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# RENEWAL STRUCTURE AND LOCAL TIME FOR DIFFUSIONS IN RANDOM ENVIRONMENT

PIERRE ANDREOLETTI, ALEXIS DEVULDER, AND GRÉGOIRE VÉCHAMBRE

ABSTRACT. We study a one-dimensional diffusion  $X$  in a drifted Brownian potential  $W_\kappa$ , with  $0 < \kappa < 1$ , and focus on the behavior of the local times  $(\mathcal{L}(t, x), x)$  of  $X$  before time  $t > 0$ . In particular we characterize the limit law of the supremum of the local time, as well as the position of the favorite sites. These limits can be written explicitly from a two dimensional stable Lévy process. Our analysis is based on the study of an extension of the renewal structure which is deeply involved in the asymptotic behavior of  $X$ .

## 1. INTRODUCTION

Let  $(X(t), t \geq 0)$  a diffusion in a random càdlàg potential  $(V(x), x \in \mathbb{R})$ , defined informally by  $X(0) = 0$  and

$$dX(t) = d\beta(t) - \frac{1}{2}V'(X(t))dt,$$

where  $\beta$  is a Brownian motion independent of  $V$ . Rigorously,  $X$  is defined by its conditional generator given  $V$ ,

$$\frac{1}{2}e^{V(x)} \frac{d}{dx} \left( e^{-V(x)} \frac{d}{dx} \right).$$

We put ourself in the case where  $V$  is a negative drifted brownian motion :  $V(x) = W_\kappa(x) := W(x) - \frac{\kappa}{2}x$ ,  $x \in \mathbb{R}$  with  $0 < \kappa < 1$  and  $W$  a two sided Brownian motion. We explain at the end of Section 1.1, what should be done to extend our results to a more general Lévy process.

In our case, the diffusion  $X$  is a.s. transient and its asymptotic behavior was first studied by K. Kawazu and H. Tanaka : if  $H(r)$  is the hitting time of  $r \in \mathbb{R}$  by  $X$

$$H(r) := \inf\{s > 0, X(s) = r\}, \tag{1.1}$$

Kawazu et al. [25] proved that, for  $0 < \kappa < 1$  under the so-called annealed probability  $\mathbb{P}$ ,  $H(r)/r^{1/\kappa}$  converges in law to a  $\kappa$ -stable distribution (see also Y. Hu et al. [24], and H. Tanaka [33]). Here we are interested in the local time of  $X$  denoted  $(\mathcal{L}(t, x), x \in \mathbb{R}, t > 0)$  until an asymptotic instant  $t$ .

For Brox's diffusion, (when  $\kappa = 0$ ) it is proved in [4] that the process local time until the instant  $t$  re-centered at the localization coordinate  $b_t$  (see [10]) converges in law, this allows the author to derive the law of the supremum of the local time before time  $t \in \mathbb{R}_+$ ,

$$\mathcal{L}^*(t) := \sup_{x \in \mathbb{R}} \mathcal{L}(t, x).$$

We recall their result below in order to compare it with what we obtain here :

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**Theorem 1.1.** [4] *If  $\kappa = 0$ , then*

$$\lim_{t \rightarrow +\infty} \frac{\mathcal{L}^*(t)}{t} = \frac{1}{\mathcal{R}_\kappa},$$

with

$$\mathcal{R}_\kappa := \int_0^{+\infty} e^{-W_\kappa^\uparrow(x)} dx + \int_0^{+\infty} e^{-\widetilde{W}_\kappa^\uparrow(x)} dx. \quad (1.2)$$

$(W_\kappa^\uparrow(x), x \geq 0)$  and  $(\widetilde{W}_\kappa^\uparrow(x), x \geq 0)$  are two independent copies of the process  $(W_\kappa(x), x \geq 0)$  Doob-conditioned to remain positive.

Extending their approach, and following the results of Shi [30], Diel [15] obtains the non-trivial normalisations for the almost sure behavior of the lim sup and lim inf of  $\mathcal{L}^*(t)$ . Notice that these results have been previously established for the discrete analogue of  $X$ , the so called Sinai's random walk in [13] and [22].

One of our aim in this paper is to extend the study of the local time in the case  $0 < \kappa < 1$ , and deduce from that the weak asymptotic behavior of  $\mathcal{L}^*(t)$ .

Before going any further let us recall to the reader what is known for the slow transient cases. When the time and space are discrete (see [26], for the seminal paper), a result of Gantert and Shi [23] states the almost sure behavior for the lim sup of the supremum of the local time  $(\mathcal{L}_S^*(n))$  of these random walks (denoted  $S$ ), before time  $n$  : a.s.  $\limsup_{n \rightarrow +\infty} \mathcal{L}_S^*(n)/n = c > 0$ . Contrarily to the recurrent case ([22]) their method, based on a link with the local time of  $S$  and a branching process in random environment, is not able to determine the law of the limit of  $\mathcal{L}_S^*(n)/n$ .

For the continuous time and space case we are treating here, the only paper dealing with  $\mathcal{L}^*(t)$  is Devulder's work [14], who proves that the  $\limsup_{t \rightarrow +\infty} \mathcal{L}^*(t)/t = +\infty$  almost surely. But once again his method can not be used to characterize the limit law of  $\mathcal{L}^*(t)/t$ .

Our motivation here is twofold, first we prove that our approach enables to characterize the limit law of  $\mathcal{L}^*(t)/t$  and open a way to determine the correct almost sure behavior of  $\mathcal{L}^*(t)$  like for Brox's diffusion. Second we make a first step on a specific way to study the local time which could be used in estimation problems with random environment [1], [2], [5], [12], [11], [20], [6].

The method we develop here is an improvement of the one used in [3] about the localization of  $X_t$  for large  $t$ .

Let us recall the main result of this paper, for that we introduce some new objects. First the notion of  $h$ -extrema, with  $h > 0$ , introduced by Neveu et al. [27] and studied more specifically in our case of drifted Brownian motion by Faggionato [19]. For  $h > 0$ , we say that  $x \in \mathbb{R}$  is a  $h$ -minimum for a given continuous process  $V$  if there exist  $u < x < v$  such that  $V(y) \geq V(x)$  for all  $y \in [u, v]$ ,  $V(u) \geq V(x) + h$  and  $V(v) \geq V(x) + h$ . Moreover,  $x$  is an  $h$ -maximum for  $V$  if  $x$  is an  $h$ -minimum for  $-V$ , and  $x$  is an  $h$ -extrema for  $V$  iff it is an  $h$ -maximum or an  $h$ -minimum for  $V$ .

As we are interested in the process  $X$  until time  $t$ , we only focus on  $h_t$ -extrema of  $W_\kappa$  where

$$h_t := \log t - \phi(t), \text{ with } 0 < \phi(t) = o(\log t), \log \log t = o(\phi(t)).$$

It is known (see [19]) that almost surely, the  $h_t$ -extrema of  $W_\kappa$  form a sequence indexed by  $\mathbb{Z}$ , unbounded from below and above, and that the  $h_t$ -minima and  $h_t$ -maxima alternate. We denote respectively by  $(m_j, j \in \mathbb{Z})$  and  $(M_j, j \in \mathbb{Z})$  the increasing sequences of  $h_t$ -minima and of  $h_t$ -maxima of  $W_\kappa$ , such that  $m_0 \leq 0 < m_1$  and  $m_j < M_j < m_{j+1}$  for every  $j \in \mathbb{Z}$ . Define

$$N_t := \max \left\{ k \in \mathbb{N}, \sup_{0 \leq s \leq t} X(s) \geq m_k \right\},$$

the number of  $h_t$ -minima visited by  $X$  until the instant  $t$ , we then have the following result

**Theorem 1.2.** [3] *Assume  $0 < \kappa < 1$ . There exists a constant  $C_1 > 0$ , such that*

$$\lim_{t \rightarrow +\infty} \mathbb{P}\left(|X(t) - m_{N_t}| \leq C_1 \phi(t)\right) = 1.$$

This result proves that the process  $X$  before the instant  $t$  visits a sequence of  $h_t$ -minima, and then gets stuck in an ultimate one. Notice that this result was proved in the discrete case model by Enriquez-Sabot-Zindy in [17]. This phenomenon is due to two facts : the first one is the appearance of a renewal structure which is composed of the time it takes to the process to move from a  $h_t$ -minima to the following one. The second is the fact that like in Brox's case, the process is trapped a significant amount of time in the neighborhood of the local minima  $m_{N_t}$ . It is the extension of this renewal structure to the sequence of local time at the  $h_t$ -minima that we study here. We now detail our results.

**1.1. Results .** Let us introduce a few notations involved in the statement of our results. Denote  $(D([0, +\infty), \mathbb{R}^2), J_1)$  the space of càdlàg functions with  $J_1$ -Skorokhod topology and denote by  $\xrightarrow{\mathfrak{L}}_s$  the convergence in law for this topology. On this space, define a 2-dimensional Lévy process  $(\mathcal{Y}_1, \mathcal{Y}_2)$  with value in  $\mathbb{R}^+ \times \mathbb{R}^+$  which is a pure positive jump process with  $\kappa$ -stable Lévy measure  $\nu$  given by

$$\forall x > 0, y > 0, \nu([x, +\infty[ \times [y, +\infty[) = \frac{1}{y^\kappa} \mathbb{E}\left[(\mathcal{R}_\kappa)^\kappa \mathbf{1}_{\mathcal{R}_\kappa \leq \frac{y}{x}}\right] + \frac{1}{x^\kappa} \mathbb{P}\left(\mathcal{R}_\kappa > \frac{y}{x}\right),$$

where  $\mathcal{R}_\kappa$  is defined in (1.2).

For a given function  $f$  in  $D([0, +\infty), \mathbb{R})$ , define for any  $s > 0, a > 0$ :

$$f^\natural(s) := \sup_{0 \leq r \leq s} (f(r) - f(r^-)), \quad f^{-1}(a) := \inf\{x > 0, f(x) > a\},$$

$f^\natural(s)$  is the largest jump of  $f$  before instant  $s$ ,  $f^{-1}(a)$  is the first time  $f$  is larger than  $a$ . Also define the couple of random variables  $(\mathcal{I}_1, \mathcal{I}_2)$

$$\mathcal{I}_1 := \mathcal{Y}_1^\natural(\mathcal{Y}_2^{-1}(1)^-), \quad \mathcal{I}_2 := (1 - \mathcal{Y}_2(\mathcal{Y}_2^{-1}(1)^-)) \times \frac{\mathcal{Y}_1(\mathcal{Y}_2^{-1}(1)) - \mathcal{Y}_1(\mathcal{Y}_2^{-1}(1)^-)}{\mathcal{Y}_2(\mathcal{Y}_2^{-1}(1)) - \mathcal{Y}_2(\mathcal{Y}_2^{-1}(1)^-)}.$$

We are now ready to state the result, the convergence in law denoted  $\xrightarrow{\mathfrak{L}}$  takes place when  $t$  goes to infinity :

**Theorem 1.3.**

$$\frac{\mathcal{L}^*(t)}{t} \xrightarrow{\mathfrak{L}} \mathcal{I} = \max(\mathcal{I}_1, \mathcal{I}_2).$$

There is an intuitive interpretation of this theorem which explains the appearance of the Lévy process  $(\mathcal{Y}_1, \mathcal{Y}_2)$ . We focus on this interpretation now.

First for any  $s > 0$ ,  $\mathcal{Y}_1(s)$  is the limit of the sum of the first  $\lfloor se^{\kappa\phi(t)} \rfloor$  normalised (by  $t$ ) local times taken specifically at the  $\lfloor se^{\kappa\phi(t)} \rfloor$  first  $h_t$ -minima.  $\mathcal{Y}_2$  plays a similar role but for the exit time of the  $\lfloor se^{\kappa\phi(t)} \rfloor$  first  $h_t$ -valleys. Where an  $h_t$ -valley is defined as a large neighborhood of an  $h_t$ -minima, see Section 2.2 for a rigorous definition as well as Figure 1.

So by definition  $\mathcal{I}_1$  is the largest jump of the process  $\mathcal{Y}_1$  before  $\mathcal{Y}_2$  is larger than 1 and can be interpreted as the largest local time (re-normalized) among the local time at the  $h_t$ -minima visited by  $X$  until time  $t$  and from where  $X$  escape. That is to say  $\mathcal{I}_1$  is the limit of the random variable  $\sup_{k \leq N_t - 1} \mathcal{L}(m_k, t)/t$ .

$\mathcal{I}_2$  is a product of two terms : the first  $(1 - \mathcal{Y}_2(\mathcal{Y}_2^{-1}(1)^-))$  corresponds to the (re-normalized) amount of time left to the process  $X$  before instant  $t$  after it has reached the ultimate visited

$h_t$ -minima,  $m_{N_t}$ . The second term corresponds to the local time of  $X$  at this ultimate  $h_t$ -minima. Intuitively  $\mathcal{Y}_2$  is construct from  $\mathcal{Y}_1$  multiplying each of its jumps by an independent copy of the variable  $\mathcal{R}_\kappa$ . Therefore this second term can be seen as an independent copy of  $1/\mathcal{R}_\kappa$  taken at the instant of the overshoot of  $\mathcal{Y}_2$  which makes it larger than 1. Notice that this variable  $\mathcal{R}_\kappa$  plays a similar role than  $\mathcal{R}_0$  of Theorem 1.1. Indeed as in the case  $\kappa = 0$ , the process  $X$  is prisoner in the neighborhood of the last  $h_t$ -minima visited before time  $t$ .

We prove this result by showing first that portions of the trajectory of  $X$  re-centered at the local  $h_t$ -minima, until the instant  $t$ , is made (in probability) with independent parts. This has been partially proved in [3] but we have to improve their results and add simultaneously the study of the local time.

Second we prove that what we seek for the supremum of the local time is, mainly, a function of the sum of theses independent parts, which converges to a Lévy process. Let us give some details about this :

Let  $(Q(s), s \geq 0)$  a canonical process, taking values in  $\mathbb{R}_+$ , with infinitesimal generator given for every  $x > 0$  by

$$\frac{1}{2} \frac{d^2}{dx^2} + \frac{\zeta}{2} \coth\left(\frac{\zeta}{2}x\right) \frac{d}{dx}.$$

This process  $Q$  can be thought of as a  $(-\zeta/2)$ -drifted Brownian motion  $W_\zeta$  Doob-conditioned to stay positive, with the terminology of [7], which is called Doob conditioned to reach  $+\infty$  before 0 in [19] (see Section 2.1 in [3] for more details). We call BES(3,  $\kappa/2$ ), the law of  $(Q(s), s \geq 0)$ . For  $a < b$ ,  $(W_\kappa^b(s), 0 \leq s \leq \tau^{W_\kappa^b}(a))$  is a  $(-\kappa/2)$ -drifted Brownian motion starting from  $b$  and killed when it first hits  $a$ . For any process  $(U(t), t \in \mathbb{R}^+)$  we denote by

$$\tau^U(a) := \inf\{t > 0, U(t) = a\},$$

the first time this process hits  $a$ , with the convention  $\inf \emptyset = +\infty$ . We now introduce functional of  $W_\kappa$  and  $Q$  :

$$F^\pm(x) := \int_0^{\tau^{Q(x)}} \exp(\pm Q(s)) ds, \quad x > 0, \quad G^\pm(a, b) := \int_0^{\tau^{W_\kappa^b}(a)} \exp(\pm W_\kappa^b(s)) ds, \quad a < b. \quad (1.3)$$

Also for any  $\delta > 0$ , and  $t > 0$ , define

$$n_t := \lfloor e^{\kappa\phi(t)(1+\delta)} \rfloor,$$

which is an upper bound of  $N_t$  as stated in Lemma 3.1.

Then let  $(S_j(t), R_j(t), \mathbf{e}_j(t), j \leq n_t)$  a sequence of i.i.d. random variables with  $S_j, R_j$  and  $\mathbf{e}_j$  independent with  $S_1 \stackrel{\mathfrak{L}}{=} F^+(h_t) + G^+(h_t, h_t/2)$ ,  $R_1 \stackrel{\mathfrak{L}}{=} F^-(h_t/2) + \tilde{F}^-(h_t/2)$  and  $\mathbf{e}_1 \stackrel{\mathfrak{L}}{=} \mathcal{E}(1/2)$  [an exponential random variable with parameter 1/2] where  $\tilde{F}^-$  an independent copy of  $F^-$  and  $F^+$  independent of  $G^+$ . Define  $\ell_j := \mathbf{e}_j S_j$ ,  $\mathcal{H}_j := \ell_j R_j$ , note that for notational simplicity we do not make appear the dependence in  $t$  in the sequel. Typically  $\ell_j$  plays the roll of the local time at the  $j$ th positive  $h_t$ -minima if the walk escape from it before the instant  $t$  and  $\mathcal{H}_j$  the roll of the time it takes for the walk to escape from the corresponding valley.

