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► **To cite this version:**

Thibault Espinasse, Paul Rochet. On relations between connected and self-avoiding walks on a graph. 2015. hal-01153191v1

HAL Id: hal-01153191

<https://hal.science/hal-01153191v1>

Preprint submitted on 19 May 2015 (v1), last revised 21 Dec 2015 (v2)

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On relations between connected and self-avoiding walks on a graph

Thibault Espinasse* and Paul Rochet†

Abstract

The characterization of a graph via the variable adjacency matrix enables one to define a partially ordered relation on the walks. Studying the incidence algebra on this poset reveals unsuspected relations between connected and self-avoiding walks on the graph. These relations are derived by considering truncated versions of the characteristic polynomial of variable adjacency matrix, resulting in a collection of matrices whose entries enumerate the self-avoiding walks of length ℓ from one vertex to another.

Keywords: Digraph; poset; characteristic polynomial; variable adjacency matrix; incidence algebra.

MSC: 05C22, 05C30, 05C38

1 Introduction

Directed graphs, or *digraphs*, have been extensively used in the literature as mathematical models to describe actual phenomena such as social interactions [4], road traffic [8], physical processes [3] or random walks [2] among many others. In most models, it is convenient to allocate weights to each edge of the graph in order to incorporate some additional information. For example, a weight w_{ij} between two nodes v_i, v_j may serve defining the transition probability of a random walk, the speed limit or the type of road in the traffic network and so on. A finite graph of N nodes is then characterized by a $N \times N$ matrix $W = (w_{ij})_{i,j=1,\dots,N}$, called *variable adjacency matrix*, which accounts for the level of interaction between edges.

A walk on the graph, consisting in a succession of contiguous states, can sometimes be used to describe the evolution of a phenomenon over time. Identifying each edge with its associated weight w_{ij} , a non-oriented walk of length $\ell \geq 1$ can be viewed as a product of ℓ entries, i.e., a degree ℓ monomial in the variables w_{ij} . The variable adjacency matrix then provides a practical tool to handle walks on the graph, as they can be derived from analytical transformations of W . For instance, the (i, j) -entry of W^2 , given by $W_{ij}^2 = \sum_k w_{ik}w_{kj}$, enumerates all connected walks of length 2 from v_i to v_j . The introduction of the variable adjacency matrix W to describe the

*Université Lyon 1 Claude Bernard, laboratoire de mathématiques Camille Jordan

†Université de Nantes, laboratoire de mathématiques Jean Leray, paul.rochet@univ-nantes.fr

walks on a graph goes back to the 60's. In [9] and [13], the spectral properties of a graph are investigated via the determinant and characteristic polynomial of W . Digraphs also provide a useful tool to compute the determinant and minors of sparse matrices, as discussed in [12]. For general results on spectral graph theory, we refer to [5, 6].

In [13], the author shows that the coefficient of degree $N - \ell$ of the characteristic polynomial of W can be interpreted in term of the self-avoiding cycles of length ℓ . In this paper, we derive a similar result concerning the self-avoiding walks (precise definitions of self-avoiding cycle and self-avoiding walk are given in Section 2). We construct a collection of polynomials of W which enumerate the self-avoiding walks of a given length, for all pairs of vertices. The polynomials are obtained as Cauchy products of the characteristic polynomial coefficients with the sequence of successive powers of W .

The analytical expression linking the self-avoiding walks and the connected walks on the graph hides a deeper connection when considering each walk individually. Precisely, the relation can be investigated in the partially ordered set, or *poset*, formed by the walks on the graph. In this context, unsuspected combinatorial properties arise when studying functions of the walks in the incidence algebra of this poset. In particular, we show that the number of different ways to travel a non-oriented connected walk can be expressed in term of its self-avoiding divisors via a Mobius-like inversion of the Dirichlet convolution on this poset. Another result, relating the multiplicity of a walk to its decompositions into self-avoiding components is then derived.

The paper is organized as follows. Definitions are introduced in Section 2 as well as the preliminary result. Section 3 is devoted to the study of the different relations between self-avoiding and connected walks in a poset, where many combinatorial properties arise. The results are investigated on specific examples in Section 4.

2 Notations and preliminary results

Let $G = (V, E)$ be a directed graph with finite vertex set $V = \{v_1, \dots, v_N\}$ and edge set E . The adjacency matrix of G is defined as the $N \times N$ matrix A with entries a_{ij} equal to one if v_i is connected to v_j and zero otherwise. A directed graph, or digraph, is a graph for which A is not symmetric, meaning that v_i can be connected to v_j without v_j being connected to v_i . In the literature, the adjacency matrix has been widely used to derive numerous properties of a graph. For instance, the power A^ℓ gives the number of connected walks of length ℓ from one vertex to another. When one is interested in each walk specifically, a useful tool is to allocate variables to each non-zero entry of the adjacency matrix. In this way, the graph G can be given a matrix representation $W = (w_{ij})_{i,j=1,\dots,N}$, setting $w_{ij} = 0$ whenever there is no edge from v_i to v_j .

The variable adjacency matrix provides a more faithful description of the graph, as it allows to distinguish between different walks. A walk of length ℓ then corresponds to a degree ℓ monomial in the variables w_{ij} that satisfies certain properties. A connected walk m of length ℓ from v_i to v_j writes as a product $m = w_{i i_1} w_{i_1 i_2} \dots w_{i_{\ell-1} j}$ of ℓ contiguous edges. The walk m is closed if $i = j$ and open otherwise. Moreover, m is *self-avoiding* if it does not cross the same vertex twice,

that is, if the indices $i, i_1, \dots, i_{\ell-1}, j$ are mutually different (with the possible exception $i = j$ if m is closed). A *simple walk*, or *path*, is a walk that is both connected and self-avoiding.

The cycle-erasing procedure of Lawler [11] shows that a connected walk from v_i to v_j can always be decomposed as the product of a simple path from v_i to v_j and cycles. However, the reverse is not true in general as a given product of simple walks may not be connected. In this paper, we use an extended definition of a walk by relaxing the connectedness condition. Precisely, a monomial is considered a walk from v_i to v_j if it is the product of a simple walk from v_i to v_j and cycles. This definition allows for instance to consider the monomial $m = w_{12}w_{34}w_{45}w_{53}$, illustrated in Figure 1, as a walk of length 4 from v_1 to v_2 , although it is not connected.

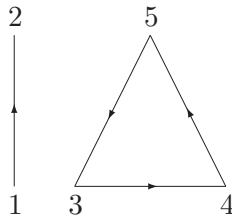


Figure 1: Non-connected walk $m = w_{12}w_{34}w_{45}w_{53}$ from v_1 to v_2 , composed of the simple path w_{12} and the simple cycle $w_{34}w_{45}w_{53}$. Its length is $\ell(m) = 4$ and its number of connected component $n(m) = 2$.

The set of self-avoiding walks (not necessarily connected) will be denoted by \mathcal{S} in the sequel. We may specify the end-vertices of the walks in index and/or their length in exponent, e.g., \mathcal{S}_{ij}^ℓ refers to the set of self-avoiding walks from v_i to v_j of length ℓ . As for connected walks, a walk is open if $i \neq j$, and closed otherwise. The number of connected components of a self-avoiding walk m is denoted by $n(m)$. Moreover, we let \mathcal{C} designate the set of self-avoiding cycles and \mathcal{M} the set of monomials. Similarly, we may specify the degree of the monomials as an exponent, e.g., \mathcal{M}^ℓ for the monomials of degree ℓ . We emphasize that while all walks of length ℓ are in \mathcal{M}^ℓ , most monomials are not walks.

Using this notational convention, the ℓ -th power of W enumerates with multiplicity the connected walks of length ℓ on the graph. Indeed, the entries of the matrix W^ℓ can be expressed as homogenous polynomials of degree ℓ in the variables w_{ij} . The coefficient associated to a monomial m in W_{ij}^ℓ corresponds to the number of ways to travel m from v_i to v_j , that is, the number of ways to write m as a succession of adjacent vertices starting from v_i and ending with v_j . In the sequel, we denote by $f_{ij}(m)$ this coefficient so that we have

$$W_{ij}^\ell = \sum_{m \in \mathcal{M}^\ell} f_{ij}(m)m. \quad (1)$$

Remark that by this definition, $f_{ij}(m)$ is zero whenever m is not a connected walk from i to j . Moreover, if m is an open walk, there is at most one couple (i, j) for which $f_{ij}(m)$ is non-zero. This property is no longer verified for a closed walk c , in which case $f_{ii}(c)$ may take different positive values for different nodes v_i crossed by c . Finally, observe that $f_{ij}(m) = 0$ for all (i, j) if m is not a walk.

