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# Cut Elimination for Monomial Proof Nets of the Purely Multiplicative and Additive Fragment of Linear Logic

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Roberto Maieli. Cut Elimination for Monomial Proof Nets of the Purely Multiplicative and Additive Fragment of Linear Logic. [Research Report] Department of Mathematics and Physics, Roma TRE University. 2007. hal-01153910

**HAL Id: hal-01153910**

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Submitted on 21 May 2015

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# Cut Elimination for Monomial Proof Nets of the Purely Multiplicative and Additive Fragment of Linear Logic\*

Roberto Maieli  
Università degli Studi “Roma Tre”  
maieli@uniroma3.it

Revisited Version: April 28, 2015

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Earlier Version: December 29, 2007

## Abstract

We present a simple cut-elimination procedure for MALL proof nets with monomial weights (*à la Girard*) and explicit contraction links, based on an almost local cut reduction steps. This procedure preserves correctness of proof nets and it is strong normalizing and confluent.

*Keywords* : Proof Theory, Sequent Calculus, Cut Elimination, Proof Nets, Linear Logic.

## Contents

<b>1</b>	<b>Introduction</b>	<b>2</b>
1.1	Multiplicative-Additive fragment of LL . . . . .	2
1.2	Monomial interpretation . . . . .	4
1.3	Polynomial interpretation . . . . .	6
1.4	Correctness Criterion for Monomial PNs . . . . .	8
1.5	Cut-elimination for Monomial PSS . . . . .	9
<b>2</b>	<b>Proof-structures with explicit binary contraction links</b>	<b>10</b>
<b>3</b>	<b>Cut-elimination</b>	<b>13</b>
3.1	Ready cut-elimination . . . . .	13
3.2	Commutative cut-elimination . . . . .	14
3.2.1	The confluence problem . . . . .	14
3.3	Stability . . . . .	19
3.4	Strong normalization . . . . .	21
3.4.1	Weak normalization . . . . .	21
3.4.2	Confluence . . . . .	21
<b>4</b>	<b>Proof-structures with explicit <math>n</math>-ary contraction links</b>	<b>26</b>
4.1	Cut-elimination . . . . .	29
4.2	Stability . . . . .	32
4.3	Strong cut-elimination . . . . .	32

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\*Partially supported by the CNR-CNRS Research Programme *Interaction Complexity* and the MIUR Research Programme *Rete Italo-Francese di Ricerca in Logica e Geometria della Computazione*. The author thanks Olivier Laurent and Paul Ruet for their remarks and suggestions.

# 1 Introduction

Proof Nets (PNs) are *parallel presentations* of sequential proofs (SP) of Linear Logic; they *quotient classes of equivalent proofs*, modulo irrelevant permutations of derivation rules. The standard key ingredients of a PN syntax are

- a graph syntax, i.e., *proof structures* (PSs);
- a *correctness criterion* defining *PNs* among PSs;
- an *interpretation* of the sequent calculus proofs;
- a *cut elimination* procedure

The **correctness criterion** should be:

- *geometrical*: an intrinsic (non-inductive) characterisation of those PS that *sequentialise* to SP (they are PN);
- *stable*: under cut elimination;
- *efficient*: *checking correctness* and *sequentialization* should be *P-time* with respect to the number of nodes.
- *Sequentializable*: (P-time), i.e., each PN must be the image of at least one SP.

The **interpretation** (translation) of SP into PS should be:

- *sound*: the PS associated to a SP, must be *correct* (a PN);
- *function*:  $SP \mapsto PN$ ;
- *canonical surjection*: SP equal up to (reasonable) commutations of rules must be identified upon translation to a PN;
- *efficient*: P-time in the size (of the proofs).
  - we should preserve the computational complexity of the interpreted proofs;
  - we should respect the notion that a *semantics* (PN) is a *structure-preserving map* or some kind of *homomorphism* from proofs.

The **cut elimination** procedure should be:

- *defined* directly on PS;
- *complete*: any cut node of a PS, reduces in one step;
- *local*: a cut elimination step only affects the nodes (immediately) connected to the reducing cut node;
- *strong normalising*: terminating and (locally) confluent;
- *efficient*: P-time in size.

## 1.1 Multiplicative-Additive fragment of LL

**Sequent Calculus** – *Formulas*  $A, B, \dots$  are built from *literals* by the binary connectives  $\otimes$  (*tensor*),  $\wp$  (*par*),  $\&$  (*with*) and  $\oplus$  (*plus*). *Negation*  $(.)^\perp$  extends to any formula by de Morgan laws:

$$\begin{aligned} (A \otimes B)^\perp &= (B^\perp \wp A^\perp) & (A \wp B)^\perp &= (B^\perp \otimes A^\perp) \\ (A \& B)^\perp &= (B^\perp \oplus A^\perp) & (A \oplus B)^\perp &= (B^\perp \& A^\perp) \end{aligned}$$

MALL (resp., MLL) *Sequents*  $\Gamma, \Delta$  are sets of formula occurrences  $A_1, \dots, A_{n \geq 1}$ , proved using the following rules (resp., only identity and multiplicative rules):

- identity:  $\frac{}{A, A^\perp} \text{ax} \quad \frac{\Gamma, A \quad \Delta, A^\perp}{\Gamma, \Delta} \text{cut}$
- multiplicatives:  $\frac{\Gamma, A \quad \Delta, B}{\Gamma, \Delta, A \otimes B} \otimes \quad \frac{\Gamma, A, B}{\Gamma, A \wp B} \wp$

- additives:  $\frac{\Gamma, A \quad \Gamma, B}{\Gamma, A \& B} \& \quad \frac{\Gamma, A}{\Gamma, A \oplus B} \oplus_1 \quad \frac{\Gamma, B}{\Gamma, A \oplus B} \oplus_2$

**Proof Structures** – The problem is to cope with the  $\&$ -rule

$$\frac{\Gamma, A \quad \Gamma, B}{\Gamma, A \& B} \&$$

for which a *superimposition* of two proof nets must be made. A *solution* is to introduce for each  $\&$ -link a *boolean variable* (called *eigen-wight*)

$$\frac{\Gamma, A \quad \Gamma, B}{\Gamma, A \& B} \&_p$$

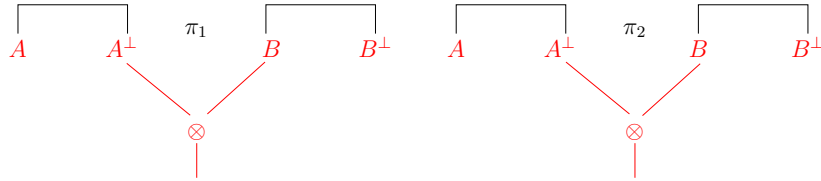
which distinguishes between two *slices* of the superimposition:

$$\frac{\overset{\bar{p} \text{ slice}}{\Gamma, A}}{\Gamma, A \& B} \&_p \quad \frac{\overset{p \text{ slice}}{\Gamma, B}}{\Gamma, A \& B} \&_p$$

But this immediately opens to the problem of which kind of superposition can be performed over already de-sequentialized PSs? Let's illustrate this by an example. Assume a sequential proof as follows:

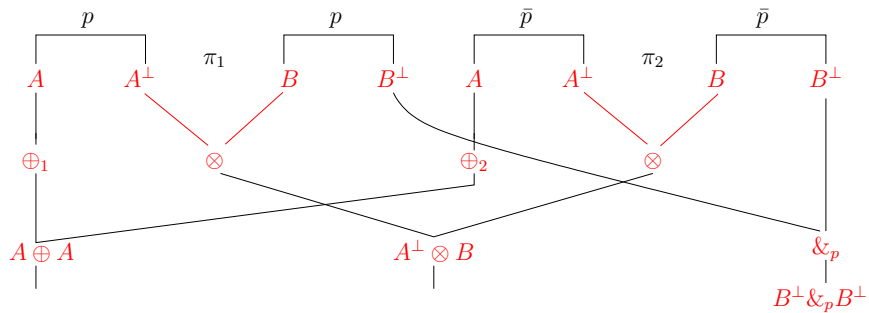
$$\begin{array}{c} \frac{\frac{\frac{A, A^\perp}{A, A^\perp \otimes B, B^\perp} \otimes \quad \frac{B, B^\perp}{A, A^\perp \otimes B, B^\perp} \otimes}{A \oplus A, A^\perp \otimes B, B^\perp} \oplus_1 \quad \frac{\frac{A, A^\perp}{A, A^\perp \otimes B, B^\perp} \otimes \quad \frac{B, B^\perp}{A, A^\perp \otimes B, B^\perp} \otimes}{A \oplus A, A^\perp \otimes B, B^\perp} \oplus_2 \\ \Pi : \frac{\quad}{A \oplus A, A^\perp \otimes B, B^\perp \&_p B^\perp} \&_p \end{array}$$

By hypothesis of induction we may assume  $\Pi_1$  de-sequentializes in to (the MLL PN)  $\pi_1$  and  $\Pi_2$  de-sequentializes in to (the MLL PN)  $\pi_2$  as follows:

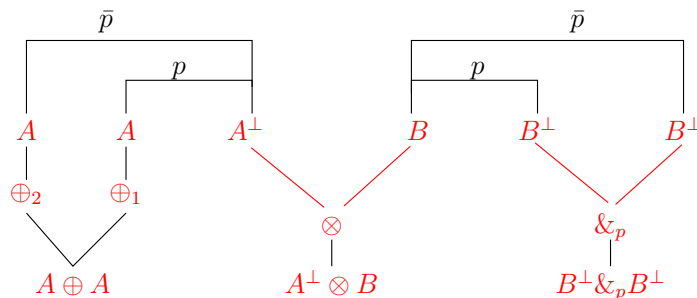


Then there are different possibilities of superposing  $\pi_1$  and  $\pi_2$  in order to get a proof structure  $\pi$  that is a de-sequentialization of  $\Pi$ .

1. (Girard's solution) minimal superposition: only conclusions superpose



2. intermediate superposition with unary  $\oplus$ -links: only some links superpose

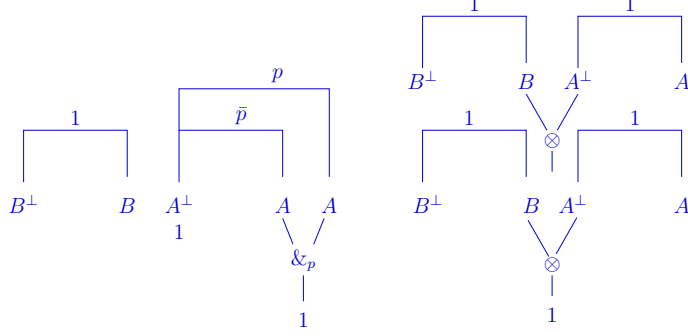




the topmost  $\&$  (with eigen-weight  $p$ ) and  $\otimes$ -rules

$$\frac{\frac{B^\perp, B}{\frac{\frac{A^\perp, A}{A^\perp, A\&A} \quad \frac{A^\perp, A}{A^\perp, A\&A}}{\frac{B^\perp, B \otimes A^\perp, A\&A}{B^\perp \& B^\perp, B \otimes A^\perp, A\&A}} \quad \otimes \quad \frac{\frac{B^\perp, B}{B^\perp, B \otimes A^\perp, A} \quad \frac{A^\perp, A}{A^\perp, A\&A}}{\frac{B^\perp, B \otimes A^\perp, A}{B^\perp, B \otimes A^\perp, A\&A}} \quad \otimes}{\frac{B^\perp, B \otimes A^\perp, A\&A}{B^\perp \& B^\perp, B \otimes A^\perp, A\&A}} \quad \otimes$$

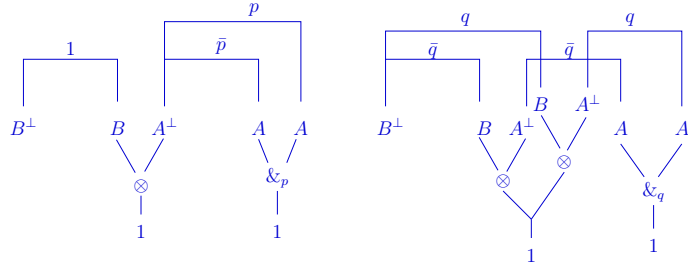
map to the following PNs



then, the middle  $\otimes$  and  $\&$ -rules (with eigen-weight  $q$ )

$$\frac{\frac{B^\perp, B}{\frac{\frac{A^\perp, A}{A^\perp, A\&A} \quad \frac{A^\perp, A}{A^\perp, A\&A}}{\frac{B^\perp, B \otimes A^\perp, A\&A}{B^\perp \& B^\perp, B \otimes A^\perp, A\&A}} \quad \otimes \quad \frac{\frac{B^\perp, B}{B^\perp, B \otimes A^\perp, A} \quad \frac{A^\perp, A}{A^\perp, A\&A}}{\frac{B^\perp, B \otimes A^\perp, A}{B^\perp, B \otimes A^\perp, A\&A}} \quad \otimes}{\frac{B^\perp, B \otimes A^\perp, A\&A}{B^\perp \& B^\perp, B \otimes A^\perp, A\&A}} \quad \&$$

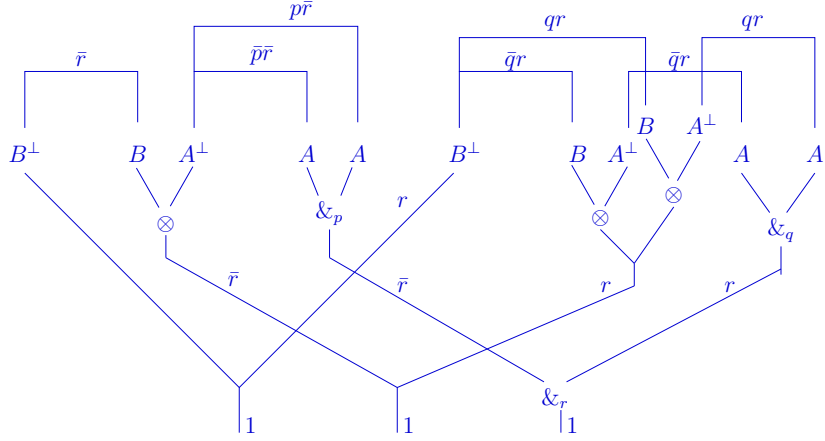
map to the following PN with only monomial weights



and, finally, the lowest  $\&$ -rule (with eigen-weight  $r$ )

$$\frac{\frac{B^\perp, B}{\frac{\frac{A^\perp, A}{A^\perp, A\&A} \quad \frac{A^\perp, A}{A^\perp, A\&A}}{\frac{B^\perp, B \otimes A^\perp, A\&A}{B^\perp \& B^\perp, B \otimes A^\perp, A\&A}} \quad \otimes \quad \frac{\frac{B^\perp, B}{B^\perp, B \otimes A^\perp, A} \quad \frac{A^\perp, A}{A^\perp, A\&A}}{\frac{B^\perp, B \otimes A^\perp, A}{B^\perp, B \otimes A^\perp, A\&A}} \quad \otimes}{\frac{B^\perp, B \otimes A^\perp, A\&A}{B^\perp \& B^\perp, B \otimes A^\perp, A\&A}} \quad \&$$

