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Nonparametric estimation of a quantile density function by wavelet methods

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Abstract: In this paper nonparametric wavelet estimators of the quantile density function are proposed. Consistency of the wavelet estimators is established under the L_p risk. A simulation study is done to compare our estimators to those proposed by Jones (1992) and Soni *et al.* (2012).

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1. Introduction

Motivation. Quantile density function, the derivative of the quantile function, comes up in the study of lifetime and survival data. The expression for the limiting variance of some nonparametric estimators of the quantile function involves the quantile density function. Nonparametric estimators of the quantile density function have been studied by Parzen (1979), Jones (1992) and Soni *et al.* (2012). Most of these estimators underperform at the tails. Hence there is a need to look at alternate estimators.

On the quantile function. Quantiles are often used for statistical modeling and analysis of data. Measures based on quantiles are less influenced by outliers. Hence they are particularly useful in the study of lifetime data and also for studying heavy tailed distributions. Sometimes the distribution function of the random variable of interest can not be expressed in a closed form. However, the quantile function can be written in an explicit form, for example, Generalised Lambda distribution (GLD) and Skew logistic distribution (see Gilchrist (2000) and Karian and Dudewicz (2000)). Quantiles are useful in modeling data as sum and product of two quantile functions is again a quantile.

The quantile function approach has been used in exploratory data analysis, applied statistics, reliability and survival analysis (see, for example, Reid (1981), Slud *et al.* (1984), Su and Wei (1993), Nair *et al.* (2008), Nair and

Sankaran (2009) and Sankaran and Nair (2009), Sankaran, Dewan and Sreedevi (2015). Soni, Dewan and Jain (2015a, 2015b) have developed multiple comparison procedures for quantile functions. It has been used for hypothesis testing in competing risks by Peng and Fine (2007), Jeong and Fine (2009) and Sankaran *et al.* (2010).

Let X be a continuous random variable with cumulative density function $F(x)$, density function $f(x)$ and hazard function $r(x)$. The quantile function of X is defined as

$$Q(x) = F^{-1}(x) = \inf\{y \in R; F(y) \geq x\}, \quad x \in [0, 1].$$

It satisfies $F(Q(x)) = x$. Parzen (1979) and Jones (1992) defined $g(x) = Q'(x)$ as the quantile density function corresponding to quantile function $Q(x)$. Differentiating (1), we get

$$g(x) = \frac{1}{f(Q(x))}, \quad x \in [0, 1]. \quad (1.1)$$

Note that the sum of two quantile density functions is again a quantile density function. This idea is useful in modeling data. Nair and Sankaran (2009) defined the hazard quantile function

$$R(x) = r(Q(x)) = \frac{f(Q(x))}{1 - F(Q(x))} = \frac{1}{(1-x)g(x)} \quad x \in (0, 1).$$

Hence $g(x)$ appears in the expression for hazard quantile function and it would be useful to study nonparametric estimators of this unknown quantile density function.

Overview of previous works. Let X_1, X_2, \dots, X_n be *iid* random variables from distribution $F(x)$ defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Jones (1992) suggested the following two smooth estimators of the quantile density function.

$$\hat{g}^{j1}(x) = \frac{1}{\hat{f}(\hat{Q}(x))}, \quad x \in [0, 1], \quad (1.2)$$

where $\hat{f}(x)$ is a kernel type density estimator of the form:

$$\hat{f}(x) = \frac{1}{nh(n)} \sum_{i=1}^n K\left(\frac{x - X_i}{h(n)}\right),$$

where $h(n)$ is the bandwidth and $K(\cdot)$ an appropriate kernel function, and $\hat{Q}(x)$ is the usual empirical estimator of $Q(x)$.

To be more specific, the kernel $K(\cdot)$ is a real valued function satisfying the following properties:

- (i) $K(u) \geq 0$ for all $u \in R$,

- (ii) $\int_{-\infty}^{\infty} K(u)du = 1$,
- (iii) $K(\cdot)$ has finite support,
- (iv) $K(\cdot)$ is symmetric about zero,
- (v) $K(\cdot)$ satisfies Lipschitz condition, that is, there exists a positive constant C such that $|K(u) - K(v)| \leq C|u - v|$.

The bandwidth $h(n)$ satisfies

- (i) $h(n) \rightarrow 0$ as $n \rightarrow \infty$,
- (ii) $nh(n) \rightarrow \infty$ as $n \rightarrow \infty$.

Another estimator of quantile density function given by Jones (1992) is

$$\begin{aligned} \hat{g}^{j2}(x) &= \sum_{i=2}^n X_{(i)} \left(K_{h(n)} \left(x - \frac{i-1}{n} \right) - K_{h(n)} \left(x - \frac{i}{n} \right) \right) \\ &= \sum_{i=2}^n (X_{(i)} - X_{(i-1)}) K_{h(n)} \left(x - \frac{i-1}{n} \right) \\ &\quad - X_{(n)} K_{h(n)}(x-1) + X_{(1)} K_{h(n)}(x), \quad x \in [0, 1], \end{aligned} \quad (1.3)$$

where $X_{(i)}$ is the i^{th} order statistic, $i = 1, 2, \dots, n$.

Jones (1992) and Soni *et al.* (2012) showed that the performance of the estimator of $g(x)$ given by (1.2) is better than that of the estimator given by (1.3).

Soni *et al.* (2012) proposed the following smooth estimator of the quantile density function:

$$\hat{g}^S(x) = \frac{1}{h(n)} \int_0^1 K \left(\frac{t-x}{h(n)} \right) \frac{1}{\hat{f}(\hat{Q}(t))} dt, \quad x \in [0, 1], \quad (1.4)$$

where $K(\cdot)$ is a kernel and $h(n)$ is the bandwidth sequence satisfying conditions listed above. Let us mention that (1.4) can be expressed as

$$\hat{g}^S(x) = \frac{1}{h(n)} \sum_{i=1}^n \frac{1}{\hat{f}(X_{(i)})} \int_{S_{i-1}}^{S_i} K \left(\frac{t-x}{h(n)} \right) dt,$$

where S_i is the proportion of observations less than or equal to $X_{(i)}$, the i^{th} order statistic. Using the mean value theorem, for small $S_i - S_{i-1}$,

$$\hat{g}^S(x) = \frac{1}{nh(n)} \sum_{i=1}^n K \left(\frac{S_i - x}{h(n)} \right) \frac{1}{\hat{f}(X_{(i)})}.$$

Contributions. In this paper, we explore a different approach by considering estimators based on projects on a wavelet basis of the (crude) form:

$$\hat{g}(x) = \sum_m \hat{a}_m e_m(x), \quad x \in [0, 1],$$

where \hat{a}_m denotes an estimator of the coefficient $a_m = \int g(x)e_m(x)dx$ and $\{e_m\}_m$ forms the wavelet basis. Such basis is of interest, thanks to its localization in space and frequency properties. For the standard nonparametric estimation problems in density, regression, . . . wavelet estimators outperform kernel estimators in representing discontinuities (edges, spikes, . . .). Basics on wavelet estimation can be found in Härdle *et al.* (1998).

In this study, we develop two kinds of wavelet estimators for the quantile density function g : a linear one based on simple projections and a nonlinear one based on a hard thresholding rule. Our wavelet hard thresholding estimator has the feature to be adaptive according to $g(x)$. Let us mention that, due to the choice of \hat{a}_m considered, our estimators belong to the family of "warped wavelet basis" estimators introduced by Kerkycharian and Picard (2004) in another statistical context - regression problem with random design. Under smoothness assumptions on $g(x)$, we determine the rates of convergence of the proposed estimators under the \mathbb{L}_p risk. The rates of convergence obtained are those established in the standard density estimation problem via wavelet hard thresholding (see Härdle *et al.* (1998)). A simulation study is done to compare our estimators to those proposed by Jones (1992) and Soni *et al.* (2012).

