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High-order integrator for sampling the invariant distribution of a class of parabolic SPDEs with additive space-time noise.

Charles-Edouard Bréhier¹ and Gilles Vilmart²

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Abstract

We introduce an integrator to sample with high-order of accuracy the invariant distribution for a class of semilinear SPDEs driven by a additive space-time noise. Using a postprocessor, the scheme is a modification with negligible overhead of the standard linearized implicit Euler-Maruyama method. It has an improved order of convergence $r + 1$, where r is the order of convergence of the original method. An analysis is provided in finite dimension for nonlinear SDE problems, and in infinite dimension in a linear case. Numerical experiments, including the stochastic nonlinear heat equation with space-time noise confirm the theoretical findings and illustrate the efficiency of the approach.

Keywords: stochastic partial differential equations, postprocessor, invariant measure, ergodicity, space-time white noise.

AMS subject classification (2010): 60H15, 60H35, 37M25

1 Introduction

We introduce an efficient integrator for the sampling of the invariant probability distribution of a class of semilinear parabolic SPDEs with additive noise written as an abstract stochastic evolution equation (in the sense of [10])

$$du(t) = Au(t) + F(u(t)) + dW^Q(t), \quad u(0) = u_0, \quad (1)$$

with solution $u(t)$ taking values in a separable infinite dimensional Hilbert space \mathcal{H} , with initial condition u_0 (assumed deterministic for simplicity). We assume that $-A$ is a positive unbounded self-adjoint linear operator with associated non-decreasing sequence of positive eigenvalues $0 < \lambda_1 < \lambda_2 < \dots$, and complete orthonormal family of eigenvectors e_1, e_2, \dots . The coefficient $F : \mathcal{H} \rightarrow \mathcal{H}$ is a Lipschitz continuous nonlinearity, which is assumed to derive from a smooth gradient $V : \mathcal{H} \rightarrow \mathbb{R}$ with $F = -DV$. We assume that $(W^Q(t))_{t \in \mathbb{R}^+}$ is a Q -Wiener process on \mathcal{H} , where the covariance operator $Q : \mathcal{H} \rightarrow \mathcal{H}$ is a bounded, non-negative self-adjoint linear operator, such that $Qe_i = q_i e_i$ for some bounded sequence of real numbers $(q_i)_{i \in \mathbb{N}^*}$.

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This abstract setting includes the well-studied stochastic semilinear heat equation where $A = \Delta = \sum_{j=1}^d \frac{\partial^2}{\partial x_j^2}$ in $\mathcal{H} = L^2(\mathcal{D})$ for some open smooth bounded domain $\mathcal{D} \subset \mathbb{R}^d$,

$$\frac{\partial u}{\partial t}(x, t) = \Delta u(x, t) + f(u(x, t)) + \frac{\partial W^Q}{\partial t}(x, t), \quad (2)$$

where $f : \mathbb{R} \rightarrow \mathbb{R}$ is a smooth Lipschitz function, with homogeneous Dirichlet boundary conditions on $\partial\mathcal{D}$.

We assume the following trace condition:

$$\bar{s} = \sup \left\{ s \in (0, 1) ; \text{Trace} \left((-A)^{-1+s} Q \right) < +\infty \right\} > 0. \quad (3)$$

Then there exists a unique mild solution of (1) on \mathbb{R}^+ (see [10]), *i.e.* a \mathcal{H} -valued continuous stochastic process $(u(t))_{t \in \mathbb{R}^+}$ which satisfies

$$u(t) = e^{tA} u^0 + \int_0^t e^{(t-s)A} F(u(s)) ds + \sigma \int_0^t e^{(t-s)A} dW^Q(s). \quad (4)$$

For instance, for the heat equation in dimension $d = 1$, one can consider space-time white noise (*i.e.* with the identity covariance operator $Q = I$, which yields $\bar{s} = 1/2$ in (3)), while for $d > 1$ a nontrivial covariance operator Q is required, yielding noise which is white in time and colored in space. Notice also that (3) is automatically satisfied with $\bar{s} = 1$ for a trace-class noise with $\text{Trace } Q < \infty$.

To study the long-time behavior of the process u , we make the additional assumption that F in (1) is Lipschitz continuous with constant $L > 0$ such that $L < \lambda_1 = \min_{p \in \mathbb{N}^*} \lambda_p$. Then (see e.g. Section 8.6 in [9]) (1) admits a unique invariant distribution μ_∞ . This means that for all (smooth and Lipschitz) test function $\phi : \mathcal{H} \rightarrow \mathbb{R}$ and for μ_∞ -almost all initial conditions u_0 ,

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \phi(u(t)) dt = \int_{\mathcal{H}} \phi(y) d\mu_\infty(y).$$

Moreover (see e.g. Section 6.3 in [9]) $u(t)$ converges in law to μ_∞ exponentially fast in the following sense: for all $t > 0$, and any test function ϕ ,

$$\left| \mathbb{E} \phi(u(t)) - \int_{\mathcal{H}} \phi(y) d\mu_\infty(y) \right| \leq C(\phi, u_0) e^{-(\lambda_1 - L)t} \quad (5)$$

where $C(\phi, u_0)$ is independent of t .

Using the assumption $F = -DV$, an explicit expression of the invariant distribution can be obtained. For instance, for the heat equation with space-time white noise ($A = \Delta$, $d = 1$, $Q = I$, $\sigma = \sqrt{2}$), we have

$$\mu_\infty(du) = \frac{1}{Z} \exp(-V(u)) \pi(du) \quad (6)$$

where Z is a normalization constant, and $\pi(du) = \mathcal{N}(0, (-A)^{-1})$ is the centered Gaussian distribution on \mathcal{H} with covariance operator $(-A)^{-1}$. However, in general, it is infeasible to compute either the (infinite dimensional) integral $\int_{\mathcal{H}} \phi(y) d\mu_\infty(y)$ – even using expressions of the invariant distribution μ_∞ such as (6) – or $\mathbb{E} \phi(u(t))$ for (large) $t \in \mathbb{R}^+$. Therefore in practice one needs to rely on a discretization of the evolution equation.

We say that an ergodic numerical scheme $\{u_0, u_1, u_2, \dots\}$ with stepsize $h > 0$ has order r for the approximation of the invariant distribution if for all h small enough and all (smooth) test function ϕ

$$|e(\phi, h)| \leq C(\phi)h^r \quad \text{with} \quad e(\phi, h) = \lim_{N \rightarrow \infty} \frac{1}{N+1} \sum_{n=0}^N \mathbb{E}\phi(u_n) - \int_{\mathcal{H}} \phi(y) d\mu_{\infty}(y), \quad (7)$$

where $C(\phi)$ is independent of the (small enough) stepsize h and of the initial condition u_0 .

The order of accuracy r for the invariant measure in (7) may be different from the standard weak order q of the method on a fixed time interval on length $T \in (0, +\infty)$. Recall that a method has weak order of accuracy q if for any h small enough and for any $t_n = nh \leq T$,

$$|\mathbb{E}(\phi(u_n)) - \mathbb{E}(\phi(u(t_n)))| \leq C(u_0, \phi)h^q. \quad (8)$$

We first consider the simplest numerical scheme, which is referred to as the linearized implicit Euler method in this article. It is defined by $v_{n+1} = v_n + hAv_{n+1} + hF(v_n) + \sigma\sqrt{h}\xi_n^Q$, equivalently

$$v_{n+1} = J_1 \left(v_n + hF(v_n) + \sigma\sqrt{h}\xi_n^Q \right) \quad (9)$$

where $v_0 = u(0) = u^0$, $J_1 = (I - hA)^{-1}$ and $\xi_n^Q = h^{-1/2}(W^Q((n+1)h) - W^Q(nh))$.

The weak order of accuracy (8) of this scheme has been analyzed in [13] for the heat equation in the case $F = 0$ and in [11] in the semilinear case; see also [38] for the case of colored noise in dimension $d > 1$ and [22] where different techniques are used. Concerning the approximation of the invariant distribution, in [4], it is proved that for $d = 1$ and $Q = I$, one can take $r = q = 1/2 - \varepsilon$, for all $\varepsilon > 0$, with $\bar{s} = 1/2$. The proof is based on the analysis of the weak approximation using the tools of [11] (expansion of the error using the backward Kolmogorov equation, Malliavin calculus), with a proof that constants $C(u_0, \phi)$ can be chosen independent of the final time T , in the spirit of [34]. In addition, one has similarly to (5), the exponential convergence property

$$\left| \mathbb{E}(\phi(u_n)) - \int_{\mathcal{H}} \phi(y) d\mu_{\infty}(y) \right| \leq K(u_0, \phi)e^{-\lambda t_n} + C(\phi)h^r \quad (10)$$

where $C(\phi), K(u_0, \phi)$ are independent of n and h , and λ is independent of ϕ, h, n . We also mention another natural integrator for (1), which is the (stochastic) trapezoidal method (also known as the Crank-Nicolson method),

$$u_{n+1} = u_n + \frac{h}{2}A(u_n + u_{n+1}) + hF(u_n) + \sigma\sqrt{h}\xi_n^Q. \quad (11)$$

However, in contrast to (9), the scheme (11) is not L -stable, a desirable property for severely stiff problems, already in the deterministic literature [14]. This makes exponential convergence estimates of the form (10) not true in general for the scheme (11).

We highlight that $p \leq r$ in general, and in many situations $p < r$, already for SDEs (*i.e.* in finite dimension in the terminology of this paper), see [25, 23, 1, 37] in the context of Brownian dynamics and [3, 2] in the context of Langevin dynamics. Inspired by these recent advances, the popular technique of processing for differential equations [7] was recently extended to the stochastic context in [37]. This idea is to enhance the accuracy of a numerical method by applying a suitable change of variables. This is a crucial ingredient

to derive the main result of this paper, which is a modification with negligible overhead of the linearized implicit Euler method (9) for the class of parabolic SPDEs (1) which permits to increase by one the order of accuracy for the invariant distribution. Notice that high order integrators in the strong sense (approximation of the trajectories instead of the distribution) for parabolic problems of the form (1) are proposed in [21, 19, 20]; however these schemes belong to the class of exponential integrators, while the method proposed in this paper avoids the computation of matrix exponentials.

Finally, since the space \mathcal{H} is infinite dimensional, a space discretization scheme is required in practice, *e.g.* finite differences or finite elements – see Section 4. In this article, we only focus on the time-discretization issue. We mention [26] where a postprocessing technique is applied to improve the spatial discretization in the strong sense.

New method The postprocessed method introduced in this paper is given by two sequences $(u_n)_{n \geq 0}$ and $(\bar{u}_n)_{n \geq 0}$ in \mathcal{H} defined by

$$u_{n+1} = J_1 \left(u_n + hF \left(u_n + \frac{1}{2} \sigma \sqrt{h} J_2 \xi_n^Q \right) + \frac{\sqrt{2}-1}{2} J_2 \xi_n^Q \right) + \frac{3-\sqrt{2}}{2} \sigma \sqrt{h} J_2 \xi_n^Q, \quad (12)$$

$$\bar{u}_n = u_n + \frac{1}{2} J_3 \sigma \sqrt{h} \xi_n^Q, \quad (13)$$

where we introduce additional operators J_2, J_3 as follows (J_3 can be obtained by a Cholesky decomposition):

$$J_2 = \left(I - \frac{3-\sqrt{2}}{2} hA \right)^{-1}, \quad J_3 J_3^T = \left(I - \frac{h}{2} A \right)^{-1}.$$

Notice that the new scheme (12)-(13) has a negligible overcost compared to (9). First, the postprocessor (13) needs not to be computed at each time step, but it can be computed only once at the end of each numerical trajectory. Second, computing $J_2 \xi_n^Q$ at each timestep is the only additional calculation.

