{2}$, $\tau(b) = 1$, $\tau(a \sqcup b) = \max(\frac{1}{2}, 1) = 1$, while $\tau(a \sqcup (a' \sqcap b)) = \max(\frac{1}{2}, \min(\frac{1}{2}, 1)) = \frac{1}{2}$. However, in the Boolean setting the two formulas $a \vee b$ and $a \vee (\neg a \wedge b)$ have the same set of models, so $\mathcal{N}(a \vee b) = \mathcal{N}(a \vee (\neg a \wedge b))$, and $\Pi(a \vee b) = \Pi(a \vee (\neg a \wedge b))$.

More generally, given a tautological Boolean formula ϕ , there exists a consistent BVP $(\bar{\nu}, \underline{\nu})$ such that $\bar{\nu}(\phi) = 1$, $\underline{\nu}(\phi) = 0$, corresponding to the third truth-value $\frac{1}{2}$. In contrast, for any possibility–necessity pair, it holds that $\mathcal{N}(\phi) = \Pi(\phi) = 1$. Conversely, given an inconsistent Boolean formula ϕ , there exists a consistent BVP $(\bar{\nu}, \underline{\nu})$ such that $\bar{\nu}(\phi) = 1$, $\underline{\nu}(\phi) = 0$. In contrast, for any possibility–necessity pair, it holds that $\mathcal{N}(\phi) = \Pi(\phi) = 0$. This indicates that possibility–necessity pairs represent incomplete information more faithfully than BVPs. So there is no equality between $\bar{\nu}$ and Π , nor between $\underline{\nu}$ and \mathcal{N} . However, a consistent BVP $(\underline{\nu}, \bar{\nu})$ determines a unique possibility–necessity pair: the former determines an orthopair (P, N) and the resulting possibility–necessity pair (Π, \mathcal{N}) is the one that is induced by the rectangular epistemic set $E_{(P, N)}$.

4.3. Possibility theory and epistemic logic

In fact, necessity–possibility pairs (\mathcal{N}, Π) , as witnessed by their characteristic axioms (8), are also pairs of dual KD modalities, and as such it is legitimate to connect them to epistemic logic [27]. This connection is made under the form of

a simplified epistemic logic called MEL [2,3]. It is based on a fragment of the KD language, with no nesting of modalities and no objective (modality-free) formulas. In other words, it is a higher-order standard propositional language with atomic formulas of the form $\Box\alpha$, where α is any propositional formula of an underlying standard propositional language. Its axioms are the usual K, D, and a necessitation axiom for propositional tautologies, on top of propositional axioms for the modal formulas. Even though one may consider MEL as a genuine special case of KD or S5, there are differences with the usual epistemic logic such as S5:

- It cannot distinguish belief from knowledge (true belief) since axiom T ($\Box\alpha \rightarrow \alpha$) cannot be expressed in the MEL language.
- MEL does not deal with introspection: it accounts for partial knowledge of the epistemic state of an external agent (e.g. the perception of what a computer knows or believes). However the present paper is not concerned with introspection either.
- Its semantics is in terms of epistemic sets $E \neq \emptyset$, not in terms of accessibility relations (in fact MEL is just a propositional logic, adopting the language of epistemic logic). Namely a modal formula in MEL is evaluated on epistemic sets, not on possible worlds.

In [10], we have shown that Kleene logic (and other three-valued logics of *unknown*) can be encoded in MEL. We then have to restrict the MEL language to atomic formulas of the form $\Box\ell$ where ℓ is a Boolean literal. The translation of $\tau(a) = 1$ is $\Box a$,² the translation of $\tau(a) = 0$ is $\Box\neg a$ and thus the translation of $\tau(a) = \frac{1}{2}$ is $\Diamond a \wedge \Diamond\neg a$. It first explains why there are no tautologies in Kleene logic for incomplete information. Indeed, $\Box a \vee \Box\neg a$ is clearly not a tautology in the modal logic KD while $a \vee \neg a$ is a tautology in propositional logic.

It also clearly highlights the limited expressive power of three-valued logics, namely a disjunction of literals such as $a \sqcup \neg b$ in Kleene logic (Kleene implication) corresponds to $\Box a \vee \Box\neg b$ in MEL, while $\Box(a \vee \neg b)$ has no counterpart in Kleene logic. In other words, the sub-logic of MEL accounting for three-valued logics of *unknown* uses a language where only literals are prefixed by modalities, and the semantic account can be restricted to rectangular epistemic sets corresponding to three-valued valuations, or equivalently, to orthopairs (P, N) .

Remark 2. A two-valued truth-functional modality reduces to a standard Boolean valuation [19], while, in the three-valued propositional setting accommodating borderline cases, such deviant modalities (where the lower necessity-like valuation distributes over disjunctions) are not trivial. More general Kleene algebras displaying such deviant modalities are studied in [8].

4.4. Supervaluations and possibility theory

Supervaluationism, proposed by Fine [22],³ is an alternative model of borderline cases as yielding truth-gaps between clearly true and clearly false. In this approach the fundamental idea is that variables taking borderline values can be indifferently assigned any of the clearcut values. Such completions correspond to a set of admissible Boolean valuations.

Definition 4.3. A sentence is *supertrue* if it is true in every admissible valuation and *superfalse* if it is false in every admissible valuation.

Accordingly, a sentence is *borderline*, if it is neither supertrue or superfalse or, in other words, if there are some admissible valuations for which the sentence is true and other ones for which it is false. Unlike three-valued logics, the supervaluationist approach is non-compositional and it also preserves Boolean logical equivalences, tautologies and contradictions.⁴ For example, even if propositional variable a and its negation $\neg a$ are both borderline, nonetheless, the sentences $a \vee \neg a$ and $a \wedge \neg a$ are always supertrue and superfalse respectively. This is in contrast to the same scenario in Kleene logic where both sentences would be allocated a borderline truth-value. This example is a particular case of what Fine [22] refers to as *penumbral connections* as corresponding to logical relations holding between borderline sentences. For instance, given a set of borderline literals, penumbral connections may result in certain logical combinations of these literals being either clearly true or clearly false. In general, its advocates argue that the supervaluationist approach is better able to represent penumbral connections than three-valued logics.

² In other words, $\tau(a) = 1$ iff $w(a) = 1$ is certainly true.

³ In fact the term *supervaluation* was originally introduced by van Fraassen [39] with regard to the situation in predicate logic when some terms of the language do not have referents in a particular interpretation.

⁴ If sentences $\theta, \phi \in \mathcal{L}$ are classically equivalent in the sense that for all Boolean valuations $w \in \Omega$, $w(\theta) = w(\phi)$, then θ is *supertrue* (*superfalse*) if and only if ϕ is *supertrue* (*superfalse*). Similarly, if ϕ is a tautology in the sense that for every Boolean valuation $w \in \Omega$, $w(\phi) = 1$ then ϕ is *supertrue*. Finally, supervaluationist truth models are not compositional since, in general, knowing the truth value of θ and of ϕ is not sufficient to determine the truth value of $\theta \wedge \phi$ or $\theta \vee \phi$.

Example 4.2. Consider the following type of penumbral connections related to the vague predicate *Tall* of Table 2. Suppose we have an increasing sequence of heights $h_1 < h_2 < \dots < h_n$ where h_1 is classed as being *clearly not tall*, whilst h_n is *clearly tall*. All other heights are taken to be borderline cases of *tall*. However, despite their borderline classification if we were to learn that height h_i was indeed considered *tall* we would immediately infer that h_j should be also considered *tall* for $j \geq i$. Now defining propositional variables such that a_i denotes the statement 'a person of height h_i is tall' for $i = 1, \dots, n$, then such penumbral connections can be captured by the epistemic set $E = \{w_2, \dots, w_n\}$ where $w_i(a_j) = 1$ if and only if $j \geq i$.

In the setting of propositional logic, a natural formalisation of supervaluationism is in terms of (non-truth-functional) lower and upper truth valuations as recently proposed by Lawry and Tang [33]. In this approach a set of admissible Boolean valuations is identified, and then for any sentence $\phi \in \mathcal{L}$, the lower valuation of ϕ is 1 if and only if ϕ is true in each of the admissible valuations i.e. ϕ is *supertrue*. Similarly, the upper valuation of ϕ is 1 if and only if ϕ is true in at least one admissible valuation, i.e., ϕ is not *superfalse*.

Definition 4.4. Let $E \subseteq \Omega$ denote the set of admissible valuations for \mathcal{L} . Then we define lower and upper valuations $\underline{v}: \mathcal{L} \rightarrow \{0, 1\}$ and $\bar{v}: \mathcal{L} \rightarrow \{0, 1\}$, referred to as a supervaluation pair [33], such that $\forall \phi \in \mathcal{L}$;

$$\underline{v}(\phi) = \min\{w(\phi) : w \in E\} \quad \text{and} \quad \bar{v}(\phi) = \max\{w(\phi) : w \in E\}$$

Here $\underline{v}(\phi) = 1$ means that ϕ is *supertrue* and $\bar{v}(\phi) = 0$ means that ϕ is *superfalse*.

This formalisation is mathematically identical to the Boolean possibility–necessity pairs described in Subsection 4.2. More specifically,

Proposition 4.2. Let $E \subseteq \Omega$ be the set of admissible valuations of the language \mathcal{L} . Consider the necessity and possibility measures defined in Eqs. (6) and (7). Then ϕ is *supertrue* if and only if $\mathcal{N}(\phi) = 1$ and *superfalse* if and only if $\mathcal{P}(\phi) = 0$.

The proof of this proposition is straightforward (see also [14]). So, the Lawry and Tang supervaluationist lower and upper valuations in the supervaluation framework then respectively correspond to necessity and possibility measures.

Despite this formal identity between supervaluations and Boolean possibility–necessity pairs there is none-the-less a subtle interpretational difference between the two models. Interestingly, this difference is inherently linked to the interpretation of the intermediate value. The notions of *conjunctive and disjunctive sets* [41,20] are fundamental to understanding these two distinct semantics. A conjunctive set is a conjunction of elements representing the value of a set-valued property, whereas a disjunctive set is a collection of mutually exclusive elements, each representing one of a number of possibilities, only one of which can actually be realised. For example, the set of people who have bought a lottery ticket is a conjunctive set, whilst the set of people who may possibly have the winning lottery ticket is a disjunctive set. From this perspective then the supervaluationist set of admissible valuations is a conjunctive set of Boolean valuations in the sense that each precisification of the given truth-model is on a par with another whilst for Boolean possibility–necessity pairs the underlying epistemic set is a disjunctive set of possible Boolean valuations only one of which captures the actual precisification. The latter disjunctive view is closer to the epistemic approach to vagueness, advocated by Williamson [40]. The conjunctive view can actually in turn be interpreted in two ways: supervaluationists consider that none of the precisifications is appropriate, while pluralvaluationists (like Smith [37]) consider all precisifications are equally good.

4.5. Bounded epistemic sets

Many of the theoretical results concerning supervaluationism can be easily translated from Lawry and Tang [33] into possibility theory. This translation highlights the relationship between Boolean possibility theory and Kleene three-valued logic of *borderline*. Such relationships are best formulated within the context of a particular restricted class of epistemic sets defined as follows:

Definition 4.5. An epistemic set $E \subseteq \Omega$ is said to be *bounded* iff $\{\underline{w}_E, \bar{w}_E\} \subseteq E$, where $\underline{w}_E(a) = \min\{w(a) : w \in E\}$ and $\bar{w}_E(a) = \max\{w(a) : w \in E\}$, for all $a \in \mathcal{A}$.