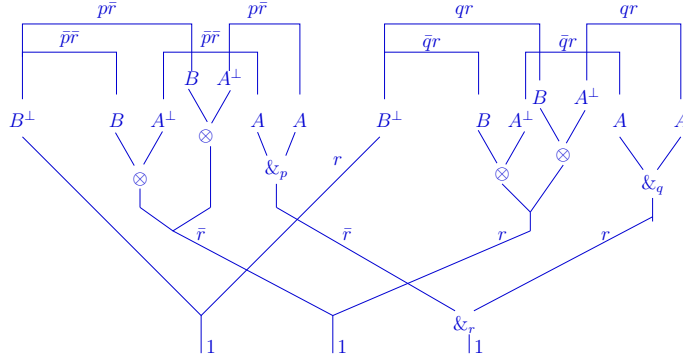
maps to the following PN with only monomial weights



Observe, this de-sequentialization is *not invariant* under the raising of the  $\wp, \otimes, \oplus, \&$  over the  $\&$ -rule. Actually, if we raise, in the previous SP, the  $\otimes$  over the  $\&_p$ -rule, as follows

$$\frac{\frac{\frac{B^\perp, B}{B^\perp, B \otimes A^\perp, A} \quad \frac{A^\perp, A}{A^\perp, A} \otimes \quad \frac{B^\perp, B}{B^\perp, B \otimes A^\perp, A} \quad \frac{A^\perp, A}{A^\perp, A}}{B^\perp, B \otimes A^\perp, A \& A} \quad \& \quad \frac{\frac{B^\perp, B}{B^\perp, B \otimes A^\perp, A} \quad \frac{A^\perp, A}{A^\perp, A}}{B^\perp, B \otimes A^\perp, A} \quad \frac{\frac{B^\perp, B}{B^\perp, B \otimes A^\perp, A} \quad \frac{A^\perp, A}{A^\perp, A}}{B^\perp, B \otimes A^\perp, A}}{B^\perp \& B^\perp, B \otimes A^\perp, A \& A}$$

we get an equivalent PS which de-sequentialise into the following different PN



When we add a  $\&$ -link, we don't know if a link  $L_1$  of  $\pi_1$  is the same as another link  $L'$  of  $\pi_2$ : in general,  $p.w_1(L) + \bar{p}.w_2(L_2)$  is not a monomial, except when  $L_1, L_2$  are conclusions.

### 1.3 Polynomial interpretation

There is a canonical surjection from MALL SP to Polynomial PN as illustrated in the following example. Assume the following SP:

$$\frac{\frac{\frac{A^\perp, A}{A^\perp, A \& A} \quad \frac{A^\perp, A}{A^\perp, A}}{B^\perp, B \otimes A^\perp, A \& A} \quad \frac{\frac{B^\perp, B}{B^\perp, B \otimes A^\perp, A} \quad \frac{A^\perp, A}{A^\perp, A}}{B^\perp, B \otimes A^\perp, A} \quad \frac{\frac{B^\perp, B}{B^\perp, B \otimes A^\perp, A} \quad \frac{A^\perp, A}{A^\perp, A}}{B^\perp, B \otimes A^\perp, A}}{B^\perp \& B^\perp, B \otimes A^\perp, A \& A}$$

assign an eigen weight to each  $\&$  in the sequent conclusion

$$\frac{\frac{\frac{A^\perp, A}{A^\perp, A \&_q A} \quad \frac{A^\perp, A}{A^\perp, A}}{B^\perp, B \otimes A^\perp, A \&_q A} \quad \frac{\frac{B^\perp, B}{B^\perp, B \otimes A^\perp, A} \quad \frac{A^\perp, A}{A^\perp, A}}{B^\perp, B \otimes A^\perp, A} \quad \frac{\frac{B^\perp, B}{B^\perp, B \otimes A^\perp, A} \quad \frac{A^\perp, A}{A^\perp, A}}{B^\perp, B \otimes A^\perp, A}}{B^\perp \&_p B^\perp, B \otimes A^\perp, A \&_q A}$$

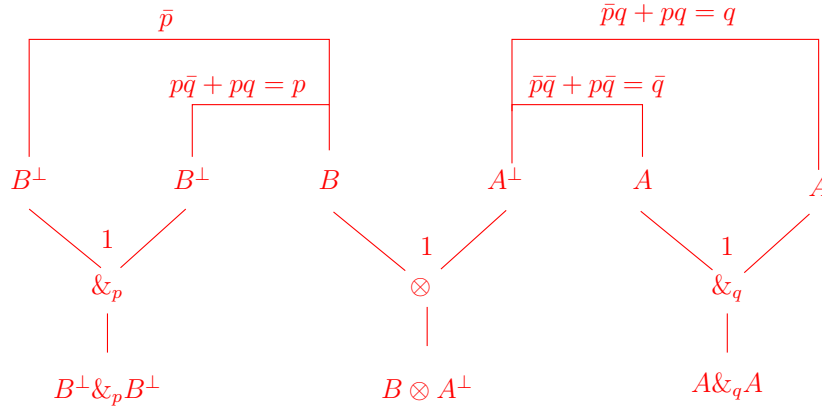
and propagate the eigen-weights upwards as follows

$$\frac{\frac{\frac{\bar{q}}{A^\perp, A} \quad \frac{q}{A^\perp, A}}{B^\perp, B} \quad \&_q \quad \frac{\frac{\bar{q}}{B^\perp, B} \quad \frac{\bar{q}}{A^\perp, A}}{B^\perp, B \otimes A^\perp, A} \quad \&_q \quad \frac{\frac{q}{B^\perp, B} \quad \frac{q}{A^\perp, A}}{B^\perp, B \otimes A^\perp, A}}{\frac{B^\perp, B \otimes A^\perp, A \&_q A}{B^\perp \&_p B^\perp, B \otimes A^\perp, A \&_q A}}$$

finally, separate, inductively (top-down), each slice by monomial weights as follows

$$\frac{\frac{\frac{\bar{p}}{B^\perp, B} \quad \frac{\bar{p}\bar{q}}{A^\perp, A} \quad \frac{\bar{p}q}{A^\perp, A}}{B^\perp, B \otimes A^\perp, A \&_q A} \quad \& \quad \frac{\frac{p\bar{q}}{B^\perp, B} \quad \frac{p\bar{q}}{A^\perp, A}}{B^\perp, B \otimes A^\perp, A} \quad \otimes \quad \frac{\frac{pq}{B^\perp, B} \quad \frac{pq}{A^\perp, A}}{B^\perp, B \otimes A^\perp, A} \quad \otimes}{\frac{B^\perp, B \otimes A^\perp, A \&_q A}{B^\perp \&_p B^\perp, B \otimes A^\perp, A \&_q A}} \&_p$$

The resulting corresponding de-sequentialized PN is a sequent forest with weighted axioms. We may replace parallel axioms  $ax_1, ax_2, \dots, ax_n$  with, resp., weights  $w_1, w_2, \dots, w_n$ , by a single  $ax$  link with weight  $w = \sum_i^n w_i$  as follows



Observe, this "polynomial interpretation" is now invariant under the raising of the  $\wp, \otimes, \oplus, \&$ -rule over  $\&$ -rule; indeed the following (equivalent) SP maps to the same (previous) Polynomial PN.

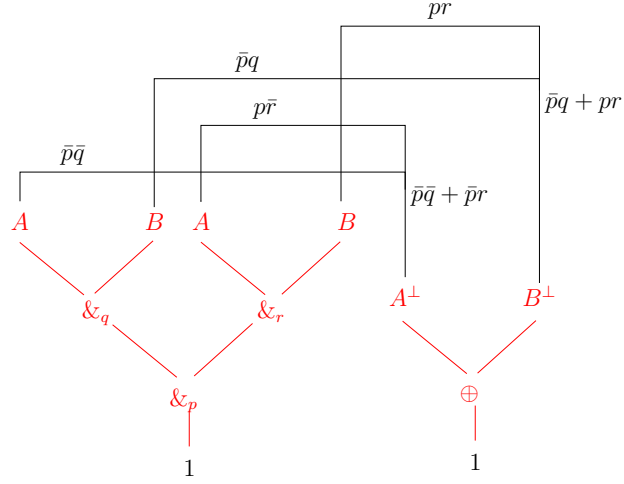
$$\frac{\frac{\frac{\bar{p}\bar{q}}{B^\perp, B} \quad \frac{\bar{p}\bar{q}}{A^\perp, A}}{B^\perp, B \otimes A^\perp, A} \quad \otimes \quad \frac{\frac{\bar{p}q}{B^\perp, B} \quad \frac{\bar{p}q}{A^\perp, A}}{B^\perp, B \otimes A^\perp, A} \quad \otimes \quad \frac{\frac{p\bar{q}}{B^\perp, B} \quad \frac{p\bar{q}}{A^\perp, A}}{B^\perp, B \otimes A^\perp, A} \quad \otimes \quad \frac{\frac{pq}{B^\perp, B} \quad \frac{pq}{A^\perp, A}}{B^\perp, B \otimes A^\perp, A}}{\frac{B^\perp, B \otimes A^\perp, A \&_q A}{B^\perp \&_p B^\perp, B \otimes A^\perp, A \&_q A}}$$

Observe that in general this surjective de-sequentialization may lead to PS labeled by polynomial weights; e.g., the next SP

$$\frac{\frac{\frac{\bar{p}\bar{q}}{A, A^\perp} \text{ ax}}{A, A^\perp \oplus B^\perp} \oplus_1 \quad \frac{\frac{\bar{p}q}{B, B^\perp} \text{ ax}}{B, A^\perp \oplus B^\perp} \oplus_2 \quad \&_q \quad \frac{\frac{p\bar{r}}{A, A^\perp} \text{ ax}}{A, A^\perp \oplus B^\perp} \oplus_1 \quad \frac{\frac{pr}{B, B^\perp} \text{ ax}}{B, A^\perp \oplus B^\perp} \oplus_2 \quad \&_r}{\frac{A \&_q B, A^\perp \oplus B^\perp}{(A \&_q B) \&_p (A \&_r B), A^\perp \oplus B^\perp} \&_p}$$

de-sequentializes into a PS with some links weighted by "non-monomial" weighted. By the way observe the conclusion or terminal links must be labeled by the monomial weight 1 as in the following PN.





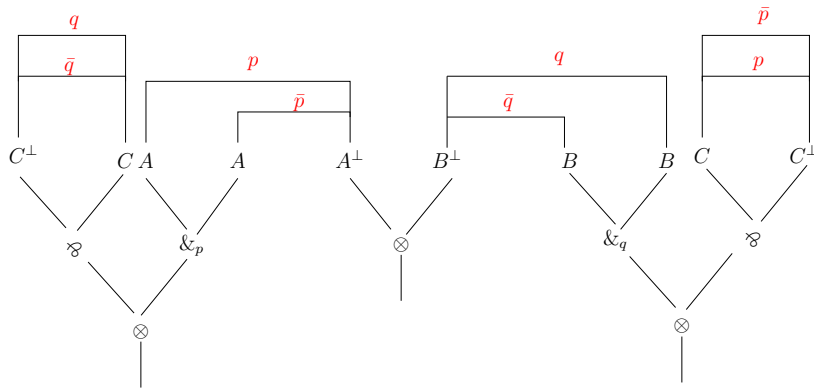
Finally, concerning the efficiency of the weight interpretations, observe that:

- both *monomial* and *polynomial* mapping are quite efficient: P-time in the size of the given sequent proof (there are  $2 \cdot n$  slices, where  $n$  is the number of  $\&$ -rules of the SP  $\pi$ );
- more efficient than *linkings mapping* (Hughes-Van Glabbeek, 2003) that is Exponential in the size of the sequent proof (there are  $2^n$ , where  $n$  is the number of the  $\&$  connective occurring in the sequent  $\Gamma$ ).

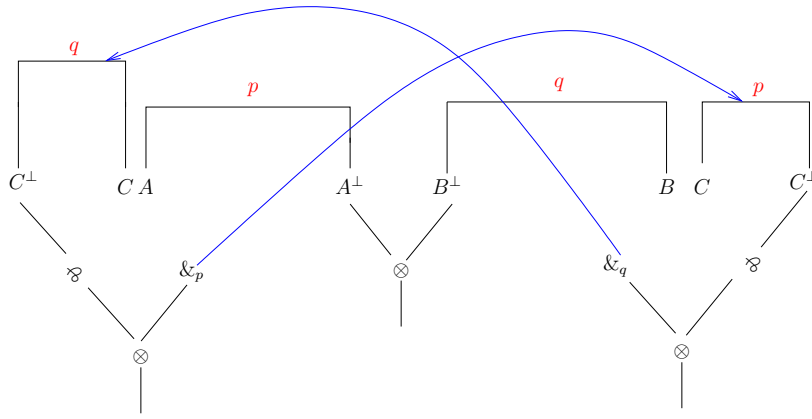
#### 1.4 Correctness Criterion for Monomial PNs

- **proof structure:** the crucial point is the *dependence condition* (“ $\&$ -boxing”): if a link  $L$  depends on a variable  $p$  then  $w(L) \leq w(\&_p)$ .
- **proof net:** every boolean valuation,  $\varphi : \text{eigenweights} \rightarrow \{0, 1\}$ , induces a (unique) slice  $S$  s.t. every *switching* on  $S$ , obtained by
  - mutilating one premise in each  $\wp$ ,
  - mutilating the unique  $\&$ -premise in  $S$ ,
  - adding a jump (an edge) from a  $\&_p$ -node to a node depending on  $p$ ,
 is an acyclic and connected (ACC) graph.

In the following we give an example of non correct PS.