Define the family of processes  $(Y_1, Y_2)^t$  indexed by  $t$ , by

$$\forall s \geq 0, (Y_1, Y_2)_s^t := \frac{1}{t} \sum_{j=1}^{\lfloor se^{\kappa\phi(t)} \rfloor} (\ell_j, \mathcal{H}_j),$$

then

**Proposition 1.4.** *We have*

$$(Y_1, Y_2)^t \xrightarrow{\mathfrak{L}^S} (\mathcal{Y}_1, \mathcal{Y}_2).$$

Once this is proved, we check that what we need for the supremum of the local time can be written as a function of  $(Y_1, Y_2)^t$ . We obtain an expression in this sens in Proposition 5.1. Then to obtain the limit, we prove the continuity (in  $J_1$ -topology) of the involved mapping and apply a continuous mapping Theorem (see Section 4.3). It appears that with this method we can obtain other asymptotics, like for the supremum of the local time before the last valley is reach (before the instant  $t$ ) and once it left it for good as well as for the position of the favorite site :

**Theorem 1.5.** *We have*

$$\frac{\mathcal{L}^*(H(m_{N_t+1}))}{t} \xrightarrow{\mathfrak{L}} \mathcal{Y}_1^\sharp(\mathcal{Y}_2^{-1}(1)),$$

$$\frac{\mathcal{L}^*(H(m_{N_t}))}{t} \xrightarrow{\mathfrak{L}} \mathcal{Y}_1^\sharp(\mathcal{Y}_2^{-1}(1)^-).$$

Let us call  $F_t^*$ , the position of the first favorite site,  $F_t^* := \inf\{s > 0, \mathcal{L}(t, s) = \mathcal{L}^*(t)\}$ , then

$$\frac{F_t^*}{X_t} \xrightarrow{\mathfrak{L}} \mathcal{B}U_{[0,1]} + 1 - \mathcal{B}$$

where  $\mathcal{B}$  is a Bernoulli with parameter  $\mathbb{P}(\mathcal{I}_1 < \mathcal{I}_2)$  independent of the uniform random variable on  $[0, 1] : U_{[0,1]}$ .

One question we may ask here is what's happen in the discret case or with a more general Lévy process ?

For the discrete case, we should have a very similar behavior as the renewal structure which appears in both cases (continuous and discrete) is very similar (see the works of Enriquez-Sabot-Zindy [17]). The main difference comes essentially from the functional  $\mathcal{R}_\kappa$  which should be replaced by a sum of exponential of a simple random walk conditioned to remain positive (see [18], [17]).

For a more general Lévy process, we think for example, of a spectrally negative Lévy process (studied in the case of diffusion in random environment by Singh [32]), more work needs to be done, especially for the environment. First to obtain a specific decomposition of the Lévy's path (similar to what is done for the drifted Brownian motion in Faggionato [19]) and also to study the more complicated functional  $\mathcal{R}_\kappa$  which is less known than in the brownian case. This is a work in preparation by Véchambre [34].

The rest of the paper is organized as follows :

*In Section 2* we recall the results of Faggionato on the path decomposition of the trajectories of  $W_\kappa$ . Also we recall from [3] the construction of specific  $h_t$ -minima which plays an important role in the appearance of independence, under  $\mathbb{P}$ , on the path of  $X$  before time  $t$ .

*In Section 3* we study the joint process of the first  $n_t$  hitting times and local times. We show that parts of the trajectory of  $X$  is not important for what we seek (this part is technical, makes use of some technical results of the paper [3] and can be omitted in the first instance). We then prove the main result of this section : Proposition 3.5. It proves that the joint process (exit time, local time) can be approximated in probability by *i.i.d* random variables, again the proofs make use of some technical parts of [3] though the main ideas are discuss in the present paper.

*In Section 4* we prove Proposition 1.4, and study the continuity of certain functional of  $(\mathcal{Y}_1, \mathcal{Y}_2)$  which appears in the expression of the law we have detailed above. This section is independent of the other, we essentially prove a basic functional limit theorem and prepare to the application of continuous mapping theorem.

Section 5 is where we make appear the renewal structure in the problem we want to solve. In particular we prove how the distribution of the supremum of the local time can be approximated by the distribution of a certain function of the couple  $(Y_1, Y_2)^t$ , the main step being Proposition 5.1.

The appendix is a reminder of some estimates on the brownian motion, Bessel processes, and functionals of both of these processes.

**1.2. Notations.** In this section we introduce typical notations for the study of diffusions in random media, as well as elementary tools for the continuous one-dimension case. We denote by  $P$  the probability measure associated to  $W_\kappa(\cdot)$ . The probability conditionally on the potential  $W_\kappa$  is denoted by  $\mathbb{P}^{W_\kappa}$  and is called the quenched probability. We also define the annealed probability as

$$\mathbb{P}(\cdot) := \int \mathbb{P}^{W_\kappa}(\cdot) P(W_\kappa \in d\omega).$$

We denote respectively by  $\mathbb{E}^{W_\kappa}$ ,  $\mathbb{E}$ , and  $E$  the expectancies with regard to  $\mathbb{P}^{W_\kappa}$ ,  $\mathbb{P}$  and  $P$ .

For any process  $(U(t), t \in \mathbb{R}^+)$  we denote by  $\mathcal{L}_U$  a bicontinuous version of the local time of  $U$  when it exists. Notice that for our main process  $X$  we simply write  $\mathcal{L}$ . We also denote by  $U^a$  the process  $U$  starting from  $a$ , and by  $P^a$  the law of  $U^a$ ; with the notation  $U = U^0$ . Now let us introduce the following functional of  $W_\kappa$ ,

$$A(r) := \int_0^r e^{W_\kappa(x)} dx, \quad r \in \mathbb{R},$$

and recall that whenever  $\kappa > 0$ ,  $A_\infty := \lim_{r \rightarrow +\infty} A(r) < \infty$  a.s. As in Brox [10], there exists a Brownian motion  $B$  independent of  $W_\kappa$ , such that  $X(t) = A^{-1}[B(T^{-1}(t))]$ , where

$$T(r) := \int_0^r \exp\{-2W_\kappa[A^{-1}(B(s))]\} ds, \quad 0 \leq r \leq \tau^B(A_\infty). \quad (1.4)$$

The local time of the process  $X$  at  $x$  until instant  $t$  simply denoted  $\mathcal{L}(t, x)$ , can be written as (see [30])

$$\mathcal{L}(t, x) = e^{-W_\kappa(x)} \mathcal{L}_B(T^{-1}(t), A(x)). \quad (1.5)$$

With these notations, we recall the following expression of  $H(r)$ , for all  $r \geq 0$ ,

$$H(r) = T[\tau^B(A(r))] = \int_{-\infty}^r e^{-W_\kappa(u)} \mathcal{L}_B[\tau^B(A(r)), A(u)] du. \quad (1.6)$$

## 2. PATH DECOMPOSITION AND VALLEYS

**2.1. Path decomposition in the neighborhood of the  $h_t$ -minima  $m_i$ .** First we recall some results for  $h$ -extrema of  $W_\kappa$ . Let

$$V^{(i)}(x) := W_\kappa(x) - W_\kappa(m_i), \quad x \in \mathbb{R}, \quad i \in \mathbb{N}^*,$$

which is the potential  $W_\kappa$  translated so that it is 0 at the local minimum  $m_i$ . We also define

$$\tau_i^-(h) := \sup\{s < m_i, V^{(i)}(x) = h\}, \quad \tau_i(h) := \inf\{s > m_i, V^{(i)}(x) = h\}, \quad h > 0.$$

The following result has been proved by Faggionato [19] [for (i) and (ii)], and the last fact comes from the strong Markov property.

**Fact 2.1.** (path decomposition of  $W_\kappa$  around the  $h_t$ -minima  $m_i$ )

(i) The truncated trajectories  $(V^{(i)}(m_i - s), 0 \leq s \leq m_i - \tau_i^-(h_t))$ ,  $(V^{(i)}(m_i + s), 0 \leq s \leq \tau_i(h_t) - m_i)$ ,  $i \geq 1$  are independent.

(ii) Let  $(Q(s), s \geq 0)$  be a process with law BES(3,  $\kappa/2$ ). All the truncated trajectories  $(V^{(i)}(m_i - s), 0 \leq s \leq m_i - \tau_i^-(h_t))$  for  $i \geq 2$  and  $(V^{(j)}(m_j + s), 0 \leq s \leq \tau_j(h_t) - m_j)$  for  $j \geq 1$  are equal in law to  $(Q(s), 0 \leq s \leq \tau^Q(h_t))$ .

(iii) For  $i \geq 1$ , the truncated trajectory  $(V^{(i)}(s + \tau_i(h_t)), s \geq 0)$  is independent of  $(W_\kappa(s), s \leq \tau_i(h_t))$  and is equal in law to  $(W_\kappa^{h_t}(s), s \geq 0)$ , that is, to a  $(-\kappa/2)$ -drifted Brownian motion starting from  $h_t$ .

## 2.2. Definition of valleys and standard $h_t$ -minima $\tilde{m}_j$ , $j \in \mathbb{N}^*$ .

We are interested in the potential around the  $h_t$ -minima  $m_i$ ,  $i \in \mathbb{N}^*$ , in fact intervals containing at least  $[\tau_i^-(1 + \kappa)h_t, M_i]$ , however, these valleys could intersect. In order to define valleys which are well separated and i.i.d., we introduce the following notations. These notations are used to define valleys of the potential around some  $\tilde{m}_i$ , which are shown in Lemma 2.2 to be equal to the  $m_i$  for  $1 \leq i \leq n_t$  with large probability.

Let

$$h_t^+ := (1 + \kappa + 2\delta)h_t,$$

and define  $\tilde{L}_0^+ := 0$ ,  $\tilde{m}_0 := 0$ , and recursively for  $i \geq 1$  (see Figure 1),

$$\begin{aligned} \tilde{L}_i^\# &:= \inf\{x > \tilde{L}_{i-1}^+, W_\kappa(x) \leq W_\kappa(\tilde{L}_{i-1}^+) - h_t^+\}, \\ \tilde{\tau}_i(h_t) &:= \inf\{x \geq \tilde{L}_i^\#, W_\kappa(x) - \inf_{[\tilde{L}_i^\#, x]} W_\kappa \geq h_t\}, \\ \tilde{m}_i &:= \inf\{x \geq \tilde{L}_i^\#, W_\kappa(x) = \inf_{[\tilde{L}_i^\#, \tilde{\tau}_i(h_t)]} W_\kappa\}, \\ \tilde{L}_i^+ &:= \inf\{x > \tilde{\tau}_i(h_t), W_\kappa(x) \leq W_\kappa(\tilde{\tau}_i(h_t)) - h_t - h_t^+\}. \end{aligned} \quad (2.7)$$

We also introduce the following random variables for  $i \in \mathbb{N}^*$ :

$$\begin{aligned} \tilde{M}_i &:= \inf\{s > \tilde{m}_i, W_\kappa(s) = \max_{\tilde{m}_i \leq u \leq \tilde{L}_i^+} W_\kappa(u)\}, \\ \tilde{L}_i^* &:= \inf\{x > \tilde{\tau}_i(h_t), W_\kappa(x) - W_\kappa(\tilde{m}_i) = 3h_t/4\}, \\ \tilde{L}_i &:= \inf\{x > \tilde{\tau}_i(h_t), W_\kappa(x) - W_\kappa(\tilde{m}_i) = h_t/2\}, \\ \tilde{\tau}_i(h) &:= \inf\{s > \tilde{m}_i, W_\kappa(x) - W_\kappa(\tilde{m}_i) = h\}, \quad h > 0, \\ \tilde{\tau}_i^-(h) &:= \sup\{s < \tilde{m}_i, W_\kappa(x) - W_\kappa(\tilde{m}_i) = h\}, \quad h > 0, \\ \tilde{L}_i^- &:= \tilde{\tau}_i^-(h_t^+). \end{aligned} \quad (2.8)$$

We stress that these r.v. depend on  $t$ , which we do not write as a subscript to simplify the notations. Notice also that  $\tilde{\tau}_i(h_t)$  is the same in definitions (2.7) and (2.8) with  $h = h_t$ . Moreover by continuity of  $W_\kappa$ ,  $W_\kappa(\tilde{\tau}_i(h_t)) = W_\kappa(\tilde{m}_i) + h_t$ . So, the  $\tilde{m}_i$ ,  $i \in \mathbb{N}^*$ , are  $h_t$ -minima, since  $W_\kappa(\tilde{m}_i) = \inf_{[\tilde{L}_{i-1}^+, \tilde{\tau}_i(h_t)]} W_\kappa$ ,  $W_\kappa(\tilde{\tau}_i(h_t)) = W_\kappa(\tilde{m}_i) + h_t$  and  $W_\kappa(\tilde{L}_{i-1}^+) \geq W_\kappa(\tilde{m}_i) + h_t$ . Moreover,

$$\tilde{L}_{i-1}^+ < \tilde{L}_i^\# \leq \tilde{m}_i < \tilde{\tau}_i(h_t) < \tilde{L}_i < \tilde{L}_i^+, \quad i \in \mathbb{N}^*, \quad (2.9)$$

$$\tilde{L}_{i-1}^+ \leq \tilde{L}_i^- < \tilde{m}_i < \tilde{\tau}_i(h_t) < \tilde{M}_i < \tilde{L}_i^+, \quad i \in \mathbb{N}^*. \quad (2.10)$$

Furthermore by induction the r.v.  $\tilde{L}_i^\#$ ,  $\tilde{\tau}_i(h_t)$  and  $\tilde{L}_i^+$ ,  $i \in \mathbb{N}^*$  are stopping times for the natural filtration of  $(W_\kappa(x), x \geq 0)$ , and so  $\tilde{L}_i$ ,  $i \in \mathbb{N}^*$ , are also stopping times. Also by induction,

$$W_\kappa(\tilde{L}_i^\#) = \inf_{[0, \tilde{L}_i^\#]} W_\kappa, \quad W_\kappa(\tilde{m}_i) = \inf_{[0, \tilde{\tau}_i(h_t)]} W_\kappa, \quad W_\kappa(\tilde{L}_i^+) = \inf_{[0, \tilde{L}_i^+]} W_\kappa = W_\kappa(\tilde{m}_i) - h_t^+, \quad (2.11)$$



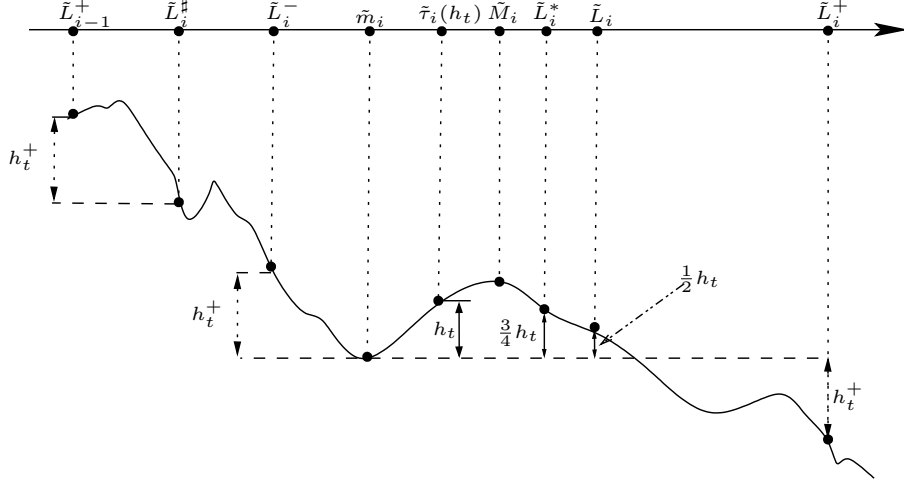


FIGURE 1. Schema of the potential between  $\tilde{L}_{i-1}^+$  and  $\tilde{L}_i^+$ , in the case  $\tilde{L}_i^\# < \tilde{L}_i^-$

for  $i \in \mathbb{N}^*$ . We also introduce the analogue of  $V^{(i)}$  for  $\tilde{m}_j$  as follows:

$$\tilde{V}^{(i)}(x) := W_\kappa(x) - W_\kappa(\tilde{m}_i), \quad x \in \mathbb{R}, \quad i \in \mathbb{N}^*.$$

We call  $i$ th  $h_t$ -valley the translated truncated potential  $(\tilde{V}^{(i)}(x), \tilde{L}_i^- \leq x \leq \tilde{L}_i)$ , for  $i \geq 1$ .