Paper organization. The rest of this paper is organized as follows. In the next section, we present our wavelet estimators. The main theoretical results are described in Section 3. Section 4 is devoted to the numerical performances of our estimators. The proofs of the technical results appear in Sections 5. We end with a concluding remark.

2. Wavelet estimators

Wavelet expansion. We define the spaces $\mathbb{L}_p([0, 1])$, $p \geq 1$, by

$$\mathbb{L}_p([0, 1]) = \left\{ h : [0, 1] \rightarrow R; \int_{[0,1]} |h(x)|^p dx < \infty \right\}.$$

For the purpose of the paper, we consider an orthonormal wavelet basis of $\mathbb{L}_2([0, 1])$ associated with the initial wavelet functions ϕ and ψ of the Daubechies wavelets $db2N$, where N denotes a positive integer. The functions ϕ and ψ are compactly supported and continuously differentiable.

For any $x \in R$,

$$\phi_{j,k}(x) = 2^{j/2}\phi(2^j x - k), \quad \psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k).$$

Then, with an appropriate treatment at the boundaries, there exists an integer τ satisfying $2^\tau \geq 2N$ such that, for any $\ell \geq \tau$, the system

$$\mathcal{W} = \{\phi_{\ell,k}; k \in \{0, \dots, 2^\ell - 1\}; \psi_{j,k}; k \in \{0, \dots, 2^j - 1\}, j \geq \ell\}$$

is an orthonormal basis of $\mathbb{L}_2([0, 1])$.

For convenience, we suppose that X_1 is compactly supported, say $X_1(\Omega) = [0, 1]$, and that F is continuous and strictly monotone from $[0, 1]$ to $[0, 1]$.

Let us suppose that $g \in \mathbb{L}_2([0, 1])$. Hence, for any integer $\ell \geq \tau$, we have the following wavelet expansion on \mathcal{W} :

$$g(x) = \sum_{k=0}^{2^\ell-1} c_{\ell,k} \phi_{\ell,k}(x) + \sum_{j=\ell}^{\infty} \sum_{k=0}^{2^j-1} d_{j,k} \psi_{j,k}(x), \quad x \in [0, 1],$$

where

$$c_{j,k} = \int_{[0,1]} g(x) \phi_{j,k}(x) dx, \quad d_{j,k} = \int_{[0,1]} g(x) \psi_{j,k}(x) dx. \quad (2.1)$$

All the technical details can be found in, e.g., Cohen *et al.* (1993) and Mallat (2009). Note that our study can be extended for any interval of the form $[a, b]$ with $a < b$ finite.

Wavelet coefficients estimators. The wavelet coefficients $c_{j,k}$ and $d_{j,k}$ are unknown and need to be estimated. Our approach is based on the following remark: by the change of variable $x = F(y)$, we can rewrite $c_{j,k}$ as

$$c_{j,k} = \int_{[0,1]} g(x) \phi_{j,k}(x) dx = \int_{[0,1]} \frac{1}{f(F^{-1}(x))} \phi_{j,k}(x) dx = \int_{[0,1]} \phi_{j,k}(F(x)) dx.$$

Similarly

$$\hat{d}_{j,k} = \int_{[0,1]} \psi_{j,k}(F(x)) dx. \quad (2.2)$$

Since F is unknown, we estimate it by the empirical estimator:

$$\hat{F}(x) = \frac{1}{n} \sum_{i=1}^n \mathbf{1}_{\{X_i \leq x\}}, \quad x \in [0, 1],$$

where $\mathbf{1}$ is the indicator function.

This leads the following integral estimator for $c_{j,k}$:

$$\hat{c}_{j,k} = \int_{[0,1]} \phi_{j,k}(\hat{F}(x)) dx. \quad (2.3)$$

The analog estimator for $d_{j,k}$ is

$$\hat{d}_{j,k} = \int_{[0,1]} \psi_{j,k}(\hat{F}(x)) dx. \quad (2.4)$$

Due to the composition of the element of the wavelet basis with $\hat{F}(x)$, $\hat{c}_{j,k}$ and $\hat{d}_{j,k}$ can be viewed as "warped wavelet basis" coefficient estimators. Such estimators was introduced by Kerkycharian and Picard (2004) in a regression with random design framework. Improvements and modern developments can be found in Chagny (2013a,b) for the same regression model, and Chagny and Lacour (2015) for the relative density estimation problem (with trigonometric basis).

Remark 2.1. Clearly, $c_{j,k}$ and $d_{j,k}$ are not unbiased estimators for $c_{j,k}$ and $d_{j,k}$. However, using the dominated convergence theorem, one can prove that they are asymptotically unbiased. Moreover, they satisfy power moments and concentration inequalities. See Lemmas 5.1, 5.2, 5.3 and 5.4 below.

Based on $\hat{c}_{j,k}$ (2.3) and $\hat{d}_{j,k}$ (2.4), we consider two kinds of wavelet estimators for $g(x)$: a linear wavelet estimator $g_L(x)$ and a hard thresholding wavelet estimator $g_H(x)$, both defined below.

Linear wavelet estimator. We define the linear wavelet estimator $\hat{g}_L(x)$ by

$$\hat{g}_L(x) = \sum_{k=0}^{2^{j_0}-1} \hat{c}_{j_0,k} \phi_{j_0,k}(x), \quad x \in [0, 1], \quad (2.5)$$

where j_0 is a positive integer chosen a posteriori (see Theorems 3.1 and 3.3).

Hard thresholding wavelet estimator. We define the hard thresholding wavelet estimator $\hat{g}_H(x)$ by

$$\hat{g}_H(x) = \sum_{k=0}^{2^\tau-1} \hat{c}_{\tau,k} \phi_{\tau,k}(x) + \sum_{j=\tau}^{j_1} \sum_{k=0}^{2^j-1} \hat{d}_{j,k} \mathbf{1}_{\{|\hat{d}_{j,k}| \geq \kappa \lambda_j\}} \psi_{j,k}(x), \quad x \in [0, 1], \quad (2.6)$$

where $\hat{c}_{j,k}$ and $\hat{d}_{j,k}$ are defined by (2.3) and (2.4), $\mathbf{1}$ is the indicator function, j_1 is a positive integer and λ_j represents a threshold. Both j_1 and λ_j will be chosen a posteriori (see Theorems 3.2 and 3.4)

The construction of $\hat{g}_H(x)$ exploits the sparse nature of the wavelet decomposition of $g(x)$: only the wavelet coefficients $d_{j,k}$ with large magnitude contain the main information (in terms of details) of g . Hence $\hat{g}_H(x)$ aims to only estimate the larger coefficients, and to remove the other (or estimate it by 0). Further aspects and explanation related to this selection techniques can be found in Antoniadis (1997), Härdle *et al.* (1998) and Vidakovic (1999).

As usual in wavelet estimation, we will suppose that the unknown function $g(x)$ belongs to Besov balls defined below.