Outline and main results This paper is organized as follows. In Section 2, we explain the derivation of the scheme (12)-(13) and prove (Theorem 2.7) that \bar{u}_n in (13) has order $r = 2$ in (7) and (10) in the case of (nonstiff) finite dimensional SDEs. We also show (Proposition 2.3) the L -stability of the scheme and its exactness for the invariant measure in some linear cases. In Section 3, we present our main results for SPDEs (1). First, we prove (Theorem 3.7) that \bar{u}_n in (13) yields the order of convergence $r = s + 1$ for the invariant measure in (7) and (10) for all $s < \bar{s}$, where \bar{s} is defined in (3) and F is a bounded linear operator. This is an improvement compared to the standard linearized implicit Euler scheme (9) with order $r = s$ for all $s < \bar{s}$. Second, we present (Proposition 3.9) a feature of the scheme which recovers the same spatial regularity as the exact solution, in contrast to method (9) which is too smooth in space. Finally, Section 4 is dedicated to numerical experiments which confirm the theoretical findings and illustrate the efficiency of the new method.

2 New high-order integrator: derivation and analysis in finite dimension

We introduce the new scheme for the more general case of a nonlinear system of SDEs (*i.e.* in finite dimension $N \in \mathbb{N}^*$), of the form

$$dX(t) = (f_1(X(t)) + f_2(X(t)))dt + \sigma dW^Q(t), \quad X(0) = X_0, \quad (14)$$

with solution $X(t)$ in \mathbb{R}^N . The nonlinearities $f_1, f_2 : \mathbb{R}^N \rightarrow \mathbb{R}^N$ are smooth and Lipschitz functions such that $f_1(x) + f_2(x) = f_0(x) = -\nabla V_0(x)$ for some $V_0 : \mathbb{R}^N \rightarrow \mathbb{R}$, where f_1 is a term to be treated implicitly and f_2 is a term to be treated explicitly. The initial condition X_0 is assumed deterministic for simplicity. We also define the Q -Wiener process $W^Q(t) = Q^{1/2}W(t)$ where Q now denotes a $N \times N$ symmetric positive definite matrix and $W(t)$ is a standard N -dimensional Wiener process.

We introduce the following new high order scheme for sampling with high order two the invariant measure of (14),

$$\begin{aligned} X_{n+1} &= X_n + hf_1 \left(X_{n+1} + \frac{-2 + \sqrt{5}}{2} J_{n,2} \sigma \sqrt{h} \xi_n^Q \right) + hf_2 \left(X_n + \frac{1}{2} J_{n,2} \sigma \sqrt{h} \xi_n^Q \right) \\ &+ \left(\frac{1 - \sqrt{2} + \sqrt{5}}{2} J_{n,1}^{-1} + \frac{1 + \sqrt{2} - \sqrt{5}}{2} \right) J_{n,2} \sigma \sqrt{h} \xi_n^Q, \\ \bar{X}_n &= X_n + \frac{1}{2} J_{n,3} \sigma \sqrt{h} \xi_n^Q, \end{aligned} \quad (15)$$

where $J_{n,1}, J_{n,2}, J_{n,3}$ are defined as

$$J_{n,1} = (I - hf_1'(X_n))^{-1}, \quad J_{n,2} = (I - \frac{3 - \sqrt{2}}{2} hf_1'(X_n))^{-1}, \quad J_{n,3} J_{n,3}^T = (I - \frac{h}{2} f_1'(X_n))^{-1}.$$

We emphasize that the matrix inverses $J_{n,1}, J_{n,2}, J_{n,3}$ are used only in the notations to define the scheme but should not be computed in practice. Indeed, in practical implementations, a LU decomposition should be used in place of computing matrix inverses. Moreover, in the semilinear case (16), this decomposition needs to be done only once and may be used for all further iterations.

Remark 2.1. *After a spatial discretization with finite differences or finite elements of the SPDE (1), in general one arrives to a system of stiff SDEs in \mathbb{R}^N with large dimension N of the form (14),*

$$dX(t) = AX(t)dt + f(X(t))dt + \sigma dW^Q(t). \quad (16)$$

where assume that $f(x) = -\nabla V(x)$ for some $V : \mathbb{R}^N \rightarrow \mathbb{R}$. It corresponds to the special case $f_1(x) = Ax$ where $-A$ now denotes a $N \times N$ symmetric positive definite matrix. In this case, the above scheme (15) simplifies to the integrator (12). Moreover, in the special case $f_1 = 0$, the method (15) reduces to

$$X_{n+1} = X_n + hf \left(X_n + \frac{1}{2} \sigma \sqrt{h} \xi_n \right) + \sigma \sqrt{h} \xi_n, \quad \bar{X}_n = X_n + \frac{1}{2} \sigma \sqrt{h} \xi_n, \quad (17)$$

a method which was first proposed in [23] and analyzed in [24], although it was constructed in another manner using a non-Markovian formulation.

For simplicity, we assume for the remaining of this section on the finite dimensional case that $Q = I$ is the identity matrix, without losing generality, applying the appropriate change of variable in (fixed) finite dimension.

2.1 Stability analysis

For the study of the stability of stochastic integrators applied to stiff SDEs with additive noise of the form (14), a widely used test problem is the following scalar SDE problem (Ornstein-Uhlenbeck process)

$$dX = -\lambda X dt + \sigma dW(t), \quad (18)$$

where $\lambda, \sigma > 0$ are fixed constants. Notice that other test equations are used in the literature in the case of multiplicative noise, see [32, 18, 33, 6, 36, 31, 5]. Consider a one step method of the form

$$X_{n+1} = \mathcal{A}(z)X_n + \mathcal{B}(z)\sqrt{h}\sigma\xi_n, \quad z = -\lambda h, \quad (19)$$

where h is the stepsize, $\mathcal{A}(z), \mathcal{B}(z)$ are analytic functions, $\xi_n \sim \mathcal{N}(0, 1)$ are independent Gaussian random variables. The SDE (18) is ergodic with unique invariant measure a Gaussian with mean zero and variance $\frac{\sigma^2}{2\lambda}$. Indeed, for any initial condition $X_0 = x$, the exact solution is a Gaussian random variable, with

$$\lim_{t \rightarrow \infty} \mathbb{E}(X(t)) = 0, \quad \lim_{t \rightarrow \infty} \mathbb{E}(|X(t)|^2) = \frac{\sigma^2}{2\lambda}.$$

The second-order moment $\mathbb{E}(|X_n|^2)$ remains bounded as $n \rightarrow \infty$ if $|\mathcal{A}(z)| < 1$, which corresponds to the mean-square stability condition. The condition $|\mathcal{A}(z)| \leq 1$ for all z with negative real part is called A -stability in the deterministic literature [14]. This is a desirable property of numerical integrators for stiff problems, because it permits to avoid a severe timestep size restriction. For $|\mathcal{A}(z)| < 1$ (stability condition), we obtain

$$\lim_{n \rightarrow \infty} \mathbb{E}(|X_n|^2) = \frac{\sigma^2}{2\lambda} \mathcal{R}(z), \quad \text{where } \mathcal{R}(z) = \frac{-2z\mathcal{B}(z)^2}{1 - \mathcal{A}(z)^2}. \quad (20)$$

We see that the method is exact (for the approximation of the invariant distribution) if and only if $\mathcal{R}(z) = 1$ for all z ; this is the case for instance for the trapezoidal method (11), which is such that $\mathcal{A}(z) = \frac{1+z/2}{1-z/2}$, $\mathcal{B}(z) = \frac{1}{1-z/2}$. However, in addition to A -stability, a desirable property of Runge-Kutta methods for very stiff problems is L -stability [14], namely $\mathcal{A}(\infty) = 0$; this is not satisfied by the trapezoidal method, for which $\mathcal{A}(\infty) = -1$. The following proposition states that for Runge-Kutta type methods, where $\mathcal{A}(z), \mathcal{B}(z)$ are rational functions, L -stability (*i.e.* $\mathcal{A}(\infty) = 0$) is incompatible with the exactness for the invariant distribution, *i.e.* $\mathcal{R}(z) \equiv 1$.

Proposition 2.2. *Consider a method of the form (19) where $\mathcal{A}(z), \mathcal{B}(z)$ are rational functions. If the method samples exactly the invariant distribution of (18) (*i.e.* $\mathcal{R}(z) \equiv 1$), then $|\mathcal{A}(\infty)| = 1$.*

Proof. If $\mathcal{A}(z)$ is a rational function with $|\mathcal{A}(\infty)| \neq 1$, there exist a constant $C \in \mathbb{R}$ and an odd integer $k \in \mathbb{Z}$ such that $\mathcal{R}(z) \sim Cz^k$ for $z \rightarrow \infty$, and thus $\mathcal{R}(z) \not\equiv 1$. \square

However, as shown below, the above barrier for L -stable Runge-Kutta methods can be circumvented by applying an appropriate postprocessor to (19) of the form

$$\bar{X}_n = \mathcal{C}(z)X_n + \mathcal{D}(z)\sqrt{h}\sigma\xi_n, \quad z = -\lambda h. \quad (21)$$

We will see that this feature serves as a crucial ingredient in Section 3 to achieve high order in the SPDE case.

Proposition 2.3. *The method (15) applied to the SDE test problem (18) with $f_1(x) = -\lambda x$ and $f_2(x) = 0$ is such that the scheme $X_n \mapsto X_{n+1}$ is L -stable and \bar{X}_n is exact for sampling the invariant measure, i.e. $e(\phi, h) = 0$ in (7) (with u_n replaced with \bar{X}_n), for all test functions ϕ and all timesteps h .*

Proof. The method is L -stable because it has the same stability function $\mathcal{A}(z) = \frac{1}{1-z}$ as the linearized implicit Euler method (9). A calculation using the above notations yields

$$\lim_{n \rightarrow \infty} \mathbb{E}(\bar{X}_n^2) = \frac{\sigma^2}{2\lambda} \bar{\mathcal{R}}(z), \quad \text{where } \bar{\mathcal{R}}(z) = \mathcal{C}(z)^2 \mathcal{R}(z) - 2z\mathcal{D}(z)^2, \quad (22)$$

with $z = -\lambda h$ and $\mathcal{R}(z)$ given by (20). Noting that $\mathcal{A}(z) = \mathcal{B}(z) = (1-z)^{-1}$, $\mathcal{C}(z) = 1$, $\mathcal{D}(z) = (1-z/2)^{-1/2}$, yields $\bar{\mathcal{R}}(z) \equiv 1$, which proves that the method is exact. \square

2.2 Construction of the method

We explain in this section how we construct method (15) with high order of accuracy for the invariant measure by using the idea of postprocessing from [37]. Consider a system of SDEs in \mathbb{R}^N of the form

$$dX(t) = f_0(X(t))dt + \sigma dW(t), \quad X(0) = X_0, \quad (23)$$

where $\sigma > 0$ and $(W(t))_{t \in \mathbb{R}^+}$ is a standard d -dimensional Wiener process.