The following lemma states that, with an overwhelming high probability, the first  $n_t + 1$  positive  $h_t$ -minima  $m_i$ ,  $1 \leq i \leq n_t + 1$ , coincide with the r.v.  $\tilde{m}_j$ ,  $1 \leq i \leq n_t + 1$ . We introduce the corresponding event  $\mathcal{V}_t := \cap_{i=1}^{n_t+1} \{m_i = \tilde{m}_i\}$ .

**Lemma 2.2.** *Assume  $0 < \delta < 1$ . For  $t$  large enough,*

$$P(\bar{\mathcal{V}}_t) \leq C_1 n_t e^{-\kappa h_t/2} = e^{[-\kappa/2 + o(1)]h_t}, \quad C_1 > 0.$$

Moreover, the sequence  $\left( (\tilde{V}^{(i)}(x + \tilde{L}_{i-1}^+), 0 \leq x \leq \tilde{L}_i^+ - \tilde{L}_{i-1}^+), i \geq 1 \right)$ , is i.i.d.

**Proof:**

This Lemma is proved in [3] : Lemma 2.3. □

The following remark is used several times in the rest of the paper.

**Remark 2.3.** *On  $\mathcal{V}_t$ , we have for every  $1 \leq i \leq n_t$ ,  $m_i = \tilde{m}_i$ , and as a consequence,  $\tilde{V}^{(i)}(x) = V^{(i)}(x)$ ,  $x \in \mathbb{R}$ ,  $\tau_i^-(h) = \tilde{\tau}_i^-(h)$  and  $\tau_i(h) = \tilde{\tau}_i(h)$  for  $h > 0$ . Moreover,  $\tilde{M}_i = M_i$ . Indeed,  $\tilde{M}_i$  is an  $h_t$ -maximum for  $W_\kappa$ , which belongs to  $[\tilde{m}_i, \tilde{m}_{i+1}] = [m_i, m_{i+1}]$  on  $\mathcal{V}_t$ , and there is exactly one  $h_t$ -maximum in this interval since the  $h_t$ -maxima and minima alternate, which we defined as  $M_i$ , so  $\tilde{M}_i = M_i$ . So in the following, on  $\mathcal{V}_t$ , we can write these r.v. with or without tilde.*

### 3. CONTRIBUTIONS FOR HITTING AND LOCAL TIMES

#### 3.1. Negligible parts for hitting times.

In the following Lemma we recall results of [3] which tell that the time spent between valleys is negligible compared to the amount of time spent to escape from the valleys. It also gives an upper bound for the number of visited valleys, and the fact that the process never backtracks in a previous visited valley. For any  $i \leq n_t$ , define

$$U_i := H(\tilde{L}_i) - H(\tilde{m}_i), \quad U_0 = 0$$

and for any  $m \geq 1$  the events

$$\mathcal{B}_1(m) := \bigcap_{k=1}^m \left\{ 0 \leq H(\tilde{m}_k) - \sum_{i=1}^{k-1} U_i < \tilde{v}_t \right\},$$

where  $\tilde{v}_t := 2t/\log h_t$ . Finally

$$\mathcal{B}_2(m) := \bigcap_{j=1}^m \{H(\tilde{L}_j) - H(\tilde{m}_j) < H^+(\tilde{L}_j^-) - H(\tilde{m}_j), H(\tilde{m}_{j+1}) - H(\tilde{L}_j) < H^+(\tilde{L}_j^*) - H(\tilde{L}_j)\},$$

with  $H^+(x_j) := \inf\{k > \tilde{m}_j, X_k = x_j\}$  for any  $x$ .

**Lemma 3.1.** *For any  $\delta$  small enough and  $t$  large enough*

$$\mathbb{P}(H(\tilde{m}_1) < \tilde{v}_t) \geq \mathbb{P}(\mathcal{B}_1(n_t)) \geq 1 - C_2 v_t, \quad (3.12)$$

with  $v_t := n_t \cdot (\log h_t) e^{-\phi(t)} = o(1)$ ,  $C_2 > 0$ . Also we have :

$$\mathbb{P}(\mathcal{B}_2(n_t)) \geq 1 - C_3 n_t e^{-\delta \kappa h_t}, \quad (3.13)$$

$$\mathbb{P}(N_t < n_t) \geq 1 - C_4 e^{-\delta \kappa \phi(t)}, \quad (3.14)$$

with  $C_3 > 0$  and  $C_4 > 0$ .

**Proof:**

The first statement is Lemma 3.7 in [3], the second one is proved in Lemmata 3.2 and 3.3 in [3]. Finally (3.14) is proved in Lemma 5.2 of the same paper.  $\square$

### 3.2. Negligible parts for local times.

We now provide estimations on the local time, more especially we prove that in the complementary of a small interval in the neighborhood of the first  $n_t$   $h_t$ -minima, the local time at each site is negligible compared to  $t$ . We split this section into two, the first one deals with coordinate away from the valleys, the second with coordinates in the valleys excluding the points near the bottom.

#### 3.2.1. Supremum of the local time outside the valleys.

The aim of this subsection is to prove that at time  $t$ , the maximum of the local time outside the valleys is negligible compared to  $t$ . More precisely, define the following events

$$\mathcal{B}_3^1(m) := \left\{ \sup_{x \in [0, \tilde{m}_1]} \mathcal{L}(H(\tilde{m}_1), x) \leq f(t) \right\} \cap \bigcap_{j=1}^{m-1} \left\{ \sup_{x \in [\tilde{L}_j, \tilde{m}_{j+1}]} \mathcal{L}(H(\tilde{m}_{j+1}), x) \leq f(t) \right\},$$

$$\mathcal{B}_3^2(m) := \bigcap_{j=1}^{m-1} \left\{ \sup_{x \leq \tilde{L}_j} \left( \mathcal{L}(H(\tilde{m}_{j+1}), x) - \mathcal{L}(H(\tilde{L}_j), x) \right) \leq f(t) \right\},$$

$$\mathcal{B}_3(m) := \mathcal{B}_3^1(m) \cap \mathcal{B}_3^2(m), \quad f(t) = t e^{[\kappa(1+3\delta)-1]\phi(t)}.$$

In this section we prove

**Lemma 3.2.** *Assume  $\delta$  small enough such that  $\kappa(1+3\delta) < 1$ . There exists  $C_5 > 0$  such that for any large  $t$*

$$\mathbb{P}(\mathcal{B}_3(n_t)) \geq 1 - C_5 w_t,$$

with  $w_t := e^{-\kappa \delta \phi(t)}$ .

The proof is based on the lemma below, let us introduce a few notation with respect to the environment.

$$\begin{aligned}\tau_1^*(h) &:= \inf\{u \geq 0, W_\kappa(u) - \inf_{[0,u]} W_\kappa \geq h\}, \quad h > 0, \\ m_1^*(h) &:= \inf\{y \geq 0, W_\kappa(y) = \inf_{[0,\tau_1^*(h)]} W_\kappa\}.\end{aligned}$$

All along this work  $C_+$  is a positive constant that may grow from line to line.

**Lemma 3.3.** *For large  $t$ ,*

$$\mathbb{P}\left(\sup_{x \in [0, m_1^*(h_t)]} \mathcal{L}[H(\tau_1^*(h_t)), x] > te^{[\kappa(1+3\delta)-1]\phi(t)}\right) \leq \frac{C_+}{n_t e^{\kappa\delta\phi(t)}}. \quad (3.15)$$

**Proof of Lemma 3.3:** Thanks to (1.5) and (1.6) there exists a Brownian motion  $(B(s), s \geq 0)$ , independent of  $W_\kappa$ , such that

$$\mathcal{L}[H(\tau_1^*(h_t)), x] = e^{-W_\kappa(x)} \mathcal{L}_B[\tau^B(A(\tau_1^*(h_t))), A(x)], \quad x \in \mathbb{R}. \quad (3.16)$$

By the first Ray–Knight theorem (see e.g. Revuz and Yor [29], chap. XI), for every  $\alpha > 0$ , there exists a Bessel processes  $Q_2$  of dimension 2 starting from 0, such that  $\mathcal{L}_B(\tau^B(\alpha), x)$  is equal to  $Q_2^2(\alpha - x)$  for every  $x \in [0, \alpha]$ . Consequently, using (3.16) and the independence of  $B$  and  $W_\kappa$ , there exists a 2-dimensional Bessel process  $Q_2$  such that

$$\mathcal{L}[H(\tau_1^*(h_t)), x] = e^{-W_\kappa(x)} Q_2^2[A(\tau_1^*(h_t)) - A(x)] \quad 0 \leq x \leq \tau_1^*(h_t). \quad (3.17)$$

In order to evaluate this quantity, the idea is to say that loosely speaking,  $Q_2^2$  grows almost linearly. More formally, we consider the functions  $k(t) := e^{2\kappa^{-1}\phi(t)}$ ,  $a(t) := 4\phi(t)$  and  $b(t) := 6\kappa^{-1}\phi(t)e^{\kappa h_t}$  and define the following events

$$\begin{aligned}\mathcal{A}_0 &:= \left\{ A_\infty := \int_0^{+\infty} e^{-W_\kappa(x)} dx \leq k(t) \right\}, \\ \mathcal{A}_1 &:= \left\{ \forall u \in (0, k(t)), Q_2^2(u) \leq 2eu[a(t) + 4 \log \log[ek(t)/u]] \right\}, \\ \mathcal{A}_2 &:= \left\{ \inf_{[0, \tau_1^*(h_t)]} W_\kappa \geq -b(t) \right\}.\end{aligned}$$

We know that  $P(A_\infty \geq y) \leq C_+ y^{-\kappa}$  for  $y > 0$  since  $2/A_\infty$  is a gamma variable of parameter  $(\kappa, 1)$  (see Dufresne [16]), so  $P(\overline{\mathcal{A}}_0) \leq C_+ k(t)^{-\kappa} = C_+ e^{-2\phi(t)}$ . Moreover,  $\mathbb{P}(\overline{\mathcal{A}}_1) \leq C_+ \exp[-a(t)/2] = C_+ e^{-2\phi(t)}$  by Lemma 6.5. Also we know that  $-\inf_{[0, \tau_1^*(h)]} W_\kappa$ , denoted by  $-\beta$  in (Faggionato [19], eq. (2.2)) is exponentially distributed with mean  $2\kappa^{-1} \sinh(\kappa h/2) e^{\kappa h/2}$  ([19], eq. (2.4)). So,

$$P(\overline{\mathcal{A}}_2) = P[-\inf_{[0, \tau_1^*(h_t)]} W_\kappa > b(t)] = \exp[-b(t)\kappa/(2 \sinh(\kappa h_t/2) e^{\kappa h_t/2})] \leq e^{-2\phi(t)}$$

for large  $t$ .

Now assume we are on  $\mathcal{A}_0 \cap \mathcal{A}_1 \cap \mathcal{A}_2$ . Due to (3.17), we have for every  $0 \leq x < \tau_1^*(h_t)$ , since  $0 < A(\tau_1^*(h_t)) - A(x) \leq A_\infty \leq k(t)$ ,

$$\mathcal{L}[H(\tau_1^*(h_t)), x] \leq e^{-W_\kappa(x)} 2e[A(\tau_1^*(h_t)) - A(x)] \{a(t) + 4 \log \log [ek(t)/[A(\tau_1^*(h_t)) - A(x)]]\}. \quad (3.18)$$

We now introduce

$$f_i := \inf\{x \geq 0, W_\kappa(x) \leq -i\} = \tau^{W_\kappa}(-i), \quad i \in \mathbb{N},$$

and let  $0 \leq x < \tau_1^*(h_t)$ . There exists  $i \in \mathbb{N}$  such that  $f_i \leq x < f_{i+1}$ . Moreover, we are on  $\mathcal{A}_2$ , so  $i \leq b(t)$ . Furthermore,  $x < f_{i+1}$ , so  $W_\kappa(x) \geq -(i+1)$  and then  $e^{-W_\kappa(x)} \leq e^{i+1} = e^{-W_\kappa(f_i)+1}$ . All this leads to

$$e^{-W_\kappa(x)} [A(\tau_1^*(h_t)) - A(x)] = e^{-W_\kappa(x)} \int_x^{\tau_1^*(h_t)} e^{W_\kappa(u)} du \leq e \int_{f_i}^{\tau_1^*(h_t)} e^{W_\kappa(u) - W_\kappa(f_i)} du. \quad (3.19)$$

To bound this, we introduce the event

$$\mathcal{A}_3 := \bigcap_{i=0}^{\lfloor b(t) \rfloor} \left\{ \int_{f_i}^{\tau_1^*(h_t)} e^{W_\kappa(u) - W_\kappa(f_i)} du \leq e^{(1-\kappa)h_t} b(t) n_t e^{\kappa\delta\phi(t)} \right\}.$$

We now consider  $\tau_1^*(u, h_t) := \inf\{y \geq u, W_\kappa(y) - \inf_{[u, y]} W_\kappa \geq h_t\} \geq \tau_1^*(h_t)$  for  $u \geq 0$ . We have

$$E\left(\int_{f_i}^{\tau_1^*(h_t)} e^{W_\kappa(u) - W_\kappa(f_i)} du\right) \leq E\left(\int_{f_i}^{\tau_1^*(f_i, h_t)} e^{W_\kappa(u) - W_\kappa(f_i)} du\right) = \beta_0(h_t),$$

by the strong Markov property applied at stopping time  $f_i$ , where  $\beta_0(h) := E\left(\int_0^{\tau_1^*(h)} e^{W_\kappa(u)} du\right)$ .

We know (see [3] eq. (3.38), in the proof of Lemma 3.6) that  $\beta_0(h) \leq C_+ e^{(1-\kappa)h}$  for large  $h$ . Hence for large  $t$  by Markov inequality,

$$\begin{aligned} P(\overline{\mathcal{A}}_3) &\leq \sum_{i=0}^{\lfloor b(t) \rfloor} P\left(\int_{f_i}^{\tau_1^*(h_t)} e^{W_\kappa(u) - W_\kappa(f_i)} du > e^{(1-\kappa)h_t} b(t) n_t e^{\kappa\delta\phi(t)}\right) \\ &\leq \frac{[b(t) + 1]\beta_0(h_t)}{e^{(1-\kappa)h_t} b(t) n_t e^{\kappa\delta\phi(t)}} \leq \frac{C_+}{n_t e^{\kappa\delta\phi(t)}}. \end{aligned}$$

Now, on  $\cap_{j=0}^3 \mathcal{A}_j$ , (3.18) and (3.19) lead to

$$\mathcal{L}[H(\tau_1^*(h_t)), x] \leq 2e^{2+(1-\kappa)h_t} b(t) n_t e^{\kappa\delta\phi(t)} \{a(t) + 4 \log \log [ek(t)/[A(\tau_1^*(h_t)) - A(x)]]\}. \quad (3.20)$$

For any  $0 \leq x \leq m_1^*(h_t)$ ,  $\inf_{[0, \tau_1^*(h_t)]} W_\kappa \geq -b(t)$  so

$$A(\tau_1^*(h_t)) - A(x) = \int_x^{\tau_1^*(h_t)} e^{W_\kappa(u)} du \geq \int_{m_1^*(h_t)}^{\tau_1^*(h_t)} e^{W_\kappa(u)} du \geq e^{-b(t)} [\tau_1^*(h_t) - m_1^*(h_t)] \geq e^{-b(t)}$$

on the event  $\cap_{i=0}^4 \mathcal{A}_i$  with  $\mathcal{A}_4 := \{\tau_1^*(h_t) - m_1^*(h_t) \geq 1\}$ . Since  $m_1 = m_1^*(h_t)$  and  $\tau_1(h_t) = \tau_1^*(h_t)$  on  $\{M_0 \leq 0\}$  by definition of  $h_t$ -extrema, we have

$$\begin{aligned} P(\overline{\mathcal{A}}_4) &\leq P(0 < M_0 < m_1) + P[\tau_1(h_t) - m_1 < 1] \\ &\leq 2\kappa h_t e^{-\kappa h_t} + P(\tau^Q(h_t) - \tau^Q(h_t/2) < 1) \leq 2\kappa h_t e^{-\kappa h_t} + C_+ \exp[-(c_-)h_t^2] \end{aligned}$$

due to ([3], eq. (2.8), coming from Faggionato [19]), Fact 2.1 (ii) and (6.85).