Besov balls. Let $M > 0$, $s > 0$, $r \geq 1$ and $q \geq 1$. For the sake of simplicity, we say that $g(x)$ belongs to the Besov ball $B_{r,q}^s(M)$ if there exists a constant $M^* > 0$ (depending on M) such that (2.1) satisfy

$$\left(\sum_{j=\tau}^{\infty} \left(2^{j(s+1/2-1/r)} \left(\sum_{k=0}^{2^j-1} |d_{j,k}|^r \right)^{1/r} \right)^q \right)^{1/q} \leq M^*,$$

with the usual modifications if $r = \infty$ or $q = \infty$.

In this expression, s is a smoothness parameter and r and q are norm parameters. Details on Besov balls can be found in Meyer (1992) and (Härdle *et al.*, 1998, Chapter 9).

3. Theoretical results

3.1. First results

This section is devoted to the study of performance of wavelet estimators $\hat{g}_L(x)$ and $\hat{g}_H(x)$.

Theorem 3.1 determines the rates of convergence attained by $\hat{g}_L(x)$ under the \mathbb{L}_p -risk over Besov balls.

Theorem 3.1. *Let $p \geq 1$. Assume that $g \in \mathbb{L}_{\max(r,2)}([0,1]) \cap B_{r,q}^s(M)$ with $s > 1/r$, $r \geq 1$ and $q \geq 1$. Set $s_* = \min(s, s - 1/r + 1/p)$ and let $\hat{g}_L(x)$ be as in (2.5) with j_0 being the integer such that*

$$n^{1/(2s_*+4)} < 2^{j_0+1} \leq 2n^{1/(2s_*+4)}.$$

Then there exists a constant $C > 0$ such that

$$\mathbb{E} \left(\int_{[0,1]} |\hat{g}_L(x) - g(x)|^p dx \right) \leq C n^{-s_* p / (2s_* + 4)}.$$

The proof is based on statistical properties of $\hat{c}_{j,k}$ and $\hat{d}_{j,k}$ (see Lemmas 5.1 and 5.2), and technical bounds related to wavelet series and the \mathbb{L}_p norm. At this stage, let us consider the rate of convergence $n^{-s_* p / (2s_* + 4)}$ as a benchmark. This aspect will be discuss later.

Theorem 3.2 explores the rates of convergence of $\hat{g}_H(x)$ under the \mathbb{L}_p -risk over Besov balls.

Theorem 3.2. *Let $p \geq 1$ and \hat{g}_H be as in (2.6) with j_1 being the integer satisfying*

$$\left(\frac{n}{\ln(n)} \right)^{1/4} < 2^{j_1+1} \leq 2 \left(\frac{n}{\ln(n)} \right)^{1/4},$$

and λ_j being the threshold:

$$\lambda_j = K 2^{3j/2} \sqrt{2p} \sqrt{\frac{\ln(n)}{n}}, \quad (3.1)$$

with $K = \sup_{x \in [0,1]} |\psi'(x)|$.

Suppose that $g \in \mathbb{L}_{\max(r,2)}([0,1]) \cap B_{r,q}^s(M)$ with $s > 1/r$, $r \geq 1$ and $q \geq 1$. Then there exists a constant $C > 0$ such that

$$\mathbb{E} \left(\int_{[0,1]} |\hat{g}_H(x) - g(x)|^p dx \right) \leq C \varphi_n,$$

where

$$\varphi_n = \begin{cases} \left(\frac{\ln(n)}{n} \right)^{sp/(2s+4)}, & \text{for } rs > 2(p-r), \\ \left(\frac{\ln(n)}{n} \right)^{(s-1/r+1/p)p/(2s-2/r+4)}, & \text{for } rs < 2(p-r), \\ \left(\frac{\ln(n)}{n} \right)^{(s-1/r+1/p)p/(2s-2/r+4)} (\ln(n))^{(p-r/q)_+}, & \text{for } rs = 2(p-r). \end{cases}$$

The proof is based on statistical properties of $\hat{c}_{j,k}$ and $\hat{d}_{j,k}$ (see Lemmas 5.3 and 5.4), and a general result on the \mathbb{L}_p risk of the hard thresholding wavelet estimator which can be proved by combining (Kerkyacharian and Picard, 2000, Theorem 5.1) and (Chesneau, 2008, Theorem 4.2).

If we do a global comparison between the results of Theorems 3.1 and 3.2, the rates of convergence achieved by $\hat{g}_H(x)$ is better than the one achieved by $\hat{g}_L(x)$. However, in comparison to the standard density estimation problem, the obtained rates of convergence in Theorems 3.1 and 3.2 are slower. To be more specific,

- for the wavelet linear estimation, the standard (and optimal one in the minimax sense) is $n^{-s_*p/(2s_*+1)}$, against $n^{-s_*p/(2s_*+4)}$ in Theorem 3.1 (see (Härdle *et al.*, 1998, Section 10.2)).
- for the wavelet nonlinear estimation, the standard one is similar to φ_n but with "+1" instead of "+4" in the exponent (see Delyon and Juditsky (1996)).

The rest of the study proves that, under additional assumptions on $g(x)$, the obtained rates of convergence can be improved and be made equal to the standard one.

3.2. Improved results but with an additional assumption

We now introduce the following Lipschitz(1/2) assumption:

(A) there exists a constant $C_* > 0$ such that, for any $(x, y) \in [0, 1]^2$,

$$|g(x) - g(y)| \leq C_* \sqrt{|x - y|}.$$

Theorems 3.3 and 3.4 below show that, under (A), one can construct more efficient linear and hard thresholding wavelet estimators than those presented in Theorems 3.1 and 3.2.

Theorem 3.3. *Let $p \geq 1$. Suppose that (A) is satisfied and $g \in B_{r,q}^s(M)$ with $s > 1/r$, $r \geq 1$ and $q \geq 1$ such that $\min(s, s - 1/r + 1/p) > 1/2$. Set $s_* = \min(s, s - 1/r + 1/p)$ and $\hat{g}_L(x)$ be as in (2.5) with j_0 being the integer such that*

$$n^{1/(2s_*+1)} < 2^{j_0+1} \leq 2n^{1/(2s_*+1)}.$$

Then there exists a constant $C > 0$ such that

$$\mathbb{E} \left(\int_{[0,1]} |\hat{g}_L(x) - g(x)|^p dx \right) \leq C n^{-s_*p/(2s_*+1)}.$$

The proof is similar to the one of Theorem 3.1. The techniques involved in the proof make use of statistical results on $\hat{c}_{j,k}$ and $\hat{d}_{j,k}$ derived by Kerkyacharian and Picard (2004) for the regression function estimation. These statistical results justify the consideration of (A) and a restriction on j_0 , i.e., $2^{j_0} \leq \sqrt{n/\ln(n)}$.

Theorem 3.4. *Let $p \geq 1$. Suppose that (A) is satisfied. Let $\hat{g}_H(x)$ be (2.6) with j_1 being the integer satisfying*

$$\sqrt{\frac{n}{\ln(n)}} < 2^{j_1+1} \leq 2\sqrt{\frac{n}{\ln(n)}},$$

and λ_j being the threshold:

$$\lambda_j = \kappa \sqrt{\frac{\ln(n)}{n}}, \quad (3.2)$$

where κ is a large enough constant (depending, among other, on C_* and ψ). Suppose that $g \in B_{r,q}^s(M)$ with $s > 1/r$, $r \geq 1$ and $q \geq 1$. Then there exists a constant $C > 0$ such that

$$\mathbb{E} \left(\int_{[0,1]} |\hat{g}_H(x) - g(x)|^p dx \right) \leq C \Psi_n,$$

where

$$\Psi_n = \begin{cases} \left(\frac{\ln(n)}{n} \right)^{sp/(2s+1)}, & \text{for } 2rs > (p-r), \\ \left(\frac{\ln(n)}{n} \right)^{(s-1/r+1/p)p/(2s-2/r+1)}, & \text{for } (r, s, p) \in \mathcal{D}, \end{cases}$$

where $\mathcal{D} = \{(r, s, q) \in [1, \infty) \times (1/r, \infty) \times [1, \infty); 1/2 + 1/r \leq s \leq (p-r)/(2r), r \leq (p-2)/(2(s-1/r)+1)\}$.