Assumption 2.4. *We assume the following.*

1. f_0 is of class C^∞ , with bounded derivatives of any order, and there exists a function $V_0 : \mathbb{R}^N \rightarrow \mathbb{R}$ such that $f_0 = -\nabla V_0$;
2. there exist $C, \beta > 0$ such that for all $x \in \mathbb{R}^N$, $x^T f_0(x) \leq -\beta x^T x + C$.

Then, system (23) is ergodic with a unique invariant measure μ_∞ (see e.g. [17]) given by

$$\mu_\infty(dy) = \rho(y)dy, \quad \text{with } \rho(y) = \frac{1}{Z} \exp\left(-\frac{2}{\sigma^2} V_0(y)\right).$$

Moreover the solution $X(t)$ of (23) satisfies an exponential ergodicity property analogous to (5).

The following theorem is the main result in [37] where postprocessed integrators for SDEs are introduced. It permits to improve the accuracy of a method of weak order q to order $r = q + 1$ for the invariant measure. Notice that this theorem remains valid for general classes of SDEs with additive or multiplicative noise in multiple dimensions. The error estimates in [37] rely on classical results from Talay and Tubaro [35] and Milstein [27] based on the backward Kolmogorov equation (see also [28, Chap. 2.2, 2.3]). In this section, we denote by $\mathcal{C}_P^\infty(\mathbb{R}^d, \mathbb{R})$ the set of C^∞ functions whose derivatives up to any order have a polynomial growth.

Theorem 2.5. *[37] Under Assumption 2.4, let X_n be an ergodic numerical solution of (23) with bounded moments of any order $M \in \mathbb{N}^*$, i.e. for all $n \geq 0$,*

$$\mathbb{E}(\|X_n\|^M) \leq C_M \quad (24)$$

where C_M is independent of h, n . Assume further that the scheme has local weak order $p \geq 1$ i.e. it satisfies for all initial condition $X_0 = x$ and any $\phi \in C_P^\infty(\mathbb{R}^d, \mathbb{R})$

$$|\mathbb{E}(\phi(X_1)) - \mathbb{E}(\phi(X(h)))| \leq C(x, \phi)h^{p+1}, \quad (25)$$

for all h sufficiently small, where $x \mapsto C(x, \phi)$ has a polynomial growth with respect to x .

Assume also that the numerical solution admits a weak Taylor series expansion of the form

$$\mathbb{E}(\phi(X_1)) = \phi(x) + hA_0\phi(x) + h^2A_1\phi(x) + \dots, \quad (26)$$

for all $\phi \in C_P^\infty(\mathbb{R}^d, \mathbb{R})$, where $A_i, i = 0, 1, 2, \dots$ are linear differential operators with smooth coefficients. Let G_n denote independent and identically distributed random maps in \mathbb{R}^d , independent of $\{X_j\}_{j \leq n}$, with $\bar{X}_n = G_n(X_n)$ having bounded moments of any order, and satisfying a weak Taylor expansion of the form

$$\mathbb{E}(\phi(G_n(x))) = \phi(x) + h^q\bar{A}_q\phi(x) + \mathcal{O}(h^{q+1}), \quad (27)$$

for all $\phi \in C_P^\infty(\mathbb{R}^d, \mathbb{R})$, where the constant in \mathcal{O} has a polynomial growth with respect to x . Assuming further

$$(A_q + [\mathcal{L}, \bar{A}_q])^* \rho = 0, \quad (28)$$

where we use the commutator notation $[A, B] = AB - BA$. Then $\bar{X}_n = G_n(X_n)$ yields an approximation of order $r = q + 1$ for the invariant measure, i.e. the postprocessed scheme \bar{X}_n satisfies

$$\left| \lim_{N \rightarrow \infty} \frac{1}{N+1} \sum_{n=0}^N \phi(\bar{X}_n) - \int_{\mathcal{H}} \phi(y) d\mu_\infty(y) \right| \leq C(\phi)h^{q+1}, \quad (29)$$

$$\left| \mathbb{E}(\phi(\bar{X}_n)) - \int_{\mathbb{R}^d} \phi(y) d\mu_\infty(y) \right| \leq K(\phi, x)e^{-\lambda t_n} + C(\phi)h^{q+1} \quad (30)$$

for all $\phi \in C_P^\infty(\mathbb{R}^d, \mathbb{R})$ with $t_n = nh$, where $\lambda, K(\phi, x), C(\phi)$ are independent of n and h assumed small enough.

Remark 2.6. We emphasize that the boundedness of moment condition (24) can be easily proved for Runge-Kutta type methods, such as the proposed method (15), following the methodology of [27] (see also [28, Chap. 2.2]) using the Lipschitz continuity of the SDE fields. We also refer to [37] where this assumption is discussed in the context of postprocessed integrators for SDEs. Notice also that the Lipschitz condition on the SDE fields and the ergodicity assumption on the numerical method could be relaxed using the the concept of rejecting exploding trajectories, as introduced in [30], see also [29] in the context of ergodic SDEs.

We are now in position to state the main result of this section. We show that the new method (15) satisfies the assumptions of Theorem 2.5 with $p = 1$, and has order two of accuracy for the invariant measure when applied to ergodic SDEs in finite dimension.

Theorem 2.7. Under Assumption 2.4, consider the method (14) with postprocessor \bar{X}_n . Assume the X_n is ergodic when applied to the system (14). Then \bar{X}_n has order two of accuracy for the invariant measure, precisely,

$$\left| \lim_{N \rightarrow \infty} \frac{1}{N+1} \sum_{k=0}^N \phi(\bar{X}_k) - \int_{\mathbb{R}^d} \phi(y) d\mu_\infty(y) \right| \leq C(\phi)h^2 \quad (31)$$

$$\left| \mathbb{E}(\phi(\bar{X}_n)) - \int_{\mathbb{R}^d} \phi(y) d\mu_\infty(y) \right| \leq K(\phi, x) e^{-\lambda t_n} + C(\phi) h^2, \quad (32)$$

for all $\phi \in \mathcal{C}_P^\infty(\mathbb{R}^d, \mathbb{R})$ with $t_n = nh$, where $\lambda, K(\phi, x), C(\phi)$ are independent of n and h assumed small enough.

Proof. Denote by \mathcal{L} the infinitesimal generator of (14) where we set $f_0 = f_1 + f_2$: for any test function $\phi \in \mathcal{C}_P^\infty(\mathbb{R}^d, \mathbb{R})$,

$$\mathcal{L}\phi = f_0 \cdot \nabla \phi + \frac{1}{2} \Delta \phi, \quad (33)$$

where $\nabla \phi$ denotes the gradient of ϕ and $\Delta \phi$ the Laplacian of ϕ . We introduce, for some real coefficients $a_1, a_2, a_3, b_1, b_2, c$ which are determined below by certain order conditions, the following variant of the linearized implicit Euler method with postprocessing

$$\begin{aligned} X_{n+1} &= X_n + h f_1 \left(X_{n+1} + a_1 \sigma \sqrt{h} \xi_n \right) + h f_2 \left(X_n + a_2 \sigma \sqrt{h} \xi_n \right) + (I + a_3 h f_1'(X_n)) \sigma \sqrt{h} \xi_n \\ \bar{X}_n &= X_n + b_1 h f_1(\bar{X}_n) + b_2 h f_2(X_n) + c \sigma \sqrt{h} \xi_n. \end{aligned} \quad (34)$$

A straightforward calculation yields the following weak expansion for all $\phi \in \mathcal{C}_P^\infty(\mathbb{R}^d, \mathbb{R})$,

$$\mathbb{E}(\phi(X_1) | X_0 = x) = \phi(x) + h \mathcal{L}\phi(x) + h^2 \mathcal{A}_1 \phi(x) + \mathcal{O}(h^3), \quad (35)$$

where the constant symbolized by \mathcal{O} is independent of h but depends on ϕ and depends on x with a polynomial growth. Here, the fourth-order linear differential operator \mathcal{A}_1 is given by¹

$$\begin{aligned} \mathcal{A}_1 \phi &= \frac{1}{2} \phi''(f_0, f_0) + \frac{\sigma^2}{2} \sum_{i=1}^d \phi'''(e_i, e_i, f_0) + \frac{\sigma^4}{8} \sum_{i,j=1}^d \phi^{(4)}(e_i, e_i, e_j, e_j) + \phi' f_1' f_0 \\ &+ \left((a_1 + 1)^2 \right) \frac{\sigma^2}{2} \phi' \sum_{i=1}^d f_1''(e_i, e_i) + (a_1 + 1 + a_3) \sigma^2 \sum_{i=1}^d \phi''(f_1' e_i, e_i) \\ &+ a_2^2 \frac{\sigma^2}{2} \phi' \sum_{i=1}^d f_2''(e_i, e_i) + a_2 \sigma^2 \sum_{i=1}^d \phi''(f_2' e_i, e_i) \end{aligned}$$

where e_1, \dots, e_d denotes the canonical basis of \mathbb{R}^d .

For any $\psi \in \mathcal{C}_P^\infty(\mathbb{R}^d, \mathbb{R})$, we denote $\langle \psi \rangle = \int_{\mathbb{R}^d} \psi(y) \mu_\infty(dy) = \int_{\mathbb{R}^d} \psi(y) \rho(y) dy$. By integration by parts, and using the identity $\nabla \rho = \frac{2}{\sigma^2} \rho f_0$, we obtain,

$$\begin{aligned} \langle \phi''(f_0, f_0) \rangle &= \langle -\phi'(f_0' f_0 + (\operatorname{div} f_0) f_0 + \frac{2}{\sigma^2} \|f_0\|^2 f_0) \rangle, \\ \langle \sigma^2 \sum_i \phi'''(f_0, e_i, e_i) \rangle &= \langle \phi'(\sigma^2 \sum_i f_0''(e_i, e_i) + 4f_0' f_0 + 2(\operatorname{div} f_0) f_0 + \frac{4}{\sigma^2} \|f_0\|^2 f_0) \rangle, \\ \langle \sigma^2 \sum_{ij} \phi^{(4)}(e_i, e_i, e_j, e_j) \rangle &= \langle -\sum_i 2\phi'''(f_0, e_i, e_i) \rangle, \\ \langle \sigma^2 \sum_i \phi''(f_1' e_i, e_i) \rangle &= \langle -\phi'(\sigma^2 \sum_i f_1''(e_i, e_i) + 2f_1' f_0) \rangle, \end{aligned} \quad (36)$$

see [1, 37] for examples of such calculations. Then

$$\langle \mathcal{A}_1 \phi \rangle = \left\langle \left(a_1^2 + 2a_1 \right) \frac{\sigma^2}{2} \phi' \sum_{i=1}^d f_1''(e_i, e_i) + \left(\frac{1}{4} + a_1 + a_3 \right) \sigma^2 \sum_{i=1}^d \phi''(f_1' e_i, e_i) \right\rangle$$