$c_-$  is a positive constant that may decrease from line to line in the sequel of the paper. Now, we have  $ek(t)/[A(\tau_1^*(h_t)) - A(x)] \leq ek(t)e^{b(t)}$  on  $\cap_{i=0}^4 \mathcal{A}_i$ , and then, on this event, (3.20) leads to

$$\begin{aligned} \mathcal{L}[H(\tau_1^*(h_t)), x] &\leq 2e^{2+(1-\kappa)h_t} b(t) n_t e^{\kappa\delta\phi(t)} \{a(t) + 4 \log \log [ek(t)e^{b(t)}]\} \\ &\leq C_+ t \phi(t) e^{[\kappa(1+\delta)-1]\phi(t)} e^{\kappa\delta\phi(t)} h_t, \end{aligned}$$

since  $\phi(t) = o(\log t)$ ,  $h_t = \log t - \phi(t)$  and  $n_t = \lfloor e^{\kappa(1+\delta)\phi(t)} \rfloor$ . We notice that for large  $t$ ,  $C_+ \phi(t) h_t \leq e^{\kappa\delta\phi(t)}$  since  $\log \log t = o(\phi(t))$ . Hence, for large  $t$ ,

$$\mathcal{L}[H(\tau_1^*(h_t)), x] \leq t e^{[\kappa(1+3\delta)-1]\phi(t)},$$

on  $\cap_{i=0}^4 \mathcal{A}_i$  for every  $0 \leq x \leq m_1^*(h_t)$ . This gives for large  $t$ ,

$$\mathbb{P}\left(\sup_{x \in [0, m_1^*(h_t)]} \mathcal{L}[H(\tau_1^*(h_t)), x] \leq t e^{[\kappa(1+3\delta)-1]\phi(t)}\right) \geq \mathbb{P}(\cap_{i=0}^4 \mathcal{A}_i) \geq 1 - \frac{C_+}{n_t e^{\kappa\delta\phi(t)}},$$

due to the previous bounds for  $\mathbb{P}(\mathcal{A}_i)$ ,  $0 \leq i \leq 4$ . This proves the lemma.  $\square$

With the help of the previous lemma, we can now prove Lemma 3.2:

**Proof of Lemma 3.2:** The method is similar to the proof of Lemma 3.7 of [3]. Recall the definition of  $\tilde{L}_i^* < \tilde{L}_i$  just above (2.8), also let

$$\begin{aligned}\tilde{\tau}_{i+1}^*(h_t) &:= \inf \{u \geq \tilde{L}_i^*, W_\kappa(u) - \inf_{[\tilde{L}_i^*, u]} W_\kappa \geq h_t\} \leq \tilde{\tau}_{i+1}(h_t), \quad i \geq 1, \\ \tilde{m}_{i+1}^*(h_t) &:= \inf \{u \geq \tilde{L}_i^*, W_\kappa(u) = \inf_{[\tilde{L}_i^*, \tilde{\tau}_{i+1}^*(h_t)]} W_\kappa\}, \quad i \geq 1, \\ \mathcal{A}_5 &:= \cap_{i=1}^{n_t-1} \{\tilde{\tau}_{i+1}^*(h_t) = \tilde{\tau}_{i+1}(h_t)\}, \\ X_i(u) &:= X(u + H(\tilde{L}_i)), \quad X_i^*(u) := X(u + H(\tilde{L}_i^*)), \quad u \geq 0.\end{aligned}\tag{3.21}$$

By the strong Markov property,  $X_i$  and  $X_i^*$  are diffusions in the environment  $W_\kappa$ , starting respectively from  $\tilde{L}_i$  and  $\tilde{L}_i^*$ . We denote respectively by  $\mathcal{L}_{X_i}$ ,  $\mathcal{L}_{X_i^*}$ ,  $H_{X_i}$  and  $H_{X_i^*}$  the local times and hitting times of  $X_i$  and  $X_i^*$ . We have for every  $x \geq \tilde{L}_i^*$ ,

$$\mathcal{L}(H(\tilde{m}_{i+1}), x) - \mathcal{L}(H(\tilde{L}_i), x) \leq \mathcal{L}(H(\tilde{m}_{i+1}), x) - \mathcal{L}(H(\tilde{L}_i^*), x) = \mathcal{L}_{X_i^*}(H_{X_i^*}(\tilde{m}_{i+1}), x).$$

Consequently, on  $\mathcal{A}_5 \cap \mathcal{A}_6$  with  $\mathcal{A}_6 := \cap_{j=1}^{n_t-1} \{H_{X_j}(\tilde{m}_{j+1}) < H_{X_j}(\tilde{L}_j^*)\}$ , for  $1 \leq i \leq n_t - 1$ ,

$$\begin{aligned}\sup_{x \in \mathbb{R}} \left( \mathcal{L}(H(\tilde{m}_{i+1}), x) - \mathcal{L}(H(\tilde{L}_i), x) \right) &= \sup_{\tilde{L}_i^* \leq x \leq \tilde{m}_{i+1}} \left( \mathcal{L}(H(\tilde{m}_{i+1}), x) - \mathcal{L}(H(\tilde{L}_i), x) \right) \\ &\leq \sup_{\tilde{L}_i^* \leq x \leq \tilde{m}_{i+1}} \mathcal{L}_{X_i^*}(H_{X_i^*}(\tilde{m}_{i+1}), x) \\ &\leq \sup_{\tilde{L}_i^* \leq x \leq \tilde{m}_{i+1}^*} \mathcal{L}_{X_i^*}(H_{X_i^*}(\tilde{\tau}_{i+1}^*(h_t)), x),\end{aligned}\tag{3.22}$$

since  $\tilde{m}_{i+1}^* = \tilde{m}_{i+1} \leq \tilde{\tau}_{i+1}(h_t) = \tilde{\tau}_{i+1}^*(h_t)$  on  $\mathcal{A}_5$ . Now, notice that the right hand side of (3.22) is the supremum of the local times of  $X_i^* - \tilde{L}_i^*$ , up to its first hitting time of  $\tilde{\tau}_{i+1}^*(h_t) - \tilde{L}_i^*$ , over all locations in  $[0, \tilde{m}_{i+1}^* - \tilde{L}_i^*]$ . Since  $X_i^* - \tilde{L}_i^*$  is a diffusion in the environment  $(W_\kappa(\tilde{L}_i^* + x) - W_\kappa(\tilde{L}_i^*), x \in \mathbb{R})$ , which has on  $[0, +\infty)$  the same law as  $(W_\kappa(x), x \geq 0)$  because  $\tilde{L}_i^*$  is a stopping time for  $W_\kappa$ , the right hand side of (3.22) has the same law, under the annealed probability  $\mathbb{P}$ , as  $\sup_{x \in [0, m_1^*(h_t)]} \mathcal{L}[H(\tau_1^*(h_t)), x]$ . Consequently,

$$\begin{aligned}&\mathbb{P} \left( \bigcup_{i=1}^{n_t-1} \left\{ \sup_{x \in \mathbb{R}} \left( \mathcal{L}(H(\tilde{m}_{i+1}), x) - \mathcal{L}(H(\tilde{L}_i), x) \right) > te^{[\kappa(1+3\delta)-1]\phi(t)} \right\} \right) \\ &\leq n_t \left[ \mathbb{P} \left( \sup_{x \in [0, m_1^*(h_t)]} \mathcal{L}[H(\tau_1^*(h_t)), x] > te^{[\kappa(1+3\delta)-1]\phi(t)} \right) + \mathbb{P}(\overline{\mathcal{A}}_5) + \mathbb{P}(\overline{\mathcal{A}}_6) \right] \\ &\leq C_+ e^{-\kappa\delta\phi(t)}\end{aligned}\tag{3.23}$$

by Lemma 3.3, since  $\mathbb{P}(\overline{\mathcal{A}}_5) \leq C_+ n_t h_t e^{-\kappa h_t}$  by ([3], eq. (3.41)),  $\mathbb{P}(\overline{\mathcal{A}}_6) \leq C_+ n_t e^{-\kappa h_t/16}$  by ([3], Lemma 3.3), and since  $\phi(t) = o(\log t)$ . Notice that, as before,  $\tilde{m}_1 = m_1 = m_1^*(h_t)$  on  $\mathcal{V}_t \cap \{M_0 \leq 0\}$ . Finally,

$$\mathbb{P} \left( \sup_{x \in [0, \tilde{m}_1]} \mathcal{L}(H(\tilde{m}_1), x) > te^{[\kappa(1+3\delta)-1]\phi(t)} \right) \leq \frac{C_+}{e^{\kappa\delta\phi(t)}} + P(\overline{\mathcal{V}}_t) + P(0 < M_0 < m_1) \leq \frac{C_+}{e^{\kappa\delta\phi(t)}}$$

also by Lemma 3.3, Lemma 2.2, and since  $P(0 < M_0 < m_1) \leq 2\kappa h_t e^{-\kappa h_t}$  due to ([3], eq. (2.8)). This and (3.23) prove the lemma.  $\square$

3.2.2. *Local time in the valley  $[\tilde{L}_j^-, \tilde{L}_j]$  but far from  $\tilde{m}_j$ .*

Let

$$\mathcal{D}_j := [\tilde{m}_j - r_t, \tilde{m}_j + r_t], \quad \text{with } r_t := (\phi(t))^2.\tag{3.24}$$

$$\mathcal{B}_4(m) := \bigcap_{j=1}^m \left\{ \sup_{x \in \bar{\mathcal{D}}_j \cap [\tau_j^-(h_t^+), \tilde{L}_j]} (\mathcal{L}(H(\tilde{L}_j), x) - \mathcal{L}(H(\tilde{m}_j), x)) < te^{-2\phi(t)} \right\}, \quad m \geq 1.$$

with  $\bar{\mathcal{D}}_j$  the complementary of  $\mathcal{D}_j$ .

**Lemma 3.4.** *Assume  $0 < \delta < 1/8$ . There exists  $C_6 > 0$  such that*

$$\mathbb{P}[\mathcal{B}_4(n_t)] \geq 1 - C_6 n_t e^{-2\phi(t)},$$

**Proof:** We make the proof replacing  $r(t) = (\phi(t))^2$  by  $r(t) = C_0 \phi(t)$  with  $C_0$  a constant large enough, this is a little more precise than what we need here and may be used for other purposes. Let  $j \in [1, n_t]$ . First under  $\mathbb{P}_{m_j}^{W_\kappa}$ , there exists a Brownian motion  $(B(s), s \geq 0)$ , independent of  $\tilde{V}^{(j)}$ , such that

$$\mathcal{L}[H(\tilde{L}_j), x] = e^{-\tilde{V}^{(j)}(x)} \mathcal{L}_B[\tau^B(A^j(\tilde{L}_j)), A^j(x)], \quad x \in \mathbb{R},$$

where  $A^j(x) := \int_{\tilde{m}_j}^x e^{\tilde{V}^{(j)}(s)} ds$ . By scaling, there exists another Brownian motion  $\tilde{B}$  that we still denote  $B$  for simplicity, independent of  $\tilde{V}^{(j)}$ , such that

$$\mathcal{L}[H(\tilde{L}_j), x] = e^{-\tilde{V}^{(j)}(x)} A^j(\tilde{L}_j) \mathcal{L}_B[\tau^B(1), A^j(x)/A^j(\tilde{L}_j)] \quad x \in \mathbb{R}. \quad (3.25)$$

In order to bound the terms  $\mathcal{L}_B[\tau^B(1), \cdot]$  and  $A^j(\tilde{L}_j)$  in (3.25), we first introduce

$$\mathcal{A}_1 := \{ \sup_{u \in \mathbb{R}} \mathcal{L}_B[\tau^B(1), u] \leq e^{2\phi(t)} \}, \quad \mathcal{A}_2 := \{ A^j(\tilde{L}_j) \leq 2e^{h_t + 2\phi(t)/\kappa} \}. \quad (3.26)$$

We have  $\mathbb{P}(\bar{\mathcal{A}}_2) \leq 2e^{-2\phi(t)}$  for large  $t$  by Lemma 6.4 eq. (6.94) and (6.95). Moreover on  $\mathcal{V}_t$ , we have by Remark 2.3 and Fact 2.1 (ii) and (iii),  $A^j(\tilde{L}_j) \leq [\tau_j(h_t) - m_j] e^{h_t} + \int_{\tau_j(h_t)}^{\tilde{L}_j} e^{V^{(j)}(s)} ds \stackrel{\mathbf{I}}{\leq} e^{h_t} \tau^Q(h_t) + G^+(h_t/2, h_t)$ , where recall that  $Q$  has law  $BES(3, \kappa/2)$  and is independent of  $G^+(h_t/2, h_t)$ , which is defined in (1.3), and with  $L_j := \inf\{s > \tau_j(h_t), V^{(j)}(s) = h_t/2\}$ . Consequently,

$$P(\bar{\mathcal{A}}_2) \leq P(\tau^Q(h_t) > e^{2\phi(t)/\kappa}) + P(G^+(h_t/2, h_t) > e^{h_t + 2\phi(t)/\kappa}) + P(\bar{\mathcal{V}}_t) \leq C_+ e^{-2\phi(t)}$$

for large  $t$  by eq. (6.86), Lemma 6.3 eq. (6.92) and Lemma 2.2, and since  $\phi(t) = o(\log t)$  and  $\log \log t = o(\phi(t))$ .

Now, we would like to bound the term  $e^{-\tilde{V}^{(j)}(x)}$  that appears in (3.25). To this aim, we define

$$\mathcal{A}_3 := \{ \tilde{\tau}_j[\kappa C_0 \phi(t)/8] \leq \tilde{m}_j + C_0 \phi(t) \}, \quad \mathcal{A}_4 := \left\{ \inf_{[\tau_j[\kappa C_0 \phi(t)/8], \tau_j(h_t)]} V^{(j)} \geq \kappa C_0 \phi(t)/16 \right\}.$$

We can prove using Fact 2.1 (see [3] Lemma 2.5 for details) that  $P(\bar{\mathcal{A}}_3) \leq C_+ e^{-[\kappa^2 C_0 \phi(t)]/(16\sqrt{2})} \leq e^{-2\phi(t)}$  if we choose  $C_0$  large enough. Moreover with Fact 2.1 again (see [3], eq. (2.31) applied with  $h = C_0 \phi(t)$ ,  $\alpha = \kappa/8$ ,  $\gamma = \kappa/16$  and  $\omega = h_t/(C_0 \phi(t))$ ) we get  $P(\bar{\mathcal{A}}_4) \leq e^{-\kappa^2 C_0 \phi(t)/16} \leq e^{-2\phi(t)}$  for large  $t$ . We notice that  $\inf_{[\tilde{m}_j + C_0 \phi(t), \tilde{\tau}_j(h_t)]} \tilde{V}^{(j)} \geq \kappa C_0 \phi(t)/16$  on  $\mathcal{A}_3 \cap \mathcal{A}_4 \cap \mathcal{V}_t$ , thanks to Remark 2.3. We prove similarly that  $P(\bar{\mathcal{A}}_5) \leq C_+ e^{-\kappa^2 C_0 \phi(t)/(16\sqrt{2})} + P(\bar{\mathcal{V}}_t) \leq 2e^{-2\phi(t)}$ , where  $\mathcal{A}_5 := \left\{ \inf_{[\tilde{\tau}_j^-(h_t), \tilde{m}_j - C_0 \phi(t)]} \tilde{V}^{(j)} \geq \kappa C_0 \phi(t)/16 \right\}$ . Also by ([3], Lemma 2.7),  $P(\bar{\mathcal{A}}_6) \leq e^{-\kappa h_t/8}$ , with  $\mathcal{A}_6 := \left\{ \inf_{[\tilde{\tau}_j^-(h_t^+), \tilde{\tau}_j^-(h_t)]} \tilde{V}^{(j)} \geq h_t/2 \right\}$ . We also know that  $\tilde{V}^{(j)}(x) \geq h_t/2$  for all  $\tilde{\tau}_j(h_t) \leq x \leq \tilde{L}_j$  by definition of  $\tilde{L}_j$ . Consequently on  $\bigcap_{i=3}^6 \mathcal{A}_i \cap \mathcal{V}_t$ , for all  $x \in \bar{\mathcal{D}}_j \cap [\tilde{\tau}_j^-(h_t^+), \tilde{L}_j]$ , we have  $e^{-\tilde{V}^{(j)}(x)} \leq e^{-\kappa C_0 \phi(t)/16}$ .