In this paper, we aim to exhibit a relation between the connected walks on the graph and the self-avoiding walks. It is widely established that self-avoiding walks play an essential part in the study of the combinatorial and spectral properties of a graph. This is explained by the simple fact that a self-avoiding cycle can be identified with a permutation σ on a subset of $\{1, \dots, N\}$, writing $c = \prod_{i: v_i \in c} w_{i\sigma(i)}$. For instance, the determinant of W can be expressed in term of self-avoiding cycles by

$$\det(W) = \sum_{\sigma \in S_N} \text{sgn}(\sigma) w_{1\sigma(1)} \dots w_{N\sigma(N)}$$

where S_N denotes the set of permutations of $\{1, \dots, N\}$ and $\text{sgn}(\cdot)$ the signature. Each monomial $c_\sigma := w_{1\sigma(1)} \dots w_{N\sigma(N)}$ can be viewed as a self-avoiding cycle of length N whose number of connected components $n(c_\sigma)$ is linked to the signature of the permutation through the identity

$$\text{sgn}(\sigma) = (-1)^{N-n(c_\sigma)} \iff (-1)^{n(c_\sigma)} = (-1)^N \text{sgn}(\sigma).$$

A more general formula, given in Theorem 1 in [13], links the coefficients ψ_k of the characteristic polynomial of W , $\chi_W(\lambda) = \det(\lambda I - W) = \sum_{k=0}^N \psi_k \lambda^{N-k}$ with the self-avoiding cycles of length k by

$$\psi_0 = 1, \quad \psi_k = \sum_{c \in \mathcal{C}^k} (-1)^{n(c)} c, \quad k = 1, \dots, N.$$

Remark that the sequence ψ_k may as well be defined for all $k \geq 0$, with ψ_k being trivially zero as soon as $k > N$. The presence of the coefficient $(-1)^{n(c)}$, reminiscent of a Mobius function, is particularly interesting. As we show further, the function μ defined over \mathcal{M} by $\mu(1) = 1$ and

$$\mu(m) := \begin{cases} (-1)^{n(m)} & \text{if } m \in \mathcal{C} \\ 0 & \text{otherwise,} \end{cases} \quad (2)$$

is deeply related to a Mobius function in a specific partially ordered set (see the discussion in Section 3 for more details). Although μ takes non-zero values only for self-avoiding cycles, some of its properties have direct repercussions on open walks. This is due to the fact that a self-avoiding open walk m between two different nodes v_i, v_j can be expressed as a particular self-avoiding cycle to which the edge w_{ji} has been removed. Actually, one verifies easily the following equivalence for $i \neq j$

$$m \in \mathcal{S}_{ij} \iff mw_{ji} \in \mathcal{C}, \quad (3)$$

where we recall that \mathcal{S}_{ij} is the set of self-avoiding walks from v_i to v_j and \mathcal{C} the set of self-avoiding cycles. In the case $i = j$, one can state a similar equivalence, namely

$$m \in \mathcal{C}_{-i} \iff mw_{ii} \in \mathcal{C}, \quad (4)$$

where \mathcal{C}_{-i} denotes the set of self-avoiding cycles that do not cross v_i . We have in this case $\mu(m) = -\mu(mw_{ii})$ due to the addition of a connected component. This leads us to considering the functions $g_{ij} : \mathcal{M} \mapsto \{-1, 0, 1\}$ defined for all $i, j = 1, \dots, N$ by

$$g_{ij}(m) := -\mu(mw_{ji}) = \begin{cases} (-1)^{n(m)} & \text{if } i = j \text{ and } m \in \mathcal{C}_{-i}, \\ (-1)^{n(m)+1} & \text{if } i \neq j \text{ and } m \in \mathcal{S}_{ij} \\ 0 & \text{otherwise.} \end{cases} \quad (5)$$

If $i \neq j$, the function g_{ij} only takes non-zero values for self-avoiding walks from v_i to v_j . In particular, $g_{ij}(1) = 0$ and $g_{ij}(m) = 1$ if m is a simple path. On the other hand, $g_{ii}(m)$ is non-zero only if m is closed and does not cross v_i . We have for instance $g_{ii}(1) = -\mu(w_{ii}) = 1$. Remark moreover that $g_{ij}(m)$ is null for all monomial m of degree $\ell(m) \geq N$ since a self-avoiding walk on the graph has maximal length N .

Similarly as W^ℓ was defined via Equation (1), we construct the matrix $X^{(\ell)}$ of homogenous, degree ℓ polynomials obtained with the coefficients g_{ij} ,

$$X_{ij}^{(\ell)} := \sum_{m \in \mathcal{M}^\ell} g_{ij}(m)m, \quad i, j = 1, \dots, N. \quad (6)$$

The matrix $X^{(\ell)}$ is defined for all $\ell \in \mathbb{N}$ with in particular $X^{(0)} = I$ and $X^{(\ell)} = 0$ for $\ell \geq N$. All the results established in this paper follow from the next lemma.

Lemma 2.1 For $\ell = 1, \dots, N$,

$$X^{(\ell)} = \sum_{k=0}^{\ell} \psi_k W^{\ell-k} \quad (7)$$

where the $\psi_k, k = 0, 1, \dots, N$ are the coefficients of the characteristic polynomial of W .

Proof. The proof relies on the identity $\text{adj}(I-W) = \det(I-W) \times (I-W)^{-1}$, where $\text{adj}(A)$ denotes the adjugate of a matrix $A = (a_{ij})_{i,j=1,\dots,N}$. We recall that the entries of the adjugate are given by

$$(\text{adj}(A))_{ij} = \det(A^{(ji)}), \quad i, j = 1, \dots, N,$$

where $A^{(ji)}$ is the matrix obtained by setting $a_{ji} = 1$, $a_{ki} = 0$ for $k \neq j$ and $a_{jk} = 0$ for $k \neq i$ in A . Assume that $\|W\| < 1$ and let $A = I - W$. We have

$$\text{adj}(A) = \text{adj}(I-W) = \det(I-W) \times (I-W)^{-1} = \sum_{k=0}^N \psi_k \times \sum_{k \geq 0} W^k = \sum_{\ell \geq 0} \sum_{k=0}^{\ell} \psi_k W^{\ell-k}, \quad (8)$$

letting $\psi_k = 0$ for $k > N$. We now need to compute the values $(\text{adj}(A))_{ij} = \det(A^{(ji)})$.

First consider the case $i = j$. Since $A = I - W$, the conditions $a_{ii} = 1$ and $a_{ki} = a_{ik} = 0$ for $k \neq i$ reduce to $w_{ii} = 0$ and $w_{ki} = w_{ik} = 0$ for $k \neq i$. Plugging these values into the identity $\det(I-W) = \sum_{c \in \mathcal{C}} \mu(c)c$ gives

$$\det((I-W)^{(ii)}) = (\text{adj}(I-W))_{ii} = \sum_{\substack{c \in \mathcal{C} \\ i \notin c}} \mu(c)c = \sum_{m \in \mathcal{M}} g_{ii}(m)m. \quad (9)$$

Now consider the case $i \neq j$. Going back to Equation (8), we see that the (i, j) entry of $\text{adj}(I-W)$ satisfies

$$(\text{adj}(I-W))_{ij} = \sum_{k=0}^N \psi_k \times \sum_{k \geq 0} (W^k)_{ij} = \sum_{c \in \mathcal{C}} \mu(c)c \times \sum_{m \in \mathcal{M}} f_{ij}(m)m = \sum_{(c,m) \in \mathcal{C} \times \mathcal{M}} \mu(c)f_{ij}(m)cm.$$

Thus, the only monomials m' with non-zero coefficient in $(\text{adj}(\mathbf{I}-W))_{ij}$ are those that can be written as a product $m' = cm$ with c a self-avoiding cycle and m a connected walk from v_i to v_j . We point out that such monomials remain walks from v_i to v_j .