¹We denote $\phi'(x) : \mathbb{R} \rightarrow \mathcal{R}$ the first derivative of ϕ at point $x \in \mathbb{R}^d$, $\phi''(x)$ the second derivative (a symmetric bilinear form on $\mathbb{R}^d \times \mathbb{R}^d$), $\phi'''(x)$ the third derivative (a symmetric bilinear form), etc.

$$+ a_2^2 \frac{\sigma^2}{2} \phi' \sum_{i=1}^d f_2''(e_i, e_i) + \left(-\frac{1}{4} + a_2 \right) \sigma^2 \sum_{i=1}^d \phi''(f_2' e_i, e_i) \Big\rangle$$

We note that the postprocessor $X_n \mapsto \bar{X}_n$ in (34) satisfies (27) with $q = 1$ and $\bar{A}_1 \phi = b_1 \phi' f_1 + b_2 \phi' f_2 + \frac{c^2}{2} \sigma^2 \Delta \phi$. We choose $b_1 = b_2 = 0$ and obtain a simple postprocessing application $X_n \mapsto \bar{X}_n$; then

$$[\mathcal{L}, \bar{A}_1] \phi = -\frac{c^2}{2} \sigma^2 \phi' \sum_{i=1}^d f_0''(e_i, e_i) - c^2 \sigma^2 \sum_{i=1}^d \phi''(f_0' e_i, e_i).$$

We deduce the following expression for $\langle A_1 \phi + [\mathcal{L}, \bar{A}_1] \phi \rangle$, which is the key quantity to compute in Theorem 2.5,

$$\begin{aligned} & \left\langle \left(a_1^2 + 2a_1 + b_1 - c^2 \right) \frac{\sigma^2}{2} \phi' \sum_{i=1}^d f_1''(e_i, e_i) + \left(a_1 + a_3 + \frac{1}{4} + b_1 - c^2 \right) \sigma^2 \sum_{i=1}^d \phi''(f_1' e_i, e_i) \right. \\ & \left. + \left(a_2^2 + b_2 - c^2 \right) \frac{\sigma^2}{2} \phi' \sum_{i=1}^d f_2''(e_i, e_i) + \left(-\frac{1}{4} + a_2 + b_2 - c^2 \right) \sigma^2 \sum_{i=1}^d \phi''(f_2' e_i, e_i) \right\rangle \end{aligned}$$

In order to fulfill (28), sufficient order conditions then read

$$a_1^2 + 2a_1 - c^2 = 0, \quad a_1 + a_3 + \frac{1}{4} - c^2 = 0, \quad a_2^2 - c^2 = 0, \quad -\frac{1}{4} + a_2 - c^2 = 0.$$

The above system has two solutions. Choosing the solution of with smallest absolute value of a_1 and a_3 , we obtain $a_1 = -a_3 = \frac{-2 + \sqrt{5}}{2}$, $a_2 = c = \frac{1}{2}$. Applying Theorem 2.5 with $q = 1$ to the scheme (34) and using the above parameters then yields for \bar{X}_n in (34) a method of weak order two for the invariant measure. Finally, it can be shown that method (15) can be obtained as a perturbation of method (34) with size $\mathcal{O}(h^2)$, i.e. method (15) satisfies the same expansions (27) (with $q = 1$) and (35) as method (34). Indeed, to recover (15), in the first line of (34) one has to replace ξ_n with $J_{n,2} \xi_n = (I + \mathcal{O}(h)) \xi_n$ in the arguments of f_1, f_2 and also to substitute $I + a_3 h f_1'(x)$ with

$$\left(\frac{1 - \sqrt{2} + \sqrt{5}}{2} J_{n,1}^{-1} + \frac{1 + \sqrt{2} - \sqrt{5}}{2} \right) J_{n,2} = I + a_3 h f_1'(x) + \mathcal{O}(h^2);$$

in the second line of (34), one has to replace ξ_n with $J_{n,3} \xi_n = (I + \mathcal{O}(h)) \xi_n$. We deduce that method (15) also has second order of accuracy for the invariant measure. This concludes the proof of Theorem 2.7. \square

Recall that the operators $J_{n,2}, J_{n,3}$ in (15) are introduced in order to have Proposition 2.3; they also play an important role for the well-posedness, the stability and the accuracy of the scheme in the SPDE case presented in Section 3.

3 Analysis in the SPDE case

3.1 Abstract setting and assumptions

The state space in the SPDE case is an infinite dimensional separable Hilbert space \mathcal{H} , for which we denote by $\langle \cdot, \cdot \rangle$ the scalar product, and by $|\cdot|$ the associated norm. Consider the

linear operator A involved in the parabolic SPDE (1). Recall that we assume that $-A$ is an unbounded self-adjoint linear operator with eigenvalues $0 < \lambda_1 \leq \dots \leq \lambda_p \leq \lambda_{p+1} \leq \dots$, such that $\lambda_p \rightarrow +\infty$, when $p \rightarrow +\infty$, and associated normalized eigenvectors e_p (such that $Ae_p = -\lambda_p e_p$), which form a complete orthonormal system in \mathcal{H} .

For any $s \in \mathbb{R}^+$, we classically define the unbounded linear operator $(-A)^{s/2}$ from \mathcal{H} to \mathcal{H} and its domain $\mathcal{H}^s \subset \mathcal{H}$ as follows:

$$(-A)^{s/2}u = \sum_{p=1}^{+\infty} \langle u, e_p \rangle \lambda_p^{s/2} e_p \quad \text{for all } u \in \mathcal{H}^s = \{u \in \mathcal{H} ; |u|_s^2 = \sum_{p=1}^{+\infty} |\langle u, e_p \rangle|^2 \lambda_p^s < +\infty\}.$$

We also define the bounded linear operator $(-A)^{-s/2}$ and the semi-group $(e^{tA})_{t \in \mathbb{R}^+}$, both as linear operators from \mathcal{H} to \mathcal{H} by

$$(-A)^{-s/2}u = \sum_{p=1}^{+\infty} \langle u, e_p \rangle \lambda_p^{-s/2} e_p, \quad e^{tA}u = \sum_{p=1}^{+\infty} \exp(-t\lambda_p) \langle u, e_p \rangle e_p.$$

It is straightforward that for any $t \in (0, +\infty)$ we have $e^{tA} \in \mathcal{L}(\mathcal{H}, \mathcal{H})$ – the space of bounded linear operators from \mathcal{H} to \mathcal{H} , endowed with the norm denoted by $\|\cdot\|$ – with $\|e^{tA}\| \leq \exp(-\lambda_1 t)$. Moreover, the following regularization property holds true: $\|(-A)^{s/2}e^{tA}\| \leq \frac{C_s}{t^{s/2}}$ where $C_s^2 = \sup_{r \in \mathbb{R}^+} \exp(-2r)r^s \in (0, +\infty)$.

Covariance operator We assume that Q is a bounded, non-negative self-adjoint linear operator from \mathcal{H} to \mathcal{H} , which satisfies $Qe_p = q_p e_p$ for any $p \in \mathbb{N}^*$, where the eigenvalues $(q_p)_{p \in \mathbb{N}^*}$ form a bounded sequence of non-negative real numbers. We assume that condition (3) is satisfied. The Q-Wiener process in (1) is then defined as follows: for any $t \geq 0$,

$$W^Q(t) = \sum_{p=1}^{+\infty} \sqrt{q_p} \beta_p(t) e_p, \tag{37}$$

where $(\beta_p)_{p \in \mathbb{N}^*}$ is a sequence of independent standard scalar Wiener processes on an underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Notice that the operators A and Q commute: $AQu = QAu$ for any $u \in \mathcal{H}^2$. This property is often assumed in the literature, and simplifies the analysis of the order of convergence made in Section 3.3. Nevertheless several arguments (especially Proposition 3.1 and the results of Section 3.2) do not require this property; in particular, the scheme remains well-defined in the non-commuting case.

Nonlinearity The nonlinear coefficient F is assumed to be a Lipschitz continuous function from \mathcal{H} to \mathcal{H} , with Lipschitz constant L satisfying the dissipation condition

$$L = \sup_{u^1 \neq u^2 \in \mathcal{H}} \frac{|F(u^1) - F(u^2)|}{|u^1 - u^2|} < \min_{p \in \mathbb{N}^*} \lambda_p = \lambda_1. \tag{38}$$

This condition ensures ergodicity of the continuous-time process (Proposition 3.1) and of its time-discretized approximations (Proposition 3.3). A typical example of such a Lipschitz function on $\mathcal{H} = L^2(\mathcal{D})$ – where \mathcal{D} is an open smooth bounded domain in \mathbb{R}^d – is the Nemytskii operator $F : u \in L^2(\mathcal{D}) \mapsto f \circ u$ where $f : \mathbb{R} \rightarrow \mathbb{R}$ is a Lipschitz function.

Under the above hypotheses, and assuming the trace condition (3), the process $(u(t))_{t \in \mathbb{R}^+}$ takes values in \mathcal{H}^s for any $s < \bar{s}$, and we recall without proof the following result of exponential convergence to a unique invariant distribution, see e.g. [9, 12].

Proposition 3.1. *Assume (3) and the above hypotheses. Then, the process $(u(t))_{t \in \mathbb{R}^+}$ solution of (1) admits a unique invariant probability distribution μ_∞ on \mathcal{H} . Moreover for all $s < \bar{s}$,*

$$\int_{\mathcal{H}} |u|_s^2 \mu_\infty(du) < +\infty,$$

and for all $\phi : \mathcal{H} \rightarrow \mathbb{R}$ Lipschitz continuous, and all $t > 0$,

$$\left| \mathbb{E}[\phi(u(t))] - \int_{\mathcal{H}} \phi(v) \mu_\infty(dv) \right| \leq C(\phi, u_0) e^{-(\lambda_1 - L)t},$$

where $C(\phi, u_0)$ is independent of t .

Condition (38) is crucial for the proof of uniqueness of numerical invariant distributions established in the next section: we compare the solutions starting from different initial conditions and driven with the same noise process, and show an exponential contraction similar to the result of Proposition 3.1. Notice that weaker conditions than (38) are known in the literature (see e.g. [12] and references therein) to ensure the ergodicity and the exponential convergence – for instance when F is bounded and Lipschitz continuous with no size restriction on L .

3.2 Numerical invariant distribution of the postprocessed method for SPDEs

In this section, we prove the existence and uniqueness of invariant distributions for the time-discretized processes defined by the numerical method, for any time-step size $h > 0$, in a general setting. Notice that the results of this section do not require the gradient assumption $F = -DV$. The results are analogous to classical results for the linearized implicit Euler method (12), as studied e.g. in [4, Remark 4.8].

It is key to observe that the sequence $(u_n, \bar{u}_{n-1})_{n \in \mathbb{N}}$ defining the new scheme (12),(13) is a Markov chain on the product space $\mathcal{H} \times \mathcal{H}$. The initial condition (u_0, \bar{u}_{-1}) is given by $u_0 = u(0) = u^0$ and an arbitrary $\bar{u}_{-1} \in \mathcal{H}$ which plays no role in the dynamics, since (u_{n+1}, \bar{u}_n) depends only on u_n and ξ_n^Q , not on \bar{u}_{n-1} .

In the following proposition, we state uniform bounds – with respect to $n \in \mathbb{N}$ and $h \in (0, 1)$ – on first-order moments for the norm $|\cdot|_s$ for $s < \bar{s}$.