Notice that an epistemic set $E_{(P,N)}$ induced by an orthopair (P, N) is a special case of a bounded epistemic set in which $\underline{w}_E = \underline{v}_{(P,N)}$ and $\bar{w}_E = \bar{v}_{(P,N)}$, the consistent BVP $(\underline{v}_{(P,N)}, \bar{v}_{(P,N)})$ induced by (P, N) . In fact we can view such states as being *bounded and complete* since $E_{(P,N)} = \{w \in \Omega : \underline{v}_{(P,N)} \leq w \leq \bar{v}_{(P,N)}\}$. There are, however, many cases of bounded epistemic sets which cannot be generated in this way.

Example 4.3. A simple example of a bounded epistemic set is $E = [(a \wedge b) \vee (\neg a \wedge \neg b)]$, where $\underline{w}_E(c) = 0, \forall c \in \mathcal{A}$, and $\bar{w}_E(c) = 1, \forall c \in \mathcal{A}$. However, $E' = [(\neg a \wedge b) \vee (a \wedge \neg b)]$ (exclusive or) is not bounded.

Example 4.4. Considering again the penumbral connection of [Example 4.2](#), E is a bounded epistemic set with $\underline{w}_E = w_n$ and $\overline{w}_E = w_2$ so that the corresponding orthopair is $(P, N) = (\{a_n\}, \{a_1\})$. However, the associated epistemic set $E_{(\{a_n\}, \{a_1\})}$ includes valuations w for which $w(a_i \wedge \neg a_j) = 1$ where $j > i$, and hence $E \subset E_{(\{a_n\}, \{a_1\})}$.

Definition 4.6. An *entirely positive* (resp. *entirely negative*) sentence is a sentence generated recursively from the propositional variables (resp. the negated propositional variables) using only \wedge and \vee . The set of entirely positive (resp. entirely negative) sentences is denoted by \mathcal{L}^+ (resp. \mathcal{L}^-).

Example 4.5. The formulas $a_1 \wedge (a_2 \vee a_3) \in \mathcal{L}^+$ and $\neg a_1 \wedge (\neg a_2 \vee \neg a_3) \in \mathcal{L}^-$.

Let Π_E and \mathcal{N}_E denote the Boolean possibility and necessity measures representing an epistemic set E as given by Eqs. (7) and (6). We recall the following results from Lawry and Tang [33], in the language of possibility theory:

Proposition 4.3. Let E be a bounded epistemic set and let $E' = \{\underline{w}_E, \overline{w}_E\}$ then $\forall \theta \in \mathcal{L}^+ \cup \mathcal{L}^-, \mathcal{N}_E(\theta) = \mathcal{N}_{E'}(\theta)$ and $\Pi_E(\theta) = \Pi_{E'}(\theta)$.

[Proposition 4.3](#) shows that for a bounded epistemic set E and for entirely positive or entirely negative sentences, the necessity and possibility values are dependent only on the bounding Boolean assignments \underline{w}_E and \overline{w}_E .

Proposition 4.4. Let E be a bounded epistemic set then $\forall \theta, \varphi \in \mathcal{L}^+$ and $\forall \theta, \varphi \in \mathcal{L}^-$ it holds that:

$$\mathcal{N}_E(\theta \vee \varphi) = \max(\mathcal{N}_E(\theta), \mathcal{N}_E(\varphi)) \quad \text{and} \quad \Pi_E(\theta \wedge \varphi) = \min(\Pi_E(\theta), \Pi_E(\varphi))$$

This result shows that possibility–necessity pairs representing bounded epistemic sets are strongly compositional when restricted either to entirely positive or entirely negative sentences. Indeed for this fragment of the logic they share the same conjunction and disjunction rules as lower and upper Boolean valuations encoding Kleene truth-values. In fact, as the following result shows, possibility–necessity pairs representing bounded epistemic sets are particularly related to the consistent BVP generated by the orthopair (P, N) where $P = \{a : \underline{w}_E(a) = 1\}$ and $N = \{a : \overline{w}_E(a) = 0\}$.

Proposition 4.5. If E is a bounded epistemic set, then there is a unique extended consistent BVP $\vec{v} = (\underline{v}, \overline{v})$ such that $\forall \theta \in \mathcal{L}^+ \cup \mathcal{L}^-, \mathcal{N}_E(\theta) = \underline{v}(\theta)$ and $\Pi_E(\theta) = \overline{v}(\theta)$, and in general $\forall \theta \in \mathcal{L}, \underline{v}(\theta) \leq \mathcal{N}_E(\theta) \leq \Pi_E(\theta) \leq \overline{v}(\theta)$. Furthermore, $\vec{v} = \vec{v}_{(P, N)}$ where $P = \{a : \underline{w}_E(a) = 1\}$ and $N = \{a : \overline{w}_E(a) = 0\}$ or alternatively $\vec{v} = \vec{v}_{RC(E)}$.

Hence, in an epistemic/partial knowledge setting involving bounded epistemic sets, we may think of Kleene valuations (in the form extended consistent BVP's) as a generally more imprecise approximation of the underlying possibility–necessity pair (see also [12]), which only agrees with the possibilistic model when restricted to entirely positive or entirely negative sentences. From the alternative perspective of truth-gaps or borderline cases induced by inherently vague propositions, the above proposition indicates that Kleene valuations can always be viewed as a semantic weakening of a strongly related supervaluationist truth-model.

5. Representations for conflicting information

In order to extend the expressive power of orthopairs of sets of variables, we may wish to relax the condition $P \cap N = \emptyset$. The epistemic understanding of this license is that if $a \in P \cap N$, it means that there are reasons to believe the truth of a and also reasons to believe a to be false. For instance, there may be agents claiming the truth of a and other agents claiming its falsity. This approach is akin to the semantics of some paraconsistent logics such as Belnap's [4]. Dubois, Konieczny and Prade [16] used such paraconsistent pairs of sets of variables for the study of a possibilistic logic counterpart of quasi-classical logic of Besnard and Hunter [5].

5.1. Paraconsistent valuations and Belnap truth-values

We call such pairs of subsets of variables $(F, G) \in 2^A \times 2^A$ with $F \cap G \neq \emptyset$ *paraconsistent*. For these pairs, it is clear that $E_{(F, G)} = \emptyset$. Another semantics is necessary for them. We use a set $\mathbf{4} = \{0, 1, u, b\}$ of truth-values, where u stands for *unknown* (it corresponds to $1/2$ in Kleene logic) and b stands for *contradictory* (both true and false). A four-valued valuation is a mapping $\mathcal{A} \rightarrow \mathbf{4}$ denoted by σ . A pair (F, G) in the paraconsistent case is closely related to Belnap [4] 4-valued logic, namely, in his terminology:

- If $a \in F \setminus G$ then $\sigma(a) = 1$: a is asserted but not negated (Belnap truth-value TRUE).
- If $a \in G \setminus F$ then $\sigma(a) = 0$: a is negated but not asserted (Belnap truth-value FALSE).
- If $a \in F \cap G$ then $\sigma(a) = b$: a is both negated and asserted (Belnap truth-value BOTH).
- If $a \notin F \cup G$ then $\sigma(a) = u$: a is neither negated nor asserted (Belnap truth-value NONE).

Table 4
Belnap disjunction and conjunction.

\sqcup	0	u	b	1
0	0	u	b	1
u	u	u	1	1
b	b	1	b	1
1	1	1	1	1

\sqcap	0	u	b	1
0	0	0	0	0
u	0	u	0	u
b	0	0	b	b
1	0	u	b	1

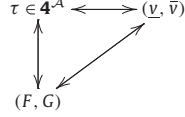


Fig. 2. Equivalences for paraconsistent information representation.

Table 5
Basic epistemic sets representable by general valuation pairs.

3	Generalised pairs	Valuation pairs	Subsets of Boolean valuations
0	(\emptyset, \mathcal{A})	$\vec{0} = (\vec{0}, \vec{0})$	$\{w_0\} = [\wedge_{a \in \mathcal{A}} \neg a]$
u	(\emptyset, \emptyset)	$(\vec{0}, \vec{1})$	Ω
1	(\mathcal{A}, \emptyset)	$\vec{1} = (\vec{1}, \vec{1})$	$\{w_1\} = [\wedge_{a \in \mathcal{A}} a]$
b	$(\mathcal{A}, \mathcal{A})$	$(\vec{1}, \vec{0})$	\emptyset

The epistemic truth set **4** is equipped with a bi-lattice structure [24], namely there is the truth-ordering $<$ such that $0 < u < 1$ and $0 < b < 1$; and the information ordering (ranging from no information (u) to too much information (b)) such that $u < 1 < b$ and $u < 0 < b$. The truth-tables are those of Kleene logic for $\{0, u, 1\}$ and $\{0, b, 1\}$ according to the truth ordering, plus $u \sqcup b = 1$, $u \sqcap b = 0$ (see Table 4). The second borderline truth-value b plays the same role as the first one with respect to 1 and 0.

Four-valued valuations can be captured by general Boolean valuation pairs. The case where $\underline{v} \not\leq \bar{v}$ exactly corresponds to pairs (F, G) of sets of variables such that $F \cap G \neq \emptyset$, letting $F = \{a \in \mathcal{A} : \underline{v}(a) = 1\}$ and $G = \{a \in \mathcal{A} : \bar{v}(a) = 0\}$. We call such pairs (\underline{v}, \bar{v}) *paraconsistent BVP's*.

Lawry and González-Rodríguez [32] have shown that truth-tables corresponding to the inductive definitions (1), (2), (3) of consistent BVP's over the language become Belnap 4-valued truth-tables when these inductive definitions are applied to all BVP's (\underline{v}, \bar{v}) without the restriction $\bar{v} \geq \underline{v}$. In particular, b corresponds to the pair $(1, 0)$, and

- $b \sqcap u$ corresponds to the componentwise Boolean conjunction of $(0, 1)$ and $(1, 0)$, which is $(0, 0)$, i.e., 0.
- $b \sqcup u$ corresponds to the componentwise Boolean disjunction of $(0, 1)$ and $(1, 0)$, which is $(1, 1)$, i.e., 1.

In this case the diagram of Fig. 1 should be updated as on Fig. 2. Table 5 completes Table 3, accounting for non-consistent valuation pairs.

Just as for Kleene logic, the truth-functionality of Belnap approach does not sound very natural in the scope of handling paraconsistent Boolean formulas other than literals [13].

5.2. Paraconsistent orthopairs as pairs of consistent orthopairs

In order to interpret his 4 truth-values, Belnap proposes to consider several conflicting but internally consistent sources of information about variables. In fact two sources of information are enough to generate the four truth values, each source corresponding to a partial model. It is then natural to try to represent paraconsistent orthopairs by means of two standard consistent orthopairs of the form $(F, G \setminus F)$ and $(F \setminus G, G)$ that conflict with each other. These are pairs of orthopairs $(P_1, N_1), (P_2, N_2)$, corresponding to the epistemic states of two conflicting agents, with $P_2 \subseteq P_1, N_1 \subseteq N_2, N_1 \cap P_2 = \emptyset, P_1 \cup N_1 = P_2 \cup N_2$. The corresponding paraconsistent pair is of the form $(F, G) = (P_1, N_2)$. It corresponds to two disjoint epistemic sets $E_i = E_{(P_i, N_i)}, i = 1, 2$.