For $i \neq j$ and $A = \mathbf{I}-W$, the conditions $a_{ji} = 1$, $a_{ki} = 0$ for $k \neq j$ and $a_{jk} = 0$ for $k \neq i$ become $w_{ji} = -1$, $w_{ki} = w_{jk} = 0$ for $k \neq i, j$ and $w_{ii} = w_{jj} = 1$. Plugging these values into a self-avoiding cycle c yields a walk from v_i to v_j if and only if w_{ji} divides c . By identification, it follows that only the walks c such that $w_{ji}|c$ are involved in the expression of $(\text{adj}(\mathbf{I}-W))_{ij}$ and the other cycles, divided by w_{ii} , w_{jj} or both, cancel out (this can also be checked explicitly by considering each situation separately). Thus, plugging these values into $\det(\mathbf{I}-W) = \sum_{c \in \mathcal{C}} \mu(c)c$ yields, in view of (3),

$$(\text{adj}(\mathbf{I}-W))_{ij} = - \sum_{\substack{c \in \mathcal{C} \\ w_{ji}|c}} \mu(c) \frac{c}{w_{ji}} = - \sum_{m \in \mathcal{S}_{ij}} \mu(mw_{ji})m = \sum_{m \in \mathcal{M}} g_{ij}(m)m. \quad (10)$$

By (9) and (10), we obtain $\text{adj}(\mathbf{I}-W) = \sum_{\ell \geq 0} X^{(\ell)}$ which, combined with (8), gives

$$\sum_{\ell \geq 0} X^{(\ell)} = \sum_{\ell \geq 0} \sum_{k=0}^{\ell} \psi_k W^{\ell-k}.$$

Identifying the terms with equal degrees, we get $X^{(\ell)} = \sum_{k=0}^{\ell} \psi_k W^{\ell-k}$, ending the proof. \square

Remark. The proof of the lemma can be derived as a direct consequence of the more general formula

$$\text{adj}(\lambda \mathbf{I}-W) = \det(\lambda \mathbf{I}-W) \times (\lambda \mathbf{I}-W)^{-1},$$

holding for suitable $\lambda \in \mathbb{R}$. This identity is quite interesting as it establishes a connection between the characteristic polynomial $\lambda \mapsto \det(\lambda \mathbf{I}-W)$ and the resolvent $(\lambda \mathbf{I}-W)^{-1}$. The result stated in the lemma follows by expanding the above equality and identifying the coefficient of λ^ℓ on both sides.

Because $g_{ij}(m)$ is trivially zero when $\ell(m) = N$, we recover Cayley-Hamilton's theorem directly from the case $\ell = N$ in the lemma. Moreover, computing the trace of $X^{(\ell)}$ in Equation (7) gives a direct proof of the identity

$$\psi_\ell = -\frac{1}{\ell} \sum_{k=0}^{\ell-1} \psi_k \text{tr}(W^{\ell-k}), \quad (11)$$

linking the coefficients ψ_ℓ of the characteristic polynomial to the trace of powers of W . Different proofs of this result can be found in [15] and [10]. To prove it using Lemma 2.1, simply observe that $X^{(\ell)}$, as defined via (5), satisfies

$$\text{tr}(X^{(\ell)} - \psi_\ell \mathbf{I}) = \sum_{i=1}^N \left(\sum_{c \in \mathcal{C}_{-i}^\ell} (-1)^{n(c)} c - \sum_{c \in \mathcal{C}^\ell} (-1)^{n(c)} c \right) = - \sum_{i=1}^N \sum_{c \in \mathcal{S}_{ii}^\ell} (-1)^{n(c)} c = -\ell \psi_\ell$$

since each cycle appears exactly ℓ times when summing. Hence, Equation (11) follows directly from the equality $X^{(\ell)} - \psi_\ell \mathbf{I} = \sum_{k=0}^{\ell-1} \psi_k W^{\ell-k}$.

3 Combinatorial properties of walks in a poset

Lemma 2.1 highlights some combinatorial properties on the graph by investigating each walk separately. The monomials form a locally finite partially ordered set, or *poset*, when endowed with division: $d \in \mathcal{M}$ divides m , denoted by $d|m$, if and only if there exists $d' \in \mathcal{M}$ such that $m = dd'$. The functions on this poset form an incidence algebra with respect to the Dirichlet convolution, defined between two functions $f, g : \mathcal{M} \rightarrow \mathbb{R}$ by

$$f * g(m) = \sum_{d|m} f(d)g\left(\frac{m}{d}\right), \quad m \in \mathcal{M},$$

where the sum is taken over all divisors d of m , including the trivial walk 1 and m itself. One verifies easily that the Dirichlet convolution is associative, commutative and distributive over addition. The function δ defined on \mathcal{M} by $\delta(1) = 1$ and $\delta(m) = 0$ for all $m \neq 1$ is the identity element for this operation as we have, for any function f on \mathcal{M} , $f * \delta = \delta * f = f$. We refer to [14] for a more comprehensive study on this subject.

Interesting combinatorial properties arise from the poset structure of monomials, after identifying each walk in Equation (7). In particular, we derive a relation between the functions f_{ij} and g_{ij} defined in the previous section.

Proposition 3.1 *For all $i, j = 1, \dots, N$, $g_{ij} = \mu * f_{ij}$.*

Proof. Recall that for all $\ell \geq 0$ and for all $i, j = 1, \dots, N$

$$X_{ij}^{(\ell)} = \sum_{m \in \mathcal{M}^\ell} g_{ij}(m)m, \quad (W^\ell)_{ij} = \sum_{m \in \mathcal{M}^\ell} f_{ij}(m)m \quad \text{and} \quad \psi_\ell = \sum_{m \in \mathcal{M}^\ell} \mu(m)m.$$

Considering each entry separately in the equality $X^{(\ell)} = \sum_{k=0}^{\ell} \psi_k W^{\ell-k}$ leads to

$$\sum_{m \in \mathcal{M}^\ell} g_{ij}(m)m = \sum_{k=0}^{\ell} \sum_{m_1 \in \mathcal{M}^k} \sum_{m_2 \in \mathcal{M}^{\ell-k}} \mu(m_1)f_{ij}(m_2)m_1m_2.$$

We now identify the coefficients of each monomial m (on the left-hand side) with its different decompositions as a product m_1m_2 (on the right-hand side). We obtain

$$g_{ij}(m) = \sum_{d|m} \mu(d)f_{ij}\left(\frac{m}{d}\right), \quad m \in \mathcal{M},$$

yielding the wanted result. □

This proposition reveals a somewhat unexpected relation between the function g_{ij} , involving the number of connected components of a self-avoiding walk, and the multiplicity f_{ij} of its connected divisors. Taking a closer look, the result is in fact not really surprising if m is self-avoiding. Indeed, a self-avoiding walk m from v_i to v_j has exactly one divisor d such that m/d is

a connected walk from v_i to v_j . Thus, the convolution $\mu * f_{ij}$ is calculated over only one non-zero element and the equality is easily verified in this case. The result is actually more interesting if m is not self-avoiding as it yields in this case the non-trivial identity

$$\forall m \in \mathcal{M} \setminus \mathcal{S}, \sum_{d|m} \mu(d) f_{ij} \left(\frac{m}{d} \right) = 0.$$

Clearly, the function μ is a key feature to understand the combinatorial properties of this poset. The fact that $\mu(1) = 1 \neq 0$ ensures it is invertible through the Dirichlet convolution, and its inverse β is the unique function characterized by $\mu * \beta = \beta * \mu = \delta$. A reversed relation, expressing f_{ij} in function of g_{ij} can then be derived easily, noticing that

$$f_{ij} = (\beta * \mu) * f_{ij} = \beta * (\mu * f_{ij}) = \beta * g_{ij}.$$

This relation turns out to be particularly important for our purposes, as we show that β satisfies interesting properties. In particular, we establish in the next proposition an expression of $\beta(m)$ that involves the number of appearances of each edge and vertex in m . Denote by \mathcal{D} the set of closed walks on the graph, i.e., walks that can be written as a product of simple cycles (walks in \mathcal{D} are not necessarily connected nor self-avoiding). We use the notation, for $c \in \mathcal{D}$,

$$\tau_{ij}(c) := \max\{p \geq 0 : w_{ij}^p | c\} \quad \text{and} \quad \tau_i(c) := \sum_{j=1}^N \tau_{ij}(c) \quad , i, j = 1, \dots, N,$$

denoting respectively the multiplicity of w_{ij} in c and the number of times v_i is traveled by c .