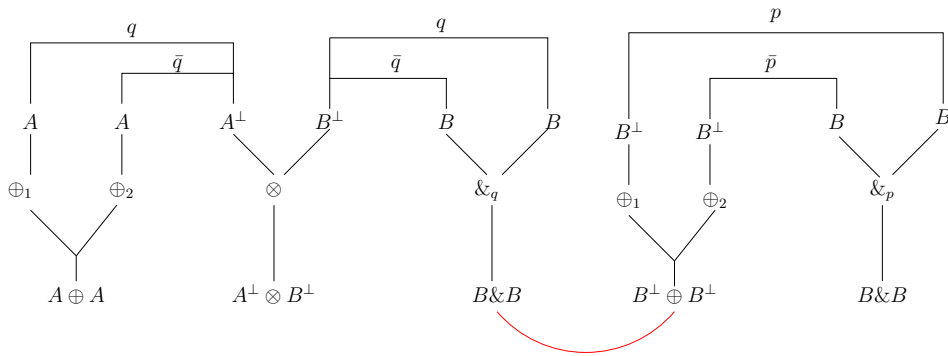


Actually, fixed a valuation  $\varphi(p) = \varphi(q) = 1$ , we get a non-ACC switching as follows.

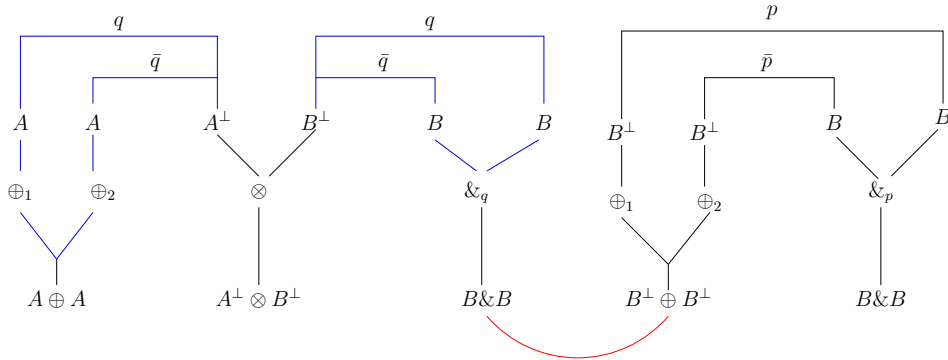


### 1.5 Cut-elimination for Monomial PSs

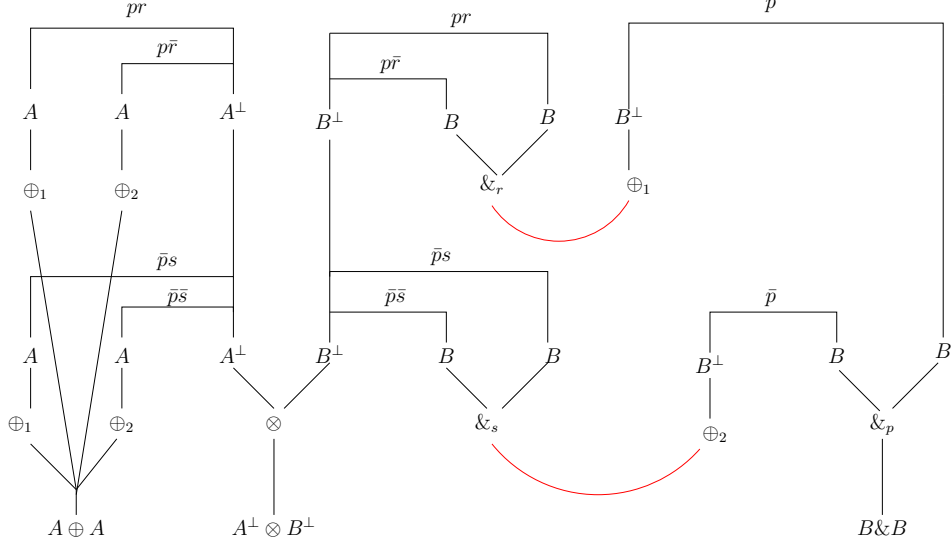
Assume a monomial PN  $\pi$  with cuts as follows.



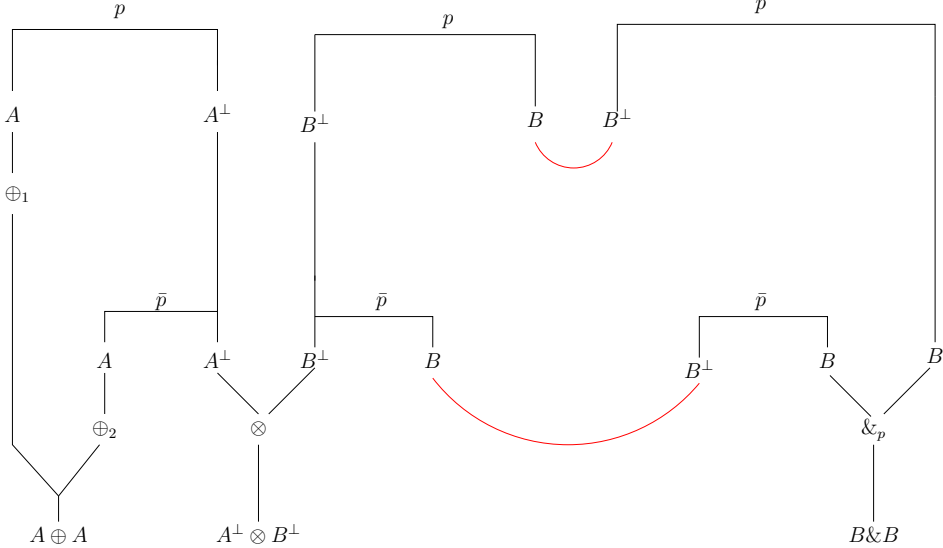
We call *dependency graph* of  $q$  w.r.t.  $\pi$  (Maieli, 2007) the (possibly disconnected) subgraph of  $\pi$  made by only those links and edges whose weight depends on  $q$ ; e.g., the dependency graph of  $q$  w.r.t.  $\pi$  is the blue sub-graph as follows:



Then  $\pi$  reduces to the following  $\pi'$  via the *duplication* of the *dependency graph* of  $q$ : we replace in  $\pi'$  the eigen-weight  $q$  of  $\pi$  by two news (fresh) eigen-weights  $r$  and  $s$  as follows



Consequently,  $\oplus_i/\&$  cuts reduce to the following PN by erasing slices  $\bar{r}$  and  $s$  (i.e., by evaluating  $\bar{r} = 0$  and  $s = 0$ ).



Cut elimination procedure is shown to be terminating and confluent but with an unknown Complexity (P-time?).

## 2 Proof-structures with explicit binary contraction links

In this section we recall the basic notions of Girard's proof-net; we adopt the syntax with explicit contractions like in [Lau99].

**Definition 1 (proof structure)** A Girard proof structure  $\pi$  of MALL, shortly proof structure (PS), is an oriented graph s.t. each edge is labelled by a MALL formula and built on the set of nodes (or vertexes) following the typing constraints of Figure 1. Pending edges are called conclusions; fixed a node, an entering edge is called premise while its unique emergent edge is called conclusion. We call link the graph made by a node together with its premise(s) and (possibly) its conclusion(s).

If  $\pi$  involves the  $\&$ -links  $L_1, \dots, L_k$  then:

1. we associate a Boolean variable  $(p, q, \dots)$ , called eigen weight, to each  $\&$ -node of  $\pi$  (eigen weights are supposed to be different; we use  $\epsilon_p$  to denote a variable  $p$  or its negation  $\bar{p}$ , and  $\bar{\epsilon}_p$  for its orthogonal);
2. we associate a weight  $w$ , i.e., a product (conjunction) of eigen weights or negations of eigen weights of  $(p, \bar{p}, q, \bar{q}, \dots)$ , to each node with the constraint that two nodes have the same weight if they have a common edge, except when the edge is the premise of a  $\&$  or  $C$ -node, in these cases we do as follows (see Figure 2):

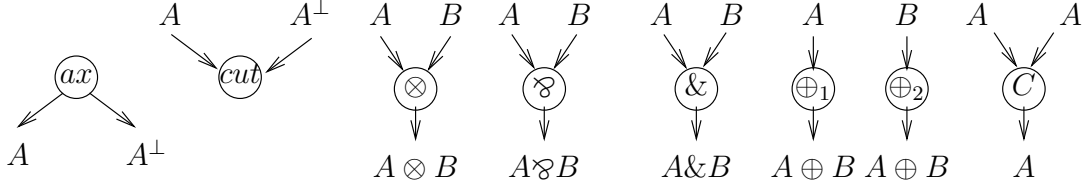


Figure 1: MALL links

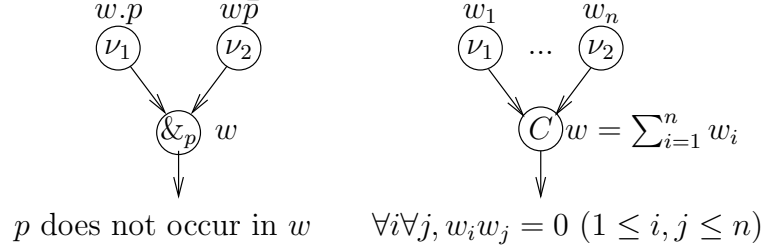


Figure 2: Weights for  $\&$  and  $C$  links

- (a) if  $w$  is the weight of a  $\&$ -link and  $p$  is its eigen weight then  $w$  does not depends on  $p$  and its premise links must have weights  $w.p$  and  $w.\bar{p}$  (we say that a weight  $w$  depends on  $p$  when  $p$  or  $\bar{p}$  occurs in  $w$ );
- (b) if  $w$  is the weight of a  $C$ -link and  $w_1, w_2$  are the weights of its premise links then we must have  $w = w_1 + w_2$  and  $w_1 w_2 = 0$ ;

3. a conclusion node has weight 1;

4. if  $w$  is the weight of a  $\&$ -link with eigen weight  $p$  and  $w'$  is a weight depending on  $p$  and appearing in the proof-structure then  $w' \leq w$ .

A node  $L$  with weight  $w$  depends on the eigen weight  $\&_p$  if  $w$  depends on  $p$  or  $L$  is a  $C$ -node and one of the weights just above it depends on  $p$ .

**Remark 1 :**

1. (splitting variable) – Observe that, since the weights associated to a PS are products (monomials) of the Boolean algebra generated by the eigen weights associated to a proof structure, then, for each weight  $w$  associated to a contraction node, there exists a unique eigen weight  $p$  that splits  $w$  into  $w_1 = wp$  and  $w_2 = w\bar{p}$ . We some times index a  $C$ -link with its splitting variable  $p$ , like in Figure 3.
2. (dependency condition) – Observe that the graph  $\pi$  of Figure 4 is not a proof structure since it violates condition 4 of Definition 1; actually, if  $w = q$  is the weight of the  $\&_p$ -link and  $w' = \bar{p}$  is a weight depending on  $p$  and appearing in the proof-structure then  $\bar{p} \not\leq q$ .

**Definition 2 (slice and switchings)** A valuation  $\varphi$  for a PS  $\pi$  is a function from the set of all weights of  $\pi$  into  $\{0, 1\}$ . Fixed a valuation  $\varphi$  for  $\pi$  then:

- the slice  $\varphi(\pi)$  is the graph obtained from  $\pi$  by keeping only those nodes with weight 1 together its emerging edges;

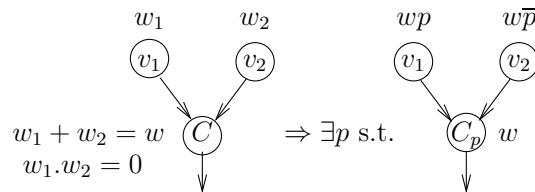


Figure 3: Splitting variable of a contraction link



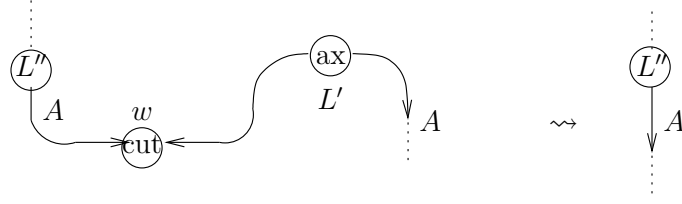


Figure 6: Axiom-cut reduction step

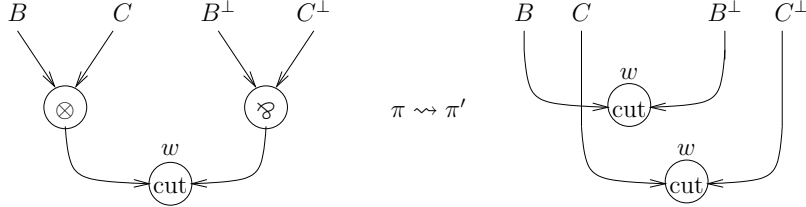


Figure 7:  $(\otimes/\wp)$ -cut reduction step

### 3 Cut-elimination

#### 3.1 Ready cut-elimination

We follow [Gir96].

**Definition 4 (ready cut reduction)** Let  $L$  be a cut in a proof net  $\pi$  whose premises  $A$  and  $A^\perp$  are the respective conclusions of links  $L', L''$  both different from the contraction  $C$ . Then we define the result  $\pi'$  (reductum) of reducing this ready cut in  $\pi$  (redex), as follows:

**Ax-cut:** if  $L'$  (resp.,  $L''$ ) is an axiom link then  $\pi'$  is obtained by removing in  $\pi$  both formulas  $A, A^\perp$  (as well as  $L$ ) and giving a new conclusion to  $L''$  (resp.,  $L'$ ), the other conclusion of  $L'$  (resp.,  $L''$ ) (see Figure 6).

**$(\otimes/\wp)$ -cut:** if  $L'$  is a  $\otimes$ -link with premises  $B$  and  $C$  and  $L''$  is a  $\wp$ -link with premises  $B^\perp$  and  $C^\perp$ , then  $\pi'$  is obtained by removing in  $\pi$  the formulas  $A$  and  $A^\perp$  as well as our cut link  $L$  with  $L'$  and  $L''$  and adding two new cut links with respective premise  $B, B^\perp$  and  $C, C^\perp$  (see Figure 7).

**$(\&/\oplus_1)$ -cut:** if  $L'$  is a  $\&_p$ -link with premises  $B$  and  $C$  and  $L''$  is a  $\oplus_1$ -link with premise  $B^\perp$ , then  $\pi'$  is obtained in three steps: first we remove in  $\pi$  both formulas  $A, A^\perp$  as well as our cut link  $L$  with  $L'$  and  $L''$ , then we replace the eigen weight  $p$  by 1 and keep only those links (vertexes and edges) that still have nonzero weight; finally we add a cut between  $B$  and  $B^\perp$  (see Figure 8).

**$(\&/\oplus_2)$ -cut:** if  $L'$  is a  $\&_p$ -link with premises  $B$  and  $C$  and  $L''$  is a  $\oplus_2$ -link, with premise  $C^\perp$ , then  $\pi'$  is obtained in three steps: first we remove in  $\pi$  both formulas  $A, A^\perp$  as well as our cut link  $L$  with  $L'$  and  $L''$ , then we replace the eigen weight  $p_\&$  by 0 and keep only those links (vertexes and edges) that still have nonzero weight; finally we add a cut between  $C$  and  $C^\perp$ .