Hence on  $\cap_{i=1}^6 \mathcal{A}_i \cap \mathcal{V}_t$ , we have under  $\mathbb{P}_{\tilde{m}_j}^{W_\kappa}$ , by (3.25) and (3.26),

$$\sup_{x \in \overline{\mathcal{D}}_j \cap [\tilde{\tau}_j^-(h_t^+), \tilde{L}_j]} \mathcal{L}[H(\tilde{L}_j), x] \leq 2te^{(1+2/\kappa)\phi(t)} e^{-\kappa C_0 \phi(t)/16} \leq te^{-2\phi(t)},$$

if we choose  $C_0$  large enough. So, conditioning by  $W_\kappa$  and applying the strong Markov property at time  $H(\tilde{m}_j)$ , we get

$$\begin{aligned} & \mathbb{P} \left[ \sup_{x \in \overline{\mathcal{D}}_j \cap [\tilde{\tau}_j^-(h_t^+), \tilde{L}_j]} (\mathcal{L}[H(\tilde{L}_j), x] - \mathcal{L}[H(\tilde{m}_j), x]) \leq te^{-2\phi(t)} \right] \\ & \geq \mathbb{E}(\mathbb{P}_{\tilde{m}_j}^{W_\kappa}(\cap_{i=1}^6 \mathcal{A}_i \cap \mathcal{V}_t)) \geq 1 - C_+ e^{-2\phi(t)} \end{aligned}$$

uniformly for large  $t$  due to the previous estimates and thanks to Lemma 2.2. This proves the lemma.  $\square$

### 3.3. Convergence of the main contributions.

In this section we make a link between the families  $\{[U_j := H(\tilde{L}_j) - H(\tilde{m}_j), \mathcal{L}(H(\tilde{L}_j), \tilde{m}_j)], j \leq n_t\}$ , and the i.i.d. sequence  $\{(\mathcal{H}_j, \ell_j), j \leq n_t\}$  described in the introduction.

Let  $F_1^\pm(a), F_2^\pm(a)$  and  $F_3^\pm(a)$  independent copies of  $F^\pm(a)$  also independent of  $G^\pm(a, b)$ .

**Proposition 3.5.** *Let  $t > 0$  large, for  $\delta > 0$  small enough (recall that  $\delta$  appears in the definitions of  $n_t$  and  $h_t^+$ ) there exists  $d_1 = d_1(\delta, \kappa) > 0$ ,  $D_1(d_1) > 0$  and a sequence  $\{(S_j(t), R_j(t), \mathbf{e}_j(t)), j \leq n_t\}$  of i.i.d. random variables with  $S_j, R_j$  and  $\mathbf{e}_j$  independent and  $S_1 \stackrel{\mathfrak{I}}{=} F_1^+(h_t) + G^+(h_t, h_t/2)$ ,  $R_1 \stackrel{\mathfrak{I}}{=} F_2^-(h_t/2) + F_3^-(h_t/2)$  and  $\mathbf{e}_1 \stackrel{\mathfrak{I}}{=} \mathcal{E}(1/2)$  such that*

$$\begin{aligned} & \mathbb{P} \left( \cap_{j=1}^{n_t} \{ |U_j - \mathcal{H}_j| \leq \epsilon_t \mathcal{H}_j, |\mathcal{L}(H(\tilde{L}_j), \tilde{m}_j) - \ell_j| \leq \epsilon_t \ell_j \} \right) \geq 1 - e^{-D_1 h_t}, \text{ with} \\ & \ell_j := S_j \mathbf{e}_j, \mathcal{H}_j := R_j \ell_j, \epsilon_t := e^{-d_1 h_t}. \end{aligned}$$

The proof of the above Proposition, which is in the spirit of the proofs of Propositions 3.4 and 4.4 in [3] makes use of the following Lemma:

**Lemma 3.6.** *Let  $t > 0$  large,*

i) *There exists a sequence  $(\mathbf{e}_j(t), j \leq n_t)$  of i.i.d. random variables with exponential law of mean 2 and independent of  $W_\kappa$  such that there exist constants  $d_- > 0$ ,  $D_- = D_-(d_-) > 0$  possibly depending on  $\kappa$  and  $\delta$  such that*

$$\mathbb{P} \left( \bigcap_{j=1}^{n_t} \{ |U_j - \mathbb{H}_j| \leq e^{-d_- h_t} \mathbb{H}_j, \mathcal{L}(H(\tilde{L}_j), \tilde{m}_j) = \mathbb{L}_j \} \right) \geq 1 - e^{-D_- h_t}, \quad (3.27)$$

where  $\mathbb{L}_j := \mathbf{e}_j \int_{\tilde{m}_j}^{\tilde{L}_j} e^{\tilde{V}^{(j)}(x)} dx$ ,  $\mathbb{H}_j := \mathbb{L}_j \mathbb{R}_j$ ,  $\mathbb{R}_j := \int_{\tilde{\tau}_j^-(h_t/2)}^{\tilde{\tau}_j^+(h_t/2)} e^{-\tilde{V}^{(j)}(x)}$ . Moreover the random variables  $\{(\mathbb{L}_j, \mathbb{H}_j), j \geq 1\}$  are i.i.d.

ii) *Also there exists a sequence of independent and identically distributed random variables  $(\mathbb{S}_j, j \leq n_t)$ , independent of  $(\mathbb{R}_j, j \leq n_t)$  and  $(\mathbf{e}_j, j \leq n_t)$  such that*

$$P \left( \bigcap_{j=1}^{n_t} \left\{ \left| \int_{\tilde{m}_j}^{\tilde{L}_j} e^{\tilde{V}^{(j)}(x)} dx - \mathbb{S}_j \right| \leq e^{-d_- h_t} \mathbb{S}_j \right\} \right) \geq 1 - e^{-D_- h_t}, \text{ and } \mathbb{S}_1 \stackrel{\mathfrak{I}}{=} F^+(h_t). \quad (3.28)$$

**Proof of Lemma 3.6**

For (3.27) : by the strong Markov property, formula (1.5) and (1.6) the sequence  $\{U_j, \mathcal{L}(H(\tilde{L}_j), \tilde{m}_j), j \leq n_t\}$  is equal to the sequence  $\{H_j(\tilde{L}_j), \mathcal{L}_j(H_j(\tilde{L}_j), \tilde{m}_j), j \leq n_t\}$ , where

$$\begin{aligned} H_j(\tilde{L}_j) &:= \int_{-\infty}^{\tilde{L}_j} e^{-\tilde{V}^{(j)}(u)} \mathcal{L}_{B^j}[\tau^{B^j}(A^j(\tilde{L}_j)), A^j(u)] du, \\ \mathcal{L}_j(H_j(\tilde{L}_j), \tilde{m}_j) &:= \mathcal{L}_{B^j}[\tau^{B^j}(A^j(\tilde{L}_j)), 0], \quad A^j(u) := \int_{\tilde{m}_j}^u e^{\tilde{V}^{(j)}(x)} dx, \end{aligned} \quad (3.29)$$

with  $(B^j, j \leq n_t)$  a sequence of independent standard Brownian motions independent of  $W_\kappa$ , such that  $B^j$  starts at  $A^j(\tilde{m}_j) = 0$  and is killed when it first hits  $A^j(\tilde{L}_j)$ . Also  $\mathcal{L}_{B^j}$  is the local time associated with  $B^j$ . Define  $\mathcal{A}_j := \{\max_{u < \tilde{L}_j^-} \mathcal{L}_{B^j}[\tau^{B^j}(A^j(\tilde{L}_j)), A^j(u)] = 0\}$ , we can prove with exactly the same method than in the proof of Lemma 3.2 in [3] (see the estimation of  $\mathbb{P}^{W_\kappa}(\bar{\mathcal{E}}_j)$ ) that there exists  $c_- > 0$  (possibly depending on  $\kappa$  and  $\delta$ ) such that  $\mathbb{P}(\cap_{j=1}^{n_t} \mathcal{A}_j) \geq 1 - e^{-c_- h_t}$ . So

$$\mathbb{P}\left(\bigcap_{j=1}^{n_t} \{H_j(\tilde{L}_j) = h_j\}\right) \geq 1 - e^{-c_- \kappa h_t},$$

with  $h_j := \int_{\tilde{L}_j^-}^{\tilde{L}_j} e^{-\tilde{V}^{(j)}(x)} \mathcal{L}_{B^j}[\tau^{B^j}(A^j(\tilde{L}_j)), A^j(u)] du$ . We also notice that  $(h_j, \mathcal{L}_j(H_j(\tilde{L}_j), \tilde{m}_j))$  are strictly function of  $B^j$  and  $(\tilde{V}^{(j)}(x + \tilde{L}_{j-1}^+), 0 \leq x < \tilde{L}_j^+ - \tilde{L}_{j-1}^+)$ , where by construction  $\tilde{L}_{i-1}^+ < \tilde{L}_j^-$  so the random variables  $(h_j, \mathcal{L}_j(H_j(\tilde{L}_j), \tilde{m}_j), j \leq n_t)$  are i.i.d by the second fact of Lemma 2.2.

By scale invariance of brownian motion  $B^j$  we have that  $(\mathcal{L}_{B^j}[\tau^{B^j}(A^j(\tilde{L}_j)), A^j(u)], \tilde{L}_j^- \leq u \leq \tilde{L}_j)$  is equal in law to  $(A^j(\tilde{L}_j) \mathcal{L}_{\tilde{B}^j}[\tau^{\tilde{B}^j}(1), A^j(u)/A^j(\tilde{L}_j)], \tilde{L}_j^- \leq u \leq \tilde{L}_j)$ , where  $\tilde{B}^j$  is a standard Brownian motion independent of  $W_\kappa$  which we still denote by  $B^j$  in the sequel. In particular

$$\begin{aligned} (h_j, \mathcal{L}_j(H_j(\tilde{L}_j), \tilde{m}_j)) &\stackrel{\mathfrak{L}}{=} (\tilde{h}_j, \tilde{\mathcal{L}}_j), \\ \tilde{h}_j &:= A^j(\tilde{L}_j) \int_{\tilde{L}_j^-}^{\tilde{L}_j} e^{-\tilde{V}^{(j)}(x)} \mathcal{L}_{B^j}[\tau^{B^j}(1), A^j(u)/A^j(\tilde{L}_j)] du, \quad \tilde{\mathcal{L}}_j := A^j(\tilde{L}_j) \mathcal{L}_{B^j}(\tau^{B^j}(1), 0). \end{aligned} \quad (3.30)$$

Then let  $\mathbf{e}_j := \mathcal{L}_{B^j}(\tau^{B^j}(1), 0)$  and recall that by the first Ray Knight theorem,  $\mathbf{e}_j$  has for law an exponential variable with parameter 1/2. Note then that  $\tilde{\mathcal{L}}_j = \mathbb{L}_j$ , so to finish the proof of 3.27 we only have to approximate  $\tilde{h}_j$  in probability. This is what is done in the proof of Lemma 4.7 of [3] : for any  $1 \leq j \leq n_t$  and  $\epsilon > 0$

$$\mathbb{P}\left(\left|\tilde{h}_j - A^j(\tilde{L}_j) \int_{\tilde{\tau}_j^-(h_t/2)}^{\tilde{\tau}_j^+(h_t/2)} e^{-\tilde{V}^{(j)}(x)} \mathbf{e}_j\right| > e^{-(1-3\epsilon)h_t/6} A^j(\tilde{L}_j) \int_{\tilde{\tau}_j^-(h_t/2)}^{\tilde{\tau}_j^+(h_t/2)} e^{-\tilde{V}^{(j)}(x)} \mathbf{e}_j\right) \leq C_+ e^{-c_- \epsilon h_t}.$$

Recall that  $C_+$  (resp.  $c_-$ ) is a positive constant that may grow (resp. decrease) from line to line along the paper. For (3.28), the proof can be found in [3] at the end of the proof of Proposition 4.4.  $\blacksquare$

**Proof of Proposition 3.5**

By the first part i) of Lemma 3.6 the sequence  $(\mathbf{e}_j, \mathbb{H}_j, \mathbb{L}_j, j \leq n_t)$  is i.i.d, moreover  $\mathbf{e}_j$  is independent of  $(\mathbb{H}_j, \mathbb{L}_j)$ . This together with part ii) yields

$$\mathbb{P}\left(\bigcap_{j=1}^{n_t} \{ |U_j - \mathbf{e}_j \mathbb{S}_j \mathbb{R}_j| \leq \epsilon_t \mathbf{e}_j \mathbb{S}_j \mathbb{R}_j, |\mathcal{L}(H(\tilde{L}_j), \tilde{m}_j) - \mathbf{e}_j \mathbb{S}_j| \leq \epsilon_t \mathbf{e}_j \mathbb{S}_j \}\right) \geq 1 - e^{-D_1 h_t},$$



with  $\mathbb{S}_j$  independent of  $\mathbb{R}_j$ . Then as  $(\tilde{m}_j, j \geq 1)$  is a subsequence of  $(m_j, j \geq 1)$ , using Fact 2.1 for any  $j$  we have  $(\mathbb{R}_j; \mathbb{S}_j) \stackrel{\mathfrak{L}}{=} (F_1^+(h_t) + G^+(h_t, h_t/2); F_2^-(h_t/2) + F_3^-(h_t/2))$ . ■

#### 4. CONVERGENCE TOWARD THE LÉVY PROCESS $(\mathcal{Y}_1, \mathcal{Y}_2)$ AND CONTINUITY

**4.1. Preliminaries.** We begin this section by the convergence of certain repartition functions, a key results that are essentially improvement of the second part of Lemma 5.1 in [3].

**Lemma 4.1.**

$$\lim_{t \rightarrow +\infty} \sup_{x \in [e^{-(1-2\epsilon)\phi(t)}, +\infty[} \left| x^\kappa e^{\kappa\phi(t)} \mathbb{P}(\mathbf{e}_1 S_1/t > x) - \mathcal{C}_2 \right| = 0, \quad (4.31)$$

$$\lim_{t \rightarrow +\infty} \sup_{y \in [e^{-(1-3\epsilon)\phi(t)}, +\infty[} \left| y^\kappa e^{\kappa\phi(t)} \mathbb{P}(\mathbf{e}_1 S_1 R_1/t > y) - \mathcal{C}_2 \mathbb{E}[(\mathcal{R}_\kappa)^\kappa] \right| = 0, \quad (4.32)$$

with  $\mathcal{C}_2 > 0$  a known constant [see below (4.39)].

For any  $\alpha > 0$ ,  $e^{\kappa\phi(t)} \mathbb{P}(\ell_1/t \geq x, \mathcal{H}_1/t \geq y)$  converges uniformly when  $t$  goes to infinity on  $[\alpha, +\infty[ \times [\alpha, +\infty[$  to  $\nu([x, +\infty[ \times [y, +\infty[)$  [see Section 1.1].