Theorem 3.2 *The function β is null over $\mathcal{M} \setminus \mathcal{D}$ and satisfies for all $c \in \mathcal{D}$,*

$$\beta(c) = \prod_{i=1}^N \frac{\tau_i(c)!}{\tau_{i1}(c)! \times \dots \times \tau_{iN}(c)!}.$$

Proof. We will prove that the function β as defined in the theorem is the inverse of μ through the Dirichlet convolution. First, let $m \notin \mathcal{D}$. Because \mathcal{D} is closed by multiplication, we know that if $d \in \mathcal{D}$ and d divides m , then $m/d \notin \mathcal{D}$. Hence, one verifies easily

$$\mu * \beta(m) = \sum_{d|m} \mu(d) \beta \left(\frac{m}{d} \right) = 0,$$

noticing that μ is null over $\mathcal{M} \setminus \mathcal{D}$. The case $c = 1$ being trivial, we now focus on the case $c \in \mathcal{D}$ with $c \neq 1$. Since $\mu(d) = 0$ if d is not a self-avoiding cycle, we can write

$$\mu * \beta(c) = \sum_{d|c} \mu(d) \beta \left(\frac{c}{d} \right) = \sum_{\substack{d|c \\ d \in \mathcal{C}}} \mu(d) \beta \left(\frac{c}{d} \right).$$

Let $d \neq 1$ be a self-avoiding cycle dividing c and denote by σ_d the permutation over $\{i : v_i \in d\}$ associated to d , i.e. such that $d = \prod_{i \in d} w_{i\sigma_d(i)}$. Since d is self-avoiding, remark that $\tau_i(c/d) =$

$\tau_i(c) - 1$ if $i \in d$ and $\tau_i(c/d) = \tau_i(c)$ if $i \notin d$. Similarly, $\tau_{i\sigma_d(i)}(c/d) = \tau_{i\sigma_d(i)}(c) - 1$ for $i \in d$ and $\tau_{ij}(c/d) = \tau_{ij}(c)$ otherwise. We obtain

$$\beta(c) = \prod_{i=1}^N \frac{\tau_i(c)!}{\tau_{i1}(c)! \times \dots \times \tau_{iN}(c)!} = \prod_{i \in d} \frac{\tau_i(c)}{\tau_{i\sigma_d(i)}(c)} \times \beta\left(\frac{c}{d}\right).$$

Including the case $d = 1$, we get

$$\sum_{d|c} \mu(d) \beta\left(\frac{c}{d}\right) = \beta(c) \left(1 + \sum_{\substack{d|c \\ d \in \mathcal{C} \setminus \{1\}}} \mu(d) \prod_{i \in d} \frac{\tau_{i\sigma_d(i)}(c)}{\tau_i(c)} \right).$$

Recall that for $d \in \mathcal{C}^k$, $\mu(d) = (-1)^k \text{sgn}(\sigma_d)$. Let $\ell = \#\{i : v_i \in c\}$ denote the number of different nodes in c . The previous equality becomes, regrouping the divisors with equal lengths,

$$\sum_{d|c} \mu(d) \beta\left(\frac{c}{d}\right) = \beta(c) \left(1 + \sum_{k=1}^{\ell} (-1)^k \sum_{\substack{d|c \\ d \in \mathcal{C}^k}} \text{sgn}(\sigma_d) \prod_{i \in d} \frac{\tau_{i\sigma_d(i)}(c)}{\tau_i(c)} \right).$$

Now consider the $\ell \times \ell$ matrix $M(c)$ with entries $\tau_{ij}(c)/\tau_i(c)$, $i, j \in c$. By identifying each self-avoiding closed divisor d of c with its corresponding permutation σ_d , we recognize in the above expression the characteristic polynomial of $M(c)$ taken at $\lambda = 1$,

$$1 + \sum_{k=1}^{\ell} (-1)^k \sum_{\substack{d|c \\ d \in \mathcal{C}^k}} \text{sgn}(\sigma_d) \prod_{i \in d} \frac{\tau_{i\sigma_d(i)}(c)}{\tau_i(c)} = \det(\mathbf{I} - M(c)).$$

Since $M(c)$ is stochastic, $\det(\mathbf{I} - M(c))$ is clearly zero, which ends the proof. \square

The coefficient $\beta(c)$ corresponds to the number of arrangements of the edges in c , regrouped by their initiating vertex. Indeed, the multinomial coefficient

$$\frac{\tau_i(c)!}{\tau_{i1}(c)! \times \dots \times \tau_{iN}(c)!}$$

counts the ways of ordering the edges initiating from v_i in c , accounting for their multiplicity $\tau_{ij}(c)$. Considering all configurations for each vertex in c recovers the coefficient $\beta(c)$. So, this result reveals that β enumerates the different ways to visit a cycle. The fact that both β and g_{ij} are easily tractable makes the identity $f_{ij} = \beta * g_{ij}$ particularly interesting from a computational point of view.

This result points out some interesting properties of β , most of which are not straightforward from its initial definition as the inverse of μ through the Dirichlet convolution. The first immediate consequence is that β is non-negative and more importantly, it is positive over the set \mathcal{D} of closed walks. Secondly, $\beta(c)$ is equal to one if c is a self-avoiding cycle, i.e. if $c \in \mathcal{C}$. This

condition is sufficient but not necessary, as we have for instance $\beta(c^2) = 1$ as soon as $\beta(c) = 1$. A third consequence is that β is non-decreasing over closed walks with respect to multiplication, which can be stated formally as: $\forall c_1, c_2 \in \mathcal{D}, \beta(c_1 c_2) \geq \max\{\beta(c_1), \beta(c_2)\}$. Finally, β is multiplicative over decompositions on distinct cycles. Indeed, if c can be written as the product of say $p \geq 2$ mutually disjoint (i.e. with no common vertex) components $c_1, \dots, c_p \in \mathcal{D}$, then $\beta(c) = \beta(c_1) \dots \beta(c_p)$. This property is reminiscent of the multiplicity of arithmetic functions over co-prime integers (see for instance [1]). In this framework, two cycles c_1, c_2 can be considered co-prime if they share no common vertex. The multiplicity property of β is then inherited from the multiplicity of its inverse μ .

Remark. The properties of μ and β suggest the existence of a deeper structure involving only the closed walks on the graph. In fact, the set \mathcal{D} is closed by division and defines a poset on which the atoms are the simple cycles. Every element of \mathcal{D} can be decomposed as a product of atoms although this decomposition is not necessarily unique. In [7], the authors consider the non-commutative operation of *nesting* to achieve the unicity of the prime decomposition. In our setting, $\beta(c)$ might correspond to the number of different non-commutative representations of c . If this is the case, μ can be defined as the inverse of the unity in the set of non-commutative representations of cycles, which justifies our notation. Nevertheless, these questions are beyond the scope of our paper and will be subject to further investigations.

A different expression for β can be derived from the inverse relation in Lemma 2.1, writing W^ℓ in terms of the $X^{(k)}, k = 0, 1, \dots, \ell$. This result is given as a corollary.

Corollary 3.3 For $\ell \in \mathbb{N}$,

$$W^\ell = \sum_{k=0}^{\ell} \phi_k X^{(\ell-k)}, \quad (12)$$

where the coefficients $\phi_0, \phi_1, \dots, \phi_N$ are defined by

$$\phi_0 = 1, \quad \phi_k = \sum_{k_1 + \dots + k_p = k} (-1)^p \psi_{k_1} \dots \psi_{k_p}, \quad k = 1, \dots, N. \quad (13)$$

Before proving the result, let us clarify that the ϕ_k 's are defined by taking the sum over all compositions of k , that is, all positive tuples (k_1, \dots, k_p) such that $k_1 + \dots + k_p = k$, for all $p = 1, \dots, k$ (two tuples composed of the same integers k_1, \dots, k_p but in different orders are to be counted twice).