**Theorem 2 (stability under ready cut reduction)** If  $\pi$  is a proof net s.t.  $\pi$  reduces to  $\pi'$  in one step of ready cut reduction, then  $\pi'$  is still a proof net.

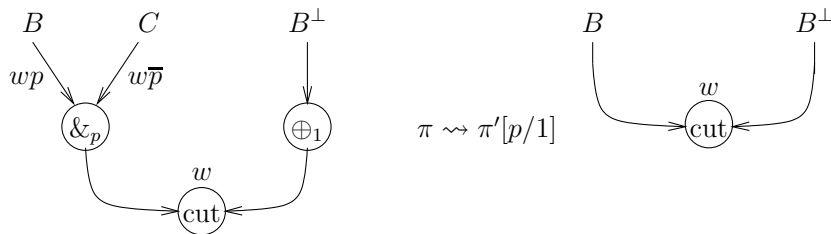


Figure 8:  $(\&/\oplus_1)$ -cut reduction step

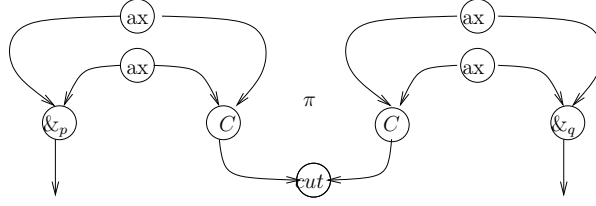


Figure 9: Reducing a commutative cut

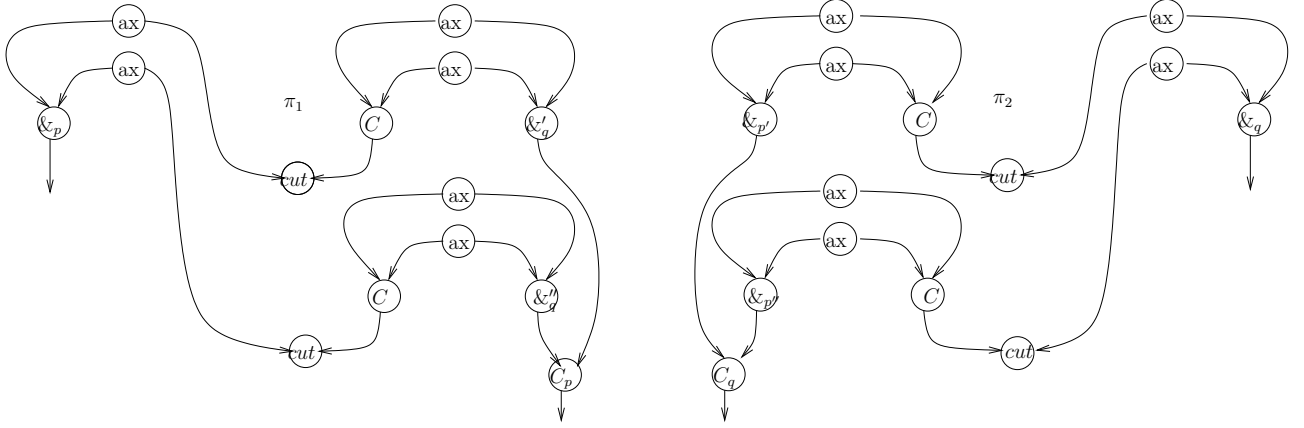


Figure 10: Non confluent, commutative, cut reduction

PROOF — See [Gir96]. □

## 3.2 Commutative cut-elimination

### 3.2.1 The confluence problem

– In general, reducing a cut involving a contraction link as (at least) one of its premises from a proof structure  $\pi$  may lead to several solutions, depending on which sub-graph of  $\pi$  we decide to duplicate. For instance, reducing the commutative cut of  $\pi$  of Figure 9 may lead to solutions  $\pi_1$  as well solution  $\pi_2$ , depicted in Figure 10, depending on which additive box ( $\&_q$  or  $\&_p$ ) we decide to duplicate. Of course these two resulting proof nets are different and there is no a-priori way to make them equal. Girard, in [Gir96], does not give any detail for the solution to this problem; indeed, a Church-Rosser procedure is only given for the ready cut-elimination.

**Definition 5 (restriction, empire, spreading)** Assume a proof structure  $\pi$ , an eigen weight  $p$  and a weight  $w$ , then:

- the restriction of  $\pi$  w.r.t.  $p$  (resp.,  $\bar{p}$ ), denoted  $\pi \upharpoonright^p$  (resp.,  $\pi \upharpoonright^{\bar{p}}$ ), is what remains of  $\pi$  when we replace  $p$  with 1 (resp.,  $\bar{p}$  with 1) and keep only those vertexes and edges whose weight is still non zero;
- the empire (or the dependency graph) of the eigen weight  $p$  w.r.t.  $\pi$ , denoted  $\mathcal{E}_p$ , is the (possibly disconnected) subgraph of  $\pi$  made by all links depending on  $p$ .
- the spreading of  $w$  over  $\pi$ , denoted by  $w[\pi]$ , is the product of  $w$  for  $\pi$ , i.e.,  $\pi$  where we replaced each weight  $v$  with the product of weights  $vw$ .

Observe that, in general, the spreading does not preserve the property of being a proof structure; moreover, it can be defined also over an empire.

**Lemma 1 (empire)** If a  $\&_p$ -node belongs to the empire of  $\mathcal{E}_q$ , then  $\mathcal{E}(p) \subset \mathcal{E}(q)$ .

PROOF — If the  $\&_p$ -node belongs to the empire of  $\mathcal{E}_q$  then the weight  $w$  of the  $\&_p$ -node depends on  $q$  (i.e.,  $w = w'q$  or  $w = w'\bar{q}$ ) then trivially, by the dependency condition 4 of Definition 1, each node  $v$  whose weight depends on  $p$  will also depends on  $q$ . □

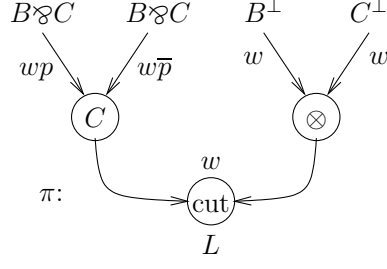


Figure 11:  $(C/\otimes)$ -redex

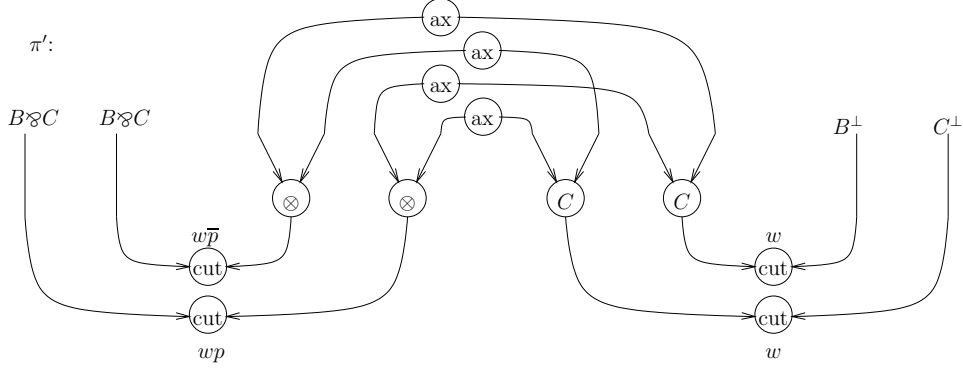


Figure 12:  $(C/\otimes)$ -reductum

**Definition 6 (commutative cut reduction)** Let  $L$  be cut link in a proof net  $\pi$  whose premises  $A$  and  $A^\perp$  are the respective conclusions of links  $L'$  and  $L''$  s. t. at least one of them is a contraction link  $C$ . Then we define the result  $\pi'$  (reductum) of reducing this commutative cut in  $\pi$  (redex) , as follows:

$(C/\otimes)$ -cut : if  $L'$  is a  $C$ -link and  $L''$  is a  $\otimes$ -link, like in Figure 11, then  $\pi$  reduces in one  $(C/\otimes)$  step to  $\pi'$ , like in Figure 12.

$(C/C)$ -cut : if both  $L'$  and  $L''$  are contraction links, then we consider two sub-cases:

- the weight  $w$  of, resp.,  $L'$  and  $L''$  is split by the same  $p$  variable, then  $\pi$  reduces in one  $(C_p/C_p)$  step to  $\pi'$  like in Figure 13;
- the weight  $w$  of  $L'$ , resp.,  $L''$ , is split by the two different variables,  $p$  and  $q$ , like in Figure 14, then  $\pi$  reduces in one  $(C_p/C_q)$  step to  $\pi'$ , like in Figure 15.

$(C/\oplus_i)$ -cut : if  $L'$  is a  $C$ -link and  $L''$  a  $\oplus_{i=1,2}$ -link, then  $\pi$  reduces in one  $(C/\oplus)$  step to  $\pi'$ , like in Figure 16.

$(C/\&)$ -cut : if  $L'$  is a  $C$ -link and  $L''$  a  $\&_p$ -link, like in Figure 17, then  $\pi$  reduces in one  $(C/\&)$  step to  $\pi'$ , like in Figure 18, with the assumptions that  $\bar{q}.\mathcal{E}'_p$  and  $q.\mathcal{E}''_p$  are obtained by spreading  $\bar{q}$ , resp.,  $q$ , over two copies of the empire of  $p$ ,  $\mathcal{E}'_p$  and  $\mathcal{E}''_p$ , where we replaced any eigen weight with a new (fresh) one.

**Example 1** Assume we want reduce the commutative  $(C/\&)$ -cut of the proof net in Figure 19 First, we calculate  $\mathcal{E}'$  and  $\mathcal{E}''$ , like in Figure 20. Then we perform the cut reduction step  $C/\&$ , like in Figure 21.