More generally we could reconstruct a paraconsistent pair from any two orthopairs $(P_1, N_1), (P_2, N_2)$ as $(F, G) = (P_1 \cup P_2, N_1 \cup N_2)$ as follows:

- If $a \in (P_1 \setminus N_2) \cup (P_2 \setminus N_1)$ then a has truth-value 1;
- If $a \in (N_1 \setminus P_2) \cup (N_2 \setminus P_1)$ then a has truth-value 0;

- If $a \in (P_1 \cap N_2) \cup (P_2 \cap N_1)$ then a has truth-value b ;
- If $a \notin P_1 \cup N_2 \cup P_2 \cup N_1$ then a has truth-value u .

Letting $(P_1, N_1) = (F, G \setminus F)$ and $(P_2, N_2) = (F \setminus G, G)$, we do recover $(P_1 \cup P_2, N_1 \cup N_2) = (F, G)$.

Whether we can define a counterpart to the simplified epistemic logic MEL using pairs of orthopairs representing conflicting epistemic states is a matter of further research. However, a modal setting for such a kind of non-truth-functional approach to Belnap-like paraconsistent reasoning is outlined in [15].

6. Kleene logic connectives and orthopairs

Lawry and González-Rodríguez [32] propose a semantics for both Kleene three-valued logic and Belnap four-valued logic in terms of pairs of sets of variables $(F, G) \in 2^{\mathcal{A}} \times 2^{\mathcal{A}}$. However the construction in [32] for Kleene logic expressions involves paraconsistent pairs. In this section, we investigate whether Kleene and Belnap logic expressions can be recursively described in terms of respectively orthopairs and pairs of sets of Boolean variables. In the following, the notation (P, N) corresponds to genuine orthopairs, while we use (F, G) in the general case.

6.1. Semantics of Kleene logic formulas in terms of orthopairs

We show here how to express Kleene valuations over the language in terms of orthopairs only. Let $\mathcal{O} = \{(P, N) \in 2^{\mathcal{A}} \times 2^{\mathcal{A}} : P \cap N = \emptyset\}$ denote the set of all orthopairs. Kleene operations of negation, intersection and union (see Section 2.2) translate on orthopairs as

$$(P, N)' := (N, P)$$

$$(P_1, N_1) \sqcap (P_2, N_2) := (P_1 \cap P_2, N_1 \cup N_2)$$

$$(P_1, N_1) \sqcup (P_2, N_2) := (P_1 \cup P_2, N_1 \cap N_2)$$

Then, for Kleene logic, we can recursively define a pair of mappings $\mathbb{T}_k : \mathcal{L}_K \rightarrow 2^{\mathcal{O}}$ and $\mathbb{F}_k : \mathcal{L}_K \rightarrow 2^{\mathcal{O}}$ from sentences to sets of orthopairs as follows:

Definition 6.1. Let $\mathbb{T}_k : \mathcal{L}_K \rightarrow 2^{\mathcal{O}}$ and $\mathbb{F}_k : \mathcal{L}_K \rightarrow 2^{\mathcal{O}}$ be defined recursively as follows. For all $\theta, \varphi \in \mathcal{L}_K$:

- If $\varphi = a \in \mathcal{A}$ then $\mathbb{T}_k(a) = \{(P, N) \in \mathcal{O} : a \in P\}$ and $\mathbb{F}_k(a) = \{(P, N) \in \mathcal{O} : a \in N\}$;
- $\mathbb{T}_k(\theta \sqcap \varphi) = \mathbb{T}_k(\theta) \cap \mathbb{T}_k(\varphi)$ and $\mathbb{F}_k(\theta \sqcap \varphi) = \mathbb{F}_k(\theta) \cup \mathbb{F}_k(\varphi)$;
- $\mathbb{T}_k(\theta \sqcup \varphi) = \mathbb{T}_k(\theta) \cup \mathbb{T}_k(\varphi)$ and $\mathbb{F}_k(\theta \sqcup \varphi) = \mathbb{F}_k(\theta) \cap \mathbb{F}_k(\varphi)$;
- $\mathbb{T}_k(\theta') = \mathbb{F}_k(\theta)$ and $\mathbb{F}_k(\theta') = \mathbb{T}_k(\theta)$.

The following proposition shows that for any sentence $\theta \in \mathcal{L}_K$, $\mathbb{T}_k(\theta)$ exactly identifies those orthopairs for which θ is clearly true given the corresponding Kleene valuation. Similarly, $\mathbb{F}_k(\theta)$ is the set of orthopairs for which θ is clearly false given the associated valuation.

Proposition 6.1. $\forall \Psi \in \mathcal{L}_K, \mathbb{T}_k(\Psi) = \{(P, N) \in \mathcal{O} : \underline{v}_{(P, N)}(\Psi) = 1\}$ and $\mathbb{F}_k(\Psi) = \{(P, N) \in \mathcal{O} : \bar{v}_{(P, N)}(\Psi) = 0\}$.

Proof. Let $\mathcal{L}_K^0 = \mathcal{A}$ and for $n \geq 1$ let $\mathcal{L}_K^n = \mathcal{L}_K^{n-1} \cup \{\theta \sqcap \varphi, \theta \sqcup \varphi, \theta' : \theta, \varphi \in \mathcal{L}_K^{n-1}\}$. Now by induction on n : Suppose $\Psi = a_i \in \mathcal{A}$ then $\underline{v}_{(P, N)}(a_i) = 1$ iff $a_i \in P$ and $\bar{v}_{(P, N)}(a_i) = 0$ iff $\underline{v}_{(P, N)}(\neg a_i) = 1$ iff $a_i \in N$ as required. Now suppose the result holds for \mathcal{L}_K^n and let $\Psi \in \mathcal{L}_K^{n+1}$ then either $\Psi \in \mathcal{L}_K^n$, in which case the result holds trivially, or one of the following holds: For $\theta, \varphi \in \mathcal{L}_K^n$:

- $\Psi = \theta \sqcap \varphi$: In this case we have by definition that $\mathbb{T}_k(\Psi) = \mathbb{T}_k(\theta \sqcap \varphi) = \mathbb{T}_k(\theta) \cap \mathbb{T}_k(\varphi) =$ (inductive hypothesis)

$$\begin{aligned} & \{(P, N) \in \mathcal{O} : \underline{v}_{(P, N)}(\theta) = 1\} \cap \{(P, N) \in \mathcal{O} : \underline{v}_{(P, N)}(\varphi) = 1\} \\ &= \{(P, N) \in \mathcal{O} : \underline{v}_{(P, N)}(\theta) = 1 \text{ and } \underline{v}_{(P, N)}(\varphi) = 1\} \\ &= \{(P, N) \in \mathcal{O} : \min(\underline{v}_{(P, N)}(\theta), \underline{v}_{(P, N)}(\varphi)) = 1\} \\ &= \{(P, N) \in \mathcal{O} : \underline{v}_{(P, N)}(\theta \sqcap \varphi) = 1\} \\ &= \{(P, N) \in \mathcal{O} : \underline{v}_{(P, N)}(\Psi) = 1\} \text{ as required.} \end{aligned}$$

Also, $\mathbb{F}_k(\Psi) = \mathbb{F}_k(\theta \sqcap \varphi) = \mathbb{F}_k(\theta) \cup \mathbb{F}_k(\varphi) =$ (by the inductive hypothesis)

$$\begin{aligned}
& \{(P, N) \in \mathcal{O} : \bar{\nu}_{(P,N)}(\theta) = 0\} \cup \{(P, N) \in \mathcal{O} : \bar{\nu}_{(P,N)}(\varphi) = 0\} \\
&= \{(P, N) \in \mathcal{O} : \bar{\nu}_{(P,N)}(\theta) = 0 \text{ or } \bar{\nu}_{(P,N)}(\varphi) = 0\} \\
&= \{(P, N) \in \mathcal{O} : \min(\bar{\nu}_{(P,N)}(\theta), \bar{\nu}_{(P,N)}(\varphi)) = 0\} \\
&= \{(P, N) \in \mathcal{O} : \bar{\nu}_{(P,N)}(\theta \sqcap \varphi) = 0\} \\
&= \{(P, N) \in \mathcal{O} : \bar{\nu}_{(P,N)}(\Psi) = 0\} \quad \text{as required.}
\end{aligned}$$

- $\Psi = \theta \sqcup \varphi$. The proof is similar to the previous case, exchanging \mathbb{T}_k and \mathbb{F}_k .
- $\Psi = \theta'$: In this case we have that $\mathbb{T}_k(\Psi) = \mathbb{T}_k(\theta') = \mathbb{F}_k(\theta) =$ (by the inductive hypothesis) $\{(P, N) \in \mathcal{O} : \bar{\nu}_{(P,N)}(\theta) = 0\} = \{(P, N) \in \mathcal{O} : \underline{\nu}_{(P,N)}(\theta') = 1\} = \{(P, N) \in \mathcal{O} : \underline{\nu}_{(P,N)}(\Psi) = 1\}$ as required. Also, $\mathbb{F}_k(\Psi) = \mathbb{F}_k(\theta') = \mathbb{T}_k(\theta) =$ (by the inductive hypothesis) $\{(P, N) \in \mathcal{O} : \underline{\nu}_{(P,N)}(\theta) = 1\} = \{(P, N) \in \mathcal{O} : \bar{\nu}_{(P,N)}(\theta') = 0\} = \{(P, N) \in \mathcal{O} : \bar{\nu}_{(P,N)}(\Psi) = 0\}$ as required. \square

Example 6.1. $\mathbb{T}_k(\neg(a_i \sqcup \neg a_j)) = \mathbb{F}_k(a_i \sqcup a_j) = \mathbb{F}_k(a_i) \cap \mathbb{F}_k(\neg a_j) = \mathbb{F}_k(a_i) \cap \mathbb{T}_k(a_j) = \{(P, N) \in \mathcal{O} : a_j \in P, a_i \in N\}$. Similarly, $\mathbb{F}_k(\neg(a_i \sqcup \neg a_j)) = \mathbb{T}_k(a_i \sqcup \neg a_j) = \mathbb{T}_k(a_i) \cup \mathbb{T}_k(\neg a_j) = \mathbb{T}_k(a_i) \cup \mathbb{F}_k(a_j) = \{(P, N) \in \mathcal{O} : a_i \in P \text{ or } a_j \in N\}$.

6.2. Semantics of Belnap logic formulas in terms of paraconsistent pairs

The characterisation of Kleene valuations in terms of orthopairs can be extended to Belnap valuations by extending the mappings described in the previous section so as to include paraconsistent pairs of sets. More specifically, a more general pair of dual mappings can be defined simply by dropping the restriction to orthopairs in [Definition 6.1](#). Let these extended dual mappings be denoted by $\mathbb{T}_b : \mathcal{L}_K \rightarrow 2^{2^{\mathcal{A}} \times 2^{\mathcal{A}}}$ and $\mathbb{F}_b : \mathcal{L}_K \rightarrow 2^{2^{\mathcal{A}} \times 2^{\mathcal{A}}}$ respectively. Now it is straightforward to see that [Proposition 6.1](#) extends to paraconsistent pairs. We can show that $\mathbb{T}_b(\theta) = \{(F, G) \in 2^{\mathcal{A}} \times 2^{\mathcal{A}} : \underline{\nu}_{(F,G)}(\theta) = 1\}$ and $\mathbb{F}_b(\theta) = \{(F, G) \in 2^{\mathcal{A}} \times 2^{\mathcal{A}} : \bar{\nu}_{(F,G)}(\theta) = 0\}$. Notice that, unlike the Kleene mappings given in [Definition 6.1](#), sets $\mathbb{T}_b(\theta)$ and $\mathbb{F}_b(\theta)$ are generally not disjoint, further highlighting the paraconsistent nature of Belnap logic. For instance, $\forall a \in \mathcal{A}$, $\mathbb{T}_b(a) \cap \mathbb{F}_b(a)$ consists of those paraconsistent pairs (F, G) for which $a \in F \cap G$.