Proof. We start from the identity

$$\sum_{\ell \geq 0} X^{(\ell)} = \sum_{k=0}^N \psi_k \times \sum_{k \geq 0} W^k \iff \sum_{\ell \geq 0} W^\ell = \frac{1}{\sum_{k=0}^N \psi_k} \sum_{k \geq 0} X^{(k)}.$$

We use the formal series expansion

$$\frac{1}{\sum_{k=0}^N \psi_k} = \frac{1}{1 + \sum_{k=1}^N \psi_k} = \sum_{p \geq 0} (-1)^p (\sum_{k=1}^N \psi_k)^p.$$

By regrouping the terms of equal degree, we get

$$\frac{1}{\sum_{k=0}^N \psi_k} = 1 + \sum_{k \geq 1} \sum_{k_1 + \dots + k_p = k} (-1)^p \psi_{k_1} \dots \psi_{k_p} = \sum_{k \geq 0} \phi_k. \quad (14)$$

Hence,

$$\sum_{\ell \geq 0} W^\ell = \sum_{k \geq 0} \phi_k \times \sum_{k \geq 0} X^{(k)} = \sum_{\ell \geq 0} \sum_{k=0}^{\ell} \phi_k X^{(\ell-k)},$$

and the result follows by identification. \square

Like the ψ_k 's, the coefficients ϕ_k are homogenous polynomials of degree k in the variables w_{ij} . While ψ_k only involves the self-avoiding cycles of length k , ϕ_k enumerates all the closed walks of length k on the graph, i.e., walks that can be written as a product of self-avoiding cycles. The formal series inversion in Equation (14) actually corresponds to the inversion of the Dirichlet convolution when identifying each walk. This means in particular that the coefficient ϕ_k can be expressed as

$$\phi_k = \sum_{d \in \mathcal{D}^k} \beta(d) d, \quad (15)$$

where \mathcal{D}^k denotes the set of closed walks of length k . One can verify this formula directly from the formal series multiplication

$$1 = \sum_{k \geq 0} \phi_k \times \sum_{k=0}^N \psi_k = \sum_{d \in \mathcal{D}} \beta(d) d \times \sum_{c \in \mathcal{C}} \mu(c) c = \sum_{m \in \mathcal{M}} \sum_{d|m} \beta(d) \mu\left(\frac{m}{d}\right) m = \sum_{m \in \mathcal{M}} \beta * \mu(m) m$$

where we used that $\beta(d) = 0$ for all $d \in \mathcal{M} \setminus \mathcal{D}$. After identifying each walk, one recovers exactly $\beta * \mu = \delta$. We deduce a new expression for β by combining Equations (13) and (15):

$$\beta(c) = \sum_{p \geq 1} \sum_{c_1 \dots c_p = c} (-1)^p \mu(c_1) \dots \mu(c_p), \quad c \in \mathcal{D}, \quad (16)$$

where for all $p \geq 1$, the sum is taken over all p -tuples (c_1, \dots, c_p) of non-empty self-avoiding cycles such that $c_1 \dots c_p = c$. This equality provides an expression of $\beta(c)$ involving the different decompositions of c into self-avoiding cycles. While this expression is presumably less practical than the previous one, it induces nevertheless interesting consequences from a combinatorial point of view, which are discussed in Section 4.

We now come to our final result, which expresses the multiplicity of an open walk in terms of its decompositions into self-avoiding components. This result will be illustrated on some examples in Section 4.

Theorem 3.4 *Let m be a walk from v_i to v_j ,*

$$f_{ij}(m) = \sum_{c_1 \dots c_p d = m} (-1)^{n'(c_1) + \dots + n'(c_p) + n'(d)}$$

setting $n'(\cdot) = n(\cdot) + 1$, where the sum is taken over all self-avoiding decompositions of m , i.e., all $(p+1)$ -tuples $(c_1, \dots, c_p, d) \in (\mathcal{C} \setminus \{1\})^p \times \mathcal{S}_{ij}$ with $p \geq 0$ such that $c_1 \dots c_p d = m$.

Proof. For $c \in \mathcal{D}$, we use the expression of $\beta(c)$ given in Equation (16) to yield

$$\beta(c) = \sum_{c_1 \dots c_p = c} (-1)^p (-1)^{n(c_1) + \dots + n(c_p)} = \sum_{c_1 \dots c_p = c} (-1)^{n'(c_1) + \dots + n'(c_p)}, \quad (17)$$

setting $n'(\cdot) = n(\cdot) + 1$. We now plug this expression into $f_{ij}(m) = g_{ij} * \beta(m) = \sum_{d|m} g_{ij}(d) \beta\left(\frac{m}{d}\right)$. For $i \neq j$, the fact that $g_{ij}(d) = 0$ for $d \notin \mathcal{S}_{ij}$ simplifies into

$$f_{ij}(m) = \sum_{\substack{d|m \\ d \in \mathcal{S}_{ij}}} (-1)^{n'(d)} \left(\sum_{c_1 \dots c_p = \frac{m}{d}} (-1)^{n'(c_1) + \dots + n'(c_p)} \right) = \sum_{c_1 \dots c_p d = m} (-1)^{n'(c_1) + \dots + n'(c_p) + n'(d)},$$

recovering the result. For $i = j$, we use that $\mathcal{C}_{-i} = \mathcal{C} \setminus \mathcal{S}_{ii}$ to get

$$f_{ii}(m) = \sum_{\substack{d|m \\ d \in \mathcal{C}_{-i}}} (-1)^{n(d)} \beta\left(\frac{m}{d}\right) = \sum_{\substack{d|m \\ d \in \mathcal{C}}} (-1)^{n(d)} \beta\left(\frac{m}{d}\right) - \sum_{\substack{d|m \\ d \in \mathcal{S}_{ii}}} (-1)^{n(d)} \beta\left(\frac{m}{d}\right).$$

The first term of the right-hand side is $-\mu * \beta(m)$ which is zero for all $m \neq 1$. The result follows by using the expression of β given in Equation (17), similarly as for $i \neq j$. \square

4 Examples

In this section, the functions f_{ij}, g_{ij}, β and μ are computed on some examples. For ease of comprehension, we first deal with explicit simple walks and then consider more general structures in the final examples.

Example 1. Let us begin with the graph represented in Figure 2 which contains only two disjoint cycles.

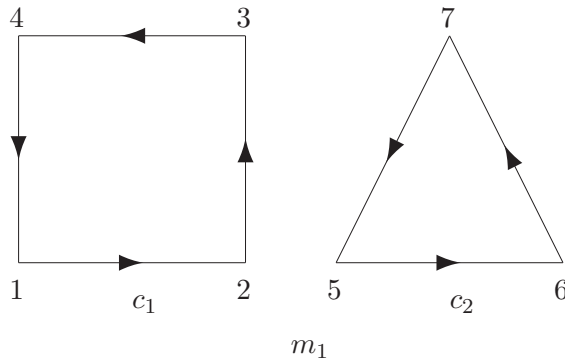


Figure 2: Disjoint cycles

This graph corresponds to the monomial $m_1 = w_{12}w_{23}w_{34}w_{41}w_{56}w_{67}w_{75}$ of W (recall that the order is not important). We obtain directly $f_{ij}(m_1) = 0$ (because m_1 is non-connected) and

$g_{ij}(m_1) = 0$ (because m_1 crosses every vertex) for all $i, j = 1, \dots, 7$. Moreover, the definitions of μ and β give in this case $\mu(m_1) = \beta(m_1) = 1$.

To check the equalities $f_{ii} * \mu = g_{ii}$ and $g_{ii} * \beta = f_{ii}$, the calculations are straightforward, since the only cycles that divide m_1 are m_1, c_1, c_2 and the void cycle 1. We get for instance,

$$f_{11} * \mu(m_1) = f_{11}(1)\mu(m_1) + f_{11}(c_1)\mu(c_2) = 0 = g_{11}(m_1),$$

using that $f_{11}(c_1) = f_{11}(1) = 1$, $f_{11}(c_2) = 0$ and $\mu(c_2) = -1$. From $g_{11}(c_1) = 0$, we also verify

$$g_{11} * \beta(m_1) = g_{11}(1)\beta(m) + g_{11}(c_2)\beta(c_1) = 0 = f_{11}(m_1).$$

Example 2. We now consider the graph given in Figure 3, composed of two cycles sharing one vertex and corresponding to the monomial $m_2 = w_{12}w_{23}w_{31}w_{24}w_{45}w_{52}$.