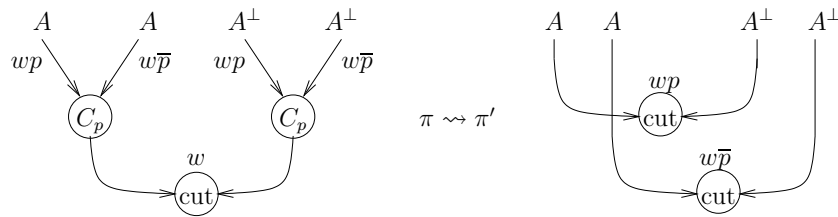


Figure 13:  $(C_p/C_p)$ -cut reduction step



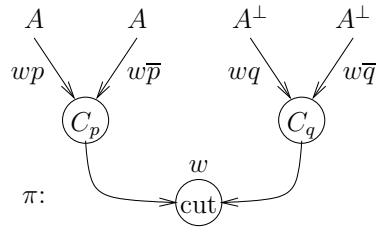


Figure 14:  $(C_p/C_q)$ -redex

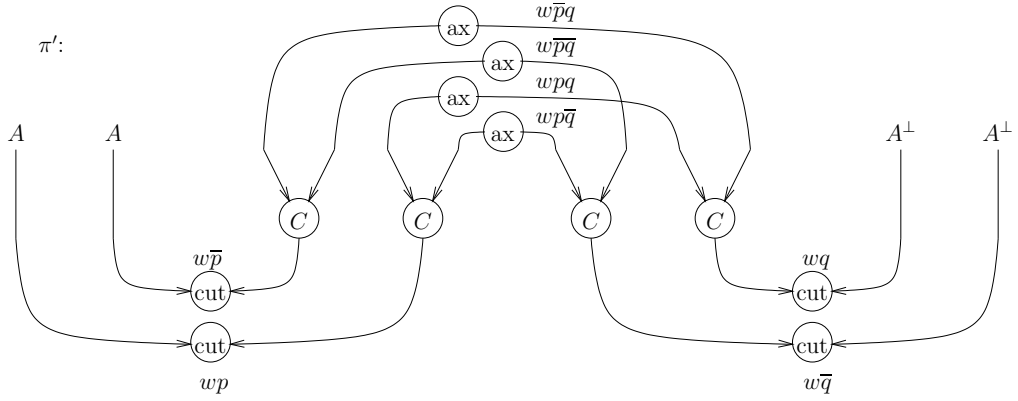


Figure 15:  $(C_p/C_q)$ -reductum

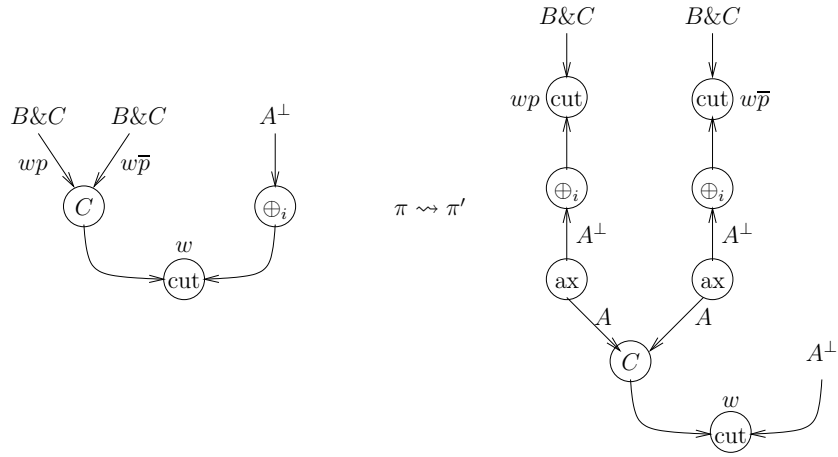


Figure 16:  $(C/\oplus_i)$ -cut reduction step

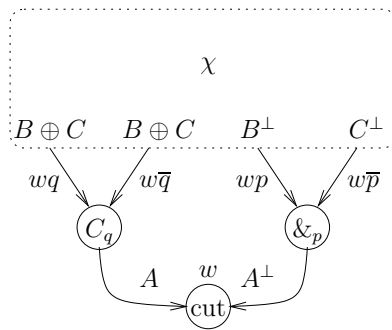


Figure 17:  $(C/\&)$ -redex

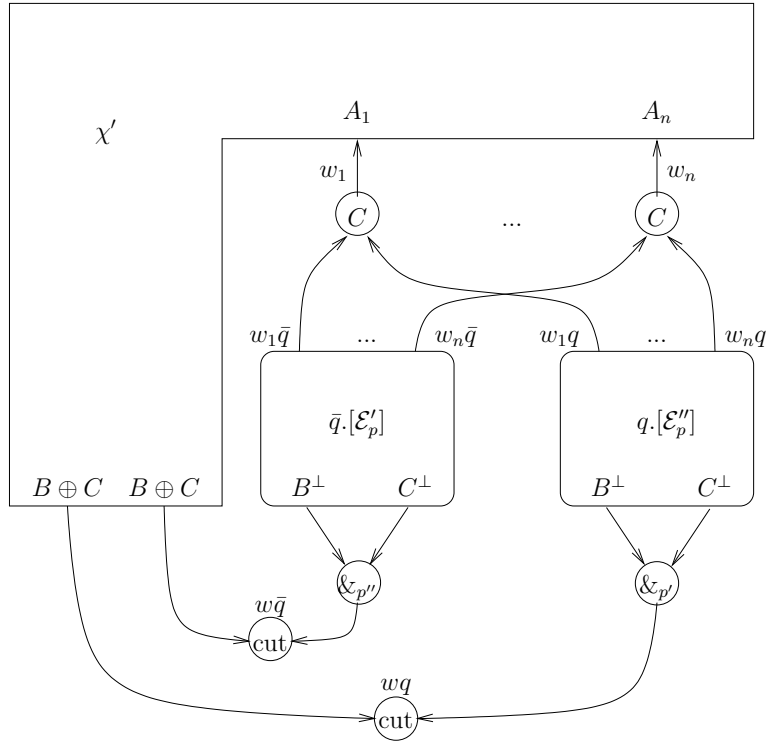


Figure 18:  $(C/\&)$ -reductum

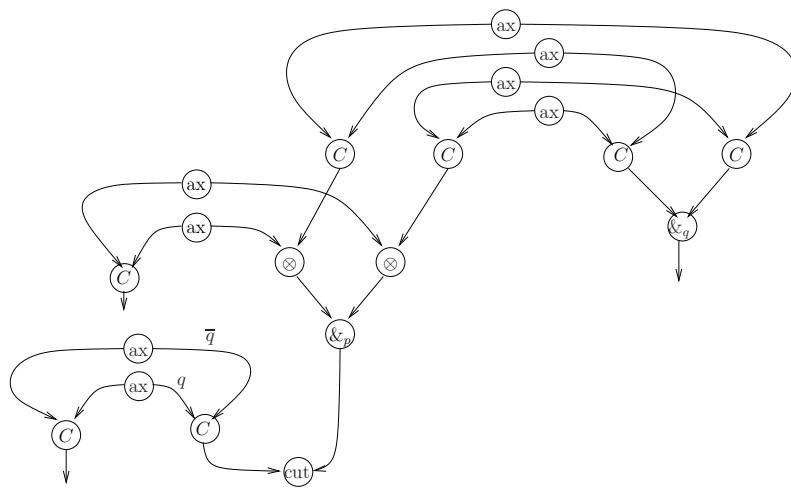


Figure 19: Redex  $\pi$

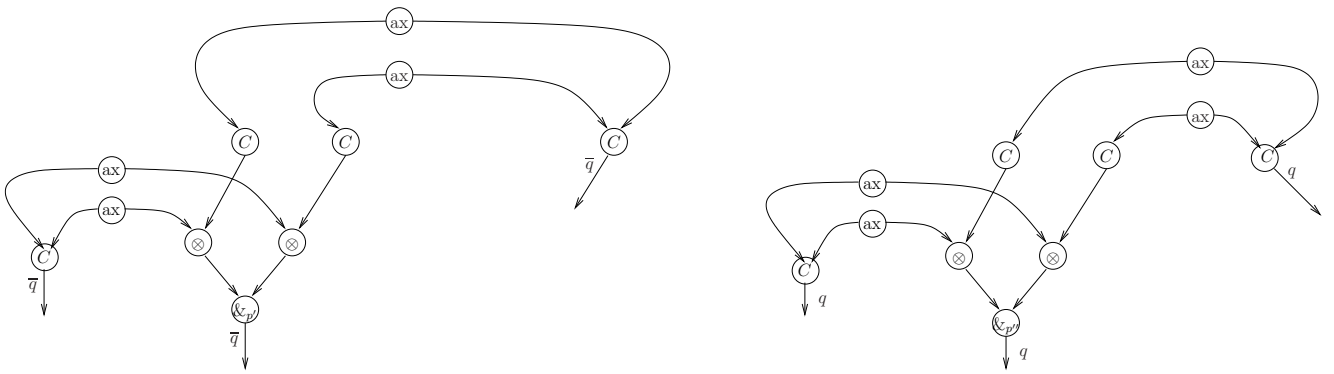


Figure 20: Graphs  $\mathcal{E}'$  and  $\mathcal{E}''$

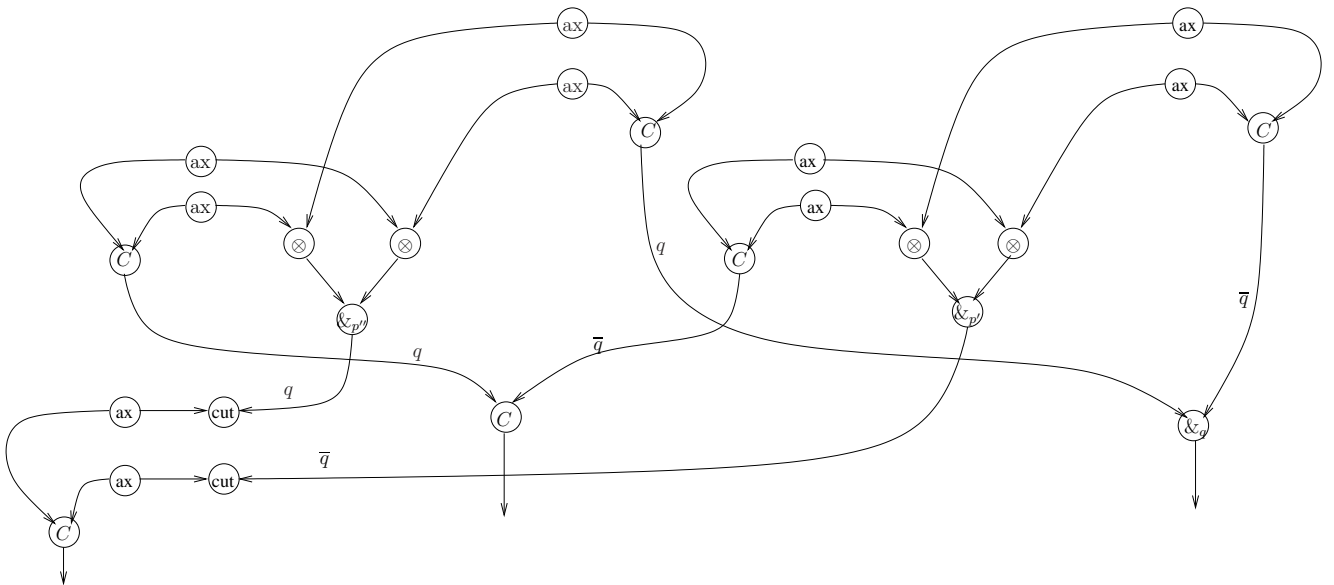


Figure 21: Reductum  $\pi'$

### 3.3 Stability

**Theorem 3 (stability under commutative cut reduction: I part)** *If  $\pi$  is a proof net s.t. it reduces to  $\pi'$  in one step of commutative cut reduction that is different from the  $C/\&$ -case, then  $\pi'$  is still a proof net.*

PROOF — (sketch) All cases are more or less immediate consequences of the next graph theoretical property (see also [Gir06], pages 250-251):

**Property 1 (Euler-Poincaré invariance)** *Given a graph  $\mathcal{G}$ , then*

$$\#CC - \#Cy = \#V - \#E$$

where  $\#CC$ ,  $\#Cy$ ,  $\#V$  and  $\#E$  denotes, respectively, the number of connected components, cycles, vertexes and edges of  $\mathcal{G}$ . □

**Lemma 2 (conservative/extensive switching)** *Assume  $\pi$  is a proof net that reduces to  $\pi'$  in one step of commutative cut reduction  $C_q/\&_p$ . We say that a switching for  $\pi'$  is extensive (resp., conservative) w.r.t.  $q$  if it makes (resp., does not make) use of at least a jump going from the  $\&_q$ -node to a node that was not previously depending on  $q$  in  $\pi$ . Then, for any conservative switching for  $\pi'$  that is not ACC we can find a corresponding switching for  $\pi$  that is not ACC too.*

PROOF — Immediate. □

**Lemma 3 (separation)** *Assume  $\pi$  is a proof net containing two nodes,  $\&_{p'}$  and  $\&_{p''}$ , with weights, resp.,  $w'q$  and  $w''\bar{q}$ , for some eigen weight  $q$  of  $\pi$ . Then, there cannot exist in  $\pi$  a node whose weight depends both on  $p'$  and on  $p''$ .*

PROOF — By absurdum, assume a node  $v$  whose weight  $w$  depends both on  $p'$  and  $p''$ , that is, for instance,  $w = w_1.p'.p''$ . Now fix an evaluation  $\varphi$  s.t.  $\varphi(w) = 1$ , then by the dependency condition 4 of Definition 1, we have both  $w_1.p'.p'' \leq w'q$  and  $w_1.p'.p'' \leq w''\bar{q}$ ; but this is only possible when  $\varphi(w) = 0$ , contradicting the assumption  $\varphi(w) = 1$ . □

**Theorem 4 (stability under commutative cut reduction: II part)** *If  $\pi$  is a proof net that reduces to  $\pi'$  in one step of cut reduction  $C/\&$ , then  $\pi'$  is a proof net too.*

PROOF — (sketch) First observe that each cut reduction step  $C_q/\&_p$  preserves the property of being a proof structure. This follows by construction of  $\pi'$ . In particular Lemma 1 ensures that we can safely rename the sets of eigen weights in  $\mathcal{E}'$  and  $\mathcal{E}''$  and get still a proof structure.

Moreover, by Lemma 2 it is enough to only verify that all the extensive switchings, w.r.t.  $q$ , for  $\pi'$  are ACC. In the rest of proof, whenever it is not explicitly declared, each switching for  $\pi'$  is meant to be extensive w.r.t.  $q$ .

*Connexion* - Assume by absurdum there exists such a switching  $S'(\pi')$  that is disconnected. Then there exists at least a node  $v_j$  occurring in a connected component  $\gamma_1$  that is separated from the component  $\gamma_2$  that contains the  $\&_q$ -node together with the jump  $\nu$  directed to a node  $v_i$  (see the picture on the left hand side of Figure 22, where  $\gamma_1$  and  $\gamma_2$  are separated by dotted lines). This means that there not exist in  $S'(\pi')$  a path from  $v_j$  to  $v_i$ . Now let  $S''(\pi')$  be an other switching that is a copy of  $S'(\pi')$  except for the jump  $\nu$  from the  $\&_q$ -node to a  $v_h$  node that was already depending on  $q$  in  $\pi$  (in other words,  $S''$  is a conservative switching, w.r.t.  $q$ , which differs from  $S'$  only for the jump  $\nu$ , like in the picture on the right hand side of Figure 22). Clearly in  $S''(\pi')$  there is no connection path between  $v_i$  and  $v_j$ , otherwise this path should go through the  $\nu$ -jump (and so through the  $\&_p$ -node) contradicting the assumption that the  $\&_q$ -node and  $v_j$  were disconnected in  $S'(\pi')$ . But this contradicts, by Lemma 2, the hypothesis that  $\pi$  was correct.

*Acyclicity* - Assume there exists a cycle in  $S'(\pi')$  going through a jump  $\nu$  from the  $\&_q$ -node to a node  $v_i$  whose weight  $w_i$  depends on  $q$ , like in Figure 23 where  $\gamma$  is the path in  $S'(\pi')$  from the conclusion of the  $\&_q$ -node to  $v_i$ . Now, the new variable  $p'$  or its negation (resp.,  $p''$  or its negation) occurs by definition of the reduction step  $C_q/\&_p$  in the weight  $w_i$  (let us say  $w_i = w'_i p' q$ ) so, by the condition 4 of Definition 1 and the connexion of  $S'(\pi')$  showed before, the  $\&_{p'}$ -node (resp., the  $\&_{p''}$ -node) and the node  $v_i$  must be connected in  $S'(\pi')$  in two possible ways:

1. either by a path going through the  $\&_q$ -node without accrossing  $\gamma$ , like a path  $\gamma_1$  in Figure 24,
2. or like a path  $\gamma'_2$  or  $\gamma''_2$  ) of Figure 24, that is, non going through the  $\&_q$ -node and possibly accrossing  $\gamma$ .

In both cases we can set a conservative switching  $S''(\pi')$  that is identical to  $S'(\pi')$  except for the jump  $\nu$  going from the  $\&_q$ -node to the node  $v_j$  whose conclusion  $A^\perp$  is the premise of the reductum cut, like in Figure 25. Then we get a cycle, contradicting, by Lemma 2, the hypothesis that  $\pi$  was correct. □

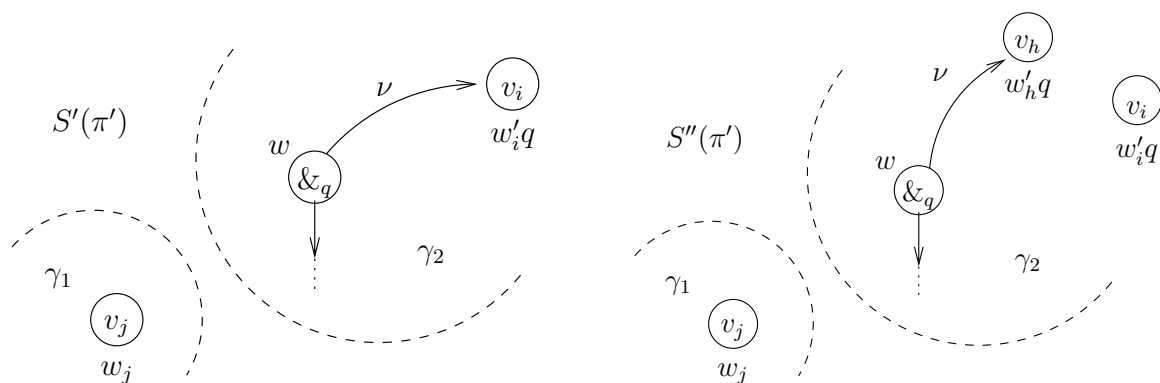


Figure 22:

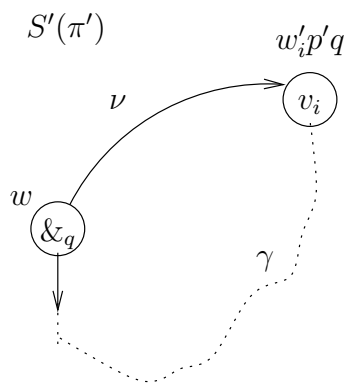


Figure 23:

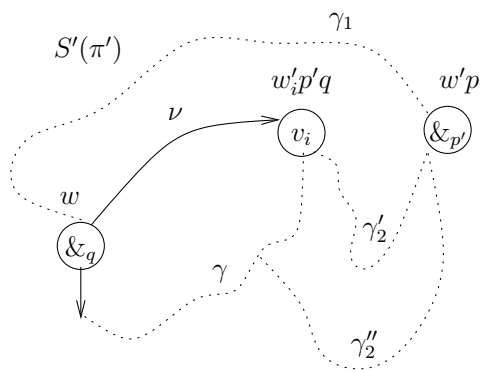


Figure 24:

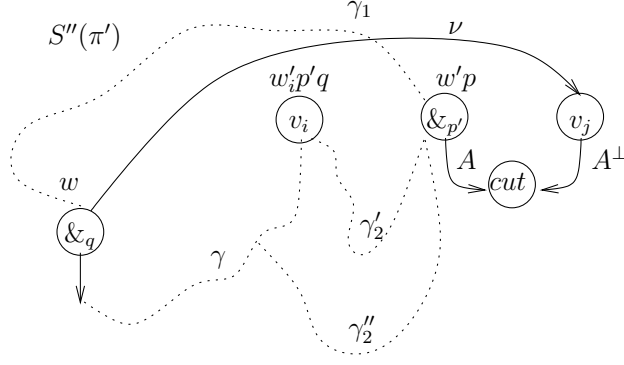


Figure 25:

### 3.4 Strong normalization

#### 3.4.1 Weak normalization

We say that a cut link  $L$  is *safe* when reducing  $L$  does not make disappear any other cut link  $L' \neq L$  from the reductum. Similarly, a cut reduction step is safe when it concerns a safe cut link. A cut reduction strategy is a finite sequence of cut reduction steps of Definition 4 and 6; a strategy is safe when it contains only safe cut reduction steps.