Proposition 3.2. *Assume the hypotheses of Section 3.1 and consider the scheme (12),(13). For all $s \in [0, \bar{s})$, assuming $u_0 = u^0 \in \mathcal{H}^s$, there exists a constant $C_s \in (0, +\infty)$ such that for all $h \in (0, 1)$,*

$$\sup_{n \in \mathbb{N}} \mathbb{E}|u_n|_s \leq C_s(1 + |u_0|_s), \quad \sup_{n \in \mathbb{N}} \mathbb{E}|\bar{u}_n|_s \leq C_s(1 + |u_0|_s).$$

Proof. Thanks to (3), $\text{Trace}(J_i Q J_i) < +\infty$ for $i \in \{1, 2, 3\}$, and thus u_n and \bar{u}_n are well-defined in \mathcal{H} for all $n \in \mathbb{N}$. The contributions of the drift part and of the stochastic perturbation are treated separately: we introduce the auxiliary process $(\ell_n)_{n \in \mathbb{N}}$, as the solution of the following equation

$$\ell_{n+1} = J_1 \ell_n + \sigma \sqrt{h} \left(\frac{\sqrt{2} - 1}{2} J_1 + \frac{3 - \sqrt{2}}{2} I \right) J_2 \xi_n^Q,$$

with $\ell_0 = 0$. Set $d_n = u_n - \ell_n$ for all $n \in \mathbb{N}$; then $d_0 = u_0$ and

$$d_{n+1} = J_1 d_n + h J_1 F(d_n + \ell_n + \frac{1}{2} J_2 \sigma \sqrt{h} \xi_n^Q).$$

The quantity ℓ_n satisfies for all $n \in \mathbb{N}^*$ the identity

$$\ell_n = \sigma \sqrt{h} \left(\frac{\sqrt{2}-1}{2} J_1 + \frac{3-\sqrt{2}}{2} I \right) J_2 J_1^{-1} \sum_{k=0}^{n-1} (J_1)^{n-k} \xi_k^Q.$$

Observe that $(\frac{\sqrt{2}-1}{2} J_1 + \frac{3-\sqrt{2}}{2} I) J_2 J_1^{-1}$ is a bounded linear operator from \mathcal{H}^s to \mathcal{H}^s with norm less than $\frac{2}{3-\sqrt{2}}$, for $s \in [0, \bar{s}]$. Since the Gaussian random variables $(\xi_k^Q)_{k \in \mathbb{N}}$ are independent, for all $0 \leq s \leq \bar{s}$,

$$\begin{aligned} \mathbb{E} |\ell_n|_s^2 &\leq \frac{2\sigma^2 h}{3-\sqrt{2}} \sum_{k=0}^{n-1} \mathbb{E} |(J_1)^{n-k} \xi_k^Q|_s^2 \leq \frac{2\sigma^2 h}{3-\sqrt{2}} \sum_{k=1}^{+\infty} \sum_{p=1}^{+\infty} \frac{q_p \lambda_p^s}{(1+\lambda_p h)^{2k}} \\ &\leq \frac{2\sigma^2}{3-\sqrt{2}} \sum_{p=1}^{+\infty} q_p \lambda_p^{s-1} \frac{\lambda_p h}{(1+\lambda_p h)^2 - 1} \leq \frac{\sigma^2}{3-\sqrt{2}} \text{Trace}((-A)^{-1+s} Q). \end{aligned}$$

Now thanks to (38), straightforward computations show that

$$\begin{aligned} \mathbb{E} |d_n| &\leq \frac{1+hL}{1+\lambda_1 h} \mathbb{E} |d_{n-1}| + \frac{h}{1+\lambda_1 h} |F(0)| + \frac{Lh}{1+\lambda_1 h} (\mathbb{E} |\ell_{n-1}| + \mathbb{E} |\frac{1}{2} J_2 \sigma \sqrt{h} \xi_{n-1}^Q|) \\ &\leq \frac{(1+hL)^n}{(1+\lambda_1 h)^n} |u_0| + \frac{|F(0)|}{\lambda_1 - L} + \frac{L}{\lambda_1 - L} \left(\sup_{k \in \mathbb{N}} \mathbb{E} |\ell_k| + \frac{1}{3-\sqrt{2}} \sigma (\text{Trace}((-A)^{-1} Q))^{1/2} \right), \end{aligned}$$

with $\frac{(1+hL)}{(1+\lambda_1 h)} \leq 1 - \frac{\lambda_1 - L}{1+\lambda_1 h} h \leq \exp\left(-\frac{(\lambda_1 - L)}{1+\lambda_1 h} h\right) < 1$.

As a consequence the claim follows for $(u_n)_{n \in \mathbb{N}}$ in the case $s = 0$. In particular, for some constant $C \in (0, +\infty)$ it comes that $\sup_{n \in \mathbb{N}} \mathbb{E} |F(u_n + \frac{1}{2} J_2 \sigma \sqrt{h} \xi_n^Q)| < C(1 + |u^0|)$.

The case $s \in (0, \bar{s})$ is treated using the estimates of Lemma 3.2 in [4]: for $n \in \mathbb{N}$

$$\begin{aligned} \mathbb{E} |d_n|_s &= \mathbb{E} |(-A)^{s/2} (J_1)^n u^0 + h \sum_{k=0}^{n-1} (-A)^{s/2} (J_1)^{n-k} F(u_k + \frac{1}{2} J_2 \sigma \sqrt{h} \xi_k^Q)| \\ &\leq \frac{1}{(1+\lambda_1 h)^n} |u^0|_s + h C (1 + |u^0|) \sum_{k=1}^n \left(\frac{\mathbb{1}_{kh \leq 1}}{(kh)^{s/2}} + \mathbb{1}_{kh > 1} \frac{C_s}{(1+\lambda_1 h)^{k-1/[h]}} \right) \\ &\leq C_s (1 + |u^0|_s). \end{aligned}$$

Finally, for $n \in \mathbb{N}$,

$$\mathbb{E} \left| \frac{1}{2} J_3 \sigma \sqrt{h} \xi_n^Q \right|_s^2 = \frac{\sigma^2 h}{4} \text{Trace} \left((I - \frac{h}{2} A)^{-1} (-A)^s Q \right) \leq \frac{\sigma^2}{2} \text{Trace}((-A)^{-1+s} Q).$$

This concludes the proof, since $\bar{u}_n = u_n + \frac{1}{2} J_3 \sigma \sqrt{h} \xi_n^Q$. \square

We now state the following existence and uniqueness result for the invariant distribution of the Markov chain $(u_n, \bar{u}_{n-1})_{n \in \mathbb{N}}$. We prove that the convergence is exponentially fast, in contrast to the trapezoidal method (11) which is not L -stable and for which such exponential estimate does not hold in general for stiff problems (even in finite dimension).

Proposition 3.3. *Assume the hypotheses of Section 3.1. For any $h \in (0, 1)$, the $\mathcal{H} \times \mathcal{H}$ -valued Markov chain $(u_n, \bar{u}_{n-1})_{n \in \mathbb{N}}$ admits a unique invariant distribution $M_\infty^h = (\mu_\infty^h, \bar{\mu}_\infty^h)$.*

Moreover, the convergence of the distributions to equilibrium is exponentially fast: for all Lipschitz test function $\varphi : \mathcal{H} \rightarrow \mathbb{R}$, and all $t_n = nh$,

$$\left| \mathbb{E}\varphi(u_n) - \int_{\mathcal{H}} \varphi d\mu_\infty^h \right| + \left| \mathbb{E}\varphi(\bar{u}_n) - \int_{\mathcal{H}} \varphi d\bar{\mu}_\infty^h \right| \leq C(\varphi, |u^0|) \exp\left(-\frac{(\lambda_1 - L)}{1 + \lambda_1 h} t_n\right).$$

Proof. Existence. The semi-group $(P^n)_{n \in \mathbb{N}}$ on $\mathcal{H} \times \mathcal{H}$ generated by the Markov chain $(u_n, \bar{u}_{n-1})_{n \in \mathbb{N}}$ satisfies the Feller property: for any $n \in \mathbb{N}$, for any bounded continuous test function $\phi : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{R}$, the map $(u_0, \bar{u}_{-1}) \mapsto P^n \phi(u_0, \bar{u}_{-1}) = \mathbb{E}\phi(u_n, \bar{u}_{n-1})$ is continuous. The claim then follows from the standard Krylov-Bogoliubov criterion (see Section 3.1 in [9]): given an arbitrary initial condition $(u_0, \bar{u}_{-1}) \in \mathcal{H} \times \mathcal{H}$, if M_n denotes the law of (u_n, \bar{u}_{n-1}) , then

- $(\frac{1}{n+1} \sum_{k=0}^n M_k)_{n \in \mathbb{N}}$ is a *tight* sequence of probability distributions on $\mathcal{H} \times \mathcal{H}$ – as a consequence of Proposition 3.2 combined with the Markov inequality, and of the observation that for any $s \in (0, \bar{s})$ and any $R > 0$, the set $\{|u|_s \leq R, |\bar{u}|_s \leq R\}$ is a compact subset of $\mathcal{H} \times \mathcal{H}$.
- every subsequence limit point M is an invariant distribution for the semi-group.

Uniqueness. Consider two initial conditions $u^{0,1}, u^{0,2} \in \mathcal{H}$, as well as $\bar{u}_{-1}^1, \bar{u}_{-1}^2 \in \mathcal{H}$ and the associated processes $(u_n^i)_{n \in \mathbb{N}}$ and $(\bar{u}_n^i)_{n \in \mathbb{N}}$, for $i = 1, 2$, defined by (12), (13), and driven by a unique noise process $(\xi_n^Q)_{n \in \mathbb{N}}$.

Then by Lipschitz continuity of F , and using the cancellations of several noise terms, computations similar to those of the proof of Proposition 3.2 yield for any $n \in \mathbb{N}$ the almost sure contraction property

$$|\bar{u}_n^1 - \bar{u}_n^2| = |u_n^1 - u_n^2| \leq \frac{1 + Lh}{1 + \lambda_1 h} |u_{n-1}^1 - u_{n-1}^2| \leq \exp\left(-\frac{(\lambda_1 - L)}{1 + \lambda_1 h} t_n\right) |u^{0,1} - u^{0,2}|.$$

Finally, taking (u_0^2, \bar{u}_{-1}^2) random, independent of the noise process $(\xi_n^Q)_{n \in \mathbb{N}}$ and distributed according to an ergodic invariant distribution M_∞^h gives the exponential convergence and the uniqueness properties. \square

3.3 Analysis of the order of convergence: a simplified linear case

It is shown in [4] (for $d = 1$ and $\bar{s} = 1/2$, associated with space-time white noise $Q = I$), that the standard linearized implicit Euler scheme (12) has order $r = 1/2 - \varepsilon$ for all $\varepsilon \in (0, 1/2)$ for the approximation of the invariant distribution μ_∞ of (1). In this section, we show that the postprocessed scheme has the improved order of convergence $\bar{s} + 1 - \varepsilon$. Since the techniques from Section 2 do not extend straightforwardly to the SPDE case, we only focus on a simplified case, where the nonlinear coefficient F is replaced with a bounded linear

operator. Numerical experiments of Section 4 show that the higher order is preserved for various examples of nonlinearities F .