It turns out that, in this context, we can give independent recursive definitions of each of \mathbb{T}_b and \mathbb{F}_b . To see this, we need to understand the relationship between the BVP's induced by the dual pairs (F, G) and (G^c, F^c) as identified in the following proposition:

Proposition 6.2. $\forall (F, G) \in 2^{\mathcal{A}} \times 2^{\mathcal{A}}, (\underline{\nu}_{(F,G)}, \bar{\nu}_{(F,G)}) = (\bar{\nu}_{(G^c, F^c)}, \underline{\nu}_{(G^c, F^c)})$.

Proof. For atomic formula the result is straightforward since $\forall a \in \mathcal{A}$, $\underline{\nu}_{(F,G)}(a) = 1$ iff $a \in F$ iff $a \notin F^c$ iff $\bar{\nu}_{(G^c, F^c)}(a) = 1$, and also $\bar{\nu}_{(F,G)}(a) = 1$ iff $a \in G^c$ iff $\underline{\nu}_{(G^c, F^c)}(a) = 1$. These equalities can then be extended to all formulas in \mathcal{L}_K by recursion on the complexity of formulas in \mathcal{L}_K along similar lines to those in the proof of [Proposition 6.1](#). \square

Corollary 6.1. $\forall \theta \in \mathcal{L}_K, \mathbb{T}_b(\theta') = \{(F, G) \in 2^{\mathcal{A}} \times 2^{\mathcal{A}} : (G^c, F^c) \in \mathbb{T}_b(\theta)\}^c$.

Proof. $\forall \theta \in \mathcal{L}_K, \mathbb{T}_b(\theta') = \{(F, G) \in 2^{\mathcal{A}} \times 2^{\mathcal{A}} : \underline{\nu}_{(F,G)}(\theta') = 1\} = \{(F, G) \in 2^{\mathcal{A}} \times 2^{\mathcal{A}} : \bar{\nu}_{(G^c, F^c)}(\theta') = 1\}$ (by [Proposition 6.2](#)) $= \{(F, G) \in 2^{\mathcal{A}} \times 2^{\mathcal{A}} : \underline{\nu}_{(G^c, F^c)}(\theta) = 0\} = \{(F, G) \in 2^{\mathcal{A}} \times 2^{\mathcal{A}} : \underline{\nu}_{(G^c, F^c)}(\theta) = 1\}^c = \{(F, G) \in 2^{\mathcal{A}} \times 2^{\mathcal{A}} : (G^c, F^c) \in \mathbb{T}_b(\theta)\}^c$. \square

As a consequence of [Corollary 6.1](#), it follows that there is an alternative recursive definition of \mathbb{T}_b which does not make direct reference to \mathbb{F}_b .

Corollary 6.2. Let $\mathbb{T}_b : \mathcal{L}_K \rightarrow 2^{2^{\mathcal{A}} \times 2^{\mathcal{A}}}$ be defined recursively as follows: $\forall \theta, \varphi \in \mathcal{L}_K$ and $\forall a \in \mathcal{A}$ then:

- $\mathbb{T}_b(a) = \{(F, G) \in 2^{\mathcal{A}} \times 2^{\mathcal{A}} : a \in F\}$.
- $\mathbb{T}_b(\theta \sqcap \varphi) = \mathbb{T}_b(\theta) \cap \mathbb{T}_b(\varphi)$.
- $\mathbb{T}_b(\theta \sqcup \varphi) = \mathbb{T}_b(\theta) \cup \mathbb{T}_b(\varphi)$.
- $\mathbb{T}_b(\theta') = \{(F, G) \in 2^{\mathcal{A}} \times 2^{\mathcal{A}} : (G^c, F^c) \in \mathbb{T}_b(\theta)\}^c$.

This corresponds to the definition proposed by Lawry and González-Rodríguez [\[32\]](#). Notice that a similar alternative definition cannot be given for \mathbb{T}_k if we restrict to orthopairs. The difficulty arises due to the fact that if $(P, N) \in \mathcal{O}$ and $N \neq P^c$ then $(N^c, P^c) \notin \mathcal{O}$.

Remark 3. It is not clear how to interpret the four-valued framework outlined above in the scope of modelling vagueness, as the two truth-values other than true and false play symmetric roles in Belnap logic. In fact, this logic aims at treating the issue of conflicting information only [\[13\]](#).

7. Order relations and aggregation operations on orthopairs

As orthopairs of variables represent epistemic states, it is natural to compare them in terms of their respective amount of information. Likewise if they account for borderline cases, it is also expected to compare them in terms of sharpness. Moreover, as orthopairs highlight negative vs. positive literals, one is led to compare them in terms of positiveness and negativeness.

In [9] and [31] some order relations and operations on orthopairs are considered that deal with some of these aspects. These ordering relations are generally related to corresponding lattice operations that are natural to merge orthopairs. In fact the meaning of the proposed ordering relations changes according to whether the aim is to model vagueness in the language or incomplete information.

This section attempts to give a complete picture of ordering relations and combination operations for orthopairs of sets of variables, and the corresponding notions using consistent BVP's (on the set of propositional variables) and three-valued functions. Aggregation operations include already met Kleene three-valued logic conjunction and disjunction, plus some which differ from these known ones. They may be used to evaluate complex non-Boolean statements, or, in the epistemic view, they recover methods for incomplete information fusion.

7.1. The truth ordering

First of all, let us consider the standard order on $\mathbf{3}$: $0 < \frac{1}{2} < 1$. Given two three-valued valuations, τ_1, τ_2 , we can let $\tau_1 \preceq_t \tau_2$ mean $\tau_1(a) \leq \tau_2(a), \forall a \in \mathcal{A}$. This ordering is known as the truth ordering [4]. It expresses the idea of a three-valued valuation making variables "not more true than" another one. If τ_1, τ_2 correspond to respective orthopairs $(P_1, N_1), (P_2, N_2)$ and consistent valuation pairs \vec{v}_1, \vec{v}_2 then $\tau_1 \preceq_t \tau_2$ reads on orthopairs as

$$P_1 \subseteq P_2, N_2 \subseteq N_1$$

On BVP's, it is the canonical extension of the order $0 < 1$ to subsets of $\{0, 1\}$:

$$\forall a \quad \underline{v}_1(a) \leq \underline{v}_2(a) \quad \bar{v}_1(a) \leq \bar{v}_2(a).$$

The corresponding join and meet operations induced by this ordering are well known (see for instance [9]) and have been met here from the start:

$$(P_1, N_1) \sqcap_t (P_2, N_2) := (P_1 \cap P_2, N_1 \cup N_2) \quad (11)$$

$$(P_1, N_1) \sqcup_t (P_2, N_2) := (P_1 \cup P_2, N_1 \cap N_2) \quad (12)$$

We retrieve the conjunction and disjunction in Kleene logic (see Table 1), since $\sqcap_t = \sqcap$ and $\sqcup_t = \sqcup$.

7.2. The information ordering

Another natural order relation \preceq_I on orthopairs is:

$$(P_1, N_1) \preceq_I (P_2, N_2) \quad \text{iff} \quad P_1 \subseteq P_2, N_1 \subseteq N_2 \quad (13)$$

This relation is known as the *knowledge ordering* [4,43] or, in the setting of vagueness modelling, *semantic precision* [31] or *sharpening relation* [36]. For valuations, it reads:

$$\vec{v}_1 \preceq_I \vec{v}_2 \quad \text{iff} \quad \forall a \quad \underline{v}_1(a) \leq \underline{v}_2(a) \quad \bar{v}_2(a) \leq \bar{v}_1(a).$$

Under an epistemic reading, it means \vec{v}_2 is at least as informative as \vec{v}_1 . In the scope of a logic of vagueness, it means that \vec{v}_2 is at least as clear-cut as (or less fuzzy than) \vec{v}_1 (the latter involves more variables taking the borderline value).

Once interpreted on $\mathbf{3}$, it can be seen that it does not generate a lattice structure but only the meet-semilattice of Fig. 3. The meet with respect to this order is defined on orthopairs as

$$(P_1, N_1) \sqcap_I (P_2, N_2) := (P_1 \cap P_2, N_1 \cap N_2) \quad (14)$$

It is pessimistic as it only retains what both orthopairs hold as being true or false, so that $(P_1, N_1) \sqcap_I (P_2, N_2)$ is less informative (epistemic reading) or less sharp (vagueness) than each of the arguments of the combination. It is akin to the union of epistemic sets. This fusion operation can be expressed via a connective \sqcap_I on consistent BVP's as follows [31]:

$$\vec{v}_1 \sqcap_I \vec{v}_2 = \vec{v}_{(P_1 \cap P_2, N_1 \cap N_2)} = (\min(\underline{v}_1, \underline{v}_2), \max(\bar{v}_1, \bar{v}_2))$$

In terms of truth-values, it corresponds to an idempotent operation \min_I on $\mathbf{3}$ such that $\min_I(1, 0) = \min_I(0, 1) = \min_I(0, \frac{1}{2}) = \min_I(\frac{1}{2}, 1) = \frac{1}{2}$ (see Table 6). Remarkably, $\min_I(x, y)$ is the median of $\{x, y, \frac{1}{2}\}$, the only associative operation $*$ such that $x \sqcap y \leq x * y \leq x \sqcup y$, known as a nullnorm [26].

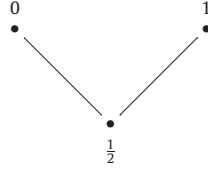


Fig. 3. The semilattice structure of $\mathbf{3}$.

Table 6
Information-ordering-based conjunction and disjunction.

\min_I	0	$\frac{1}{2}$	1	\max_I	0	$\frac{1}{2}$	1
0	0	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	\times
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	1
1	$\frac{1}{2}$	$\frac{1}{2}$	1	1	\times	1	1

Clearly, the corresponding join can be naturally defined in terms of orthopairs as

$$(P_1, N_1) \sqcup_I (P_2, N_2) := (P_1 \cup P_2, N_1 \cup N_2) \quad (15)$$

It does not always yield an orthopair as the result may become paraconsistent. It corresponds to the *optimistic combination operator* \sqcup_I on consistent valuation pairs [31]:

$$\vec{v}_1 \sqcup_I \vec{v}_2 = \vec{v}_{(P_1 \cup P_2, N_1 \cup N_2)} = (\max(\underline{v}_1, \underline{v}_2), \min(\bar{v}_1, \bar{v}_2)).$$

where the resulting valuation pairs may fail to be consistent. In terms of an operation \max_I on $\mathbf{3}$, it is a partially defined idempotent one such that $\max_I(0, \frac{1}{2}) = 0$, $\max_I(\frac{1}{2}, 1) = 1$, and where $\max_I(0, 1)$ is undefined. Interestingly, it is the restriction to $\{0, \frac{1}{2}, 1\}$ of the (associative) uninorm $\frac{xy}{xy+(1-x)(1-y)}$ [26] on $[0, 1]$, which is indeed undefined for $(0, 1)$ and $(1, 0)$.

It is optimistic as it retains what at least one orthopair holds as being true or false, so that $(P_1, N_1) \sqcup_I (P_2, N_2)$ is more informative (epistemic reading) or sharper (vagueness) than each of the arguments of the combination, when it is defined. It is akin to the consistent intersection of epistemic sets, and is undefined when this intersection is empty.