The proof uses arguments similar to those in Theorem 3.2. Again, the main originality is the use of new statistical results on $\hat{c}_{j,k}$ and $\hat{d}_{j,k}$ derived to results proved by Kerkyacharian and Picard (2004).

Let us remark that Ψ_n correspond to the standard rate of convergence for the standard density problem via the hard thresholding wavelet method (see Delyon and Juditsky (1996)).

The next section explores the numerical performances of our wavelet estimators and compare them with the estimators developed by Jones (1992) and Soni *et al.* (2012).

4. Simulation Study

We consider performance of linear wavelets estimator $\hat{g}_L(x)$ and the hard thresholding wavelets estimator $\hat{g}_L(x)$ presented in Section 3 and compare them with a linear wavelet estimator after local linear smoothing proposed by Ramirez and Vidakovic (2010). The new smooth linear wavelet estimator has been adapted from Fan (1992, 1993). Chesneau *et al.* (2012, 2014), Shirazi *et al.* (2012, 2013) and Abbaszadeh *et al.* (2012, 2013) showed that the performances of this version of wavelet estimators perform better in different applications.

In this section we consider the mean integrated square error (MISE) of the smooth linear estimator for Beta distributions and Generalized Lambda Distributions (GLD). It is easy to see when parameters in beta distribution were chosen from $(0, 1)$ the corresponding quantile density satisfies all conditions required to prove the results. On the otherhand GLD has four different parameters, *i.e.*, $\lambda_1, \lambda_2, \lambda_3$ and λ_4 . The parameters λ_1 and λ_2 are, respectively, location and scale parameters, while λ_3 and λ_4 determine the skewness and kurtosis of the $\text{GLD}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$, see Karian and Dudewicz (2000). Freimer *et al.* (1988, section 3) noted that GLD is very rich in the variety of density and tail shapes. It contains unimodal, U-shaped, J-shaped and monotone probability density functions. These can be symmetric or asymmetric and their tails can be smooth, abrupt, or truncated, and long, medium or short depending on the choice of parameters. Karian and Dudewicz (2000, page 43) show that when $\lambda_3 > 0$ and $\lambda_4 > 0$ the support of GLD is $[\lambda_1 - 1/\lambda_2, \lambda_1 + 1/\lambda_2]$. On the other hand, it is easy to see from the closed form of the quantile function of GLD that when $\lambda_3 > 1$ and $\lambda_4 > 1$, we have $g \in \mathbb{L}_2([0, 1])$.

In this section we compare the performance of our proposed wavelet based estimators with two competitors $\hat{g}^{j1}(x)$ and $\hat{g}^S(x)$ given by (1.2) and (1.4), respectively. Integrals in wavelets coefficients estimators introduced in (2.3) and (2.4) have been approximated by Simpson's rule. Following Soni *et al.* (2012) the kernel function chosen is Triangular, $k(x) = (1 - |x|)\mathbf{1}_{\{|x| \leq 1\}}$, and the bandwidth is $h(n) = 0.15$.

Figures 1, 2 and 3 show the results of simulation for the Generalized Lambda Distributions : $\text{GLD}(0,7,7,7)$, $\text{GLD}(0.5,1,2,6)$ and the Beta distribution : Beta $(0.5, 0.5)$, respectively. The black curve is the true quantile density function, the blue line is the linear wavelet estimator, the red line is threshold wavelet estimator, the green line is smooth version of our estimator (SL), the yellow line is Jones' estimator and the magenta line is Soni-Dewan-Jain's estimator (SDJ). From these two figures we conclude that the smooth version of the proposed wavelet estimators is closer to the unknown quantile density function as compared to other four estimators we have studied.

Tables 1, 2 and 3 show mean square errors (MSEs) for different choices of fixed $x \in [0, 1]$ and different bandwidth parameters : $h_n = 0.15, 0.19, 0.25$. We calculate the MSE of these estimators by generating 500 samples from $\text{GLD}(0,7,7,7)$, $\text{GLD}(0.5,1,2,6)$ and Beta $(0.5,0.5)$, respectively. The sample size is $n = 200$.

Table 4 shows the estimation of MISE for GLDs for four different choices of parameters and Beta $(0.5, 0.5)$. The results in this Table 4 are based on 500 replications and sample size $n = 200$ and $n = 500$.

From Figures and Tables we conclude that

- (i) our wavelet based estimator perform well in estimation of unknown quantile density. The performance of smooth version of linear wavelet estimator works the best.
- (ii) while the competitors of (1.2) and (1.4) suffer from boundary effect, wavelet estimators perform well in the tails.
- (iii) in majority of the cases, the MSEs for smooth version of wavelet estimator

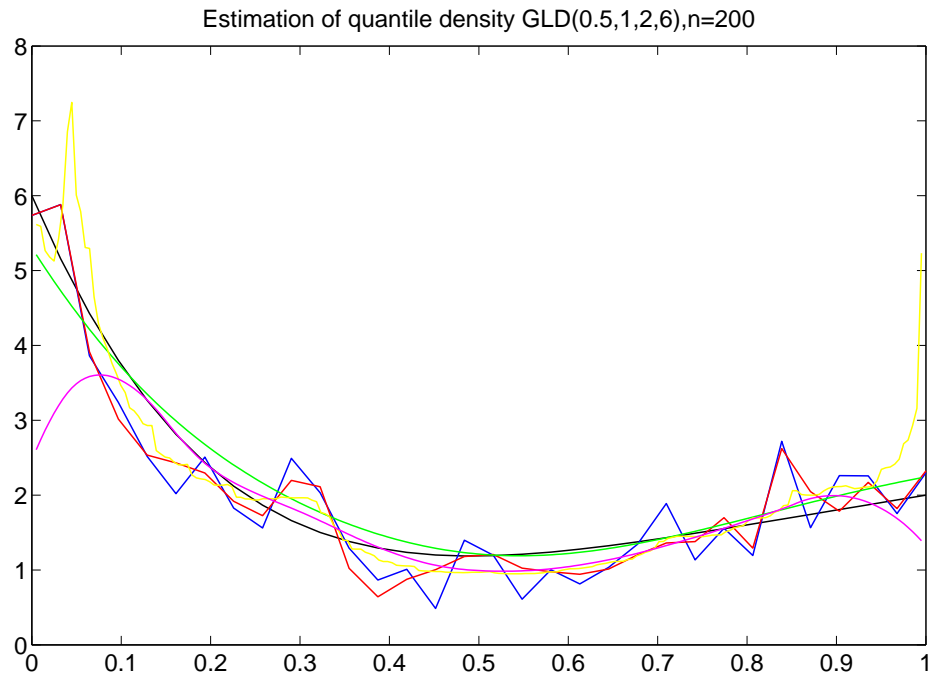


FIG 1. Estimation of quantile density. The black curve is the true GLD, the blue line is linear wavelet estimator, the red line is threshold wavelet estimator, the green line is smooth version of our estimator, the yellow line is Jones' estimator and the magenta line is Soni-Dewan-Jain's estimator.

are smaller than those of competitors.