In addition to the hypotheses of Section 3.1, assume that the coefficient F is given by a linear mapping: for any $u \in \mathcal{H}$, $F(u) = Bu$ where $B \in \mathcal{L}(\mathcal{H})$ satisfies $Be_p = -b_p e_p$ for all $p \in \mathbb{N}^*$, with real eigenvalues $b_p \in (-\lambda_1, \lambda_1)$ (due to condition (38)):

$$du(t) = Au(t)dt + Bu(t)dt + \sigma dW^Q(t) \quad , \quad u(0) = u^0. \quad (39)$$

In this situation, the components $\langle u(t), e_p \rangle$ in the basis $\{e_p\}_{p \in \mathbb{N}^*}$ of the solution $(u(t))_{t \in \mathbb{R}^+}$ of the SPDE (1) are independent Ornstein-Uhlenbeck processes. Similarly, the components in the basis $\{e_p\}_{p \in \mathbb{N}^*}$ of the discrete-time processes $(v_n)_{n \in \mathbb{N}}$ (resp. $(u_n)_{n \in \mathbb{N}}$, resp. $(\bar{u}_n)_{n \in \mathbb{N}}$) are also independent processes.

As a consequence, explicit expressions for the invariant distributions μ_∞ , ν_∞^h , μ_∞^h and $\bar{\mu}_\infty^h$ are available: they are centered Gaussian probability measures on \mathcal{H}

$$\begin{aligned} \mu_\infty &= \mathcal{N}(0, Q_\infty) \quad , \quad Q_\infty = \frac{\sigma^2}{2} Q(-A - B)^{-1}, \\ \nu_\infty^h &= \mathcal{N}(0, Q_{\infty, \nu}^h) \quad , \quad Q_{\infty, \nu}^h = \frac{\sigma^2}{2} Q(-A - B)^{-1} (I - h \frac{A - B}{2})^{-1}, \\ \mu_\infty^h &= \mathcal{N}(0, Q_\infty^h) \quad , \quad \bar{\mu}_\infty^h = \mathcal{N}(0, \bar{Q}_\infty^h). \end{aligned}$$

The expressions for Q_∞^h and \bar{Q}_∞^h being more complicated are not included, see (43) below.

The following lemma is a key elementary tool in order to exhibit the order of convergence as $h \rightarrow 0$ of the approximating measures towards μ_∞ .

Lemma 3.4. *Let $\pi_j = \mathcal{N}(0, Q_j)$, $j \in \{1, 2\}$ be two Gaussian probability distributions on the Hilbert space \mathcal{H} . Assume that for any $p \in \mathbb{N}^*$, $j = 1, 2$, $Q_j e_p = q_{j,p} e_p$. Let $\varphi \in \mathcal{C}^2(\mathcal{H}, \mathbb{R})$, such that $\sup_{v \in \mathcal{H}} \|D^2 \varphi(v)\| < +\infty$. Then*

$$\left| \int_{\mathcal{H}} \varphi d\pi_2 - \int_{\mathcal{H}} \varphi d\pi_1 \right| \leq \frac{\sup_{v \in \mathcal{H}} \|D^2 \varphi(v)\|}{2} \sum_{p=1}^{+\infty} |q_{2,p} - q_{1,p}|. \quad (40)$$

Remark 3.5. *Assume that $q_{2,p} \geq q_{1,p}$. Then the above Lemma 3.4 yields optimal orders of convergence. Indeed choosing the test function $\varphi_{\text{opt}}(u) = \exp(-|u|^2)$, Lemma 9.5 in [22] yields that*

$$\int_{\mathcal{H}} \varphi_{\text{opt}} d\pi_1 - \int_{\mathcal{H}} \varphi_{\text{opt}} d\pi_2 \geq \frac{\text{Trace}(Q_2 - Q_1)}{\exp(6\text{Trace}(Q_2))}.$$

This means that the quantity $\text{Trace}(Q_2 - Q_1)$ also provides a lower bound for the error between the invariant distributions π_1 and π_2 .

Proof of Lemma 3.4. Let $(\gamma_p)_{p \in \mathbb{N}^*}$ and $(\delta_p)_{p \in \mathbb{N}^*}$ be two independent sequences of *i.i.d.* standard real valued Gaussian random variables – centered and with variance 1. Set $X_2 = X_1 + R_{1,2}$, where $X_1 = \sum_{p \in \mathbb{N}^*} \sqrt{q_{1,p}} \gamma_p e_p$ and $R_{1,2} = R_{1,2}^+ - R_{1,2}^-$, with $R_{1,2}^\pm = \sum_{p \in \mathbb{N}^*} \sqrt{\max(\pm(q_{2,p} - q_{1,p}), 0)} \delta_p e_p$. Observe that X_1 , $R_{1,2}^+$ and $R_{1,2}^-$ are independent random variables, and that $X_1 \sim \pi_1$ and $X_2 \sim \pi_2$. $X_1 \sim \pi_1$, $X_2 \sim \pi_2$. A second-order Taylor expansion then yields

$$\left| \int_{\mathcal{H}} \varphi d\pi_2 - \int_{\mathcal{H}} \varphi d\pi_1 \right| = \left| \mathbb{E}[\varphi(X_2)] - \mathbb{E}[\varphi(X_1)] \right|$$

$$\begin{aligned}
&= \left| \mathbb{E}[D\varphi(X_1) \cdot R_{1,2}] + \mathbb{E}\left[\int_0^1 (1-\theta)D^2\varphi(\theta X_2 + (1-\theta)X_1) \cdot (R_{1,2}, R_{1,2})d\theta\right] \right| \\
&\leq \frac{\sup_{v \in \mathcal{H}} \|D^2\varphi(v)\|}{2} \mathbb{E}|R_{1,2}|^2,
\end{aligned}$$

where the first-order term cancels out due to the independence of X_1 and $R_{1,2}$ and using $\mathbb{E}[R_{1,2}] = 0$. Finally $\mathbb{E}|R_{1,2}|^2 = \mathbb{E}|R_{1,2}^+|^2 + \mathbb{E}|R_{1,2}^-|^2 = \sum_{p=1}^{+\infty} |q_{2,p} - q_{1,p}|$. \square

We now explain how Lemma 3.4 permits to find the order of convergence of $\bar{\mu}_\infty^h$ to μ_∞ . We define, for $p \in \mathbb{N}^*$, the component processes $u \cdot (p)$ and $\bar{u} \cdot (p)$ by projecting on the eigenvector e_p : for any $n \in \mathbb{N}$, $u_n(p) = \langle u_n, e_p \rangle$, $\bar{u}_{n-1}(p) = \langle \bar{u}_{n-1}, e_p \rangle$. Then (12),(13) applied to (39) rewrites as a system of independent equations, decoupled with respect to $p \in \mathbb{N}^*$,

$$\begin{aligned}
u_{n+1}(p) &= \mathcal{A}(-\lambda_p h, -b_p h)u_n(p) + \sigma\sqrt{h}\sqrt{q_p}\mathcal{B}(-\lambda_p h, -b_p h)\xi_{n,p} \\
\bar{u}_n(p) &= \mathcal{C}(-\lambda_p h)u_n(p) + \sigma\sqrt{h}\sqrt{q_p}\mathcal{D}(-\lambda_p h)\xi_{n,p},
\end{aligned} \tag{41}$$

where $\sqrt{q_p}\xi_{n,p} = \langle \xi_n^Q, e_p \rangle$: thus $(\xi_{n,p})_{n \in \mathbb{N}, p \in \mathbb{N}^*}$ are independent standard Gaussian random variables, and the rational functions $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ satisfy for any $z \in (-\infty, 0)$ and $\beta \in (-1, \min(1, |z|))$,

$$\mathcal{A}(z, \beta) = \frac{1 + \beta}{1 - z} \quad \mathcal{B}(z, \beta) = \frac{1 + \frac{\beta}{2} - \frac{3 - \sqrt{2}}{2}z}{(1 - z)(1 - \frac{3 - \sqrt{2}}{2}z)}, \quad \mathcal{C}(z) = 1, \quad \mathcal{D}(z) = \frac{1}{2(1 - z/2)^{1/2}}. \tag{42}$$

Since for $\beta \in (-1, \min(1, |z|))$ the stability condition $|\mathcal{A}(z, \beta)| < 1$ is satisfied, straightforward computations yield

$$\begin{aligned}
Q_\infty^h e_p &= \lim_{n \rightarrow +\infty} \mathbb{E}|u_n(p)|^2 e_p = \frac{\sigma^2 q_p}{2(\lambda_p + b_p)} \mathcal{R}(-\lambda_p h, -b_p h) e_p \\
\bar{Q}_\infty^h e_p &= \lim_{n \rightarrow +\infty} \mathbb{E}|\bar{u}_n(p)|^2 e_p = \frac{\sigma^2 q_p}{2(\lambda_p + b_p)} \bar{\mathcal{R}}(-\lambda_p h, -b_p h) e_p
\end{aligned} \tag{43}$$

where $\mathcal{R}(z, \beta) = \frac{-2(z+\beta)\mathcal{B}(z,\beta)^2}{1-\mathcal{A}(z,\beta)^2}$ and

$$\bar{\mathcal{R}}(z, \beta) = \mathcal{C}(z)^2 \mathcal{R}(z, \beta) - 2(z + \beta)\mathcal{D}(z)^2 = 1 + \beta z \frac{P_1(z)\beta + P_2(z)}{(2 + \beta - z)P_3(z)}$$

with polynomial functions $P_1(z) = 10 - 4\sqrt{2} - (11 - 6\sqrt{2})z$, $P_2(z) = 20 - 8\sqrt{2} - (44 - 24\sqrt{2})z + (11 - 6\sqrt{2})z^2$ and $P_3(z) = (2 - z)(2 - (3 - \sqrt{2})z)^2$.

The following result is crucial to obtain an improved order for the convergence of $\bar{\mu}_\infty^h$ to μ_∞ when $h \rightarrow 0$.

Lemma 3.6. *For all $z \leq 0$ and all $\beta \in (-1, \min(1, |z|))$, we have*

$$|1 - \bar{\mathcal{R}}(z, \beta)| \leq |z\beta| \frac{(15 - 6\sqrt{2})}{4(1 - z)^2}.$$

Proof. Observe that for all $z \leq 0$, we have $P_2(z) \geq P_1(z) \geq 0$ and $(2 + \beta - z) > 0$, $P_3(z) > 0$. Since $(2 + \beta - z)^{-1} \leq (1 - z)^{-1}$, we obtain

$$|1 - \overline{\mathcal{R}}(z, \beta)| \leq |z\beta| \frac{P_1(z) + P_2(z)}{(1 - z)P_3(z)}.$$

The estimate then follows by observing that $(1 - z) \frac{P_1(z) + P_2(z)}{P_3(z)}$ is an increasing function of $z \leq 0$, with maximum at $z = 0$, given by $\frac{P_1(0) + P_2(0)}{P_3(0)} = \frac{(15 - 6\sqrt{2})}{4}$. \square

We are in position to state our main convergence result, which yields the order of convergence $r = s + 1$ for any $s < \bar{s}$ for the invariant distribution with postprocessing $\overline{\mu}_\infty^h$.