The cases where the result exists then correspond to the situation where the two orthopairs are logically consistent with each other, that is: $P_1 \cap N_2 = P_2 \cap N_1 = \emptyset$. Generalising both orderings to paraconsistent pairs yields a bilattice structure revealed by Belnap [4], corresponding to the bilattice $\mathbf{4}$ (and then $\max_I(0, 1) = b$).

7.3. One-sided ordering relations

Now, by relaxing the requirements of the information ordering \leq_I , we can obtain two other ordering relations on orthopairs, which generate two lattice operations. They are one-sided in the sense that the negative and positive literals do not play the same role. In one case, we keep the inclusion condition on the negative part of the orthopair and on the positive part in the other case. These order relations on orthopairs are:

$$(P_1, N_1) \leq_N (P_2, N_2) \quad \text{iff} \quad N_1 \subseteq N_2 \text{ and } P_1 \cup N_1 \subseteq P_2 \cup N_2, \quad (16)$$

$$(P_1, N_1) \leq_P (P_2, N_2) \quad \text{iff} \quad P_1 \subseteq P_2, \text{ and } P_1 \cup N_1 \subseteq P_2 \cup N_2, \quad (17)$$

In fact, $(P_1, N_1) \leq_N (P_2, N_2)$ means that (P_2, N_2) is at least as negative as (P_1, N_1) and not less informed than the latter and likewise for \leq_P , replacing negative by positive. It can be easily seen that the information ordering \leq_I implies orders \leq_N and \leq_P but not conversely. Together, however, they reconstruct the information ordering⁵ as follows:

$$(P_1, N_1) \leq_I (P_2, N_2) \iff (P_1, N_1) \leq_N (P_2, N_2) \text{ and } (P_1, N_1) \leq_P (P_2, N_2). \quad (18)$$

Fig. 4 represents these three orderings on orthopairs.

On pairs of valuations the two orders are translated as:

⁵ Condition $P_1 \cup N_1 \subseteq P_2 \cup N_2$ becomes redundant.

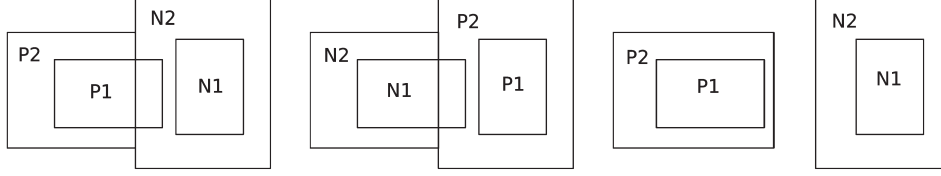


Fig. 4. Representations of orders \leq_N , \leq_P and \leq_I .

$$\begin{aligned} \vec{v}_1 \leq_N \vec{v}_2 \quad \text{iff} \quad & \min(1 - \underline{v}_2(a), \overline{v}_2(a)) \leq \min(1 - \underline{v}_1(a), \overline{v}_1(a)) \\ & \text{and} \quad \overline{v}_2(a) \leq \overline{v}_1(a) \end{aligned}$$

$$\begin{aligned} \vec{v}_1 \leq_P \vec{v}_2 \quad \text{iff} \quad & \min(1 - \underline{v}_2(a), \overline{v}_2(a)) \leq \min(1 - \underline{v}_1(a), \overline{v}_1(a)) \\ & \text{and} \quad \underline{v}_1(a) \leq \underline{v}_2(a). \end{aligned}$$

In terms of truth-values this corresponds to:

$$\tau_1 \leq_N \tau_2 \quad \text{iff} \quad \forall x \tau_1(x) = 0 \Rightarrow \tau_2(x) = 0 \text{ and } \tau_1(x) \neq \frac{1}{2} \Rightarrow \tau_2(x) \neq \frac{1}{2}$$

$$\tau_1 \leq_P \tau_2 \quad \text{iff} \quad \forall x \tau_1(x) = 1 \Rightarrow \tau_2(x) = 1 \text{ and } \tau_1(x) \neq \frac{1}{2} \Rightarrow \tau_2(x) \neq \frac{1}{2}$$

the order relations (16) and (17) do not correspond to the standard ordering on numbers, but are expressed as follows:

$$\tau_1 \leq_N \tau_2 \text{ iff } \forall x \tau_1(x) \leq_N \tau_2(x) \quad \text{where} \quad \frac{1}{2} \prec_N 1 \prec_N 0. \quad (19)$$

$$\tau_1 \leq_P \tau_2 \text{ iff } \forall x \tau_1(x) \leq_P \tau_2(x) \quad \text{where} \quad \frac{1}{2} \prec_P 0 \prec_P 1. \quad (20)$$

These two linear orderings on $\mathbf{3}$ are naturally associated with the following meet and join operations on orthopairs:

$$(P_1, N_1) \sqcup_N (P_2, N_2) := (P_1 \setminus N_2 \cup P_2 \setminus N_1, N_1 \cup N_2)$$

$$(P_1, N_1) \sqcap_N (P_2, N_2) := ((P_1 \cap (P_2 \cup N_2)) \cup (P_2 \cap (P_1 \cup N_1)), N_1 \cap N_2)$$

$$(P_1, N_1) \sqcup_P (P_2, N_2) := (P_1 \cup P_2, N_1 \setminus P_2 \cup N_2 \setminus P_1)$$

$$(P_1, N_1) \sqcap_P (P_2, N_2) := (P_1 \cap P_2, (N_1 \cap (P_2 \cup N_2)) \cup (N_2 \cap (P_1 \cup N_1)))$$

Clearly, the orthopair $(P_1, N_1) \sqcup_N (P_2, N_2)$ is always more negatively balanced than its operands, while $(P_1, N_1) \sqcap_N (P_2, N_2)$ is less negatively balanced. Likewise, $(P_1, N_1) \sqcup_P (P_2, N_2)$ is always more positively balanced than its operands, while $(P_1, N_1) \sqcap_P (P_2, N_2)$ is less positively balanced.

Proposition 7.1. *The two following equivalences hold:*

1. $(P_1, N_1) \leq_N (P_2, N_2)$ if and only if $(P_1, N_1) \sqcup_N (P_2, N_2) = (P_2, N_2)$ if and only if $(P_1, N_1) \sqcap_N (P_2, N_2) = (P_1, N_1)$;
2. $(P_1, N_1) \leq_P (P_2, N_2)$ if and only if $(P_1, N_1) \sqcup_P (P_2, N_2) = (P_2, N_2)$ if and only if $(P_1, N_1) \sqcap_P (P_2, N_2) = (P_1, N_1)$.

Proof. \sqcup_N : Let us prove that if $N_1 \subseteq N_2$ and $N_1 \cup P_1 \subseteq N_2 \cup P_2$ then $(P_1 \setminus N_2) \cup (P_2 \setminus N_1) = P_2$.

- $P_2^c \cap ((P_1 \setminus N_2) \cup (P_2 \setminus N_1)) = P_2^c \cap (P_1 \setminus N_2)$ since $P_2^c \cap P_2$ is empty. Moreover, $P_2^c \cap (P_1 \setminus N_2) = \emptyset$. Indeed, proceeding by refutation, if there exists $a \in P_2^c \cap (P_1 \setminus N_2)$ then $a \notin P_2$, $a \notin N_2$ and $a \in P_1$ implies $a \notin N_1$, which violates the condition $N_1 \cup P_1 \subseteq N_2 \cup P_2$. Hence $(P_1 \setminus N_2) \cup (P_2 \setminus N_1) \subseteq P_2$.
- Now, $P_2 \cap ((P_1 \setminus N_2) \cup (P_2 \setminus N_1))^c = P_2 \cap (P_1^c \cup N_2) \cap (P_2^c \cup N_1) = \emptyset$, since $P_2 \cap (P_2^c \cup N_1) = P_2 \cap N_1 = \emptyset$. Indeed, $N_1 \subseteq N_2$ and (P_2, N_2) is an orthopair. So $P_2 \subseteq (P_1 \setminus N_2) \cup (P_2 \setminus N_1)$. Hence $(P_1 \setminus N_2) \cup (P_2 \setminus N_1) = P_2$.

Conversely, we must prove that if $N_1 \subseteq N_2$ and $(P_1 \setminus N_2) \cup (P_2 \setminus N_1) = P_2$ then $N_1 \cup P_1 \subseteq N_2 \cup P_2$. Consider an element a such that $a \in N_1 \cup P_1$. There are two cases:

- $a \in N_1$. Then $N_1 \subseteq N_2$ leads immediately to $a \in N_2 \cup P_2$.
- $a \in P_1$. Either $a \in P_1 \setminus N_2$ hence $a \in P_2$ hence $a \in N_2 \cup P_2$. Or $a \in P_1 \cap N_2$, hence $a \in N_2 \cup P_2$.

The other cases can be proved similarly. \square

Expressed as operations on $\mathbf{3}$, \sqcup_N and \sqcup_P become \max_N and \max_P as shown in Table 7. These operations are not new. They are the only two everywhere defined uninorms on $\{0, \frac{1}{2}, 1\}$ (semi-groups with identity $\frac{1}{2}$ [26]); \max_N and \max_P also

Table 7
Disjunctive and Conjunctive uninorms.

\max_P	0	$\frac{1}{2}$	1	\max_N	0	$\frac{1}{2}$	1
0	0	0	1	0	0	0	0
$\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{1}{2}$	0	$\frac{1}{2}$	1
1	1	1	1	1	0	1	1

Table 8
Operations corresponding to \sqcap_P and \sqcap_N .

\min_P	0	$\frac{1}{2}$	1	\min_N	0	$\frac{1}{2}$	1
0	0	$\frac{1}{2}$	0	0	0	$\frac{1}{2}$	1
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
1	0	$\frac{1}{2}$	1	1	1	$\frac{1}{2}$	1

coincide with \max_I when the latter is defined. In the probabilistic literature, the operation \max_N is named *quasi-conjunction* [1] and we call \max_P *quasi-disjunction*. They are used in the three-valued logic of conditional events [17] to combine conditionals (which are indeed orthopairs of examples and counterexamples). Actually, these operations first appeared in Sobocinski's three-valued logic [38]. For orthopairs, \sqcup_N looks like a disjunction, but the truth-table of \max_N (right-hand side of Table 7) makes it clear it is a generalised conjunction.

Expressed as operations on $\mathbf{3}$, \sqcap_N and \sqcap_P become \min_N and \min_P as shown in Table 8. Interestingly, the table of operation \min_I (Table 6) can be obtained as the arithmetic mean of the tables of operation \min_N and \min_P , which reflects property (18). We note that \sqcap_P corresponds to Kleene weak conjunction and \sqcap_N to Kleene weak disjunction [29], where the third value is interpreted as *undefined*. Again, \sqcap_N looks like a conjunction on orthopairs and its truth-table (right-hand side of Table 8) makes it clear it is a generalised disjunction.

If we apply the two meet operations \sqcap_N, \sqcap_P to two orthopairs, one dominating the other in the sense of \leq_N or \leq_P , then they both reduce to \sqcap_I .