**Proof:**

*Proof of (4.31)*

We first prove that  $x^\kappa e^{\kappa\phi(t)} \mathbb{P}(S_1/t > x)$  converges uniformly in  $x \in [e^{-(1-\epsilon)\phi(t)}, +\infty[$  to a constant  $c_3$ , that is, we prove that

$$\lim_{t \rightarrow +\infty} \sup_{x \in [e^{-(1-\epsilon)\phi(t)}, +\infty[} \left| x^\kappa e^{\kappa\phi(t)} \mathbb{P}(S_1/t > x) - c_3 \right| = 0. \quad (4.33)$$

For that, let  $y = e^{(1-\epsilon)\phi(t)}x$ , we have to prove that

$$\lim_{t \rightarrow +\infty} \sup_{y \in [1, +\infty[} \left| y^\kappa e^{\kappa\epsilon\phi(t)} \mathbb{P}(S_1/e^{h_t+\epsilon\phi(t)} > y) - c_3 \right| = 0, \quad (4.34)$$

but this is equivalent to prove that for any function  $f : ]0, +\infty[ \rightarrow [1, +\infty[$ ,

$$\lim_{t \rightarrow +\infty} (f(t))^\kappa e^{\kappa\epsilon\phi(t)} \mathbb{P}(S_1/e^{h_t+\epsilon\phi(t)} > f(t)) = c_3. \quad (4.35)$$

First by definition (see Proposition 3.5),  $S_1$  can be written as a sum of two independent functional, that we denote, for simplicity, as the sum of two generic functionals

$$S_1/t = (F_1^+(h_t) + G^+(h_t, h_t/2)) / t = e^{-\phi(t)} \left( e^{-h_t} F_1^+(h_t) + e^{-h_t} G^+(h_t, h_t/2) \right) \quad (4.36)$$

Since we know an asymptotic for the Laplace transform of  $F^+(h_t)/e^{h_t}$  and  $G^+(h_t/2, h_t)$ , the proof of (4.35) is similar to the proof of a Tauberian theorem. First by (6.82) and (6.83) we have

$$\forall \gamma > 0, \omega_{f,t}(\gamma) := \left( 1 - \mathbb{E} \left[ e^{-\gamma S_1 / (f(t) e^{h_t + \epsilon\phi(t)})} \right] \right) (\gamma)^{-1} \underset{t \rightarrow +\infty}{\sim} c_4 \gamma^{\kappa-1} (f(t))^{-\kappa} e^{-\kappa\epsilon\phi(t)}, \quad (4.37)$$

with  $c_4$  a positive constant. Now as  $\omega_{f,t}$  is the Laplace transform of the measure  $U_{f,t} := \mathbf{1}_{\mathbb{R}_+}(z) \mathbb{P}(S_1 / (f(t) e^{h_t + \epsilon\phi(t)}) > z) dz$ , from (4.37) we have

$$\forall \gamma > 0, \frac{\omega_{f,t}(\gamma)}{\omega_{f,t}(1)} \underset{t \rightarrow +\infty}{\longrightarrow} \gamma^{\kappa-1}.$$

From this we can follow the same line as in the proof of a classical Tauberian theorem (see for example [21] volume 2, section XIII.5, page 442). So as for the proof of Theorem 1 in [21] we

can deduce that

$$\forall z > 0, \frac{U_{f,t}([0, z])}{\omega_{f,t}(1)} \xrightarrow{t \rightarrow +\infty} \frac{z^{1-\kappa}}{\Gamma(2-\kappa)}.$$

Then as in the proof of Theorem 4 of the same reference page 446, we deduce from the monotony of the densities of measures  $U_{f,t}$  that

$$\forall z > 0, \frac{\mathbb{P}(S_1/(f(t)e^{h_t+\epsilon\phi(t)}) > z)}{\omega_{f,t}(1)} \xrightarrow{t \rightarrow +\infty} z^{-\kappa} \frac{1-\kappa}{\Gamma(2-\kappa)}.$$

Considering this convergence with  $z = 1$  we get exactly (4.35) for  $c_3 = (1-\kappa)/c_4\Gamma(2-\kappa)$ , so (4.33) follows. Let  $a_t := e^{\epsilon\phi(t)}$ , for any  $x > 0$

$$x^\kappa e^{\kappa\phi(t)} \mathbb{P}(\mathbf{e}_1 S_1/t > x, \mathbf{e}_1 < a_t) = 2^{-1} \int_0^{a_t} (x/u)^\kappa e^{\kappa\phi(t)} \mathbb{P}(S_1/t > x/u) u^\kappa e^{-u/2} du,$$

because  $\mathbf{e}_1$  has law  $\mathcal{E}(1/2)$ . Taking  $x$  arbitrary in  $[e^{-(1-2\epsilon)\phi(t)}, +\infty[$ , we have  $x/u \in [e^{-(1-\epsilon)\phi(t)}, +\infty[$ ,  $\forall u \in ]0, a_t]$ , so thanks to (4.33) we get

$$\lim_{t \rightarrow +\infty} \sup_{x \in [e^{-(1-2\epsilon)\phi(t)}, +\infty[} \left| x^\kappa e^{\kappa\phi(t)} \mathbb{P}(\mathbf{e}_1 S_1/t > x, \mathbf{e}_1 < a_t) - 2^{-1} c_3 \int_0^{+\infty} u^\kappa e^{-u/2} du \right| = 0. \quad (4.38)$$

Now for any  $x > 0$  and  $t$  large enough such that  $\forall y \geq 1, y^\kappa e^{\kappa\phi(t)} \mathbb{P}(S_1/t > y) < 2c_3$ , we have

$$\begin{aligned} & \left| x^\kappa e^{\kappa\phi(t)} \mathbb{P}(\mathbf{e}_1 S_1/t > x, \mathbf{e}_1 < a_t) - x^\kappa e^{\kappa\phi(t)} \mathbb{P}(\mathbf{e}_1 S_1/t > x) \right| \\ &= x^\kappa e^{\kappa\phi(t)} \mathbb{P}(\mathbf{e}_1 S_1/t > x, \mathbf{e}_1 \geq a_t) \\ &= 2^{-1} \int_{a_t}^{+\infty} x^\kappa e^{\kappa\phi(t)} \mathbb{P}(S_1/t > x/u) e^{-u/2} du \\ &= 2^{-1} \int_{a_t}^{+\infty} u^\kappa (x/u)^\kappa e^{\kappa\phi(t)} \mathbb{P}(S_1/t > x/u) \mathbf{1}_{x \leq u} e^{-u/2} du \\ &+ 2^{-1} \int_{a_t}^{+\infty} u^\kappa (x/u)^\kappa e^{\kappa\phi(t)} \mathbb{P}(S_1/t > x/u) \mathbf{1}_{x > u} e^{-u/2} du \\ &\leq 2^{-1} e^{\kappa\phi(t)} \int_{a_t}^{+\infty} u^\kappa e^{-u/2} du + c_3 \int_{a_t}^{+\infty} u^\kappa e^{-u/2} du. \end{aligned}$$

For the second term in the third equality we used the fact that  $(x/u)^\kappa e^{\kappa\phi(t)} \mathbb{P}(S_1/t > x/u) < 2c_3$  when  $x \geq u$ . Since  $a_t = e^{\epsilon\phi(t)}$ , the last quantities converges to 0 when  $t$  goes to infinity. Combining this with (4.38) we get

$$\lim_{t \rightarrow +\infty} \sup_{x \in [e^{-(1-2\epsilon)\phi(t)}, +\infty[} \left| x^\kappa e^{\kappa\phi(t)} \mathbb{P}(\mathbf{e}_1 S_1/t > x) - 2^{-1} c_3 \int_0^{+\infty} u^\kappa e^{-u/2} du \right| = 0, \quad (4.39)$$

and this is exactly (4.31) with  $C_2 := 2^{-1} c_3 \int_0^{+\infty} u^\kappa e^{-u/2} du$ .

*Proof of (4.32)*

Let  $\mu_{R_1}$  the distribution of  $R_1$ , for any  $y, a > 0$  and  $t > 0$  by independence of  $\mathbf{e}_1 S_1$  and  $R_1$

$$y^\kappa e^{\kappa\phi(t)} \mathbb{P}(\mathbf{e}_1 S_1 R_1/t > y, R_1 < a) = \int_0^a (y/u)^\kappa e^{\kappa\phi(t)} \mathbb{P}(\mathbf{e}_1 S_1/t > y/u) u^\kappa \mu_{R_1}(du).$$

Taking  $a = a_t = e^{\epsilon\phi(t)}$  and  $y$  arbitrary in  $[e^{-(1-3\epsilon)\phi(t)}, +\infty[$ , we have  $y/u \in [e^{-(1-2\epsilon)\phi(t)}, +\infty[$ ,  $\forall u \in ]0, a_t]$ , so, thanks to (4.39) we get

$$\lim_{t \rightarrow +\infty} \sup_{y \in [e^{-(1-3\epsilon)\phi(t)}, +\infty[} \left| y^\kappa e^{\kappa\phi(t)} \mathbb{P}(\mathbf{e}_1 S_1 R_1 / t > y, R_1 < a_t) - \mathcal{C}_2 \int_0^{+\infty} u^\kappa \mu_{R_1}(du) \right| = 0.$$

Also, as  $\int_0^{+\infty} u^\kappa \mu_{R_1}(du)$  converges to  $\mathbb{E}[(\mathcal{R}_\kappa)^\kappa]$ , when  $t$  goes to infinity

$$\lim_{t \rightarrow +\infty} \sup_{y \in [e^{-(1-3\epsilon)\phi(t)}, +\infty[} \left| y^\kappa e^{\kappa\phi(t)} \mathbb{P}(\mathbf{e}_1 S_1 R_1 / t > y, R_1 < a_t) - \mathcal{C}_2 \mathbb{E}[(\mathcal{R}_\kappa)^\kappa] \right| = 0.$$

Finally, as the family  $(R_1)_{t>0}$  is bounded in all  $L^p$  spaces, we can proceed as before to remove the event  $R_1 < a_t$  and we thus get

$$\lim_{t \rightarrow +\infty} \sup_{y \in [e^{-(1-3\epsilon)\phi(t)}, +\infty[} \left| y^\kappa e^{\kappa\phi(t)} \mathbb{P}(\mathbf{e}_1 S_1 R_1 / t > y) - \mathcal{C}_2 \mathbb{E}[(\mathcal{R}_\kappa)^\kappa] \right| = 0, \quad (4.40)$$

which is (4.32).

We now prove the last assertion. For any  $x, y, a$  and  $t > 0$ , we have

$$\begin{aligned} & e^{\kappa\phi(t)} \mathbb{P}(\mathbf{e}_1 S_1 / t > x, \mathbf{e}_1 S_1 R_1 / t > y, R_1 < a) \\ &= \int_0^a e^{\kappa\phi(t)} \mathbb{P}(\mathbf{e}_1 S_1 / t > x, \mathbf{e}_1 S_1 / t > y/u) \mu_{R_1}(du), \\ &= \int_0^{a \wedge (y/x)} e^{\kappa\phi(t)} \mathbb{P}(\mathbf{e}_1 S_1 / t > y/u) \mu_{R_1}(du) + \int_{a \wedge (y/x)}^a e^{\kappa\phi(t)} \mathbb{P}(\mathbf{e}_1 S_1 / t > x) \mu_{R_1}(du), \\ &= \frac{1}{y^\kappa} \int_0^{a \wedge (y/x)} e^{\kappa\phi(t)} (y/u)^\kappa \mathbb{P}(\mathbf{e}_1 S_1 / t > y/u) u^\kappa \mu_{R_1}(du) \\ &+ \frac{1}{x^\kappa} \int_{a \wedge (y/x)}^a e^{\kappa\phi(t)} x^\kappa \mathbb{P}(\mathbf{e}_1 S_1 / t > x) \mu_{R_1}(du). \end{aligned}$$

Taking  $a = a_t = e^{\epsilon\phi(t)}$  and  $x, y$  arbitrary in  $[\alpha, +\infty[$  (for some  $\alpha > 0$ ), we have  $y/u \in [e^{-(1-2\epsilon)\phi(t)}, +\infty[$ ,  $\forall u \in ]0, a_t]$  whenever  $t$  is large enough, so, thanks to (4.39) we get that  $e^{\kappa\phi(t)} \mathbb{P}(\mathbf{e}_1 S_1 / t > x, \mathbf{e}_1 S_1 R_1 / t > y, R_1 < a_t)$  converges uniformly in  $(x, y) \in [\alpha, +\infty[ \times [\alpha, +\infty[$  toward

$$x^{-\kappa} \mathbb{P}(\mathcal{R}_\kappa > y/x) + y^{-\kappa} \mathbb{E}((\mathcal{R}_\kappa)^\kappa \mathbf{1}_{\mathcal{R}_\kappa \leq y/x}).$$

Then as before we can remove the event  $\{R_1 < a_t\}$ , we get the last assertion.  $\square$

## 4.2. Proof of Proposition 1.4.

We start with the finite dimensional convergence

**Lemma 4.2.** *For any  $k \in \mathbb{N}$  and  $s_i > 0, i \leq k$ ,  $((Y_1, Y_2)_{s_i}^t, i \leq k)$  converge in law as  $t$  goes to infinity to  $((\mathcal{Y}_1, \mathcal{Y}_2)_{s_i}, i \leq k)$ .*

**Proof:**

Proof is basic here, however we give some details as we deal with a two dimensional walk which increments depend on  $t$  itself. As  $Y_1^t(s)$  and  $Y_2^t(s)$  are sums of i.i.d sequence we only have to prove the convergence in law for the couple  $(Y_1, Y_2)_s^t$  for any  $s > 0$ . Define  $(Y_1^{>b}, Y_2^{>b})$  obtained from  $(Y_1, Y_2)$  keeping the increments larger than  $b$ ,  $Y_1^{>b}(s) := \frac{1}{t} \sum_{j=1}^{\lfloor s e^{\kappa\phi(t)} \rfloor} \ell_j \mathbf{1}_{\ell_j > b}$ , and  $Y_2^{>b}(s) := \frac{1}{t} \sum_{j=1}^{\lfloor s e^{\kappa\phi(t)} \rfloor} \mathcal{H}_j \mathbf{1}_{\mathcal{H}_j > b}$ . Also let  $Y_i^{\leq b} := Y_i - Y_i^{>b}$ . We first prove that for any  $s > 0$ ,

$$\lim_{\epsilon \rightarrow 0} \lim_{t \rightarrow +\infty} \mathbb{P}(\|(Y_1^{\leq \epsilon}, Y_2^{\leq \epsilon})_s^t\| > \epsilon^{1-\kappa(2-\kappa)}) = 0, \quad (4.41)$$

where for any  $a \in \mathbb{R}^2$ ,  $\|a\| := \max_{i \leq 2} |a_i|$ , and  $1 - \kappa(2 - \kappa) > 0$  as  $\kappa < 1$ .

We compute the first moment of  $Y_1^{\leq \epsilon}(s)$  and  $Y_2^{\leq \epsilon}(s)$ . Let  $\eta > 0$  such that  $\kappa - (1 - 3\eta) < 0$ , applying Fubini we have

$$\begin{aligned} e^{\kappa\phi(t)} \mathbb{E} \left( \frac{\ell_1}{t} \mathbb{1}_{\ell_1/t \leq \epsilon} \right) &= e^{\kappa\phi(t)} \mathbb{E} \left[ \frac{\mathbf{e}_1 S_1}{t} \mathbb{1}_{\mathbf{e}_1 S_1/t \leq \epsilon} \right] \\ &\leq \int_0^\epsilon e^{\kappa\phi(t)} \mathbb{P}(\mathbf{e}_1 S_1/t > x) dx \\ &= \int_0^{e^{-(1-2\eta)\phi(t)}} e^{\kappa\phi(t)} \mathbb{P}(\mathbf{e}_1 S_1/t > x) dx + \int_{e^{-(1-2\eta)\phi(t)}}^\epsilon e^{\kappa\phi(t)} \mathbb{P}(\mathbf{e}_1 S_1/t > x) dx \\ &\leq e^{(\kappa - (1-2\eta))\phi(t)} + \int_{e^{-(1-2\eta)\phi(t)}}^\epsilon x^{-\kappa} x^\kappa e^{\kappa\phi(t)} \mathbb{P}(\mathbf{e}_1 S_1/t > x) dx. \end{aligned}$$

The first term converges to 0 when  $t$  goes to infinity because  $\kappa - (1 - 2\eta) < 0$  and, according to (4.31), for  $t$  large enough, we have

$$x^\kappa e^{\kappa\phi(t)} \mathbb{P}(\mathbf{e}_1 S_1/t > x) \leq 2\mathcal{C}_2, \quad \forall x \geq e^{-(1-2\eta)\phi(t)}.$$

For such  $t$ , the second term is less than

$$2\mathcal{C}_2 \int_0^\epsilon x^{-\kappa} dx = 2\mathcal{C}_2 \frac{\epsilon^{1-\kappa}}{1-\kappa},$$

so we get that

$$\forall t \geq 1, \forall \epsilon \in ]0, 1], \quad e^{\kappa\phi(t)} \mathbb{E} \left( \frac{\ell_1}{t} \mathbb{1}_{\ell_1/t \leq \epsilon} \right) \leq e^{(\kappa - (1-2\eta))\phi(t)} + C_+ \epsilon^{1-\kappa}. \quad (4.42)$$

Using the same method and applying this time (4.31), we get that

$$\forall t \geq 1, \forall \epsilon \in ]0, 1], \quad e^{\kappa\phi(t)} \mathbb{E} \left( \frac{\mathcal{H}_1}{t} \mathbb{1}_{\mathcal{H}_1/t \leq \epsilon} \right) \leq e^{(\kappa - (1-3\eta))\phi(t)} + C_+ \epsilon^{1-\kappa}. \quad (4.43)$$

We thus obtain

$$\mathbb{E} \left( Y_1^{\leq \epsilon}(s) \right) \leq s e^{(\kappa - (1-2\eta))\phi(t)} + C_+ s \epsilon^{1-\kappa}, \quad (4.44)$$

$$\mathbb{E} \left( Y_2^{\leq \epsilon}(s) \right) \leq s e^{(\kappa - (1-3\eta))\phi(t)} + C_+ s \epsilon^{1-\kappa}, \quad (4.45)$$

then a Markov inequality yields (4.41).