Theorem 3.7. *Consider the method (12)-(13) applied to (39). Let $\varphi \in \mathcal{C}^2(\mathcal{H}, \mathbb{R})$, such that $\sup_{v \in \mathcal{H}} \|D^2\varphi(v)\| < +\infty$. For any $s < \bar{s}$ there exists $C_s \in (0, +\infty)$ such that for any $h \in (0, 1/L)$ we have*

$$\left| \int_{\mathcal{H}} \varphi d\mu_\infty - \int_{\mathcal{H}} \varphi d\overline{\mu}_\infty^h \right| \leq C_s \sup_{v \in \mathcal{H}} \|D^2\varphi(v)\| \sigma^2 h^{1+s}. \quad (44)$$

Moreover, if $B = 0$ (i.e. $b_p = 0$ for all $p \in \mathbb{N}^*$) then the method is exact: $\overline{\mu}_\infty^h = \overline{\mu}_\infty$.

Proof. Thanks to Lemma 3.4, it is sufficient to control

$$\begin{aligned} \sum_{p=1}^{+\infty} | \langle (Q_\infty - \overline{Q}_\infty^h) e_p, e_p \rangle | &= \sum_{p=1}^{+\infty} \frac{\sigma^2 q_p}{2(\lambda_p + b_p)} | \overline{\mathcal{R}}(-\lambda_p h, -b_p h) - 1 | \\ &\leq C \lambda_1 \sigma^2 \sum_{p=1}^{+\infty} \frac{q_p |b_p|}{\lambda_p} \frac{\lambda_p h^2}{(1 + \lambda_p h)^2} \leq C \lambda_1^2 \sigma^2 \sum_{p=1}^{+\infty} q_p \lambda_p^{-1+s} \frac{(\lambda_p h)^{1-s} h^{1+s}}{(1 + \lambda_p h)^2} \\ &\leq C_s \lambda_1^2 \sigma^2 \text{Trace}((-A)^{-1+s} Q) h^{1+s}, \end{aligned}$$

which gives the order of convergence $s + 1$ for all $s < \bar{s}$. Moreover, it is clear that $\overline{\mathcal{R}}(z, 0) = 1$ for any $z \leq 0$, so that if $b_p = 0$ for all $p \in \mathbb{N}^*$ then $\overline{Q}_\infty^h = Q_\infty$. \square

Remark 3.8. *We see in Lemma 3.6 that the error is zero if $z = 0$ or $\beta = 0$. This is related to Proposition 2.3 which shows that the error of the postprocessed method is zero in the linear case when $\beta = 0$. This feature permits to gain one power of h and thus one order of accuracy in the proof above. In contrast, notice that the standard linearized implicit Euler scheme has the lower order s for all $s < \bar{s}$. Indeed, under the hypotheses of Theorem 3.7, then for all h small enough, $Q_\infty - Q_{\infty, \nu}^h$ is nonnegative and using Lemma 3.4 we only need to control*

$$\begin{aligned} \text{Trace}(Q_\infty - Q_{\infty, \nu}^h) &= \frac{\sigma^2}{2} \sum_{p=1}^{+\infty} \frac{q_p}{\lambda_p + b_p} \frac{(\lambda_p + b_p)h}{2 + \lambda_p h + b_p h} \\ &= \frac{\sigma^2 h^s}{2} \sum_{p=1}^{+\infty} q_p \lambda_p^{-1+s} \frac{\lambda_p}{\lambda_p + b_p} \frac{\frac{\lambda_p + b_p}{\lambda_p^s} h^{1-s}}{2 + \lambda_p h + b_p h} \end{aligned}$$

$$\leq \frac{\sigma^2 h^s}{2} C_s \text{Trace}((-A)^{-1+s} Q).$$

Thus for all $s < \bar{s}$ that there exists $C_s \in (0, +\infty)$ such that for all h small enough

$$\left| \int_{\mathcal{H}} \varphi d\mu_\infty - \int_{\mathcal{H}} \varphi d\nu_\infty^h \right| \leq C_s \sup_{v \in \mathcal{H}} \|D^2 \varphi(v)\| \sigma^2 h^s. \quad (45)$$

It is also possible to prove that the above Theorem 3.7 (resp. (45)) gives optimal order of convergence, namely that (44) does not hold true for all $h > 0$ if $s > \bar{s} + 1$ (resp. (45) does not hold true for any $h > 0$ if $s > \bar{s}$). This fact is also supported by the numerical simulations of Figure 3 in Section 4.

3.4 Spatial regularity analysis

We show in this section that the postprocessed method yields a solution which has the same regularity in space as the exact solution, in contrast to the standard linearized implicit Euler method, which yields a solution that is too smooth. The action of the postprocessing thus not only increases the order of the convergence, but also provides a qualitatively better approximation with the correct regularity. Given a random variable $v \in \mathcal{H}$, we define the regularity of v , and of its probability distribution μ_v , by the supremum of s such that the norm $|\cdot|_s$ of \mathcal{H}^s is integrable with respect to μ_v :

$$\text{reg}(v) = \text{reg}(\mu_v) = \sup\{s \in \mathbb{R}; \int_{\mathcal{H}} |u|_s^2 \mu_v(du) < \infty\}.$$

Notice that instead of quantifying the regularity in terms of the Sobolev space \mathcal{H}^s , one could state similar results in terms of Hölder regularity. We refer to [8] for such a study of the θ -method applied to the stochastic heat equation with finite differences.

We focus for simplicity on the case $F = 0$, $\sigma = 1$, and with the initial condition $u_0 = 0$, but we emphasize that the extension to the general semilinear situation is straightforward, the regularity being determined only by the stochastic terms in our setting. First, notice that for the exact solution $u(t)$, the regularity parameter is $\text{reg}(u(t)) = \text{reg}(\mu_\infty) = \bar{s}$ for all $t > 0$, (see also Proposition 3.1). Indeed, using (4), and the Itô formula yields

$$\mathbb{E}|u(t)|_s^2 = \sum_{p=1}^{+\infty} \frac{q_p}{2\lambda_p^{1-s}} (1 - \exp(-2\lambda_p t)) \begin{cases} < +\infty & \text{if } s < \bar{s} \\ = +\infty & \text{if } s > \bar{s} \end{cases}.$$

The following proposition shows that at the discrete-time level, that the standard linearized implicit Euler method v_n in (9) and the method without processing u_n in (12) have the regularity parameter $\bar{s} + 1$, whereas the postprocessor \bar{u}_n in (13) has the correct regularity parameter \bar{s} .

Proposition 3.9. *Consider (1) with $F = 0$, $\sigma = 1$, $u_0 = 0$ and assume (3). Then, for all $h > 0$ and all $n \in \mathbb{N}^*$, $\text{reg}(v_n) = \text{reg}(u_n) = \text{reg}(\nu_\infty^h) = \text{reg}(\mu_\infty^h) = \bar{s} + 1$, whereas $\text{reg}(\bar{u}_n) = \text{reg}(\bar{\mu}_\infty^h) = \bar{s}$.*

Proof. Inspecting the proof of Proposition 3.2, we have $u_n = \ell_n$ and for all $s < \bar{s} + 1$, $h > 0$, $n \in \mathbb{N}^*$

$$\mathbb{E}|\ell_n|_{s+1}^2 \leq \frac{C}{h} \sum_{p=1}^{+\infty} q_p \lambda_p^s \frac{(\lambda_p h)^2}{(1 + \lambda_p h)^2 - 1} \leq \frac{C}{h} \text{Trace}((-A)^{s-1} Q);$$

this yields $\text{reg}(u_n) = \text{reg}(\mu_\infty^h) \geq \bar{s} + 1$. The reverse inequality is obtained with a similar lower bound. The proof for the standard linearized implicit Euler scheme v_n is similar. Now, adding the postprocessing, $\bar{u}_n = u_n + \frac{1}{2}J_3\sigma\sqrt{h}\xi_n^Q$, we obtain using the definition of J_3 ,

$$\mathbb{E}|J_3\sigma\sqrt{h}\xi_n^Q|_s^2 = h\text{Trace}((I - \frac{h}{2}A)^{-1}(-A)^s Q) = \sum_{p=1}^{+\infty} \frac{q_p}{\lambda_p^{1-s}} \frac{\lambda_p h}{1 + \frac{1}{2}\lambda_p h} \begin{cases} < +\infty & \text{if } s < \bar{s} \\ = +\infty & \text{if } s > \bar{s} \end{cases}.$$

The term $\frac{1}{2}J_3\sigma\sqrt{h}\xi_n^Q$ thus has exactly the same regularity \bar{s} as the exact solution.

This concludes the proof of $\text{reg}(\bar{u}_n) = \text{reg}(\bar{\mu}_\infty^h) = \bar{s}$. \square

4 Numerical experiments

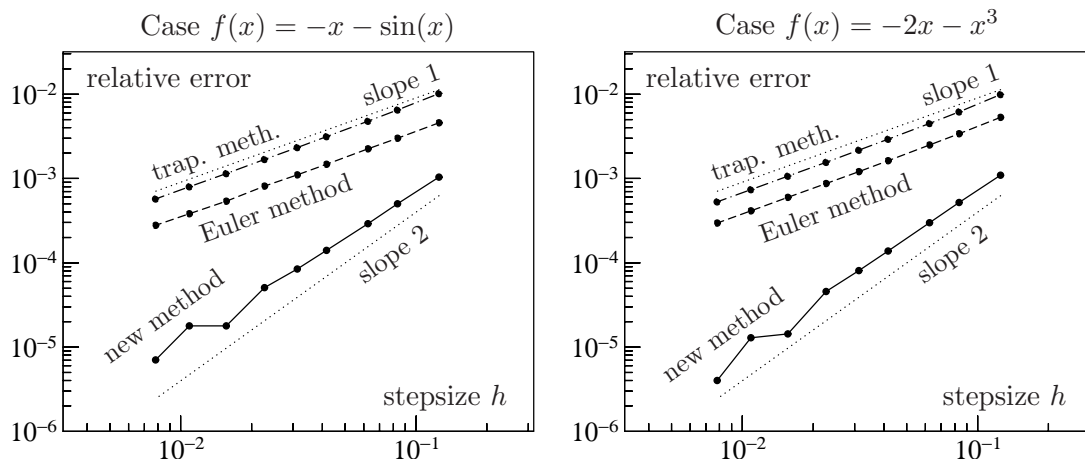


Figure 1: Comparison of the new method (solid lines) with the standard linearized implicit Euler method (dashed lines) and the trapezoidal method (dashed-dotted lines) for the scalar SDE with nonlinearity $f(u)$. Error for $\mathbb{E}(\exp(-X(T)^2))$ at final time $T = 1$ versus the stepsize h , where $1/h = 8, 12, 16, 24, 32, 44, 64, 92, 128$. Averages over 10^{10} samples.

In this section, we compare numerically the performances of the new postprocessed method (15) with the standard linearized implicit Euler method (9), and the trapezoidal method (11), both in finite and infinite dimensions.