Let us note that we can express the two conjunctions of orthopairs in the following way:

$$\sqcap_N: ((P_1 \cap P_2) \cup [(P_1 \cap N_2) \cup (P_2 \cap N_1)], N_1 \cap N_2) \quad (21)$$

$$\sqcap_P: (P_1 \cap P_2, (N_1 \cap N_2) \cup [(N_1 \cap P_2) \cup (N_2 \cap P_1)]) \quad (22)$$

from which we better understand that we “add” something to the intersection of positive (resp., negative) parts. The corresponding operations on valuation pairs are:

$$\bar{v}_1 \wedge_N \bar{v}_2 = (\max(\min(\underline{v}_1, \underline{v}_2), \min(\underline{v}_1, 1 - \bar{v}_2), \min(\underline{v}_2, 1 - \bar{v}_1)), \max(\bar{v}_1, \bar{v}_2)) \quad (23)$$

$$\bar{v}_1 \wedge_P \bar{v}_2 = (\min(\underline{v}_1, \underline{v}_2), \min(\max(\bar{v}_1, \bar{v}_2), \max(\bar{v}_1, 1 - \underline{v}_2), \max(\bar{v}_2, 1 - \underline{v}_1))) \quad (24)$$

$$\bar{v}_1 \vee_N \bar{v}_2 = (\max(\min(\underline{v}_1, \bar{v}_2), \min(\underline{v}_2, \bar{v}_1)), \min(\bar{v}_1, \bar{v}_2)) \quad (25)$$

$$\bar{v}_1 \vee_P \bar{v}_2 = (\max(\underline{v}_1, \underline{v}_2), \min(\max(\bar{v}_1, \underline{v}_2), \max(\bar{v}_2, \underline{v}_1))) \quad (26)$$

Thus, orderings \leq_P and \leq_N are less demanding from the “knowledge” point of view than the information ordering, but they have the advantage of generating a lattice structure, with the possibility to define genuine operations of conjunction and disjunction on $\mathbf{3}$.

7.4. Fusion of orthopairs by consensus

In [31], a *difference* operation is equivalently defined in three formats by:

$$\begin{aligned} \bar{v}_1 \ominus \bar{v}_2 &:= (\min(\underline{v}_1, \bar{v}_2), \max(\bar{v}_1, \underline{v}_2)) \\ (P_1, N_1) \ominus (P_2, N_2) &:= (P_1 \setminus N_2, N_1 \setminus P_2) \\ (\tau_1 \ominus \tau_2)(x) &:= \begin{cases} 1 & \tau_1 = 1, \tau_2 \neq 0 \\ 0 & \tau_1 = 0, \tau_2 \neq 1 \\ \frac{1}{2} & \text{otherwise} \end{cases} \end{aligned}$$

On orthopairs, it consists in removing from (P_1, N_1) the literals that make this partial model inconsistent with (P_2, N_2) . This can be viewed as a contraction operation, that is a prerequisite for a consistent fusion step.

Table 9
Consensus operation.

\odot	0	$\frac{1}{2}$	1
0	0	0	$\frac{1}{2}$
$\frac{1}{2}$	0	$\frac{1}{2}$	1
1	$\frac{1}{2}$	1	1

Table 10
Mutual translations ($\max_I(x, y)$ is not always defined).

3	Orthopairs
$0 < \frac{1}{2} < 1$	\leq_t
$\frac{1}{2} \leq_I 1; \frac{1}{2} \leq_I 0$	\leq_I
$\frac{1}{2} \leq_N 1 \leq_N 0$	\leq_N
$\frac{1}{2} \leq_P 0 \leq_P 1$	\leq_P
$\min(\sqcap)$	\sqcap_t
$\max(\sqcup)$	\sqcup_t
$\min_I(x, y) = \text{med}(x, y, \frac{1}{2}) = \frac{\min_N(x, y) + \min_P(x, y)}{2}$	\sqcap_I
$\max_I(x, y) = \frac{xy}{xy + (1-x)(1-y)}$ on $\{0, \frac{1}{2}, 1\}$	\sqcup_I
$x \odot y = \frac{\max_N(x, y) + \max_P(x, y)}{2}$	$(\cdot \sqcup_N \cdot) \sqcap_I (\cdot \sqcup_P \cdot)$

If we consider the so-called *consensus* operation \odot in [31], defined as

$$(P_1, N_1) \odot (P_2, N_2) = (P_1 \setminus N_2 \cup P_2 \setminus N_1, N_1 \setminus P_2 \cup N_2 \setminus P_1)$$

we can see that it is a sequence of difference operations followed by operation \sqcup_I that increases information. The difference operations are used to symmetrically eliminate the discrepancies between the orthopairs (P_1, N_1) and (P_2, N_2) prior to merging, eliminating conflicting literals. This is akin to variable forgetting [30] in propositional logic for the handling of inconsistency: the truth-value of conflicting variables (in $(P_1 \cap N_2) \cup (P_2 \cap N_1)$) is forgotten. So we have that

$$(P_1, N_1) \odot (P_2, N_2) = ((P_1, N_1) \ominus (P_2, N_2)) \sqcup_I ((P_2, N_2) \ominus (P_1, N_1)).$$

We can also relate the consensus operation to the asymmetric relations \sqcup_N and \sqcup_P . Namely it can be checked that:

$$(P_1, N_1) \odot (P_2, N_2) = ((P_1, N_1) \sqcup_N (P_2, N_2)) \sqcap_I ((P_2, N_2) \sqcup_P (P_1, N_1)). \quad (27)$$

Thus we can think of interpreting \sqcup_N and \sqcup_P as *one-sided consensus* operations where in \sqcup_N both agents give up their conflicts on the positive parts and in \sqcup_P on the negative parts of the orthopairs.

The consensus operation can be expressed in terms of valuation pairs as $(\vec{v}_1 \ominus \vec{v}_2) \sqcup_I (\vec{v}_2 \ominus \vec{v}_1)$:

$$\vec{v}_1 \odot \vec{v}_2 = (\max(\min(\underline{v}_1, \bar{v}_2), \min(\bar{v}_1, \underline{v}_2)), \min(\max(\underline{v}_1, \bar{v}_2), \max(\bar{v}_1, \underline{v}_2))) \quad (28)$$

On **3** the consensus operator is based on the following three-valued commutative connective:

$$(\tau_1 \odot \tau_2)(x) = \begin{cases} 1 & \text{if } \tau_1(x) = 1, \tau_2(x) \neq 0 \text{ or } \tau_2(x) = 1, \tau_1(x) \neq 0 \\ 0 & \text{if } \tau_1(x) = 0, \tau_2(x) \neq 1 \text{ or } \tau_2(x) = 0, \tau_1(x) \neq 1 \\ \frac{1}{2} & \text{otherwise} \end{cases}$$

which is pictured on Table 9. It lays bare the fact that it is a kind of degenerated qualitative counterpart of an average. It has $\frac{1}{2}$ as an identity and differs from the conjunctive and disjunctive uninorms only for the result of $1 \odot 0 = 0 \odot 1 = \frac{1}{2}$, hence it is not the extension of a Boolean connective. Moreover, it is not associative. It can be obtained as the arithmetic average of the two uninorms \max_P and \max_N of Table 7, which is consistent with the abstract construction of \odot from \sqcup_N and \sqcup_P in Eq. (27).

Table 10 summarises the results of this section in terms of mutual translation of orderings and connectives between 3-valued logic and orthopairs. For consistent BVP's, not all the above results apply over the whole language, involving more complex formulas. The problem seems to lie in the formulas containing negations. However, this issue will need a further in-depth study.

8. From orthopairs to epistemic sets of interpretations

We now investigate whether it is feasible to translate all the above orderings and operations so as to express them in terms of comparison and combinations of subsets of Ω , understood as possibility distributions representing epistemic states.

8.1. Order relations

The order relation \leq_I is just the subethood relation of 2^Ω , i.e., $E_{(P_1, N_1)} \leq_I E_{(P_2, N_2)}$ iff $E_{(P_2, N_2)} \subseteq E_{(P_1, N_1)}$. In the case of orders \leq_t, \leq_N, \leq_p , we only get a partial overlap of the two subsets, which displays no special features. This is due to the fact that the representation in terms of orthopairs highlights the polarity of the known variables (positive or negative) while the standard propositional setting represents an epistemic set by any subset of Boolean interpretations where positive or negative literals play no specific roles.

Let us consider order \leq_N . We have $N_1 \subseteq N_2$ and $P_1 \cup N_1 \subseteq P_2 \cup N_2$. So, we can also derive $P_1 \subseteq P_2 \cup N_2$. That is, for all $p \in P_1$, agent 2 may believe that $w(p) = 1$ (like agent 1) or that $w(p) = 0$ (contrary to agent 1). A similar explanation can be given in the case of \leq_p . The ordering \leq_t is similar. This order requires that agent 1 has more knowledge on the negative part ($N_2 \subseteq N_1$) but less on the positive one ($P_1 \subseteq P_2$). So, we can have the case where a valuation belongs to the epistemic set E_1 but not to E_2 and another one to the epistemic set E_2 but not to E_1 . So, most of the orderings introduced previously are dedicated to orthopairs of sets of variables and hardly extend to the general propositional setting.

8.2. Intersection and union

Given that for possibility distributions we can only represent the order relation \leq_I we take into account here only the operations derived from it: \sqcap_I and \sqcup_I . As we have seen, they are the meet and join (when it exists) on orthopairs. Generally, the join and meet with respect to the subethood relation are the intersection and union of sets, respectively. However, \sqcap_I does not correspond to the union between possibility distributions, i.e., $E_{(P_1, N_1) \sqcap_I (P_2, N_2)} \neq E_{(P_1, N_1)} \cup E_{(P_2, N_2)}$. Indeed $E_{(P_1, N_1)} \cup E_{(P_2, N_2)}$ is generally not rectangular. We can just prove:

Proposition 8.1.

$$E_{(P_1, N_1)} \cup E_{(P_2, N_2)} \subseteq E_{((P_1, N_1) \sqcap_I (P_2, N_2))}.$$

Proof. We have that $(P_1 \cap P_2) \subseteq P_1$ and $(N_1 \cap N_2) \subseteq N_1$. That is, $(P_1, N_1) \sqcap_I (P_2, N_2) \leq_I (P_1, N_1)$. So, $E_{(P_1, N_1)} \subseteq E_{(P_1, N_1) \sqcap_I (P_2, N_2)}$.

Likewise for $E_{(P_2, N_2)}$. Hence $E_{(P_1, N_1)} \cup E_{(P_2, N_2)} \subseteq E_{((P_1, N_1) \sqcap_I (P_2, N_2))}$. \square

The other direction does not hold. Indeed we can only prove the following: consider the valuation w^* characterised by the orthopair $(P_1 \cap P_2, (P_1 \cap P_2)^c)$: then $w^* \in E_{(P_1, N_1) \sqcap_I (P_2, N_2)}$ but $w^* \notin E_{(P_1, N_1)} \cup E_{(P_2, N_2)}$.

This behaviour is due to the impossibility of representing non-rectangular regions using orthopairs. Indeed, we can only represent the smallest hyper-rectangle which contains $E_{(P_1, N_1)}$ and $E_{(P_2, N_2)}$ (the rectangular closure of their union), which corresponds indeed to $E_{((P_1, N_1) \sqcap_I (P_2, N_2))}$.

Proposition 8.2. $E_{((P_1, N_1) \sqcap_I (P_2, N_2))} = \times_i (\text{Proj}_i E_{(P_1, N_1)} \cup \text{Proj}_i E_{(P_2, N_2)}) = \text{RC}(E_{(P_1, N_1)} \cup E_{(P_2, N_2)})$.