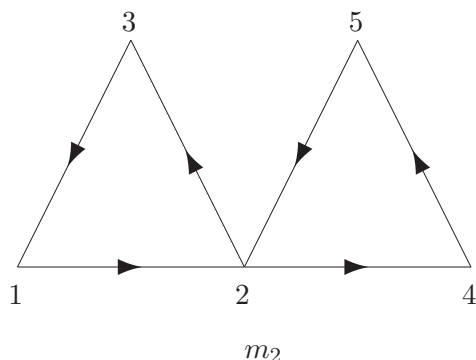


Figure 3: 2 cycles with 1 common vertex

Since m_2 is closed, $f_{ij}(m_2)$ is null for all $i \neq j$. Moreover, there are two ways to travel across m_2 starting from v_2 , depending on which side is visited first, and one way for every other vertex. We deduce $f_{22}(m_2) = 2$ and $f_{ii}(m_2) = 1$ for $i = 1, 3, 4, 5$. Since m_2 is not self-avoiding $\mu(m_2) = 0$ and Theorem 3.2 gives $\beta(m_2) = 2$.

To check the formulas, we now consider all the decompositions of m into two subgraphs. Actually, we only need to look at the decompositions into a product of cycles since both g_{ii} and μ vanish for non cycles. The two non-trivial subcycles of m_2 are shown in Figure 4.

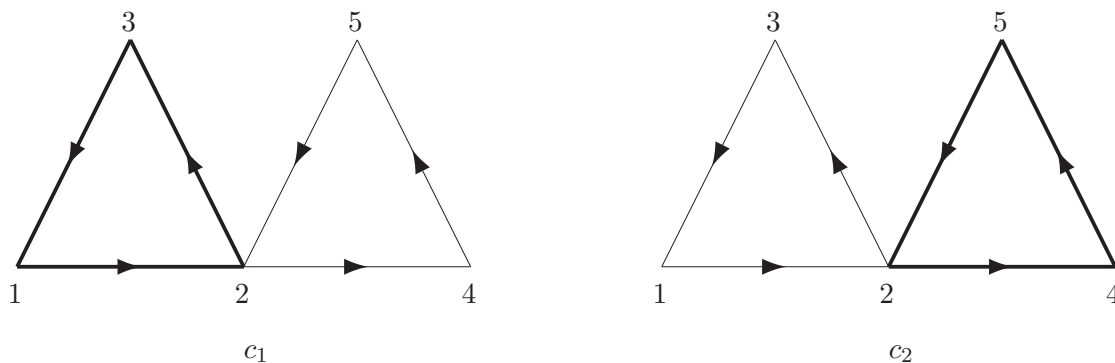


Figure 4: Subcycles of m_2

We verify for instance,

$$f_{22} * \mu(m_2) = f_{22}(m_2)\mu(1) + f_{22}(c_1)\mu(c_2) + f_{22}(c_2)\mu(c_1) = 0 = g_{22}(m_2)$$

$$g_{11} * \beta(m_2) = g_{11}(1)\beta(m_2) + g_{11}(c_2)\beta(c_1) = 1 = f_{11}(m_2)$$

Example 3. This example deals with the walk $m_3 = w_{12}w_{23}w_{23}w_{34}w_{41}w_{62}w_{25}w_{54}w_{46}$ composed of two cycles sharing two vertices, represented in Figure 5. The non-trivial subcycles of m_3 are detailed in Figure 6.

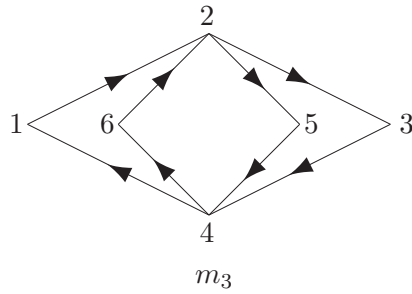


Figure 5: 2 cycles with 2 common vertices

Direct computation gives $f_{ij}(m_3) = 0$ for $i \neq j$, $f_{11}(m_3) = f_{33}(m_3) = f_{55}(m_3) = f_{66}(m_3) = 2$ and $f_{22}(m_3) = f_{44}(m_3) = 4$. Here again, $g_{ij}(m_3)$ is null for all $i, j = 1, \dots, 6$ as well as $\mu(m_3)$, while $\beta(m_3) = 4$.

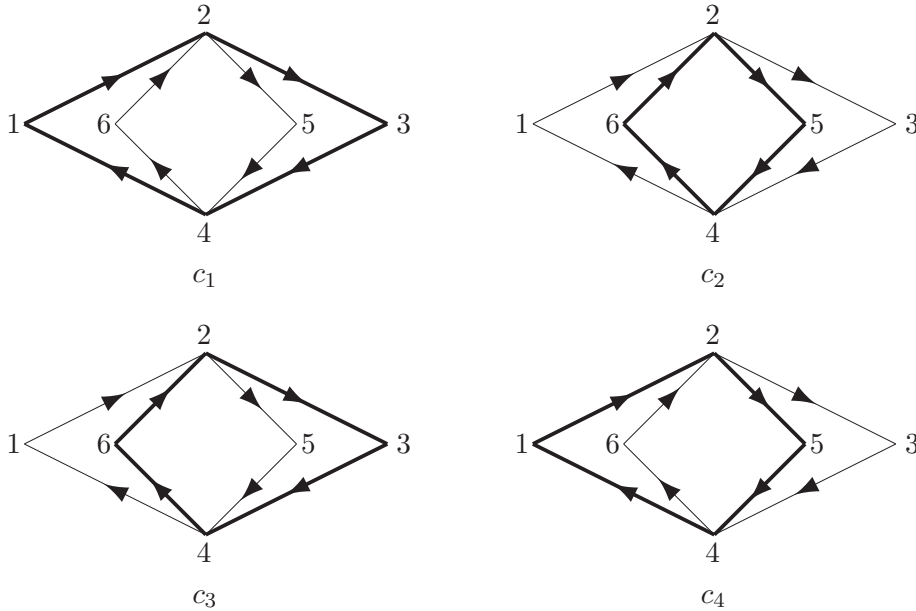


Figure 6: Subcycles of m_3

We then verify easily the convolution equalities $g_{ii}(m_3) = \mu * f_{ii}(m_3)$ and $f_{ii}(m_3) = \beta * g_{ii}(m_3)$ for arbitrary vertices, e.g.,

$$\begin{aligned} f_{11} * \mu(m_3) &= f_{11}(m_3)\mu(1) + f_{11}(c_1)\mu(c_2) + f_{11}(c_4)\mu(c_3) = 0 = g_{11}(m_3) \\ g_{22} * \beta(m_3) &= g_{22}(1)\beta(m_3) = 4 = f_{22}(m_3) \end{aligned}$$

Example 4. Let us consider the walk $m_4 = w_{12}w_{23}w_{35}w_{56}w_{64}w_{41}w_{25}w_{54}w_{42}$, illustrated in Figure 7, composed of 2 cycles sharing 3 vertices. In this case, note that the orientation of the two cycles has an impact on the values of f_{ii} . We do not deal with the walk obtained by changing the orientation of the arrow of the inside triangle, since it can be viewed as a particular case of the next example.

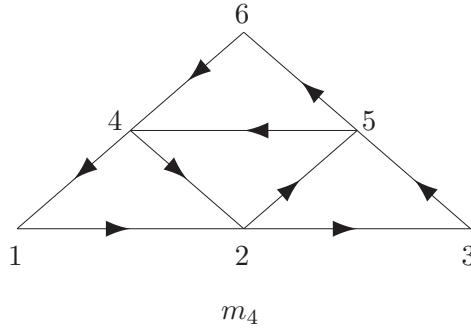


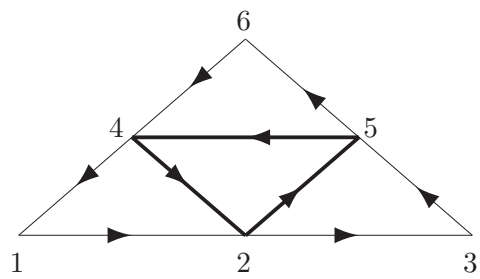
Figure 7: 2 cycles with 3 common vertices

As in the previous examples, we find easily $f_{ij}(m_4) = 0$ for $i \neq j$, $g_{ij}(m_4) = 0$ for all $i, j = 1, \dots, 6$, $\mu(m_4) = 0$ and $\beta(m_4) = 8$. We compute the values $f_{ii}(m) = 4$ by enumerating all connected paths that cross each edge once. For instance, the connected walks starting from the first vertex are