- (iv) the MSEs and MISE decrease as the sample size increases.
- (v) the MSEs are bigger for x closed to the tails.
- (vi) the MISE of smooth version of wavelet estimators are smaller than the MISE of the competitors.

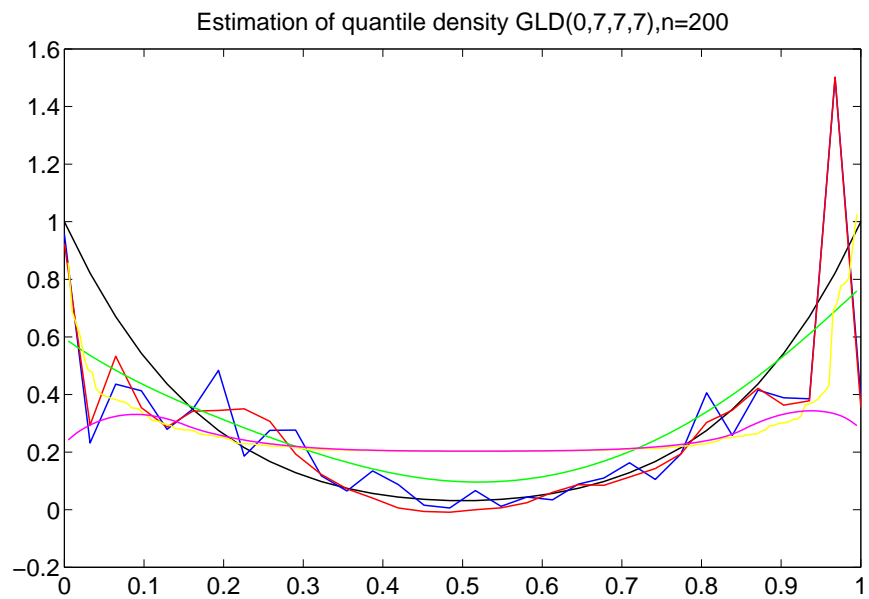


FIG 2. Estimation of quantile density. The black curve is the true GLD, the blue line is linear wavelet estimator, the red line is threshold wavelet estimator, the green line is smooth version of our estimator, the yellow line is Jones' estimator and the magenta line is Soni-Dewan-Jain's estimator.

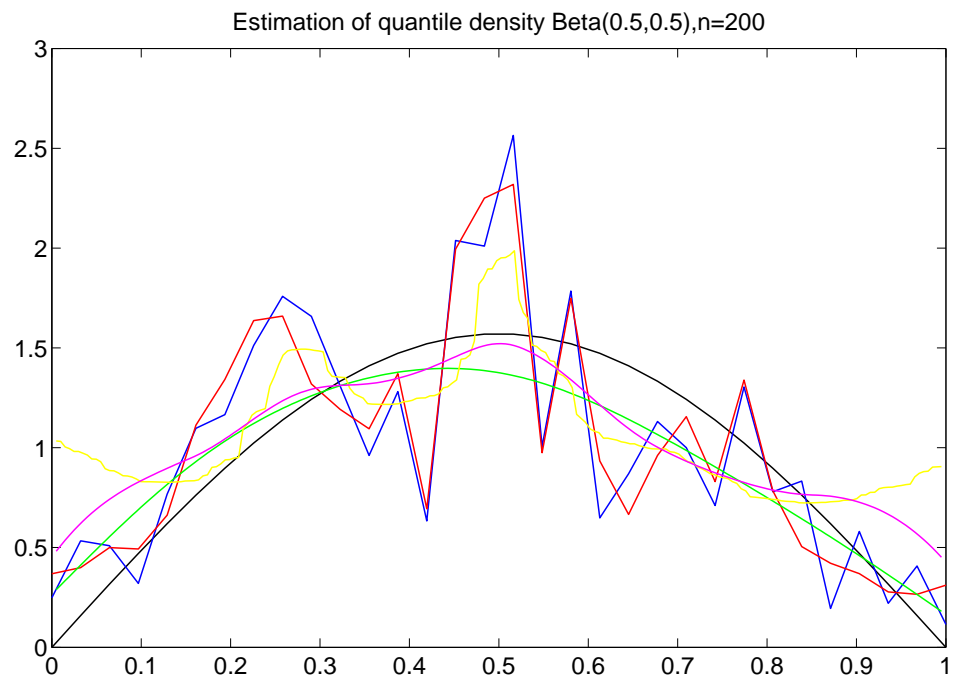


FIG 3. Estimation of quantile density. The black curve is the true Beta (0.5,0.5), the blue line is linear wavelet estimator, the red line is threshold wavelet estimator, the green line is smooth version of our estimator, the yellow line is Jones' estimator and the magenta line is Soni-Dewan-Jain's estimator.

Table 1 MSEs (Standard Deviation) based on 500 replications and $n=200$, $GLD(0,7,7,7)$

		h(n)		
x	estimate	0.15	0.19	0.25
0.01	<i>SL</i>	0.0581(0.0425)	0.0741(0.0391)	0.1084(0.0375)
	<i>Jones</i>	0.3120(0.0349)	0.1223(0.0337)	0.1648(0.0229)
	<i>SDJ</i>	0.4556(0.0339)	0.5246(0.0189)	0.5338(0.0116)
0.1045	<i>SL</i>	0.0021(0.0029)	0.0026(0.0031)	0.0048(0.0041)
	<i>Jones</i>	0.0524(0.0125)	0.0426(0.0087)	0.0261(0.0054)
	<i>SDJ</i>	0.0408(0.0099)	0.0492(0.0068)	0.0480(0.0043)
0.2040	<i>SL</i>	0.0022(0.0015)	0.0035(0.0017)	0.0045(0.0020)
	<i>Jones</i>	0.0014(0.0007)	0.0001(0.0001)	0.0030(0.0009)
	<i>SDJ</i>	0.0008(0.0006)	0.0003(0.0003)	0.0044(0.0011)
0.4030	<i>SL</i>	0.0028(0.0014)	0.0068(0.0019)	0.0151(0.0025)
	<i>Jones</i>	0.0218(0.0014)	0.0340(0.0016)	0.0587(0.0019)
	<i>SDJ</i>	0.0220(0.0015)	0.0344(0.0017)	0.0596(0.0020)
0.6020	<i>SL</i>	0.0038(0.0017)	0.0081(0.0022)	0.0165(0.0024)
	<i>Jones</i>	0.0212(0.0014)	0.0333(0.0016)	0.0578(0.0018)
	<i>SDJ</i>	0.0214(0.0015)	0.0337(0.0017)	0.0587(0.0020)
0.8010	<i>SL</i>	0.0039(0.0018)	0.0049(0.0017)	0.0053(0.0018)
	<i>Jones</i>	0.0020(0.0009)	0.0002(0.0002)	0.0021(0.0007)
	<i>SDJ</i>	0.0013(0.0008)	0.0001(0.0002)	0.0032(0.0009)
0.9005	<i>SL</i>	0.0023(0.0034)	0.0031(0.0039)	0.0055(0.0044)
	<i>Jones</i>	0.0563(0.0141)	0.0476(0.0102)	0.0304(0.0058)
	<i>SDJ</i>	0.0478(0.0108)	0.0572(0.0076)	0.0558(0.0046)
0.99	<i>SL</i>	0.782(0.0534)	0.0877(0.0464)	0.1155(0.0370)
	<i>Jones</i>	0.0324(0.0374)	0.1275(0.0376)	0.1667(0.0240)
	<i>SDJ</i>	0.4416(0.0365)	0.5134(0.0200)	0.5240(0.0121)