The next step is to prove that  $(Y_1^{> \epsilon}, Y_2^{> \epsilon})_s^t$  can be written as the integral of a point process which converge to the desired limit. We have

$$(Y_1^{> \epsilon}, Y_2^{> \epsilon})_s^t = \left( \int_{x > \epsilon} \int_0^s x \mathcal{P}_t^1(dx, dv), \int_{x > \epsilon} \int_0^s x \mathcal{P}_t^2(dx, dv) \right)$$

where the measures  $\mathcal{P}_t^1$  and  $\mathcal{P}_t^2$  are defined by  $\mathcal{P}_t^1 := \sum_{i=1}^{+\infty} \delta_{t^{-1}\ell_i, e^{-\kappa\phi(t)}i}$ , and the same for  $\mathcal{P}_t^2$  replacing  $\ell$  by  $\mathcal{H}$ . Recall that  $\mathcal{P}_t^1$  and  $\mathcal{P}_t^2$  are dependent and now prove that  $(\mathcal{P}_t^1, \mathcal{P}_t^2)$  converge to a Poisson point measure. For that just use Lemma 4.1 together with Proposition 3.1 in [28] after discretization, it implies that  $(\mathcal{P}_t^1, \mathcal{P}_t^2)$  converge weakly to the Poisson random measure denoted  $(\mathcal{P}^1, \mathcal{P}^2)$  with intensity measure given by  $ds \times \nu$ .

Then using that for any  $\epsilon > 0$ , and  $T < +\infty$ , on  $[0, T) \times (\epsilon, +\infty) \times (\epsilon, +\infty)$   $ds\nu$  is finite, we have that  $(Y_1^{> \epsilon}, Y_2^{> \epsilon})_s^t$  converge weakly to

$$(\mathcal{Y}_1^{> \epsilon}, \mathcal{Y}_2^{> \epsilon})_s := \left( \int_{x > \epsilon} \int_0^s x \mathcal{P}^1(dx, dv), \int_{x > \epsilon} \int_0^s x \mathcal{P}^2(dx, dv) \right).$$

We are left to prove that  $(\mathcal{Y}_1^{>\epsilon}, \mathcal{Y}_2^{>\epsilon})$  converge to  $(\mathcal{Y}_1, \mathcal{Y}_2)$  when  $\epsilon \downarrow 0$ . This is a straightforward computation, that we detail for completeness. Let  $\nu_1([x, +\infty[) := \int_0^{+\infty} \nu([x, +\infty[ \times [y, +\infty[) dy$ , we have

$$\mathbb{E} \left( \int_{x \leq \epsilon} \int_0^s x \mathcal{P}^1(dx, dv) \right) = s \int_{x \leq \epsilon} x \nu_1(x) = C \epsilon^{2-\kappa},$$

Then a Markov inequality, proves that for any  $s > 0$ , the process  $\int_{x \leq \epsilon} \int_0^s x \mathcal{P}^1(dx, dv)$  converge to zero (when  $\epsilon$  goes to zero) in probability. The same is true for  $\int_{x \leq \epsilon} \int_0^s x \mathcal{P}^2(dx, dv)$ , so we obtain that in probability  $(\mathcal{Y}_1^{>\epsilon}, \mathcal{Y}_2^{>\epsilon})_s$  converge to  $(\mathcal{Y}_1, \mathcal{Y}_2)_s$  when  $\epsilon \rightarrow 0$ .  $\square$

We now prove tightness of the family measures of the processes  $(Y_1, Y_2)^t$  denoted  $(\mathcal{D}(Y_1, Y_2)^t)_t$ .

**Lemma 4.3.** *The family of laws  $(\mathcal{D}(Y_1, Y_2)^t)_t$  is tight on  $(D([0, +\infty), \mathbb{R}^2), J_1)$ .*

**Proof:**

We only have to prove that the family law of the restriction of the process to the interval  $[0, T]$ ,  $((Y_1, Y_2)^t|_{[0, T]})_t$  is tight. To prove this we use the following restatement of Theorem 1.8 in [8] using Aldous's tightness criterion (see Condition 1, and equation 16.22 page 176 in [8]) also used in [9] page 100. We have to check the two following statements:

- 1) for any  $\epsilon > 0$ , there exists  $a$  such that for any  $t$  large enough  $\mathbb{P}(\sup_{s \in [0, T]} \|(Y_1, Y_2)_s^t\| \geq a) \leq \epsilon$ .
- 2) for any  $\epsilon > 0$ , and  $\eta > 0$  there exists  $\delta$ ,  $0 < \delta < T$  and  $t_0 > 0$  such that for  $t > t_0$ ,

$$\mathbb{P}(\omega((Y_1, Y_2)^t, \delta, T) \geq \eta) \leq \epsilon,$$

with  $\omega((Y_1, Y_2)^t, \delta, T) := \sup_{0 \leq r \leq T} \omega((Y_1, Y_2)^t, \delta, T, r)$ , and  $\omega((Y_1, Y_2)^t, \delta, T, r)$  is defined by  $\sup_{0 \vee (r-\delta) \leq u_1 < u_2 \leq (r+\delta) \wedge T} \{\min(\|(Y_1, Y_2)_{u_2}^t - (Y_1, Y_2)_u^t\|, \|(Y_1, Y_2)_u^t - (Y_1, Y_2)_{u_1}^t\|)\}$ . Also

$$\mathbb{P}(v((Y_1, Y_2)^t, 0, \delta, T) \geq \eta) \leq \epsilon, \text{ and } \mathbb{P}(v((Y_1, Y_2)^t, T, \delta, T) \geq \eta) \leq \epsilon,$$

where  $v((Y_1, Y_2)^t, u, \delta, T) := \sup_{(u-\delta) \vee 0 \leq u_1 \leq u_2 \leq (u+\delta) \wedge T} \{\|(Y_1, Y_2)_{u_1}^t - (Y_1, Y_2)_{u_2}^t\|\}$ .

We first check 1) since the process is monotone increasing,

$$\mathbb{P}(\sup_{s \in [0, T]} \|(Y_1, Y_2)_s^t\| \geq a) = \mathbb{P}(\|(Y_1, Y_2)_T^t\| \geq a) \leq \mathbb{P}(Y_1(T) \geq a) + \mathbb{P}(Y_2(T) \geq a). \quad (4.46)$$

Define  $Y_1^{>b}$  obtained from  $Y_1$  where we remove the increments  $\ell_j/t$  smaller than  $b$ . That is to say  $Y_1^{>b}(s) := \frac{1}{t} \sum_{j=1}^{\lfloor se^{\kappa\phi(t)} \rfloor} \ell_j \mathbf{1}_{\ell_j/t > b}$ . Also let  $Y_1^{\leq b} := Y_1 - Y_1^{>b}$  and  $N_u^{>b} := \sum_{i=1}^{\lfloor ue^{\kappa\phi(t)} \rfloor} \mathbf{1}_{\ell_j/t > b}$ . Let  $0 < \delta_1 < 1$ , Markov inequality yields

$$\begin{aligned} \mathbb{P}(Y_1^t(T) \geq a) &\leq \mathbb{P}\left(Y_1^{\leq 1}(T) \geq \frac{a}{2}\right) + \mathbb{P}\left(Y_1^{>1}(T) \geq \frac{a}{2}\right) \\ &\leq \frac{2}{a} \mathbb{E}\left[Y_1^{\leq 1}(T)\right] + \frac{1}{a^{\delta_1}} \mathbb{E}(N_T^{>1}) + \mathbb{P}\left(Y_1^{>1}(T) \geq \frac{a}{2}, N_T^{>1} \leq a^{\delta_1}\right). \end{aligned} \quad (4.47)$$

On  $\{N_T^{>1} \leq a^{\delta_1}\}$  there is at most  $a^{\delta_1}$  terms in the sum  $Y_1^{>1}(T)$  so

$$\begin{aligned} \mathbb{P}\left(Y_1^{>1}(T) > a/2, N_T^{>1} \leq a^{\delta_1}\right) &\leq \sum_{1 \leq i \leq a^{\delta_1}} \mathbb{P}\left(\ell_i/t \geq (a^{1-\delta_1}/2) | \ell_i/t \geq 1\right) \\ &\leq a^{\delta_1} \mathbb{P}\left(\ell_1/t \geq (a^{1-\delta_1}/2) | \ell_1/t \geq 1\right) \\ &\leq a^{\delta_1} 2 \frac{\mathcal{C}_2 e^{-\kappa\phi(t)} a^{-\kappa(1-\delta_1)} 2^\kappa}{\mathcal{C}_2 e^{-\kappa\phi(t)}} = 2^{1+\kappa} a^{\delta_1 - \kappa(1-\delta_1)}, \end{aligned} \quad (4.48)$$

for all  $t$  large enough thanks to (4.31) and  $\delta_1$  such that  $\delta_1 - \kappa(1 - \delta_1) < 0$ .

Also, as for any positive  $b$ ,  $N_T^{>b}$  follows a binomial law with parameter  $(\lfloor Te^{\kappa\Phi(t)} \rfloor, \mathbb{P}(\ell_1/t > b))$  using (4.31) again, and (4.44) we obtain for  $t$  is large enough

$$\mathbb{E}(N_T^{>b}) \leq 2\mathcal{C}_2 T b^{-\kappa}, \quad \mathbb{E} \left[ Y_1^{\leq b}(T) \right] \leq 2\mathcal{C}_2 T b^{1-\kappa}. \quad (4.49)$$

Collecting (4.48), (4.49) and (4.47) we get the existence of  $t_1 > 0$  such that

$$\lim_{a \rightarrow +\infty} \sup_{t \geq t_1} \mathbb{P}(Y_1(T) \geq a) = 0. \quad (4.50)$$

The same arguments holds for  $Y_2$  (using (4.32) instead of (4.31) and (4.45) instead of (4.44)) so (4.50) also holds for  $Y_2$  instead of  $Y_1$ . We conclude the proof of 1) by putting (4.50) and its analogous for  $Y_2$  in (4.46).

We now check 2) First as usual we write

$$\{\omega((Y_1, Y_2)^t, \delta, T) \geq \eta\} \subset \{\omega((Y_1^{\leq b}, Y_2^{\leq b})^t, \delta, T) \geq \eta/2\} \cup \{\omega((Y_1^{>b}, Y_2^{>b})^t, \delta, T) \geq \eta/2\}.$$

For  $Y^{\leq b}$  we have

$$\mathbb{P}(\omega((Y_1^{\leq b}, Y_2^{\leq b})^t, \delta, T) \geq \eta/2) \leq \mathbb{P}(\omega(Y_1^{\leq b}, \delta, T) \geq \eta/2) + \mathbb{P}(\omega(Y_2^{\leq b}, \delta, T) \geq \eta/2)$$

moreover by positivity of the increments

$$\begin{aligned} \mathbb{P} \left( \omega(Y_1^{\leq b}, \delta, T) \geq \eta/2 \right) &\leq \mathbb{P} \left( \bigcup_{k \leq \lfloor T/\delta \rfloor} \{Y_1^{\leq b}((k+1)\delta) - Y_1^{\leq b}(k\delta) \geq \eta/2\} \right) \\ &\leq \sum_{k \leq \lfloor T/\delta \rfloor} \mathbb{P} \left( Y_1^{\leq b}((k+1)\delta) - Y_1^{\leq b}(k\delta) \geq \eta/2 \right). \end{aligned} \quad (4.51)$$

For any  $k$ ,  $Y_1^{\leq b}((k+1)\delta) - Y_1^{\leq b}(k\delta)$  is the sum of at most  $\lfloor \delta e^{\kappa\Phi(t)} \rfloor + 1$  i.i.d. random variables having the same law as  $\ell_1/t$ . We get that for any integer  $k$

$$\mathbb{P} \left( Y_1^{\leq b}((k+1)\delta) - Y_1^{\leq b}(k\delta) \geq \eta/2 \right) \leq \mathbb{P} \left( Y_1^{\leq b}(2\delta) \geq \eta/2 \right) \leq 8\mathcal{C}_2 \delta b^{1-\kappa} / \eta,$$

where the first inequality holds for  $t$  large enough so that  $\delta e^{\kappa\Phi(t)} \geq 1$  and the second from the second expression in (4.49) (replacing  $T$  by  $2\delta$ ). Combining with (4.51) we get for large  $t$

$$\mathbb{P} \left( \omega(Y_1^{\leq b}, \delta, T) \geq \eta/2 \right) \leq 8\mathcal{C}_2 T (1 + \delta) b^{1-\kappa} / \eta, \quad (4.52)$$

[note that  $\delta$  will be chosen later (and will be less than 1)].  $T$  and  $\eta$  are fixed so we choose  $b$  small enough so that the right hand side of (4.51) is less than  $\epsilon/4$ . A similar estimate can be proved for  $\mathbb{P}(\omega(Y_2^{\leq b}, \delta, T) \geq \eta/2)$ .

For  $Y^{>b}$ , again we have

$$\mathbb{P}(\omega((Y_1^{>b}, Y_2^{>b})^t, \delta, T) \geq \eta/2) \leq \mathbb{P}(\omega(Y_1^{>b}, \delta, T) \geq \eta/2) + \mathbb{P}(\omega(Y_2^{>b}, \delta, T) \geq \eta/2).$$

Let us decrease  $b$  in order to get  $b < \eta/2$  so that  $\{\omega(Y_1^{>b}, \delta, T) > \eta/2\}$  implies that two jumps larger than  $b$  occur in an interval smaller than  $2\delta$ . That is  $\{\omega(Y_1^{>b}, \delta, T) > \eta/2\} \subset \bigcup_{j=1}^{\lfloor Te^{\kappa\Phi(t)} \rfloor} \bigcup_{i>j, (i-j)/e^{\kappa\Phi(t)} \leq 2\delta} \{\ell_j \wedge \ell_i/t > b\}$ . Applying (4.31) for  $t$  large enough

$$\mathbb{P} \left( \bigcup_{j=1}^{\lfloor Te^{\kappa\Phi(t)} \rfloor} \bigcup_{i>j, (i-j)/e^{\kappa\Phi(t)} \leq 2\delta} \{\ell_j \wedge \ell_i/t > b\} \right) \leq 8\mathcal{C}_2 \delta T b^{-2\kappa},$$

which can be small choosing this time  $\delta = \delta(b)$  properly. Again the same argument can be used for  $\omega(Y_2^{>b}, \delta, T)$ . To finish the proof, we have to deal with  $v()$ , as again our processes are increasing,

$$\mathbb{P}(v((Y_1, Y_2)^t, 0, \delta, T) \geq \eta) \leq \mathbb{P}(\|(Y_1, Y_2)_\delta^t\| \geq \eta)$$

we can then proceed as for 1) decreasing  $\delta$  if needed, this also applies to  $\mathbb{P}(v((Y_1, Y_2)^t, T, \delta, T) \geq \eta)$ .  $\square$

**4.3. Continuity of certain functionals of  $(\mathcal{Y}_1, \mathcal{Y}_2)$ , in  $J_1$ .** In this section, we study the continuity of functionals of the Lévy processes  $(\mathcal{Y}_1, \mathcal{Y}_2)$ .

For our purpose we are interested in the following mappings, first the two we have already mention in the introduction which are the basics

$$J : D(\mathbb{R}_+, \mathbb{R}) \longrightarrow D(\mathbb{R}_+, \mathbb{R}) \quad I : (D(\mathbb{R}_+, \mathbb{R}), J_1) \longrightarrow (D(\mathbb{R}_+, \mathbb{R}), U)$$

$$f \longmapsto f^\sharp \quad f \longmapsto f^{-1}$$

Then we also need the compositions of these two : let  $J_I$  and  $J_I^-$

$$J_I : D(\mathbb{R}_+, \mathbb{R}^2) \longrightarrow \mathbb{R} \quad J_I^- : D(\mathbb{R}_+, \mathbb{R}^2) \longrightarrow \mathbb{R}$$

$$f = (f_1, f_2) \longmapsto f_1^\sharp(f_2^{-1}(1)) \quad f = (f_1, f_2) \longmapsto f_1^\sharp(f_2^{-1}(1)^-)$$

$J_I$  (respectively  $J_I^-$ ) produces the largest jump of  $f_1$ , just after (respectively before)  $f_2$  reach 1. Finally let  $F^*$  define by

$$F^* : D(\mathbb{R}_+, \mathbb{R}^2) \longrightarrow \mathbb{R}$$

$$f = (f_1, f_2) \longmapsto \inf \left\{ s \in [0, f_2^{-1}(1)), \Delta f_1(s) = f_1^\sharp(f_2^{-1}(1)^-) \right\},$$

we need this variable for the characterization of the favorite sites.