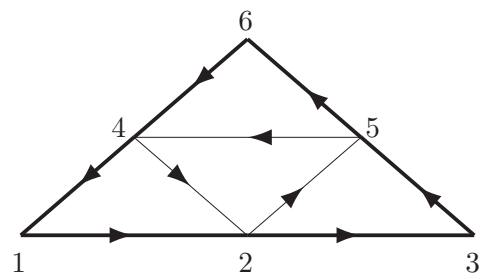
$$\begin{aligned} &1 \rightarrow 2 \rightarrow 3 \rightarrow 5 \rightarrow 6 \rightarrow 4 \rightarrow 2 \rightarrow 5 \rightarrow 4 \rightarrow 1 \\ &1 \rightarrow 2 \rightarrow 3 \rightarrow 5 \rightarrow 4 \rightarrow 2 \rightarrow 5 \rightarrow 6 \rightarrow 4 \rightarrow 1 \\ &1 \rightarrow 2 \rightarrow 5 \rightarrow 6 \rightarrow 4 \rightarrow 2 \rightarrow 3 \rightarrow 5 \rightarrow 4 \rightarrow 1 \\ &1 \rightarrow 2 \rightarrow 5 \rightarrow 4 \rightarrow 2 \rightarrow 3 \rightarrow 5 \rightarrow 6 \rightarrow 4 \rightarrow 1 \end{aligned}$$

which gives $f_{11}(m_4) = 4$. We obtain similarly $f_{33}(m_4) = f_{66}(m_4) = 4$ and $f_{22}(m_4) = f_{44}(m_4) = f_{55}(m_4) = 8$. The walk m_4 contains 8 non-trivial subcycles, detailed in Figure 8. We recover the correct values from the identities $g_{ii}(m_4) = f_{ii} * \mu(m_4)$ and $f_{ii}(m_4) = g_{ii} * \mu(m_4)$. Keeping only the non-zero values in the convolution gives, for instance

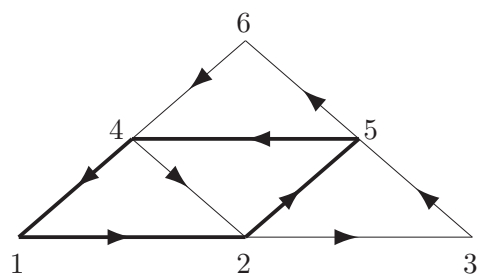
$$\begin{aligned} g_{22}(m_4) &= f_{22}(m_4)\mu(1) + f_{22}(c_1)\mu(c_2) + f_{22}(c_2)\mu(c_1) + f_{22}(c_3)\mu(c_4) + f_{22}(c_4)\mu(c_3) \\ &\quad + f_{22}(c_5)\mu(c_6) + f_{22}(c_6)\mu(c_5) + f_{22}(c_7)\mu(c_8) + f_{22}(c_8)\mu(c_7) \\ &= 8 - 1 - 1 - 1 - 1 - 1 - 1 - 1 - 1 - 1 \\ &= 0 \\ f_{11}(m_4) &= g_{11}(1)\beta(m_4) + g_{11}(c_1)\beta(c_2) + g_{11}(c_4)\beta(c_3) + g_{11}(c_5)\beta(c_6) + g_{11}(c_7)\beta(c_8) \\ &= 8 - 1 - 1 - 1 - 1 \\ &= 4. \end{aligned}$$



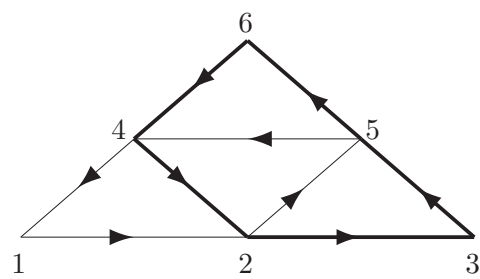
c_1



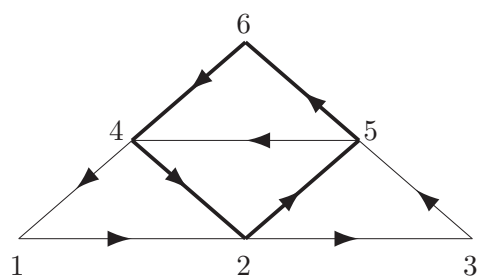
c_2



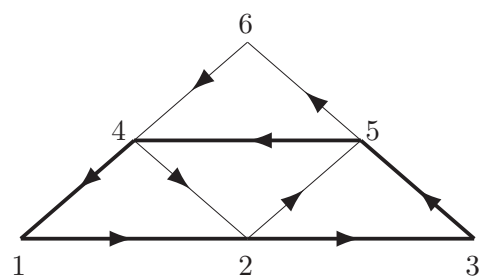
c_3



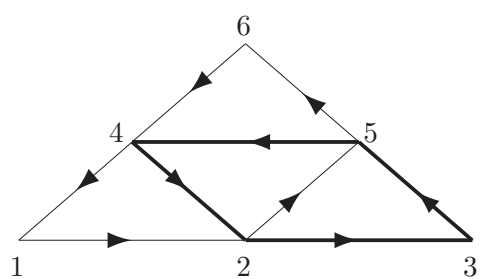
c_4



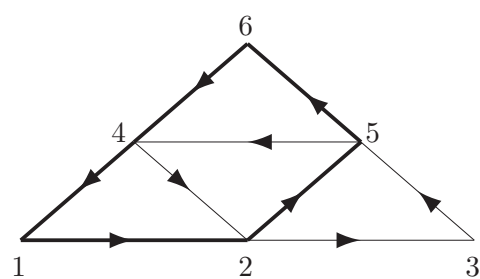
c_5



c_6



c_7



c_8

Figure 8: Subgraphs of m_4

Example 5. Consider the more general situation obtained by 2 cycles crossing n times. This example can be represented as n cycles placed one after the other. As we observed in the previous examples, the length of the cycles does not impact the values of the functions f_{ii}, g_{ii}, μ and β so that we can take cycles of length 2 without loss of generality, considering for instance the monomial $w_{12}w_{21}w_{23}w_{32} \cdots w_{n1}w_{1n}$ illustrated in Figure 9.

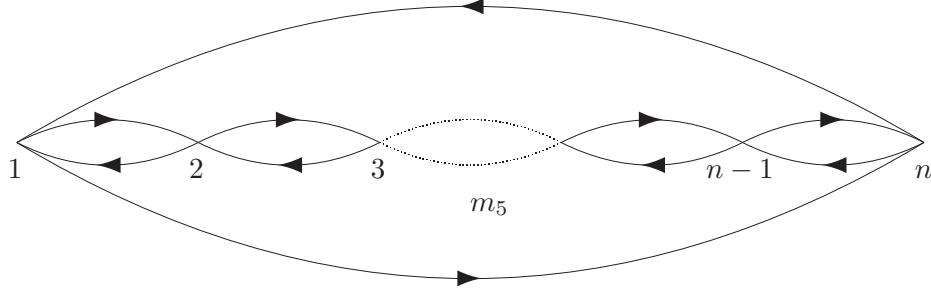


Figure 9: 2 cycles with n common vertices

We find $f_{ij}(m_5) = 0$ for $i \neq j$, $f_{ii}(m_5) = 2n$, $g_{ij}(m_5) = 0$ for all $i, j = 1, \dots, n$, $\mu(m_5) = 0$ and $\beta(m_5) = 2^n$.