Proof. It is clear that $E_{((P_1, N_1) \sqcap_I (P_2, N_2))} = E_{(P_1 \cap P_2, N_1 \cap N_2)}$ contains all the valuations w that have value 1 on $P_1 \cap P_2$ and value zero on $N_1 \cap N_2$. Now, if we consider the Cartesian product of the union of the projections of $E_{(P_1, N_1)}$ and $E_{(P_2, N_2)}$, we have that, for each variable a_i , a_i must be equal to 1 iff both $E_{(P_1, N_1)}$ and $E_{(P_2, N_2)}$ assume value 1 for a_i . This is only possible if $a_i \in P_1 \cap P_2$. The same applies to the negative part. Thus, $\times_i (\text{Proj}_i E_{(P_1, N_1)} \cup \text{Proj}_i E_{(P_2, N_2)})$ is the set of all the valuations w that have value 1 on $P_1 \cap P_2$ and value zero on $N_1 \cap N_2$. \square

As pointed out before, the disjunction of two rectangular possibility distributions is generally not representable by one orthopair, but by several orthopairs, since it corresponds to the disjunction of two partial models. We can denote this disjunction of orthopairs as $(P_1, N_1) + (P_2, N_2)$ with the meaning that it refers the set $\{(P_1, N_1), (P_2, N_2)\}$ of orthopairs, and computes the disjunction of the corresponding partial models:

$$E_{((P_1, N_1) + (P_2, N_2))} = E_{(P_1, N_1)} \cup E_{(P_2, N_2)}$$

Alternatively, we can interpret this as collecting together all the situations where at least one of two agents is right, without specifying which one. Of course, we cannot express this operation on three-valued functions, since it is not closed: it just collects together the two valuations representing the two orthopairs $\tau_{(P_1, N_1)} + \tau_{(P_2, N_2)} = \{\tau_{(P_1, N_1)}, \tau_{(P_2, N_2)}\}$. Similarly in the case of valuations pairs.

If we consider the case of two consistent orthopairs, then the operation \sqcup_I is defined, and we have:

Proposition 8.3. *If (P_1, N_1) and (P_2, N_2) are consistent orthopairs, then $E_{(P_1, N_1)} \cap E_{(P_2, N_2)} = E_{(P_1, N_1) \sqcup_I (P_2, N_2)} = E_{(P_1, N_1) \odot (P_2, N_2)}$.*

Proof. Under consistency we have that $(P_1, N_1) \sqcup_I (P_2, N_2) = (P_1, N_1) \odot (P_2, N_2)$. Then, $E_{(P_1, N_1) \sqcup_I (P_2, N_2)} = E_{((P_1 \cup P_2), (N_1 \cup N_2))}$, the latter being equal by definition to: $\{w \in \Omega : w(p) = 1, \forall p \in P_1 \cup P_2 \text{ and } w(p) = 0, \forall p \in N_1 \cup N_2\} = \{w \in \Omega :$

$w(p) = 1, \forall p \in P_1$ and $w(p) = 1, \forall p \in P_2$ and $w(p) = 0, \forall p \in N_1$ and $w(p) = 0, \forall p \in N_2$, from which $E_{(P_1, N_1)} \cap E_{(P_2, N_2)} = E_{(P_1, N_1) \sqcup_I (P_2, N_2)}$ follows. \square

8.3. Consensus and difference

By a straightforward application of its definition, we get that the consensus $E_{(P_1, N_1) \odot (P_2, N_2)}$ contains the intersection of the two epistemic subsets $E_{(P_1, N_1)}$ and $E_{(P_2, N_2)}$, and is also contained in $E_{(P_1, N_1) \sqcap_I (P_2, N_2)}$.

Proposition 8.4.

$$E_{(P_1, N_1)} \cap E_{(P_2, N_2)} \subseteq E_{(P_1, N_1) \odot (P_2, N_2)} \subseteq E_{(P_1, N_1) \sqcap_I (P_2, N_2)}.$$

However, it is incomparable with their union $E_{(P_1, N_1)} \cup E_{(P_2, N_2)}$, that is there can exist $w \in E_{(P_1, N_1) \odot (P_2, N_2)}$ such that $w \notin E_{(P_1, N_1)} \cup E_{(P_2, N_2)}$ and conversely.

Example 8.1. Consider the Boolean valuation w_1 , corresponding to the orthopair $(P_1 \setminus N_2 \cup P_2 \setminus N_1, (P_1 \setminus N_2 \cup P_2 \setminus N_1)^c)$. It belongs to $E_{(P_1, N_1) \odot (P_2, N_2)}$ (by definition) and not to $E_{(P_1, N_1)} \cup E_{(P_2, N_2)}$, since it belongs neither to $E_{(P_1, N_1)}$ nor to $E_{(P_2, N_2)}$.

Conversely, the Boolean valuation w_2 encoded by $((N_1 \cup (P_2 \setminus P_1))^c, N_1 \cup (P_2 \setminus P_1)) = (N_1^c \cap (P_2^c \cup P_1), N_1 \cup (P_2 \setminus P_1))$ belongs to $E_{(P_1, N_1)} \cup E_{(P_2, N_2)}$ (since it lies in $E_{(P_1, N_1)}$) and not to $E_{(P_1, N_1) \odot (P_2, N_2)}$, since any $a \in P_1^c \cap P_2^c \cap N_1^c \cap N_2$ is such that $w_2(a) = 1$ but note that $\forall w \in E_{(P_1, N_1) \odot (P_2, N_2)}, w(a) = 0$ since $a \in (N_1 \setminus P_2) \cup (N_2 \setminus P_1)$ (the negative literals of $(P_1, N_1) \odot (P_2, N_2)$).

Furthermore, by [Propositions 8.1 and 8.4](#) we can derive

$$E_{(P_1, N_1)} \cup E_{(P_2, N_2)} \cup E_{((P_1, N_1) \odot (P_2, N_2))} \subseteq E_{(P_1, N_1) \sqcap_I (P_2, N_2)}.$$

But not the opposite direction, as can be seen by considering the valuation w^* above, characterised by the orthopair $(P_1 \cap P_2, (P_1 \cap P_2)^c)$.

If we write $(P_1, N_1) \odot (P_2, N_2)$ in terms of unions between orthopairs, namely $(P_1, N_1) \odot (P_2, N_2) = (P_1 \setminus N_2, N_1 \setminus P_2) \sqcup_I (P_2 \setminus N_1, N_2 \setminus P_1)$, by [Proposition 8.3](#), we get

$$E_{((P_1, N_1) \odot (P_2, N_2))} = E_{(P_1 \setminus N_2, N_1 \setminus P_2)} \cap E_{(P_2 \setminus N_1, N_2 \setminus P_1)},$$

which expresses the already outlined relation between consensus and difference. The latter operation can be expressed in terms of possibility distributions associated with $E_{(P_i, N_i)}$. Consider an orthopair (P, N) and a valuation $w \in \Omega$ such that $w \notin E_{(P, N)}$. Then define:

$$E_{(P, N)}^{+w} = RC(E_{(P, N)} \cup \{w\})$$

Now clearly we obtain: $E_{(P, N)}^{+w} = E_{(P \cap P_w, N \cap P_w^c)}$, where $P_w = \{a : w(a) = 1\}$, and $\forall w_1 \neq w_2$ it holds that $E_{(P, N)}^{+w_1} \cap E_{(P, N)}^{+w_2} \neq \emptyset$ (it contains $E_{(P, N)}$).

Proposition 8.5. Let (P_1, N_1) and (P_2, N_2) be mutually inconsistent orthopairs (i.e. $E_{(P_1, N_1)} \cap E_{(P_2, N_2)} = \emptyset$). Then

$$E_{(P_1, N_1) \odot (P_2, N_2)} = E_{(P_1 \setminus N_2, N_1 \setminus P_2)} = \bigcap_{w \in E_{(P_2, N_2)}} E_{(P_1, N_1)}^{+w}$$

Proof. By [Proposition 8.3](#) it follows that:

$$\bigcap_{w \in E_{(P_2, N_2)}} E_{(P_1, N_1)}^{+w} = \bigcap_{w \in E_{(P_2, N_2)}} E_{(P_1 \cap P_w, N_1 \cap P_w^c)} = E_{\sqcup_I w \in E_{(P_2, N_2)} (P_1 \cap P_w, N_1 \cap P_w^c)}$$

Now,

$$\begin{aligned} \bigcap_{w \in E_{(P_2, N_2)}} (P_1 \cap P_w, N_1 \cap P_w^c) &= \left(P_1 \cap \bigcup_{w \in E_{(P_2, N_2)}} P_w, N_1 \cap \bigcup_{w \in E_{(P_2, N_2)}} P_w^c \right) \\ &= (P_1 \cap \{a_i : \exists w \in E_{(P_2, N_2)}, w(a_i) = 1\}, N_1 \cap \{a_i : \exists w \in E_{(P_2, N_2)}, w(a_i) = 0\}) \\ &= (P_1 \cap N_2^c, N_1 \cap P_2^c) \end{aligned}$$

as required. \square

Table 11
Translation between epistemic sets and orthopairs.

Orthopairs	Epistemic sets
$(P_1, N_1) \leq_l (P_2, N_2)$	$E_{(P_1, N_1)} \supseteq E_{(P_2, N_2)}$
$(P_1, N_1) \sqcap_l (P_2, N_2)$	$RC(E_{(P_1, N_1)} \cup E_{(P_2, N_2)})$
$(P_1, N_1) \sqcup_l (P_2, N_2)$	$E_{(P_1, N_1)} \cap E_{(P_2, N_2)}$ (if consistency)
$(P_1, N_1) \odot (P_2, N_2)$	$(\bigcap_{w \in E_{(P_2, N_2)}} E_{(P_1, N_1)}^{+w}) \cap (\bigcap_{w \in E_{(P_1, N_1)}} E_{(P_2, N_2)}^{+w})$
$(P_1, N_1) \ominus (P_2, N_2)$	$E_{(P_1 \setminus N_2, N_1 \setminus P_2)} = \bigcap_{w \in E_{(P_2, N_2)}} E_{(P_1, N_1)}^{+w}$ (Proposition 8.5)
$(P_1, N_1) + (P_2, N_2)$	$E_{(P_1, N_1)} \cup E_{(P_2, N_2)}$
$(P, N)' = (N, P)$	$\times_{p_i \in \mathcal{V}} (\text{Proj}_i E)^c$
$\mathcal{E}((P, N)')$	Complement $E_{(P, N)}^c$

Finally, from this proposition it follows that:

$$E_{(P_1, N_1) \odot (P_2, N_2)} = \left(\bigcap_{w \in E_{(P_2, N_2)}} E_{(P_1, N_1)}^{+w} \right) \cap \left(\bigcap_{w \in E_{(P_1, N_1)}} E_{(P_2, N_2)}^{+w} \right)$$

8.4. Negations vs. complements

Consider now the case of Kleene negation $(P, N)' = (N, P)$. The set of models of the negation of a Boolean proposition with set of models E is its set complement E^c . For a partial model (P, N) , we obtain:

$$(E_{(P, N)})^c = \{w : \exists p \in P, w(p) = 0 \text{ or } \exists p \in N, w(p) = 1\}$$

which is clearly different from the epistemic set we obtain from the Kleene (involutive) negation of (P, N) :

$$E_{(P, N)'} = E_{(N, P)}$$

which swaps positive and negative literals in the orthopair. The latter comes down to considering the projections of $E_{(P, N)}$ on each variable domain and complementing them with respect to these variable domains. This operation cannot be extended to general epistemic sets.