The *complexity* of a cut node is the logical complexity of its cut-formula<sup>1</sup>. We say that a cut-node with weight  $w$  has *depth*  $n$  if  $|w| = n$ , where  $|w|$  is the number of variables of negations of variables occurring in  $w$ .

**Lemma 4 (safe reduction)** *If  $\pi$  is a proof net with only cut links of type  $\&/\oplus_i$ , then at least one of them is safe.*

PROOF — By absurdum, assume  $\pi$  contains only  $\&/\oplus_i$  cut links (at least one) and assume none of them is safe; then by the dependence condition we can easily find a switching with a cycle.  $\square$

**Theorem 5 (weak normalization)** *If  $\pi$  is a non cut-free proof net, then there exists a safe reduction strategy for  $\pi$  terminating with a cut-free (normal) proof net.*

PROOF — Assume a safe reduction strategy consisting in applying a safe reduction step  $\&/\oplus_i$  only when no other reduction choice is possible. By Lemma 4 we know that such a strategy exists. Then, termination of this such a safe strategy follows by induction on the lexicographic order of the *cut complexity sequence*  $\#1, \dots, \#n$  of  $\pi$ , where  $n$  is the number of eigen-weight variables occurring in  $\pi$  and  $\#i$ , with  $1 \leq i \leq n$ , is the sum of the *complexities* of all cuts whose *depth* is  $i$ . It is immediate to check that the complexity sequence decreases at each reduction step, except for the reduction step  $C/\&$ , where in order to apply the hypothesis of induction we have to ensure that the length  $n$  of the complexity sequence does not increase. In other words we have to show that there cannot exist in  $\pi'$  a cut whose weight has depth greater than  $n$ . Now, assume by absurdum that, after one step of cut reduction  $C/\&_p$ ,  $\pi$  reduces to  $\pi'$  with a cut node  $v_i$  whose depth is  $n + 1$ , so the cut complexity sequence of  $\pi'$  is  $\#1, \dots, \#n, \#(n + 1)$ . This could only be consequence of the fact that, after the  $C/\&_p$ -reduction step, the eigen weight  $p$  of  $\pi$  has been replaced in  $\pi'$  by two new eigen weights  $p'$  and  $p''$ . In other words, the weight  $w_i$  of  $v_i$  will depend both on  $p'$  and  $p''$ , like in Figure 26, with for instance  $w'_i p' p'' \leq w_1 q$  and  $w'_i p' p'' \leq w_1 \bar{q}$ , contradicting the Separation Lemma 3.  $\square$

#### 3.4.2 Confluence

**Theorem 6 (confluence)** *Assume  $\pi$  is proof net s.t. it reduces in one step  $\alpha$  to  $\pi'$  ( $\pi \rightsquigarrow_\alpha \pi'$ ) and it reduces in an other step  $\beta$  to  $\pi''$  ( $\pi \rightsquigarrow_\beta \pi''$ ); then, there exists a proof net  $\sigma$  such that  $\pi'$  reduces, in a certain number of steps, to  $\sigma$  ( $\pi' \rightsquigarrow^* \sigma$ ) and  $\pi''$  reduces, in a certain number of steps, to  $\sigma$  ( $\pi'' \rightsquigarrow^* \sigma$ ).*

PROOF — (sketch) Assume  $\pi \rightsquigarrow_\alpha \pi'$  and  $\pi \rightsquigarrow_\beta \pi''$ , then we proceed by cases according to  $\alpha$  and  $\beta$  and we show that there always exists such a  $\sigma$  which both  $\pi'$  and  $\pi''$  reduce to.

**Case 1** - If neither  $\alpha$  nor  $\beta$  is a reduction step involving a  $\&$ -node as a cut premise node, then it is easy to check that we get the confluence to  $\sigma$  by two more single reduction steps,  $\pi' \rightsquigarrow_\beta \sigma$  and  $\pi'' \rightsquigarrow_\alpha \sigma$ , like in Figure 27 (the *diamond composition*).

<sup>1</sup>The *logical complexity* of a formula is inductively defined as follows: atoms have complexity 1; the complexity of  $A^\perp$  is the same as the complexity of  $A$ ; the complexity of  $B \bullet C$ , where  $\bullet$  is any binary connective, is the sum of the complexities of  $B$  and  $C$  plus 1.

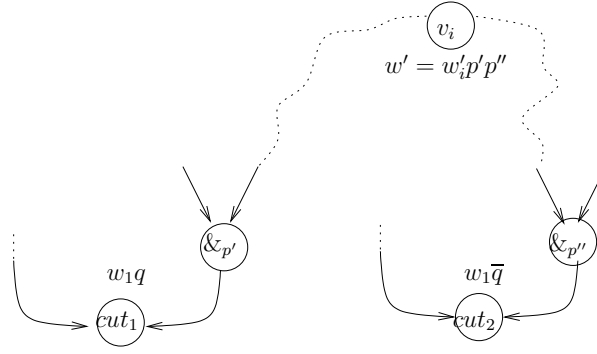


Figure 26:

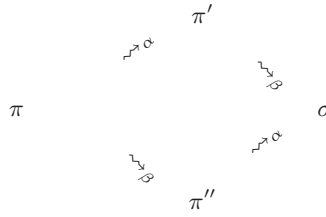


Figure 27: Diamond composition

**Case 2** - Otherwise, if  $\alpha$  or  $\beta$  is a reduction step involving a  $\&$ -node as a cut premise node, then we split our reasoning in two main sub-cases:

*Case 2.1* -  $\alpha$  or  $\beta$  is a ready cut  $\&/\oplus_i$ . In these cases we get the confluence to  $\sigma$  by (possibly) two more single reduction steps, like in Figure 27. First, observe that the diamond composition can be partial, in the case that both  $\alpha$  and  $\beta$  concern two  $\&/\oplus_i$  cuts s.t. one of them depends on the eigen weight of the  $\&$ -node that is premise of the other cut. Second, observe the case when  $\alpha$  is a reduction step of a  $(C/\&_p)$ -cut link  $L_1$  and  $\beta$  is a reduction step of a  $(C/\&_p q)$ -cut link  $L_2$ , with  $L_1$  depending on  $q$  and  $L_2$  depending on  $p$ , is excluded by the correctness of  $\pi$ .

*Case 2.2* - Both  $\alpha$  and  $\beta$  are two commutative reduction steps, then of course at least one of them must be a commutative reduction  $C/\&$ . (Observe the case when  $\alpha$  is a reduction step  $C_q/\&_p$  (with  $C$  depending on  $q$ ) and  $\beta$  is a reduction step  $C_p/\&_q$  (with  $C$  depending on  $p$ ) is excluded by the correctness of  $\pi$ . We consider the remaining sub-cases:

1. Assume  $\alpha$  and  $\beta$  concern the reduction of, respectively, cut  $L_1$  and cut  $L_2$  of Figure 28. Then, the two sequences of reductions,  $\pi \rightsquigarrow_\alpha \pi_1 \rightsquigarrow_\beta \pi'_1$  of Figure 29 and  $\pi \rightsquigarrow_\beta \pi_2 \rightsquigarrow_\alpha \pi'_2$  of Figure 30, converge to  $\pi'_1 = \sigma$  by means of a sequence of two axiom reductions starting from  $\pi'_2$  (i.e.,  $\pi'_2 \rightsquigarrow_{AX}^* \pi'_1 = \sigma$ ).
2. Assume  $\alpha$  and  $\beta$  concern the reductions of, respectively, cut  $L_1$  and cut  $L_2$  of Figure 31. Then, the two reduction sequences,  $\pi \rightsquigarrow_\alpha \pi_1 \rightsquigarrow_\beta \pi'_1$  of Figure 32 and  $\pi \rightsquigarrow_\beta \pi_2 \rightsquigarrow_\alpha \pi'_2$  of Figure 33, can converge in to  $\pi'_2 = \sigma$  by means of two more commutative reductions,  $C_{p'}/C_{p'}$  and  $C_{p''}/C_{p''}$  applied to  $\pi'_1$  (i.e.,  $\pi'_1 \rightsquigarrow_{(C/C)}^* \pi'_2 = \sigma$ ). We omit the simpler particular sub-case when the reduction  $\beta$  concerns a commutative  $C_p/C_p$  reduction whose weight  $w_2$  depends on  $q$ .
3. Assume  $\alpha$  and  $\beta$  concern the reductions of, respectively, cut  $L_1$  and cut  $L_2$  of Figure 34. First observe that,

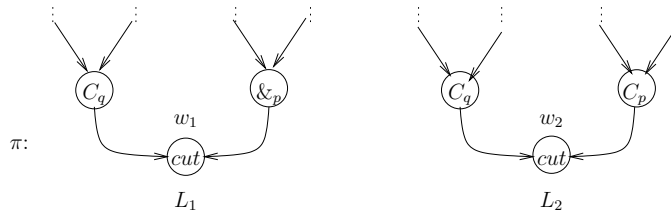


Figure 28:

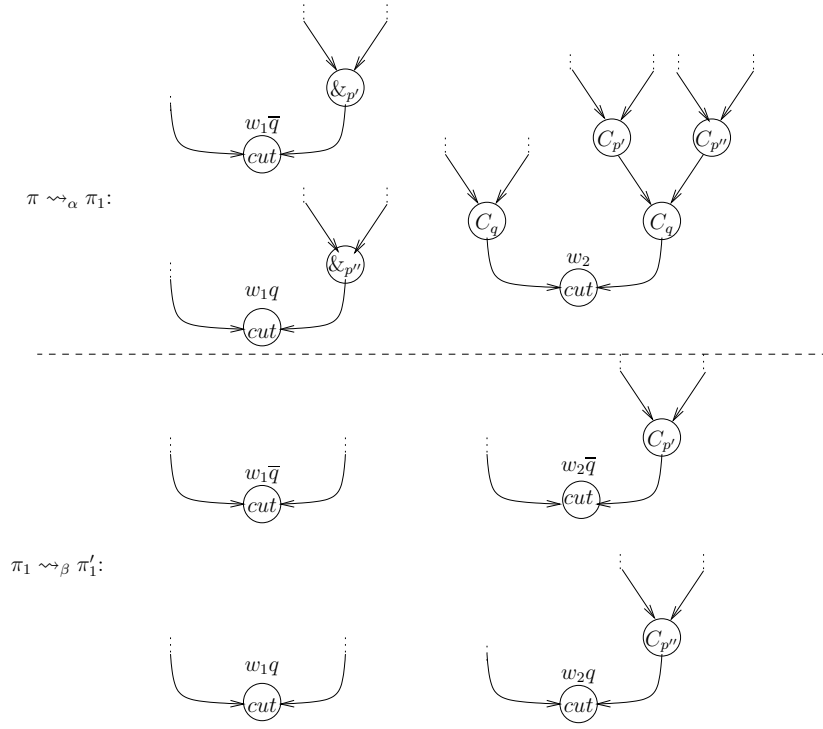


Figure 29:

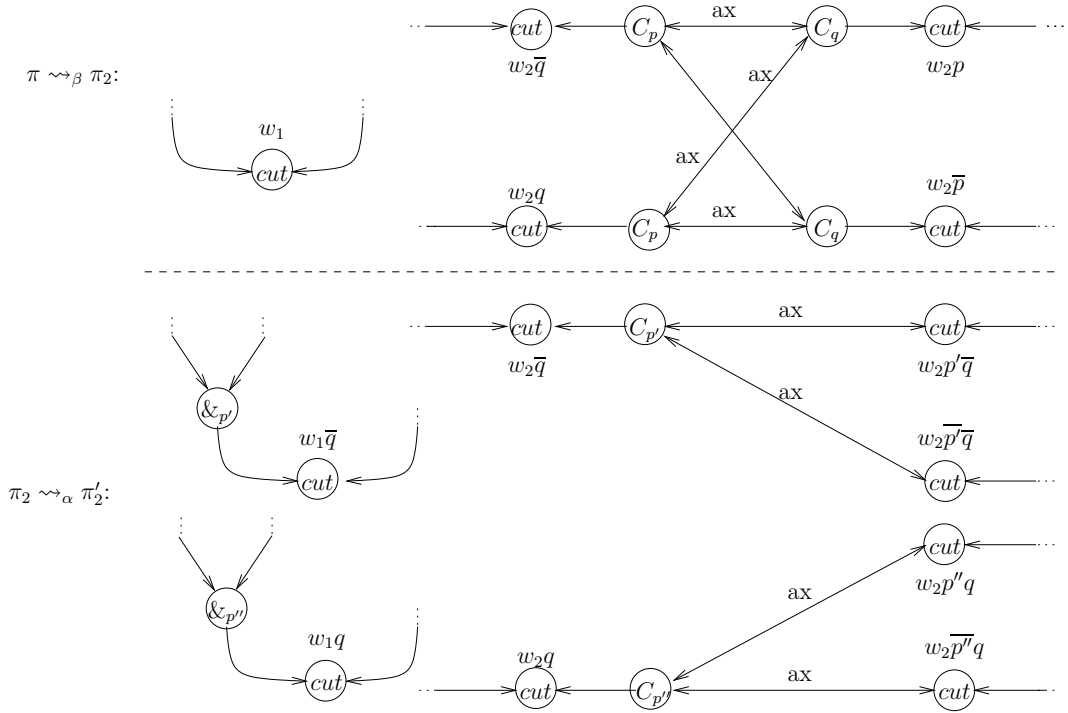


Figure 30:



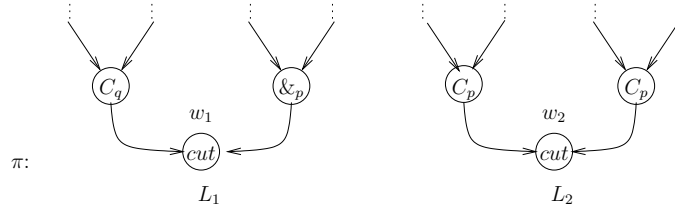


Figure 31:

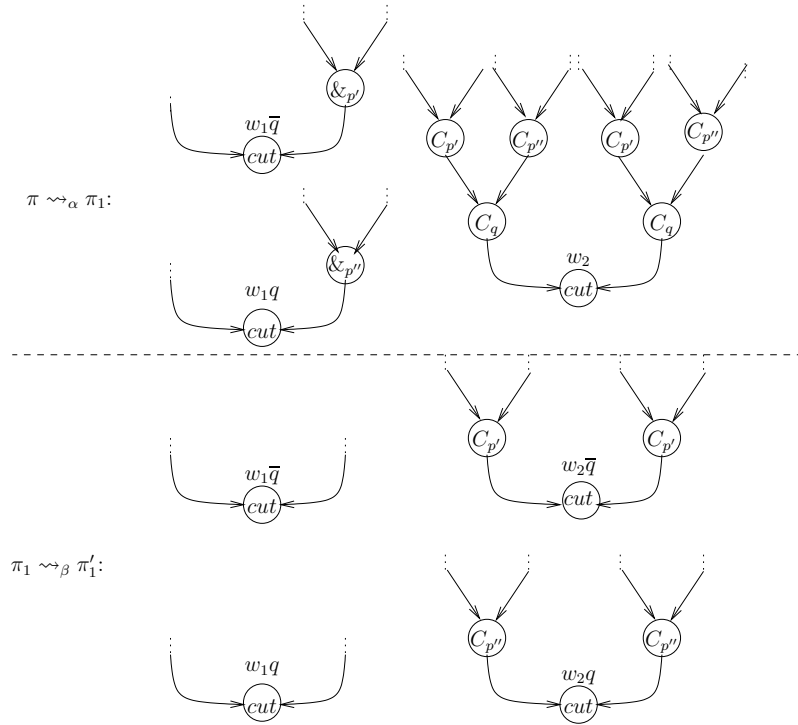


Figure 32:

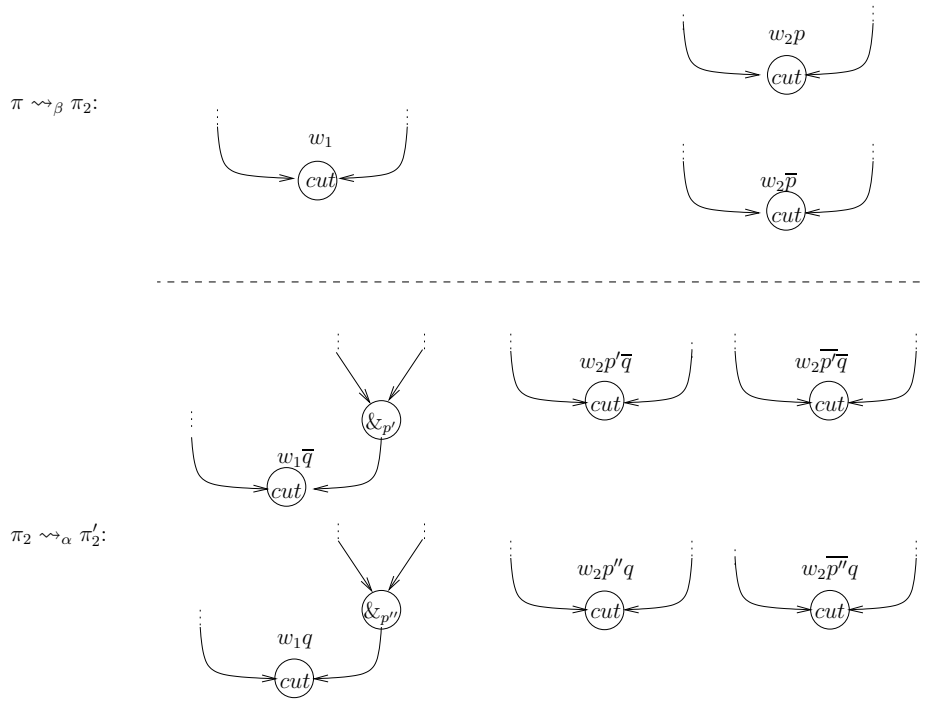


Figure 33:

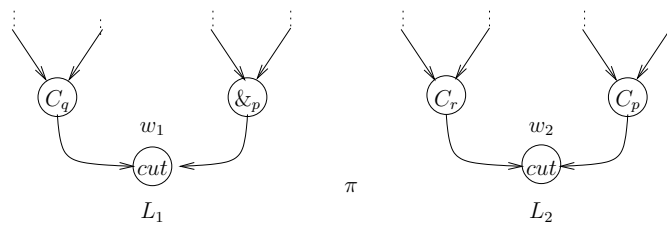


Figure 34:

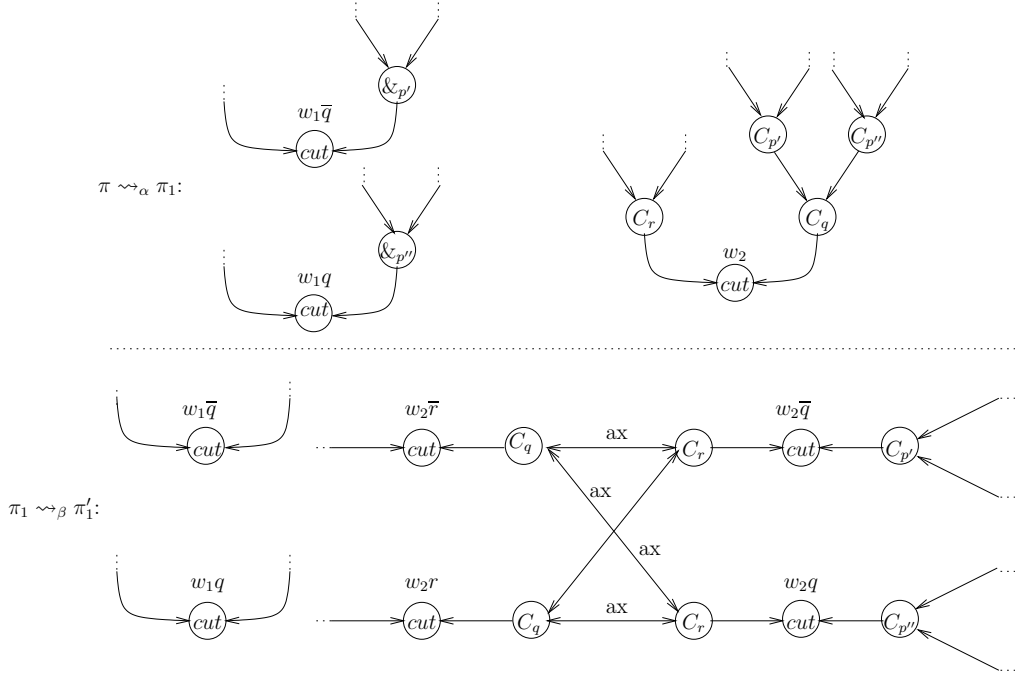


Figure 35:

by the dependency condition 4 of Definition 1, the  $\&_r$ -node cannot occur in the empire  $\mathcal{E}(p)$ , so the reduction  $\alpha$  does not imply any duplication of the eigen weight  $r$  in  $\pi'$ . Now, consider the two reduction sequences,  $\pi \rightsquigarrow_\alpha \pi_1 \rightsquigarrow_\beta \pi'_1$  of Figure 35 and  $\pi \rightsquigarrow_\beta \pi_2 \rightsquigarrow_\alpha \pi'_2$  of Figure 36. Clearly  $\pi'_1$  converges to  $\pi'_2 = \sigma$  by means of two more commutative reduction steps,  $(C_r/C_{p'})$  and  $(C_r/C_{p''})$ , modulo some axiom reductions. We omit the particular (simpler) sub-cases when the weight  $w_2$  of cut  $L_2$  depends on  $q$ .

4. In all the remaining cases, illustrated in

- (a) Figure 37,
- (b) Figure 38,
- (c) Figure 39

the confluence is obtained as simple diamond composition (see Figure 27). □

**Example 2** According to our reduction system rules, the problematic (non confluent) reduction of the commutative cut of  $\pi$  of Figure 9 (see Section 3.2.1), has got now a (unique solution): the confluent reduction sequence is illustrated by Figure 40 and Figure 41.

**Theorem 7 (strong normalization)** Cut-elimination is strong normalizing.

PROOF — Weak-normalization (Theorem 5) and confluence (Theorem 6) imply strong normalization. □

## 4 Proof-structures with explicit $n$ -ary contraction links

Proof-structures with explicit  $n$ -ary contraction links differ from those one defined in Definition 1 only for the use of  $n$ -ary contractions (defined in Figure 42) instead of the binary ones. Then, the notion of slices and switchings (Definitions 2) and proof-nets (Definitions 3) remain unchanged as well the (De-)Sequentialization Theorem 1.

**Remark 3 (binary syntax vs.  $n$ -ary syntax for  $C$ -links)** Observe that in general the syntax with  $n$ -ary contraction is not equivalent to the former one with binary contraction, at least for the following two reasons:

1. Figure 43 shows an example of proof-net with an  $n$ -ary contraction link whose monomial sum,  $\overline{pqr} + \overline{pr} + \overline{qr} + p\overline{q} + pqr$ , cannot be associated in such a way to reduce this  $n$ -ary contraction into a sum of only monomial binary sums (contractions);

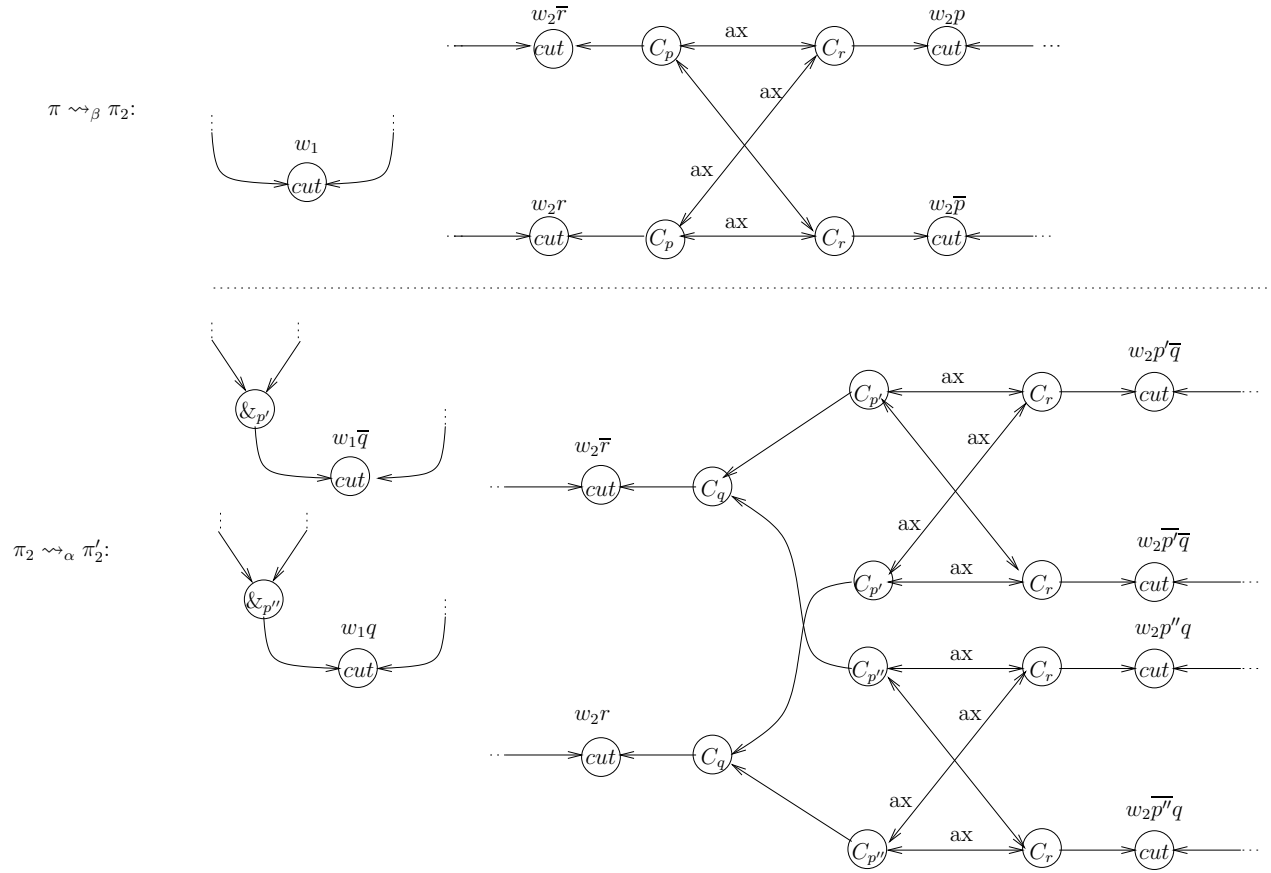


Figure 36:

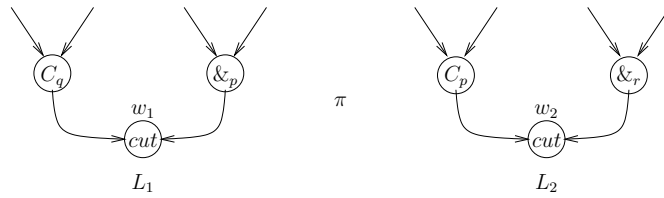


Figure 37:

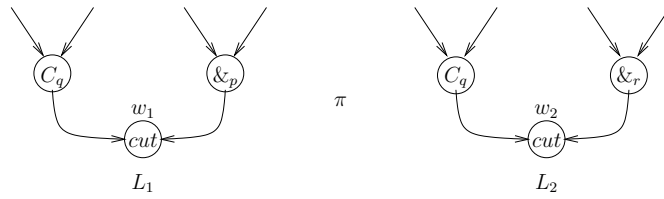


Figure 38:

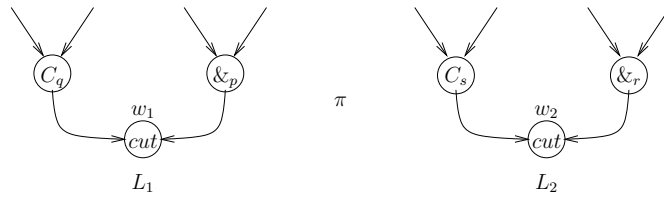


Figure 39:

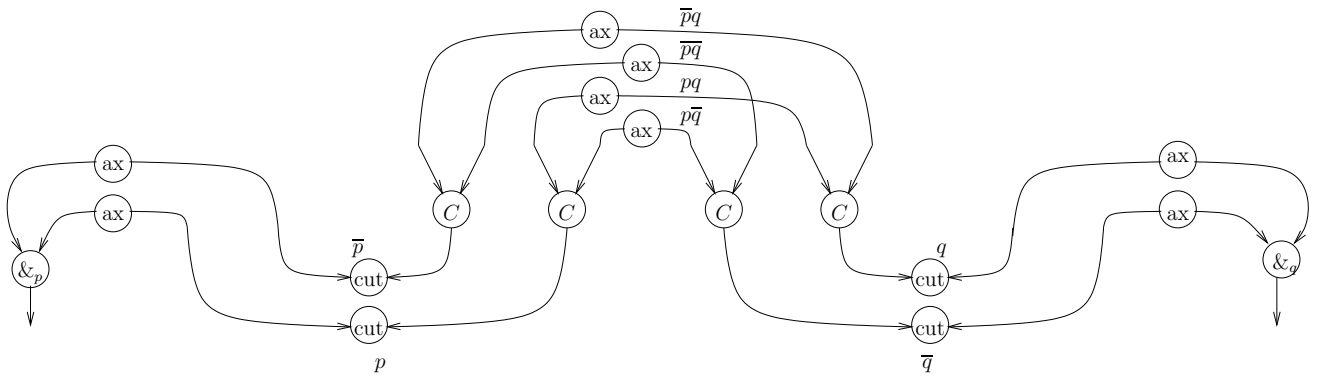


Figure 40:

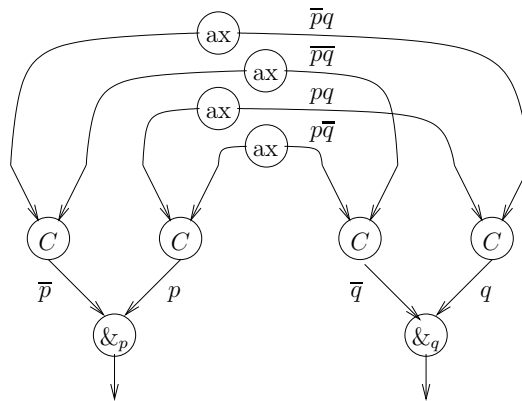


Figure 41:

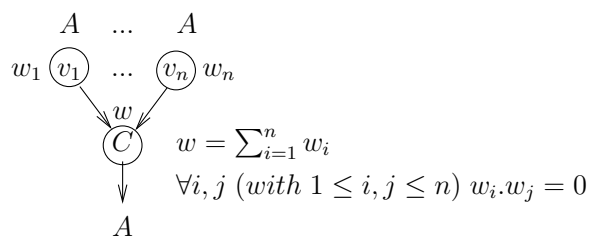


Figure 42:  $n$ -ary contraction link

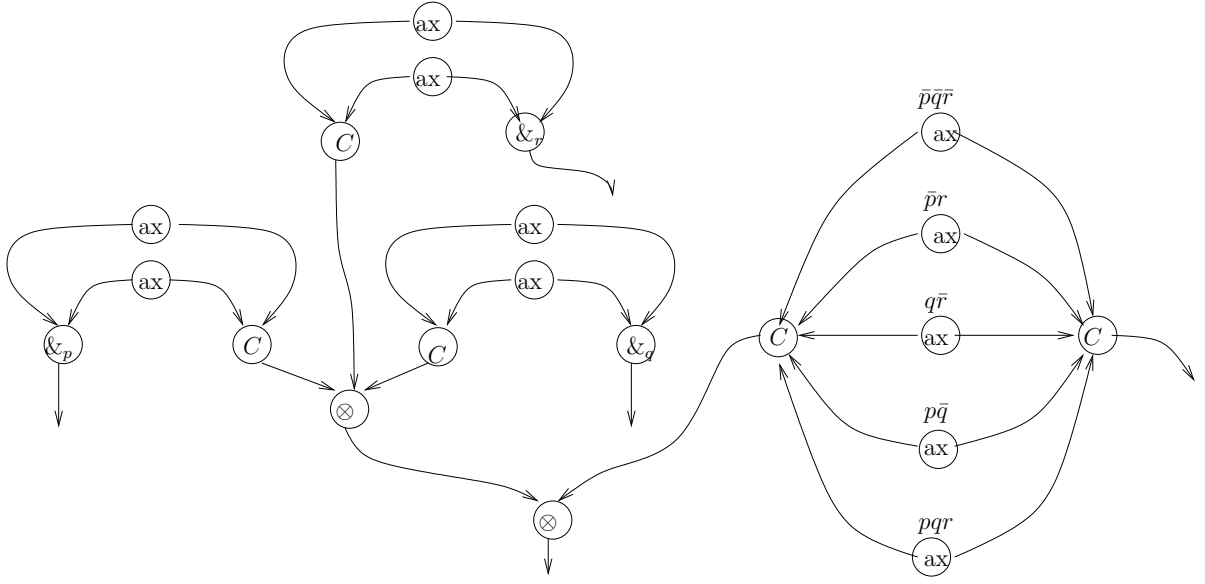


Figure 43: Proof-net with  $n$ -ary contraction

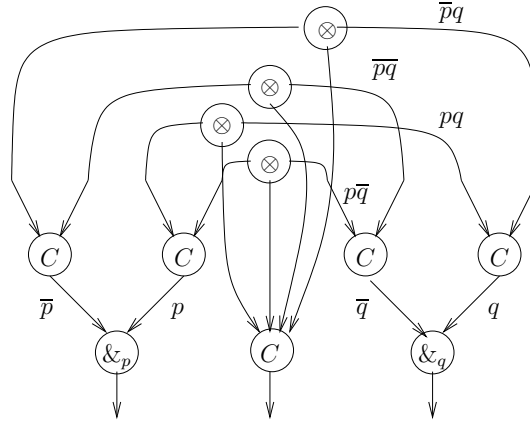


Figure 44: Proof-net with  $n$ -ary contraction

2. moreover Figure 44 shows an example of proof-net that as soon its  $n$ -ary contraction is factorized into only binary contraction, the resulting proof-net will quotients less proofs than the previous one (in other words the new proof-net has less sequentialization than the original one with the  $n$ -ary contraction).

## 4.1 Cut-elimination

We only consider here the reduction steps of commutative cuts, since the reduction of ready-cuts remains unchanged w.r.t. that one of Section 3.1

- the  $C/\otimes$ -cut of Figure 45, reduces in one step like in Figure 46.
- the  $C/C$ -cut, like in Figure 47, reduces in one step like in Figure 48 and we keep only those nodes and edges with nonzero weight.
- the  $C/\oplus$ -cut of Figure 49 is reduced in a way that is a trivial generalization of the simpler one illustrated in Figure 16.
- the  $C/\&$ -cut of Figure 50 reduces in one step like in Figure 51, where  $w_i.[\mathcal{E}_p^i]$ ,  $1 \leq i \leq n$  is obtained by spreading  $w_i$  over a copy of the empire of  $p$ ,  $\mathcal{E}_p$ , and replacing in it each eigen weight with a new (fresh) one (observe that  $w = \sum_{i=1}^n w_i$  and by the dependency condition it must be  $\forall j, w \subset v_{j=1, \dots, m}$ , i.e. there exists a prefix  $\sigma$  s.t.  $w\sigma = v_j$ , therefore  $v_j w = v_j$ ).

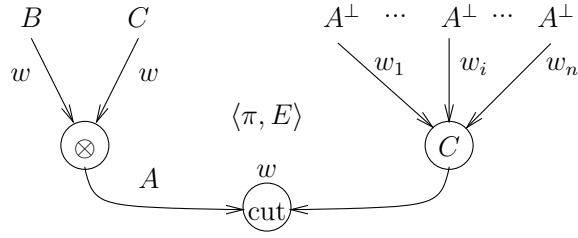


Figure 45:

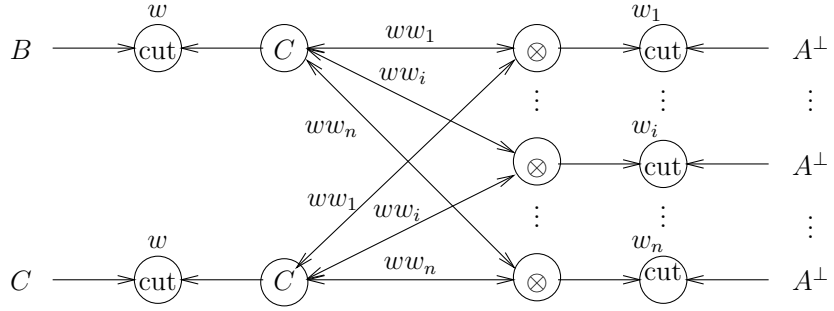


Figure 46:

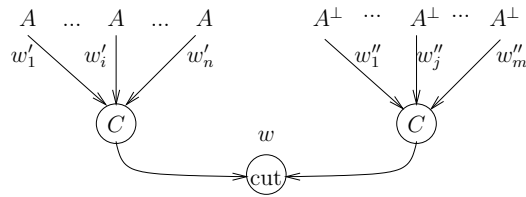


Figure 47:

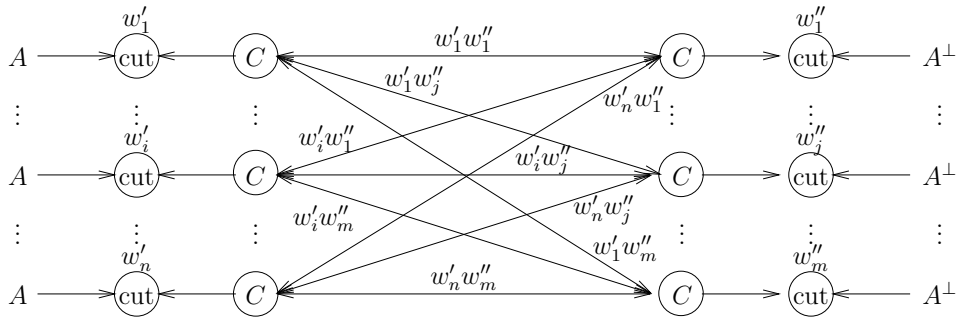


Figure 48:

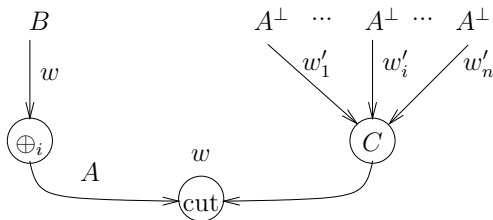


Figure 49:

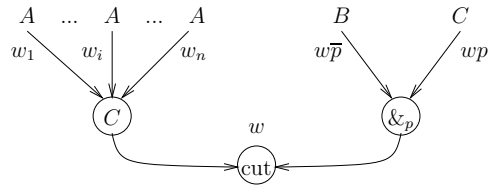


Figure 50:

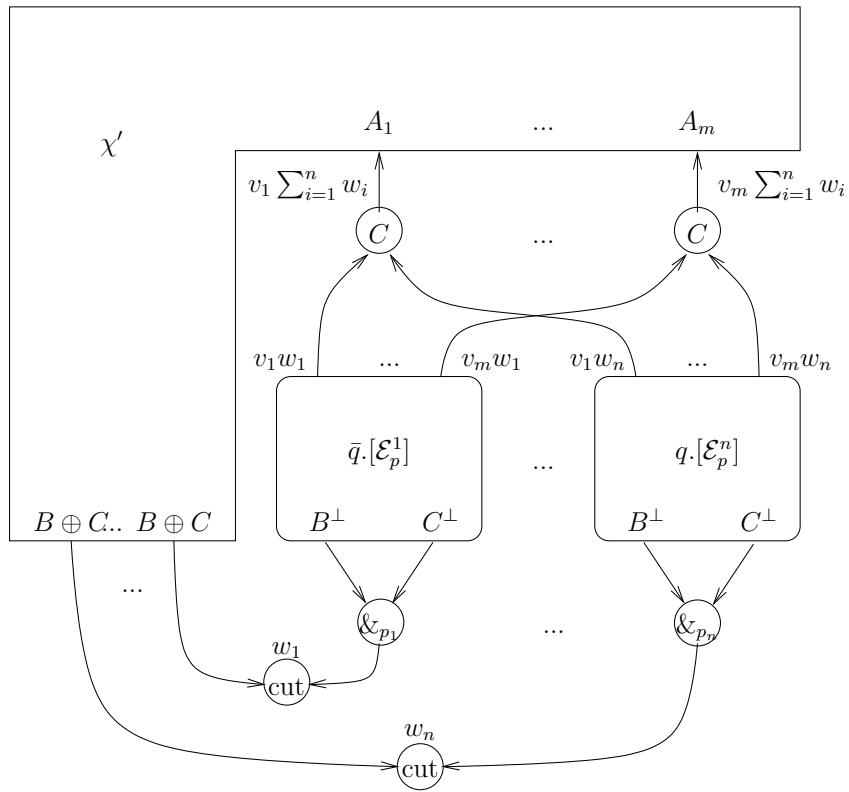


Figure 51:



## 4.2 Stability

Naively, we show that if there exists an extensive switching  $S(\pi)$ , with  $n$  extensive jumps. that is not ACC, then (by a reasoning like in Theorem 4) we can find an other non ACC switching  $S'(\pi)$  with a number of extensive jumps strictly smaller than  $n$ . So, by iterating t reasoning for  $S'$  we get (t the end, when  $n = 0$ ) a conservative switching  $S * (\pi)$  (the base of the induction) that is not ACC, contradicting the (analogous) of Theorem 3.

## 4.3 Strong cut-elimination

Similar to Theorem 7.

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