Let $a_k = w_{kk+1}w_{k+1k}$, $k = 1, \dots, n-1$, $a_n = w_{n1}w_{1n}$, $d_0 = w_{12}w_{23} \cdots w_{n-1n}w_{n1}$ and $d_1 = w_{21}w_{32} \cdots w_{nn-1}w_{1n}$ denote the simple cycles (or atoms) dividing m_5 . The non-trivial subcycles of m_5 are d_0, d_1 (which satisfy $d_0d_1 = m_5$) and every product $a_{k_1} \cdots a_{k_p}$ obtained for a subset $\{k_1, \dots, k_p\}$ of $[1, n]$. Using that f_{11} is non-zero only for connected walks passing through v_1 and μ vanishes for non self-avoiding cycles, we obtain by keeping only the non-zero terms in the Dirichlet convolution

$$f_{11} * \mu(m_5) = f_{11}(m_5)\mu(1) + f_{11}(d_0)\mu(d_1) + f_{11}(d_1)\mu(d_0) + \sum_{k=2}^n \mu(a_k)f_{11}\left(\frac{m_5}{a_k}\right)$$

which recovers ultimately $f_{11} * \mu(m_5) = 2n - 2 - 2(n-1) = 0 = g_{11}(m_5)$. The reverse relation $f_{11}(m_5) = g_{11} * \beta(m_5)$ is less trivial. To compute it, we have to enumerate for any p , the sets $\{k_1, \dots, k_p\} \subset [1, n]$ such that $g_{11}(a_{k_1} \cdots a_{k_p}) \neq 0$, i.e., such that $a_{k_1} \cdots a_{k_p}$ is a self-avoiding cycle that does not cross v_1 . For each such cycle, the complement $m_5/a_{k_1} \cdots a_{k_p}$ is composed of p disjoint connected components, with one of them containing a_1a_n . Moreover, the component that contains a_1a_n can be divided into two connected components by setting a separation line between a_1 and a_n . Thus, each cycle $a_{k_1} \cdots a_{k_p}$ such that $g_{11}(a_{k_1} \cdots a_{k_p}) \neq 0$ can be associated with a composition of $n-p$ containing $p+1$ elements, which there are $\binom{n-p-1}{p}$ of them. For each $a_{k_1} \cdots a_{k_p}$, the coefficient β of the complement equals 2^{n-2p} , as every cycle a_i removed divides the coefficients by 4. We recover the formula

$$g_{11} * \beta(m_5) = \sum_{p=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-p-1}{p} (-1)^p 2^{n-2p} = 2n = f_{11}(m_5).$$

Example 6. Consider a self-avoiding cycle m_6 composed of $k \geq 2$ simple connected components a_1, \dots, a_k (we may assume without loss of generality that each connected component is of length 1, taking for instance $a_i = w_{ii}$ as illustrated in Figure 10). From the first expression of β given in Theorem 3.2, it is clear that $\beta(m_6) = 1$. On the other hand, the number of ways to decompose m_6 into a product of $p \leq k$ non-empty self-avoiding cycles writes as the sum of the multinomial coefficients over all positive compositions k_1, \dots, k_p of k . Since for any self-avoiding decomposition c_1, \dots, c_p , the product $\mu(c_1)\dots\mu(c_p)$ always equals $\mu(m_6) = (-1)^k$, combining the two expressions of $\beta(m_6)$ yields

$$\beta(m_6) = 1 = \sum_{p=1}^k (-1)^p \sum_{k_1+\dots+k_p=k} (-1)^k \frac{k!}{k_1!\dots k_p!}.$$

Alternatively, this equality can be obtained by identifying the coefficient of x^k in the power series expansions of $e^{-x} = 1/e^x$.

Since m_6 is not connected, we know that $f_{ii}(m_6) = 0$ for all i . To compute the expression of $f_{ii}(m_6)$ from Theorem 3.4, we consider the self-avoiding decompositions of the form $c = c_1\dots c_p d$ with $c_1, \dots, c_p \in \mathcal{C} \setminus \{1\}$ and $d \in \mathcal{S}_{ii}$. To verify that this expression gives $f_{ii}(m_6) = 0$ in this case simply observe that any self-avoiding decomposition c_1, \dots, c_p, d such that $d \neq w_{ii}$ cancels out with the decomposition $c_1, \dots, c_p, d/w_{ii}, w_{ii}$ in view of

$$(-1)^{n'(c_1)+\dots+n'(c_p)+n'(d)} = -(-1)^{n'(c_1)+\dots+n'(c_p)+n'(d/w_{ii})+n'(w_{ii})}.$$

Thus, summing over all self-avoiding decompositions recovers $f_{ii}(m_6) = 0$.

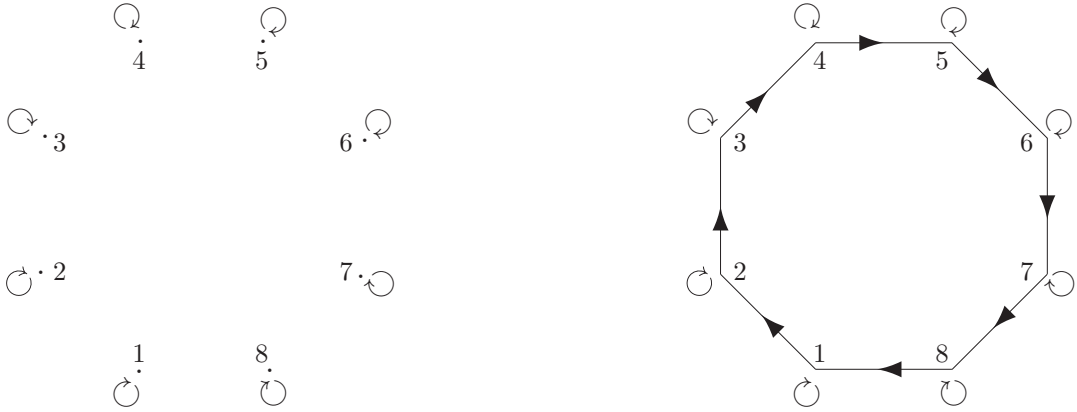


Figure 10: Illustration of the walks considered in Examples 6 (left) and 7 (right) for $k = 8$.

Example 7. We now consider the cycle m_7 constructed from the previous example with an extra cycle passing through each vertex, e.g. $m_7 = w_{11}\dots w_{kk} \times w_{12}w_{23}\dots w_{k-1k}w_{k1} := m_6 \times c_0$ (see Figure 10). In this example, the cycle c_0 is isolated in every self-avoiding decomposition since it shares a common node with all the other cycles dividing m_7 . Thus, the different ways to express m_7 as a product of self-avoiding cycles can be obtained from the previous example, inserting the

cycle c_0 wherever possible. Precisely, for a decomposition $m_6 = c_1 \dots c_p$ of m_6 into $p \leq k$ non-empty self-avoiding cycles, there are exactly $p + 1$ possibilities to insert c_0 . Moreover, remark that $\mu(c_1) \dots \mu(c_p) \mu(c_0) = \mu(m_6) \mu(c_0) = (-1)^{k+1}$ is constant over all self-avoiding decompositions. Combining the two expressions of $\beta(m_7)$ thus recovers the formula

$$\sum_{p=1}^k (-1)^{p+1} (p+1) \sum_{k_1 + \dots + k_p = k} (-1)^{k+1} \frac{k!}{k_1! \dots k_p!} = 2^k.$$

In this example, there are two ways of visiting the whole walk from one vertex v_i to itself, depending on whether the loop at v_i is traveled at the start or at the end. Thus, we know that $f_{ii}(m_7) = 2$ for all $v_i \in m_7$. In a self-avoiding decomposition with $c_1, \dots, c_p \in \mathcal{C} \setminus \{1\}$ and $d \in \mathcal{S}_{ii}$ we can distinguish the cases $d = c_0$, $d = w_{ii}$ and $d \neq w_{ii}, c_0$. Clearly, the sum over all self-avoiding decompositions $m_7 = c_1 \dots c_p d$ such that $d = c_0$ yields $\beta(m_6)$ since $(-1)^{n'(c_0)} = 1$. Moreover, the sum over all self-avoiding decompositions with $d = w_{ii}$ recovers $\beta(m_7/w_{ii}) = 2^{k-1}$. Finally, for a self-avoiding decomposition $m_6 = c_1 \dots c_p d$ of m_6 with $d \neq w_{ii}$, there are p possibilities to insert c_0 , yielding

$$f_{ii}(m_7) = \beta(m_6) + \beta\left(\frac{m_7}{w_{ii}}\right) + \sum_{p=1}^{k-1} (-1)^{p+1} p \sum_{k_1 + \dots + k_p = k-1} (-1)^{k+1} \frac{(k-1)!}{k_1! \dots k_p!} = 1 + 2^{k-1} - 2^{k-1} + 1 = 2.$$

Acknowledgments

The authors are grateful to Pierre-Louis Giscard for his helpful comments and his suggestion to consider the poset structure of walks, which helped improve this paper.

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