In contrast, the $+$ operation, that concatenates orthopairs (see Subsection 8.2) can express the complement of an epistemic set corresponding to an orthopair. Let us define the unary operations $\mathcal{E}(P, N) = (+_{p_i \in P}(\{p_i\}, \emptyset)) + (+_{n_i \in N}(\emptyset, \{n_i\}))$, that is, $\mathcal{E} = \{(\{p_i\}, \emptyset), (\emptyset, \{n_i\}) : p_i \in P, n_i \in N\}$. Then:

$$(E_{(P, N)})^c = E_{\mathcal{E}((P, N)')} = \left(\bigcup_{p_i \in P} E_{(\emptyset, \{p_i\})} \right) \cup \left(\bigcup_{n_i \in N} E_{(\{n_i\}, \emptyset)} \right).$$

The operation \mathcal{E} on three-valued valuations is a multimapping that collects all the following three-valued valuations $\mathcal{E}(\tau) = \{\tau_i^+ : p_i \in P\} \cup \{\tau_i^- : n_i \in N\}$ where:

$$\tau_i^+(x) = \begin{cases} 1 & x = p_i \\ \frac{1}{2} & \text{otherwise} \end{cases} \quad \text{or} \quad \tau_i^-(x) = \begin{cases} 0 & x = n_i \\ \frac{1}{2} & \text{otherwise} \end{cases}$$

and we can define the 3-valued logic counterpart of the complement of an epistemic set as $\tau^c = \mathcal{E}(\tau') = \{(\tau_i^+)' : p_i \in \mathcal{A}, \tau(p_i) = 1\} \cup \{(\tau_i^-)' : n_i \in \mathcal{A}, \tau(n_i) = 0\}$. Likewise on Boolean valuations, we must collect all pairs such that:

$$(\underline{v}_i(x), \bar{v}_i(x)) = \begin{cases} (1, 1) & x = p_i \\ (0, 1) & \text{otherwise} \end{cases} \quad \text{or} \quad (\underline{v}_i(x), \bar{v}_i(x)) = \begin{cases} (0, 0) & x = n_i \\ (0, 1) & \text{otherwise.} \end{cases}$$

Table 11 summarises the possible translations from the language of orthopairs and three-valued valuations into the language of general epistemic sets. Not all three-valued connectives can be expressed in terms of general epistemic sets. The impossibility of expressing \sqcup_l and \sqcap_l , or likewise, \sqcup_N and \sqcap_N , \sqcup_P and \sqcap_P in terms of general subsets of valuation is due to the fact that the former operations do not consider positive and negative literals on a par. Other impossible direct translations come from the fact that the set-union of hyper-rectangles of interpretations is generally not an hyper-rectangle, or in other words, that the disjunction of partial models is not a partial model.

9. Conclusion

The bijection between three-valued valuations and partial models has often been a source of confusion. In particular, the truth-functionality assumption of three-valued logic, for which there is a case as an elementary modelling of vagueness, becomes counterintuitive when applied to incomplete Boolean information. Another well-known example of such confusion

between truth and certainty is the alleged inconsistency of fuzzy set theory, claimed by Elkan [21], on the grounds that degrees of uncertainty cannot be compositional.

This paper has presented a comparative overview of the simplest representation frameworks for the truth-functional account of vagueness or incomplete information, namely three-valued logics where the third truth-value means *borderline* or *unknown*. We have shown the connections existing between pairs of sets of literals, pairs of Boolean valuations, and three-valued valuations. We have explored their differences with non-truth-functional accounts using possibility theory or, in the case of vagueness, supervaluations. We have shown that supervaluations play the same role with respect to truth-functional approaches to vagueness as do possibility and necessity functions with respect to partial or Kleene logic approaches to incomplete information. In addition, we have highlighted the difference in expressive power between partial model approaches and the use of general epistemic sets for handling incomplete information.

Finally, we have described various natural orderings and aggregation operations in the three-valued setting, in terms of orthopairs and Boolean valuation pairs. We have shown that some of these aggregations, expressed on the three-valued truth-set are already known in the literature. The systematic assumption that knowledge pertains to individual variables has led to asymmetric ordering relations where positive and negative literals do not play the same role (as in logic programming, or logic-based diagnosis). As a consequence these polarity-sensitive orderings between partial models as well as the aggregation operations they induce make sense for partial models but not to the same extent for general epistemic sets, where positive literals do not play a specific role.

Hopefully, the analysis given in this paper will contribute to bridging the gap between various communities (databases, logic programming, epistemic logic, possibilistic logic, vagueness) that use different languages for handling similar issues in the modelling of vague predicates or incomplete information.

References

- [1] E. Adams, *The Logic of Conditionals: An Application of Probability to Deductive Logic*, Reidel, Dordrecht, 1975.
- [2] M. Banerjee, D. Dubois, A simple modal logic for reasoning about revealed beliefs, in: *Proceedings ECSQARU*, 2009, pp. 805–816.
- [3] M. Banerjee, D. Dubois, A simple logic for reasoning about incomplete knowledge, *Int. J. Approx. Reason.* 55 (2014) 639–653.
- [4] N.D. Belnap, A useful four-valued logic, in: J.M. Dunn, G. Epstein (Eds.), *Modern Uses of Multiple-Valued Logic*, D. Reidel Publishing Company, 1977, pp. 8–37.
- [5] P. Besnard, A. Hunter, Quasi-classical logic: Non-trivializable classical reasoning from inconsistent information, in: C. Froidevaux, J. Kohlas (Eds.), *ECSQARU*, in: *Lecture Notes in Computer Science.*, vol. 946, Springer, 1995, pp. 44–51.
- [6] S. Blamey, Partial logic, in: D.M. Gabbay, F. Guenther (Eds.), *Handbook of Philosophical Logic*, vol. 3, D. Reidel Publishing Company, 1985, pp. 1–70.
- [7] R.E. Bryant, Symbolic boolean manipulation with ordered binary-decision diagrams, *ACM Comput. Surv.* 24 (3) (1992) 293–318.
- [8] G. Cattaneo, D. Ciucci, D. Dubois, Algebraic models of deviant modal operators based on De Morgan and Kleene lattices, *Inf. Sci.* 181 (19) (2011) 4075–4100.
- [9] D. Ciucci, Orthopairs: A simple and widely used way to model uncertainty, *Fundam. Inform.* 108 (3–4) (2011) 287–304.
- [10] D. Ciucci, D. Dubois, A modal theorem-preserving translation of a class of three-valued logics of incomplete information, *J. Appl. Non-Class. Log.* 23 (4) (2013) 321–352.
- [11] E.F. Codd, Extending the database relational model to capture more meaning, *ACM Trans. Database Syst.* 4 (4) (1979) 397–434.
- [12] G. De Cooman, From possibilistic information to Kleene strong multi-valued logics, in: *Fuzzy Sets, Logics and Reasoning about Knowledge*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1999, pp. 315–323.
- [13] D. Dubois, On ignorance and contradiction considered as truth-values, *Log. J. IGPL* 16 (2008) 195–216.
- [14] D. Dubois, Have fuzzy sets anything to do with vagueness? (with discussion) in: P. Cintula, C. Fermüller (Eds.), *Understanding Vagueness – Logical, Philosophical and Linguistic Perspectives*, in: *Studies in Logic*. College Publications, vol. 36, 2012, pp. 317–346, <http://www.collegepublications.co.uk/>.
- [15] D. Dubois, Reasoning about ignorance and contradiction: many-valued logics versus epistemic logic, *Soft Comput.* 16 (11) (2012) 1817–1831.
- [16] D. Dubois, S. Konieczny, H. Prade, Quasi-possibilistic logic and its measures of information and conflict, *Fundam. Inform.* 57 (2–4) (2003) 101–125.
- [17] D. Dubois, H. Prade, Conditional objects as nonmonotonic consequence relationships, *IEEE Trans. Syst. Man Cybern.* 24 (12) (1994) 1724–1740.
- [18] D. Dubois, H. Prade, Possibility theory: qualitative and quantitative aspects, in: *Handbook of Defeasible Reasoning and Uncertainty Management Systems*, 1, Kluwer, 1998, pp. 169–226.
- [19] D. Dubois, H. Prade, Possibility theory, probability theory and multiple-valued logics: A clarification, *Ann. Math. Artif. Intell.* 32 (2001) 35–66.
- [20] D. Dubois, H. Prade, Gradualness, uncertainty and bipolarity: making sense of fuzzy sets, *Fuzzy Sets Syst.* 192 (2012) 3–24.
- [21] C. Elkan, The paradoxical success of fuzzy logic, *IEEE Expert* 9 (4) (1994) 3–8.
- [22] K. Fine, Vagueness, truth and logic, *Synthese* 30 (1975) 265–300.
- [23] M. Fitting, A Kripke–Kleene semantics for logic programs, *J. Log. Program.* 2 (4) (1985) 295–312.
- [24] M. Fitting, Bilattices and the semantics of logic programming, *J. Log. Program.* 11 (1–2) (1991) 91–116.
- [25] Y. Gentilhomme, Les ensembles flous en linguistique, in: *Cahiers de Linguistique Théorique et Appliquée*, vol. 5, Editions de l’Académie de la République Populaire Roumaine, Bucarest, 1968, pp. 47–63.
- [26] M. Grabisch, J.-L. Marichal, R. Mesiar, E. Pap, *Aggregation Functions*, Cambridge University Press, 2009.
- [27] J.Y. Halpern, R. Fagin, Y. Moses, M. Vardi, *Reasoning About Knowledge*, MIT Press, 2003 (Revised paperback edition).
- [28] R. Keefe, P. Smith, *Vagueness: A Reader*, MIT Press, Cambridge, MA, 2007.
- [29] S.C. Kleene, *Introduction to Metamathematics*, North-Holland Pub. Co., Amsterdam, 1952.
- [30] J. Lang, P. Marquis, Reasoning under inconsistency: a forgetting-based approach, *Artif. Intell.* 174 (12–13) (2010) 799–823.
- [31] J. Lawry, D. Dubois, A bipolar framework for combining beliefs about vague propositions, in: G. Brewka, T. Eiter, S.A. McIlraith (Eds.), *Principles of Knowledge Representation and Reasoning: Proceedings of the Thirteenth International Conference, AAAI Press*, 2012, pp. 530–540.
- [32] J. Lawry, I. González-Rodríguez, A bipolar model of assertability and belief, *Int. J. Approx. Reason.* 52 (1) (2011) 76–91.
- [33] J. Lawry, Y. Tang, On truth-gaps, bipolar belief and the assertability of vague propositions, *Artif. Intell.* 191–192 (2012) 20–41.
- [34] G. Priest, The logic of paradox, *J. Philos. Log.* 8 (1979) 219–241.
- [35] R. Sainsbury, Concepts without boundaries, in: *Vagueness: A Reader*, MIT Press, Cambridge, MA, 1997, pp. 251–264.
- [36] S. Shapiro, *Vagueness in Context*, Oxford University Press, 2006.
- [37] N. Smith, *Vagueness and Degrees of Truth*, Oxford University Press, Oxford, UK, 2008.

- [38] B. Sobocinski, Axiomatization of a partial system of three-value calculus of propositions, *J. Comput. Syst.* 1 (1952) 23–55.
- [39] B. van Fraassen, Singular terms, truth-value gaps, and free logic, *J. Philos.* 63 (1966) 481–495.
- [40] T. Williamson, *Vagueness*, Routledge, London, 1994.
- [41] R. Yager, Set-based representations of conjunctive and disjunctive knowledge, *Inf. Sci.* 41 (1987) 1–22.
- [42] R.R. Yager, A foundation for a theory of possibility, *Cybern. Syst.* 10 (1–3) (1980) 177–204.
- [43] Y. Yao, Interval sets and interval-set algebras, in: *Proceedings of the 8th IEEE International Conference on Cognitive Informatics*, 2009, pp. 307–314.
- [44] L. Zadeh, Fuzzy sets, *Inf. Control* 8 (1965) 338–353.
- [45] L. Zadeh, Fuzzy sets as a basis for a theory of possibility, *Fuzzy Sets Syst.* 1 (1978) 3–28.