Table 2 MSEs (Standard Deviation) based on 500 replications and $n=200$, $GLD(0.5,1,2,6)$

		h(n)		
x	estimate	0.15	0.19	0.25
0.01	<i>SL</i>	1.5244(1.4414)	1.9204(1.4104)	2.5767(1.3743)
	<i>Jones</i>	9.1864(21.656)	10.467(22.298)	10.115(20.877)
	<i>SDJ</i>	9.4850(2.2242)	10.418(1.7664)	11.719(1.4807)
0.1045	<i>SL</i>	0.1027(0.1373)	0.1263(0.1635)	0.1595(0.1787)
	<i>Jones</i>	0.9224(1.8540)	0.6914(1.2891)	0.4624(0.9149)
	<i>SDJ</i>	0.1396(0.1615)	0.3028(0.2442)	0.7100(0.2998)
0.2040	<i>SL</i>	0.0609(0.0633)	0.0904(0.0727)	0.1278(0.0897)
	<i>Jones</i>	0.2605(0.3657)	0.2045(0.3190)	0.1622(0.2456)
	<i>SDJ</i>	0.0767(0.1019)	0.0536(0.0748)	0.0556(0.0595)
0.4030	<i>SL</i>	0.0682(0.0601)	0.1515(0.0758)	0.2929(0.0776)
	<i>Jones</i>	0.0387(0.0638)	0.0322(0.0579)	0.0253(0.0460)
	<i>SDJ</i>	0.0261(0.0405)	0.0313(0.0473)	0.0437(0.0513)
0.6020	<i>SL</i>	0.0165(0.0219)	0.0296(0.0263)	0.0580(0.0324)
	<i>Jones</i>	0.0358(0.0548)	0.0293(0.0442)	0.0222(0.0423)
	<i>SDJ</i>	0.0205(0.0270)	0.0173(0.0248)	0.0145(0.0234)
0.8010	<i>SL</i>	0.0127(0.0176)	0.0127(0.0176)	0.0134(0.0190)
	<i>Jones</i>	0.0829(0.1376)	0.0566(0.0882)	0.0358(0.0540)
	<i>SDJ</i>	0.0287(0.0361)	0.0183(0.0249)	0.0107(0.0147)
0.9005	<i>SL</i>	0.0478(0.0615)	0.0527(0.0672)	0.0553(0.0673)
	<i>Jones</i>	0.0853(1403)	0.0600(0.0858)	0.0405(0.0953)
	<i>SDJ</i>	0.0365(0.0499)	0.0431(0.0549)	0.0785(0.0634)
0.99	<i>SL</i>	0.1761(0.2068)	0.1566(0.1840)	0.1333(0.1587)
	<i>Jones</i>	2.2551(2.7919)	2.2150(2.9676)	0.4694(2.3519)
	<i>SDJ</i>	0.5055(0.2449)	0.5793(0.2201)	0.6505(0.1897)

Table 3 MSEs (Standard Deviation) based on 500 replications and $n=200$, Beta(0.5,0.5)

x	estimate	h(n)		
		0.15	0.19	0.25
0.01	<i>SL</i>	0.0477(0.0678)	0.0665(0.0816)	0.1253(0.1202)
	<i>Jones</i>	0.7728(2429)	0.9684(0.2820)	1.2762(0.3435)
	<i>SDJ</i>	0.1847(0.0532)	0.2204(0.0577)	0.2780(0.0649)
0.1045	<i>SL</i>	0.0231(0.0357)	0.0255(0.0399)	0.0414(0.0556)
	<i>Jones</i>	0.0937(0.0580)	0.1663(0.0792)	0.3074(0.1169)
	<i>SDJ</i>	0.0860(0.0619)	0.1083(0.0660)	0.1403(0.0700)
0.2040	<i>SL</i>	0.0142(0.0190)	0.0128(0.0174)	0.0125(0.0174)
	<i>Jones</i>	0.0332(0.0409)	0.0175(0.0278)	0.0114(0.0197)
	<i>SDJ</i>	0.0193(0.0252)	0.0151(0.0247)	0.0253(0.0347)
0.4030	<i>SL</i>	0.0288(0.0291)	0.0516(0.0309)	0.0912(0.0287)
	<i>Jones</i>	0.0738(0.0412)	0.0471(0.0591)	0.0421(0.0565)
	<i>SDJ</i>	0.0350(0.0446)	0.0342(0.0396)	0.0416(0.0403)
0.6020	<i>SL</i>	0.0365(0.0352)	0.0571(0.0338)	0.0999(0.0364)
	<i>Jones</i>	0.0646(0.0888)	0.0502(0.0616)	0.0478(0.0663)
	<i>SDJ</i>	0.0364(0.0457)	0.0317(0.0379)	0.0385(0.0405)
0.8010	<i>SL</i>	0.0169(0.0218)	0.0157(0.0212)	0.0151(0.0207)
	<i>Jones</i>	0.0319(0.0507)	0.0132(0.0175)	0.0113(0.0178)
	<i>SDJ</i>	0.0195(0.0297)	0.0176(0.0255)	0.0289(0.0333)
0.9005	<i>SL</i>	0.0136(0.0240)	0.0170(0.0250)	0.0210(0.0277)
	<i>Jones</i>	0.1110(0.0618)	0.1994(0.0862)	0.3351(0.1155)
	<i>SDJ</i>	0.0989(0.0658)	0.1308(0.0693)	0.1538(0.0643)
0.99	<i>SL</i>	0.0218(0.0367)	0.0386(0.0521)	0.0684(0.0628)
	<i>Jones</i>	0.7654(0.2416)	0.9902(0.2863)	0.2495(0.2813)
	<i>SDJ</i>	0.2062(0.0585)	0.2484(0.0633)	0.2954(0.0603)

TABLE 1
MISE for different estimators based on 500 replications

Estimator	GLD(0,7,7,7)		GLD(0.5,2,1.5,1.5)		GLD(0.1.5,1.5,1.5)		GLD(0.5,1,2,6)		Beta(0.5,0.5)	
	n=200	n=500	n=200	n=500	n=200	n=500	n=200	n=500	n=200	n=500
SL	0.0078	0.0067	0.0101	0.0050	0.0189	0.0093	0.1149	0.0886	0.0252	0.0167
Jones	0.0268	0.0273	0.0648	0.0521	0.1088	0.0797	0.4718	0.2798	0.1259	0.1011
SDJ	0.0512	0.0530	0.0127	0.0064	0.0256	0.0127	0.3758	0.3343	0.0604	0.0459

5. Proofs

5.1. Auxiliary results

The following lemmas show several statistical properties (moments and concentration inequalities) of the estimators $\hat{c}_{j,k}$ (2.3) and $\hat{d}_{j,k}$ (2.4). The proofs or appropriate references are given in the next sub-section.