We consider first in Figure 1 the scalar nonlinear SDE (16) with dimension $N = 1$, parameters $\lambda = \sigma = 1$ and the initial condition $X(0) = 0$. We consider the nonlinearities $f(x) = -x - \sin(x)$ and $f(x) = -2x - x^3$, respectively, and we compute the averages over 10^{10} independent trajectories with final time $T = 1$ and compare for many time stepsizes the accuracy for $\mathbb{E}(\exp(-X(T)^2)) = \int_{-\infty}^{+\infty} \exp(-x^2)\rho(x)dx$. The final time $T = 1$ is chosen large enough so that the equilibrium is reached and the exponentially decaying term $e^{-\lambda T}$ in (32) is negligible. In the left picture of Figure 1 where the nonlinearity $f(x) = -x - \sin(x)$ is Lipschitz, we observe as shown in Theorem 2.7 the expected order 2 of convergence for the new method, while the standard methods exhibit order 1 of convergence (see the reference lines with slopes 1, 2). Although our analysis in Section 2.2 applies only to globally-Lipschitz

vector fields, we observe that the excellent performances of the new method persist also in the example with non-Lipschitz nonlinearity $f(u) = -2u - u^3$ (right picture of Figure 1).

We next consider a standard finite-difference approximation $U_j(t) \simeq u(j\Delta x, t)$ of the 1D heat equation (2) with zero Dirichlet boundary conditions on a uniform grid with size $\Delta x = 1/(N + 1)$. This yields the following system of SDEs in dimension N ,

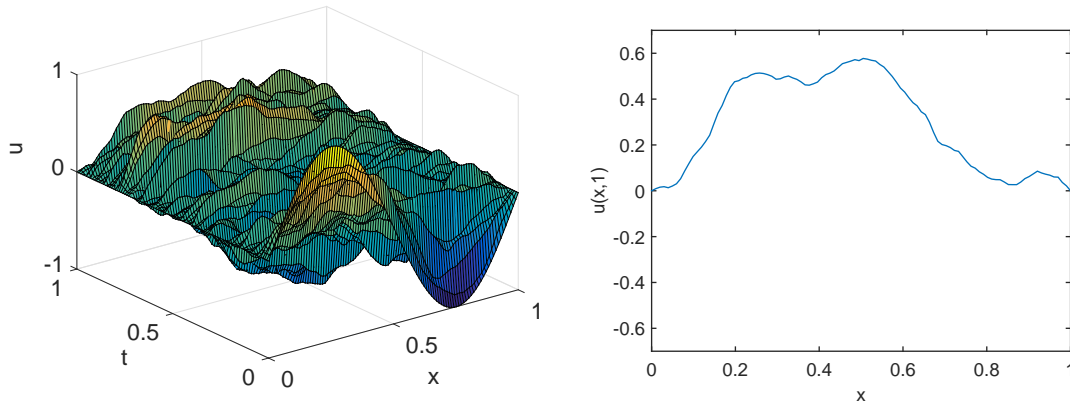
$$dX(t) = \frac{1}{\Delta x^2} \begin{pmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & & 1 & -2 \end{pmatrix} \begin{pmatrix} X^1(t) \\ X^2(t) \\ \vdots \\ X^N(t) \end{pmatrix} dt + \begin{pmatrix} f(X^1(t)) \\ f(X^2(t)) \\ \vdots \\ f(X^N(t)) \end{pmatrix} dt + \frac{1}{\sqrt{\Delta x}} \begin{pmatrix} dW^1(t) \\ dW^2(t) \\ \vdots \\ dW^N(t) \end{pmatrix},$$

where W^1, \dots, W^N are independent one-dimensional standard Wiener processes.

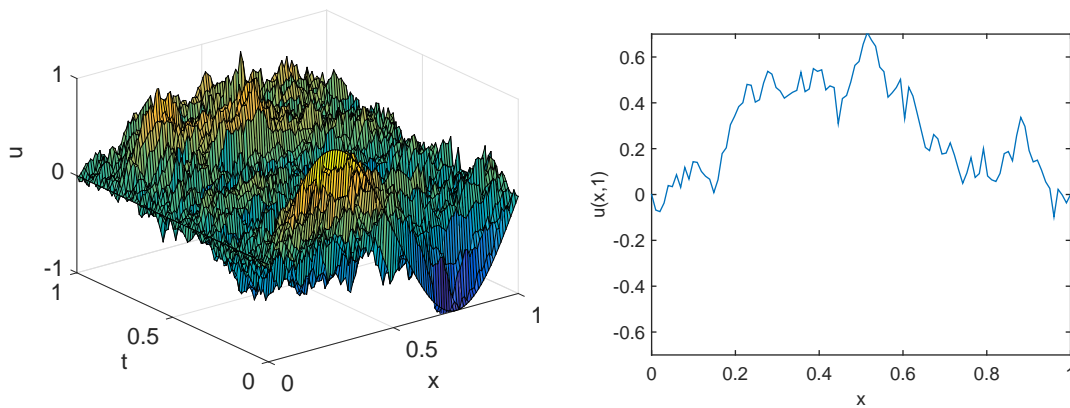
Considering $N = 100$ grid points for the space discretization, we take the initial condition $u(x, 0) = \sin(2\pi x)$ and plot in Figure 2 a sample trajectory on the time interval $(0, 1)$ for the standard linearized implicit Euler method (9) and the new method with postprocessor \bar{X}_n , using the same sets of generated random numbers for both methods. We observe that the solution of the standard linearized implicit Euler method (Fig. 2(a)) is qualitatively too smooth compared to the new method (Fig. 2(b)) with the postprocessor applied at each timestep, which corroborates the statement of Proposition 3.9 in Section 3.4. Notice that the trajectory for the new method without applying the postprocessor would look very similar to that of the standard linearized implicit Euler method. The spatial regularity observed in Figure 2(b) is the same as the one of a diffusion process driven by Brownian motion, conditioned to be zero at initial and final times. This property is not surprising, see [16, 15] for a study of the link between the law of a conditioned diffusion and invariant distributions of SPDEs.

We finally plot in Figure 3 the error at final time $T = 1$ for the quantity $\mathbb{E}(\exp(-\|u\|_{L^2(0,1)}^2))$ where we use the approximation $\|u\|_{L^2(0,1)}^2 \simeq \Delta x \sum_{j=1}^N (X^j)^2$. We use arbitrarily the initial condition $u(x, 0) = 0$ and we compute the averages over 10^9 independent trajectories so that the Monte-Carlo errors become negligible. The reference solution is computed using the new method with stepsize $h = 1/512$. We consider respectively, the cases of various nonlinearities $f(u) = 0$, $f(u) = -u$, $f(u) = -u - \sin(u)$, $f(u) = -2u - u^3$. We observe in all cases an order of convergence 1/2 for the standard linearized implicit Euler method, while the trapezoidal method has order 1. In contrast to the standard linearized implicit Euler method, the new method with postprocessor has a much better accuracy by a factor 15 – 250 in the range of stepsizes considered. It has a zero bias for $f(u) = 0$ and order 3/2 in the linear case $f(u) = -u$, as proved in Theorem 3.7. We observe that the order of convergence persists in the nonlinear case $f(u) = -u - \sin(u)$, and the excellent accuracy persists in the non-Lipschitz case $f(u) = -2u - u^3$, although an order reduction can be observed.

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(a) Standard linearized implicit Euler method, sample trajectory $u(x, t)$ and corresponding profile at final time $t = 1$.



(b) New method with postprocessor, sample trajectory $u(x, t)$ and corresponding profile at final time $t = 1$.

Figure 2: Samples of realisation of the stochastic nonlinear heat equation (2) with $f(u) = -u - \sin(u)$ using the standard linearized implicit Euler method and the new method with postprocessor. With $N = 100$ space grid points and timestep size $h = 1/100$.

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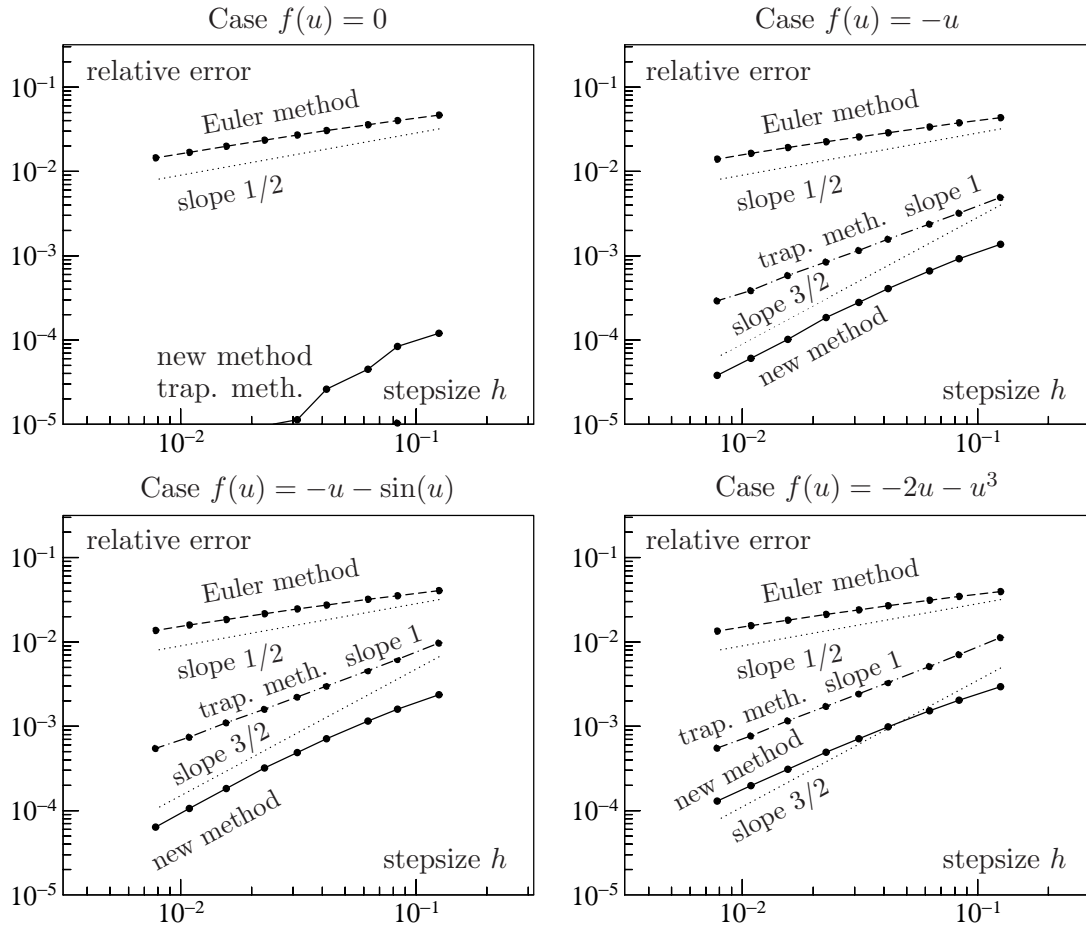


Figure 3: Comparison of the new method (solid lines) with the standard linearized implicit Euler method (dashed lines) and the trapezoidal method (dashed-dotted lines) for the stochastic heat equation (2) with nonlinearity $f(u)$ discretized in space with $N = 100$ grid points. Error for $\mathbb{E}(\exp(-\|u(T)\|^2))$ at final time $T = 1$ versus the stepsize h , where $1/h = 8, 12, 16, 24, 32, 44, 64, 92, 128$. Averages over 10^9 samples.

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