Lemma 5.1. *Let $p \geq 1$, $\hat{c}_{j,k}$ be defined by (2.3) and $\hat{d}_{j,k}$ be defined by (2.4). Then*

- there exists a constant $C > 0$ such that

$$\mathbb{E}((\hat{c}_{j,k} - c_{j,k})^{2p}) \leq C 2^{3jp} \frac{1}{n^p}.$$

- there exists a constant $C > 0$ such that

$$\mathbb{E}((\hat{d}_{j,k} - d_{j,k})^{2p}) \leq C 2^{3jp} \frac{1}{n^p}.$$

Lemma 5.2. *Let $p \geq 1$, $\hat{d}_{j,k}$ be defined by (2.4) and λ_j be defined by (3.1). Then*

$$\mathbb{P}\left(|\hat{d}_{j,k} - d_{j,k}| \geq \lambda_j\right) \leq 2 \left(\frac{\ln(n)}{n}\right)^p.$$

Lemma 5.3. *Suppose that **(A)** is satisfied. Let $p \geq 1$, j such that $2^j \leq \sqrt{n/\ln(n)}$, $\hat{c}_{j,k}$ be defined by (2.3) and $\hat{d}_{j,k}$ be defined by (2.4). Then*

- there exists a constant $C > 0$ such that

$$\mathbb{E}((\hat{c}_{j,k} - c_{j,k})^{2p}) \leq C \frac{1}{n^p}.$$

- there exists a constant $C > 0$ such that

$$\mathbb{E}((\hat{d}_{j,k} - d_{j,k})^{2p}) \leq C \frac{1}{n^p}.$$

Lemma 5.4. *Suppose that **(A)** is satisfied. Let $p \geq 1$, j such that $2^j \leq \sqrt{n/\ln(n)}$, $\hat{d}_{j,k}$ be defined by (2.4) and λ_j be defined by (3.2). Then*

$$\mathbb{P}\left(|\hat{d}_{j,k} - d_{j,k}| \geq \lambda_j\right) \leq 2 \left(\frac{\ln(n)}{n}\right)^p.$$

In the rest of the paper, we use C to denote positive constants whose value may change from line to line.

5.2. Proofs of the auxiliary results

Proof of Lemma 5.1. Note that $(\phi_{j,k}(x))' = 2^{3j/2}\phi'(2^jx - k)$ and $K = \sup_{x \in [0,1]} |\phi'(x)| < \infty$. The mean value theorem gives, for any $(x, y) \in [0, 1]^2$,

$$|\phi_{j,k}(x) - \phi_{j,k}(y)| \leq 2^{3j/2}K|x - y|.$$

Therefore

$$\begin{aligned} |\hat{c}_{j,k} - c_{j,k}| &= \left| \int_{[0,1]} \left(\phi_{j,k}(\hat{F}(x)) - \phi_{j,k}(F(x)) \right) dx \right| \\ &\leq \int_{[0,1]} |\phi_{j,k}(\hat{F}(x)) - \phi_{j,k}(F(x))| dx \\ &\leq C2^{3j/2} \int_{[0,1]} |\hat{F}(x) - F(x)| dx. \end{aligned} \quad (5.1)$$

Using the Hölder inequality and the Fubini theorem, we have

$$\begin{aligned} \mathbb{E}((\hat{c}_{j,k} - c_{j,k})^{2p}) &\leq C2^{3jp} \mathbb{E} \left(\left(\int_{[0,1]} |\hat{F}(x) - F(x)| dx \right)^{2p} \right) \\ &\leq C2^{3jp} \left(\int_{[0,1]} \mathbb{E} \left((\hat{F}(x) - F(x))^{2p} \right) dx \right). \end{aligned}$$

The Rosenthal inequality (see Rosenthal (1970)) yields

$$\begin{aligned} &\mathbb{E} \left((\hat{F}(x) - F(x))^{2p} \right) \\ &\leq C \left(\frac{(1 - F(x))^{2p} F(x) + (F(x))^{2p} (1 - F(x))}{n^{2p-1}} + \frac{(F(x)(1 - F(x)))^p}{n^p} \right) \\ &\leq C \frac{1}{n^p}. \end{aligned}$$

Hence

$$\mathbb{E}((\hat{c}_{j,k} - c_{j,k})^{2p}) \leq C2^{3jp} \frac{1}{n^p}.$$

Working with ψ instead of ϕ , we show that

$$\mathbb{E}((\hat{d}_{j,k} - d_{j,k})^{2p}) \leq C2^{3jp} \frac{1}{n^p}.$$

The proof of Lemma 5.1 is complete. \square

Proof of Lemma 5.2. By (5.1) with ψ instead of ϕ , we obtain

$$\begin{aligned} |\hat{d}_{j,k} - d_{j,k}| &\leq K2^{3j/2} \int_{[0,1]} |\hat{F}(x) - F(x)| dx \\ &\leq K2^{3j/2} \sup_{x \in [0,1]} |\hat{F}(x) - F(x)|, \end{aligned}$$

with $K = \sup_{x \in [0,1]} |\psi'(x)|$. It follows from the Dvoretzky-Kiefer-Wolfowitz inequality (see Dvoretzky *et al.* (1959)) and the definition of λ_j (3.1) that

$$\begin{aligned} &\mathbb{P} \left(|\hat{d}_{j,k} - d_{j,k}| \geq \frac{\lambda_j}{2} \right) \\ &\leq \mathbb{P} \left(K2^{3j/2} \sup_{x \in [0,1]} |\hat{F}(x) - F(x)| \geq \frac{1}{2} K2^{3j/2} \sqrt{2p} \sqrt{\frac{\ln(n)}{n}} \right) \\ &\leq \mathbb{P} \left(\sup_{x \in [0,1]} |\hat{F}(x) - F(x)| \geq \sqrt{\frac{p}{2}} \sqrt{\frac{\ln(n)}{n}} \right) \\ &\leq 2 \exp \left(-2n \left(\sqrt{\frac{p}{2}} \sqrt{\frac{\ln(n)}{n}} \right)^2 \right) = 2 \frac{1}{n^p} \leq 2 \left(\frac{\ln(n)}{n} \right)^p. \end{aligned}$$

This ends the proof of Lemma 5.2. \square

Proof of Lemmas 5.3 and 5.4. By the change of variable $x = Q(y)$, let us observe that

$$\begin{aligned} \hat{d}_{j,k} - d_{j,k} &= \int_{[0,1]} \left(\psi_{j,k}(\hat{F}(x)) - \psi_{j,k}(F(x)) \right) dx \\ &= \int_{[0,1]} \left(\psi_{j,k}(\hat{U}(x)) - \psi_{j,k}(x) \right) g(x) dx, \end{aligned}$$

with

$$\hat{U}(x) = \frac{1}{n} \sum_{i=1}^n \mathbf{1}_{\{U_i \leq x\}}, \quad U_i = F(X_i).$$

Then the proofs of Lemmas 5.3 and 5.4 follow from the technical part of (Kerkycharian and Picard, 2004, Subsection 9.2.2. pages 1093 - 1098) with "g" instead of "f(G⁻¹)". Let us mention that for the validity of results we need to suppose **(A)** and a restriction on j considered in our study, i.e., $2^j \leq \sqrt{n/\ln(n)}$. \square

5.3. Proofs of the main results

Proof of Theorem 3.1. We expand g on the wavelet basis \mathcal{W} as

$$g(x) = \sum_{k=0}^{2^{j_0}-1} c_{j_0,k} \phi_{j_0,k}(x) + \sum_{j=j_0}^{\infty} \sum_{k=0}^{2^j-1} d_{j,k} \psi_{j,k}(x),$$

where $c_{j_0,k} = \int_{[0,1]} g(x) \phi_{j_0,k}(x) dx$ and $d_{j,k} = \int_{[0,1]} g(x) \psi_{j,k}(x) dx$. Hence

$$\hat{g}_L(x) - g(x) = \sum_{k=0}^{2^{j_0}-1} (\hat{c}_{j_0,k} - c_{j_0,k}) \phi_{j_0,k}(x) - \sum_{j=j_0}^{\infty} \sum_{k=0}^{2^j-1} d_{j,k} \psi_{j,k}(x).$$

Using the inequality: $|x + y|^p \leq 2^{p-1}(|x|^p + |y|^p)$, $(x, y) \in \mathbb{R}^2$, we obtain

$$\mathbb{E} \left(\int_{[0,1]} |\hat{g}_L(x) - g(x)|^p dx \right) \leq 2^{p-1}(F + G), \quad (5.2)$$

where

$$F = \mathbb{E} \left(\int_{[0,1]} \left| \sum_{k=0}^{2^{j_0}-1} (\hat{c}_{j_0,k} - c_{j_0,k}) \phi_{j_0,k}(x) \right|^p dx \right)$$

and

$$G = \int_{[0,1]} \left| \sum_{j=j_0}^{\infty} \sum_{k=0}^{2^j-1} d_{j,k} \psi_{j,k}(x) \right|^p dx.$$

Using a \mathbb{L}_p norm result on wavelet series (see (Härdle *et al.*, 1998, Proposition 8.3)), Lemma 5.1 and the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} F &\leq C 2^{j_0(p/2-1)} \sum_{k=0}^{2^{j_0}-1} \mathbb{E} (|\hat{c}_{j_0,k} - c_{j_0,k}|^p) \\ &\leq C 2^{j_0(p/2-1)} \sum_{k=0}^{2^{j_0}-1} (\mathbb{E} ((\hat{c}_{j_0,k} - c_{j_0,k})^{2p}))^{1/2} \\ &\leq C 2^{j_0(p/2-1)} 2^{j_0} 2^{3j_0 p/2} \frac{1}{n^{p/2}} = C \left(\frac{2^{4j_0}}{n} \right)^{p/2}. \end{aligned} \quad (5.3)$$

On the other hand, using $g \in B_{r,q}^s(M)$ and proceeding as in (Donoho *et al.*, 1996, eq (24)), we have immediately

$$G \leq C 2^{-j_0 s_* p}. \quad (5.4)$$

It follows from (5.2), (5.3), (5.4) and the definition of j_0 that

$$\mathbb{E} \left(\int_{[0,1]} |\hat{g}_L(x) - g(x)|^p dx \right) \leq C \left(\left(\frac{2^{4j_0}}{n} \right)^{p/2} + 2^{-j_0 s_* p} \right) \leq C n^{-s_* p / (2s_* + 4)}.$$

This ends the proof of Theorem 3.1. □

Proof of Theorem 3.2. Theorem 3.2 is a consequence of Theorem 5.1 below with $\nu = 3/2$ and Lemmas 5.1 and 5.2 above.

Theorem 5.1. *Let $h \in \mathbb{L}_2([0, 1])$ be an unknown function to be estimated from n observations and consider its wavelet decomposition. Let $\hat{c}_{j,k}$ and $\hat{d}_{j,k}$ be estimators of these wavelet coefficients $c_{j,k}$ and $d_{j,k}$ respectively. We suppose that there exist three constants $\nu > 0$, $C > 0$ and $\kappa > 0$ such that the following inequalities hold:*

Moments inequalities: for any $j \geq \tau$ such that $2^j \leq n$ and $k \in \{0, \dots, 2^j - 1\}$,

$$\mathbb{E} \left((\hat{c}_{j,k} - c_{j,k})^{2p} \right) \leq C 2^{2\nu j p} \left(\frac{\ln(n)}{n} \right)^p$$

and

$$\mathbb{E} \left((\hat{d}_{j,k} - d_{j,k})^{2p} \right) \leq C 2^{2\nu j p} \left(\frac{\ln(n)}{n} \right)^p.$$

Concentration inequality: for any $j \geq \tau$ such that $2^j \leq n/\ln(n)$ and $k \in \{0, \dots, 2^j - 1\}$,

$$\mathbb{P} \left(|\hat{d}_{j,k} - d_{j,k}| \geq \frac{\kappa}{2} 2^{\nu j} \sqrt{\frac{\ln(n)}{n}} \right) \leq C \left(\frac{\ln(n)}{n} \right)^p.$$

Let us define the hard thresholding wavelet estimator of h by

$$\hat{h}(x) = \sum_{k=0}^{2^\tau-1} \hat{c}_{\tau,k} \phi_{\tau,k}(x) + \sum_{j=\tau}^{j_1} \sum_{k=0}^{2^j-1} \hat{d}_{j,k} \mathbf{1}_{\left\{ |\hat{d}_{j,k}| \geq \kappa 2^{\nu j} \sqrt{\frac{\ln(n)}{n}} \right\}} \psi_{j,k}(x), \quad x \in [0, 1],$$

where j_1 is the integer satisfying

$$\left(\frac{n}{\ln(n)} \right)^{1/(2\nu+1)} < 2^{j_1+1} \leq 2 \left(\frac{n}{\ln(n)} \right)^{1/(2\nu+1)}.$$

Suppose that $h \in B_{r,q}^s(M)$ with $s > 0$, $r \geq 1$ and $q \geq 1$. Then there exists a constant $C > 0$ such that

$$\mathbb{E} \left(\int_{[0,1]} |\hat{h}(x) - h(x)|^p dx \right) \leq C \Theta_{n,\nu},$$

where

$$\Theta_{n,\nu} = \begin{cases} \left(\frac{\ln(n)}{n} \right)^{sp/(2s+2\nu+1)}, & \text{for } rs > (\nu + 1/2)(p - r), \\ \left(\frac{\ln(n)}{n} \right)^{(s-1/r+1/p)p/(2s-2/r+2\nu+1)}, & \text{for } rs < (\nu + 1/2)(p - r), \\ \left(\frac{\ln(n)}{n} \right)^{(s-1/r+1/p)p/(2s-2/r+2\nu+1)} (\ln(n))^{(p-r/q)_+}, & \text{for } rs = (\nu + 1/2)(p - r). \end{cases}$$

The general of form of Theorem 5.1 can be proved using arguments similar to (Kerkyacharian and Picard, 2000, Theorem 5.1) for a bound of the \mathbb{L}_p -risk and (Chesneau, 2008, Theorem 4.2) for the determination of the rates of convergence. \square

Proofs of Theorem 3.3. The proof follows step by step the one of Theorem 3.1 with the use of Lemma 5.3 instead of Lemma 5.1. Thanks to the new definition of j_0 , we finally obtain

$$\mathbb{E} \left(\int_{[0,1]} |\hat{g}_L(x) - g(x)|^p dx \right) \leq C \left(\left(\frac{2^{j_0}}{n} \right)^{p/2} + 2^{-j_0 s^* p} \right) \leq C n^{-s^* p / (2s^* + 1)}.$$

\square

Proofs of Theorem 3.4. The proof follows the one of Theorem 3.2 with the used of Lemmas 5.3 and 5.4 instead of Lemmas 5.1 and 5.2, and an adjustment on the rates of convergence as in (Kerkyacharian and Picard, 2004, Proposition 2). This minor modification is due to our definition of j_1 . \square

6. Conclusions

We have proposed wavelet based estimators for the quantile density function enjoying nice theoretical and practical properties. For the choice of parameters considered, there is significant improvement in the tails - especially for very small and very large values of the quantiles. We have used empirical estimator of the distribution function for estimating the wavelet coefficients. One could possibly use other smooth estimators of the distribution function, say the kernel type estimator, and define another class of estimators.

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