**Lemma 4.4.**  *$J$  is continuous in the  $J_1$  topology.*

**Proof:**

This fact is basic, but as we have not found a proof in the literature, we give some details. To prove the continuity on  $D(\mathbb{R}_+, \mathbb{R})$ , we only have to prove it for every compact subset of  $\mathbb{R}_+$ , (see [35] Theorem 12.9.1). So let  $f \in D(\mathbb{R}_+, \mathbb{R})$  and  $T > 0$  for which  $f$  is continuous, let us prove that  $J_T$  defined by

$$J_T : D([0, T], \mathbb{R}) \longrightarrow D([0, T], \mathbb{R})$$

$$g \longmapsto g^\sharp$$

is continuous at the restriction  $f|_{[0, T]}$ . Let  $\epsilon > 0$  and  $g \in D([0, T], \mathbb{R})$  such that  $d_T(f|_{[0, T]}, g) \leq \frac{\epsilon}{2}$ .  $d_T$  is the usual metric  $d$  of the  $J_1$ -topology restricted to the interval  $[0, T]$ . By definition of  $d_T$  there exists a strictly increasing continuous mapping of  $[0, 1]$  onto itself,  $e : [0, T] \longrightarrow [0, T]$  such that

$$\sup_{s \in [0, T]} |e(s) - s| \leq \frac{\epsilon}{2}, \quad \text{and} \quad \sup_{s \in [0, T]} |g(e(s)) - f|_{[0, T]}(s)| \leq \frac{\epsilon}{2}.$$

So for every  $s \in [0, T]$  we have

$$|\Delta g(e(s)) - \Delta f|_{[0, T]}(s)| = |(g(e(s)) - g(e(s)^-)) - (f|_{[0, T]}(s) - f|_{[0, T]}(s^-))|$$

$$\leq |g(e(s)) - f|_{[0, T]}(s)| + |g(e(s)^-) - f|_{[0, T]}(s^-)|$$

$$\leq 2 \frac{\epsilon}{2} = \epsilon,$$

where  $\Delta h(s) = h(s) - h(s^-)$ . This implies  $d_T(J_T(f|_{[0, T]}), J_T(g)) \leq \epsilon$ .  $\square$

**Lemma 4.5.** *The mapping  $J_I^-$  and  $J_I$  are continuous for  $J_1$ -topology for every couple  $(f_1, f_2) \in D(\mathbb{R}_+, \mathbb{R}^2)$  such that*

- (1) For any  $\epsilon > 0$ ,  $f_1$  has a finite number of jumps greater than  $\epsilon$  on every compact subset of  $\mathbb{R}_+^*$ ,
- (2)  $f_2$  is strictly increasing, with a limit equal to  $+\infty$ ,
- (3)  $f_2(0) = 0$ ,
- (4)  $f_2$  has a jump at  $I(f_2)(1)$  and  $I(f_2)(1^-) < 1 < I(f_2)(1)$ .

**Proof:**

This fact may also be known as we are looking at randomly stopped process, but once again we did not find what we need in the literature ([31],[35]).

Let  $(f_n^1, f_n^2)_n$  a sequence of  $D(\mathbb{R}^+, \mathbb{R})$  which converges to  $(f^1, f^2)$  for the  $J_1$  topology, to prove continuity we prove that the sequence  $(J_I^-(f_n^1, f_n^2))_n$  converges to  $J_I^-(f^1, f^2)$ , and the equivalent for  $J_1$ .

The first hypothesis guaranties that there exists neighborhoods of  $I(f^2)(1)$  for which  $f^1$  makes no jump greater than  $1/4$  times its higher previous jump, that is to say there exists  $\delta \in ]0, I(f^2)(1)[$  (notice that  $I(f^2)(1)$  exists thanks to (2) and is positive thanks to (3)) such that  $f^1$  makes no jump greater than  $J(f^1)(I(f^2)(1) - \delta)/4$  on  $[I(f^2)(1) - \delta, I(f_2)(1)[$  and on  $]I(f^2)(1), I(f^2)(1) + \delta]$ . Note also that  $J(f^1)$  is constant on  $[I(f^2)(1) - \delta, I(f_2)(1)[$  and on  $]I(f^2)(1), I(f^2)(1) + \delta]$ .

Also  $\delta$  can be made smaller (if needed) in such a way that  $I(f^2)(1) + \delta$  is a point of continuity of  $(f^1, f^2)$  and  $(f_n^1, f_n^2)_n$  for every  $n \in \mathbb{N}$ . By hypothesis  $d((f_n^1, f_n^2), (f^1, f^2)) \xrightarrow{n \rightarrow +\infty} 0$  so

$$d_n := d_{[0, I(f^2)(1) + \delta]}((f_n^1, f_n^2)_{|[0, I(f^2)(1) + \delta]}, (f^1, f^2)_{|[0, I(f^2)(1) + \delta]}) \xrightarrow{n \rightarrow +\infty} 0,$$

where  $_{|[0, I(f^2)(1) + \delta]}$  in index means restriction to  $[0, I(f^2)(1) + \delta]$ . Also by continuity of  $J$  (see Lemma 4.4) we also have  $d(J(f_n^1), J(f^1)) \xrightarrow{n \rightarrow +\infty} 0$  and therefore

$$d'_n := d_{[0, I(f^2)(1) + \delta]}((J(f_n^1))_{|[0, I(f^2)(1) + \delta]}, (J(f^1))_{|[0, I(f^2)(1) + \delta]}) \xrightarrow{n \rightarrow +\infty} 0.$$

Let  $h^-$  (respectively  $h^+$ ) the largest jump of  $f^1$  just before (resp. just after)  $I(f^2)(1)$ . By definition of  $\delta$  we have

$$h^- = J(f^1)(I(f^2)(1) - \delta), h^+ = J(f^1)(I(f^2)(1) + \delta).$$

We have two cases, either  $J(f^1)$  is continuous at  $I(f^2)(1)$  or it makes a jump.

*Case  $J(f^1)$  makes a jump*, in this case the size of the jump is  $h^+ - h^- > 0$ .

Let  $\alpha = 8^{-1} \min(h^-, \delta, 1 - f^2(I(f^2)(1)^-), f^2(I(f^2)(1)) - 1)$ , and  $n_0$  such that for any  $n \geq n_0$ ,  $d_n < \alpha$  and  $d'_n < \alpha$ .  $T = I(f^2)(1) + \delta$ , there exist two homeomorphisms  $e_n, e'_n : [0, T] \rightarrow [0, T]$  such that :

- $\sup_{s \in [0, T]} |e_n(s) - s| \leq d_n$ ,
- $\sup_{s \in [0, T]} \left\| (f_n^1(e_n(s)), f_n^2(e_n(s)))_{|[0, I(f^2)(1) + \delta]} - (f^1(s), f^2(s))_{|[0, I(f^2)(1) + \delta]} \right\|_\infty \leq d_n$ .
- $\sup_{s \in [0, T]} |e'_n(s) - s| \leq d'_n$ ,
- $\sup_{s \in [0, T]} \left| (J(f_n^1))_{|[0, I(f^2)(1) + \delta]}(e'_n(s)) - (J(f^1))_{|[0, I(f^2)(1) + \delta]}(s) \right| \leq d'_n$ .

The second inequality implies that for any  $n \geq n_0$ ,

$$f_n^2(e_n(I(f^2)(1)^-)) < 1 < f_n^2(e_n(I(f^2)(1))),$$

so as we also have  $f_n^2(I(f_n^2)(1)^-) < 1 < f_n^2(I(f_n^2)(1))$  we get

$$I(f_n^2)(1) = e_n(I(f^2)(1)). \tag{4.53}$$



The fourth point implies that for any  $n \geq n_0$ ,

$$J(f_n^1) \left( e'_n \left( I(f^2)(1) - \frac{1}{2}\delta \right) \right) \geq J(f^1) \left( I(f^2)(1) - \frac{1}{2}\delta \right) - \alpha = h^- - \alpha > \frac{1}{2}h^-. \quad (4.54)$$

The second point and the argument of the previous proof imply that for  $n \geq n_0$ , each jump of  $f_n^1$  on  $[e_n(I(f^2)(1) - \delta), e_n(I(f^2)(1))]$  is  $2\alpha$ -close to a jump of  $f^1$  on  $[I(f^2)(1) - \delta, I(f^2)(1)]$ , but such jumps are less than  $h^-/4$  because of the definition of  $\delta$ . Thus,  $f_n^1$  makes no jump larger than  $h^-/2$  on  $[e_n(I(f^2)(1) - \delta), e_n(I(f^2)(1))]$ . Moreover, the increases of  $e'_n$  and the first and third points imply that

$$e_n(I(f^2)(1) - \delta) \leq e'_n(I(f^2)(1) - \delta/2) \leq e_n(I(f^2)(1)),$$

so, combining with (4.54) we get that  $J(f_n^1)$  is constant on  $[e'_n(I(f^2)(1) - \delta/2), e_n(I(f^2)(1))]$ .

Now by definition of  $J_I^-$ , with (4.53) and then collecting what have just done above yields

$$\begin{aligned} \forall n \geq n_0, J_I^-((f_n^1, f_n^2)) &= J(f_n^1)(I(f_n^2)(1)^-) = J(f_n^1)(e_n(I(f^2)(1))^-) \\ &= J(f_n^1)(e'_n(I(f^2)(1) - \delta/2)). \end{aligned} \quad (4.55)$$

From definition of  $J_I^-$  and the constantness of  $J(f^1)$  on  $[I(f^2)(1) - \delta, I(f^2)(1)]$  we also have

$$J_I^-(f^1, f^2) := J(f^1)(I(f^2)(1)^-) = J(f^1)(I(f^2)(1) - \delta/2). \quad (4.56)$$

Combining (4.55), (4.56) and the fourth point we obtain that  $J_I^-((f_n^1, f_n^2))$  converges to  $J_I^-((f^1, f^2))$  as  $n$  goes to infinity.

For  $J_I$ , we prove in a similar way as above that  $J(f_n^1)$  is constant on  $[e_n(I(f^2)(1)), e'_n(I(f^2)(1) + \delta/2)]$  so, as in (4.55) we have for  $n$  large enough

$$J_I((f_n^1, f_n^2)) = J(f_n^1)(e'_n(I(f^2)(1) + \delta/2)),$$

which, combined with the analogous of (4.56)

$$J_I(f^1, f^2) = J(f^1)(I(f^2)(1) + \delta/2)$$

allows us to conclude, using the fourth point, that  $J_I((f_n^1, f_n^2))$  converges to  $J_I((f^1, f^2))$  as  $n$  goes to infinity. Therefore, both  $J_I^-$  and  $J_I$  are continue at  $(f^1, f^2)$ .  $\square$

**Lemma 4.6.** *For any  $(f^1, f^2)$  in  $D(\mathbb{R}^+, \mathbb{R}^2)$  that satisfy the hypothesis of lemma 4.5 and such that the sizes of the jumps of  $f^1$  are all distinct,  $F^*$  is continuous at  $(f^1, f^2)$  in the  $J_1$  topology.*

**Proof:**

The proof follows mainly the steps of Lemma 4.5, we keep the same notations. The jump which takes place at the instant  $F^*(f^1, f^2)$  has value  $h^-$ . With the additional hypothesis that the values of the jumps for  $f^1$  are all different we have unicity for the value  $h^-$ . Let us define  $h'$ , the second highest jump  $f^1$  before instant  $I(f^2)(1)$ . With the additional condition that  $\alpha < \frac{1}{8}(h^- - h')$  we have with the same arguments than in the proof of the continuity of  $J$  that for any  $n \geq n_0$ ,  $f_n^1$  effectuates at  $e_n(F^*(f^1, f^2))$  a jump larger than  $h^- - 2\alpha$ , and larger than all the other jumps of  $f_n^1$  before  $e_n(I(f^2)(1)^-) = I(f_n^2)(1)$  which are smaller than  $h' + 2\alpha$ . So for  $n \geq n_0$ , the largest jump of  $f^1$  before  $I(f_n^2)(1)$  is obtained for  $e_n(F^*(f^1, f^2))$ , that is to say for any  $n \geq n_0$ ,

$$F^*((f_n^1, f_n^2)) = e_n(F^*(f^1, f^2)),$$

this implies  $F^*((f_n^1, f_n^2)) \xrightarrow{n \rightarrow \infty} F^*(f^1, f^2)$ .  $\square$

## 5. SUPREMUM OF THE LOCAL TIME - AND OTHER FUNCTIONALS

## 5.1. Supremum of the local time (proof of Theorem 1.3).

We start with the proof of the following proposition which makes a link between the supremum of the local time and the process  $(Y_1, Y_2)^t$ .

**Proposition 5.1.** *For any  $\epsilon > 0$  and large  $t$ ,*

$$\mathcal{P}_1^- - v(\epsilon, t) \leq \mathbb{P} \left( \sup_{x \in \mathbb{R}} \mathcal{L}(t, x)/t \leq \alpha \right) \leq \mathcal{P}_1^+ + v(\epsilon, t)$$

with

$$\mathcal{P}_1^\pm := P \left[ (1 - \bar{\mathcal{H}}_{\mathcal{N}_t^\epsilon - 1}) \frac{\bar{\ell}_{\mathcal{N}_t^\epsilon} - \bar{\ell}_{\mathcal{N}_t^\epsilon - 1}}{(\bar{\mathcal{H}}_{\mathcal{N}_t^\epsilon} - \bar{\mathcal{H}}_{\mathcal{N}_t^\epsilon - 1})} \leq \alpha_t^\pm, \max_{1 \leq j \leq \mathcal{N}_t^\epsilon - 1} \frac{\ell_j}{t} \leq \alpha_t^\pm \right],$$

with  $\bar{\mathcal{H}}_k := Y_2(ke^{-\kappa\phi(t)}) = \frac{1}{t} \sum_{i=1}^k \mathcal{H}_i$ ,  $\bar{\ell}_k := Y_1(ke^{-\kappa\phi(t)}) = \frac{1}{t} \sum_{i=1}^k \ell_i$ ,  $\mathcal{N}_t^\epsilon := \inf\{m \geq 1, \bar{\mathcal{H}}_m > 1 - \epsilon\}$ ,  $\alpha_t^\pm := \alpha(1 \pm (\log \log t)^{-1/2})$ , and  $v$  a positive function such that  $\lim_{t \rightarrow +\infty} v(\epsilon, t) \leq \text{const} \times \epsilon^{\kappa \wedge (1-\kappa)}$ .

The proof of this Proposition needs the three following Lemmata, the first one deals with the local time at the  $h_t$ -minima for which the process has enough time to escape from it. The second deals with the local time at the last  $h_t$ -minima in the remaining time before the instant  $t$ . Finally the last one is a technical point.

**Lemma 5.2.** *For any large  $t > 0$ ,  $2 \leq k \leq n_t$ , any  $0 < x \leq 1$  and  $\gamma > 0$  possibly depending on  $t$ , define the repartition function*

$$F_\gamma(x) := \mathbb{P} \left( \max_{1 \leq j \leq k-1} \mathcal{L}(H(\tilde{L}_j), \tilde{m}_j) \leq \gamma t, \sum_{i=1}^{k-1} U_i \leq x \right),$$

then

$$F_\gamma^-(x) - e^{-D_1 h t} \leq F_\gamma(x) \leq F_\gamma^+(x) + e^{-D_1 h t},$$

where  $F_\gamma^\pm(x) := \mathbb{P} \left( \max_{1 \leq j \leq k-1} \ell_j \leq \gamma_t^\pm t, \sum_{i=1}^{k-1} \mathcal{H}_i \leq x_t^\pm t \right)$  with  $\gamma_t^\pm := \gamma(1 \pm 2\epsilon_t)$ ,  $x_t^\pm := x(1 \pm 2\epsilon_t)$ ,  $\epsilon_t$  and  $D_1$  are given in Proposition 3.5.

**Lemma 5.3.** *For any  $t > 0$ , define for every  $\gamma > 0$  and  $0 < x < 1$  possibly depending on  $t$ ,*

$$f_\gamma(x) := E \left[ \mathbb{P}_{\tilde{m}_1}^{W_\kappa} (\mathcal{L}_{X'}(t(1-x), \tilde{m}_1) \leq \gamma t, H'(\tilde{L}_1) > t(1-x), H'(\tilde{L}_1) < H'(\tilde{L}_1^-)) \right],$$

there exist  $c_2 > 0$  such that

$$f_\gamma^-(x) - o(n_t^{-1}) \leq f_\gamma(x) \leq f_\gamma^+(x) + o(n_t^{-1}) \quad (5.57)$$

with  $f_\gamma^\pm(x) := \mathbb{P} \left( \frac{1}{R_1} \leq \frac{\gamma}{1-x}(1 \pm \epsilon'_t), \mathcal{H}_1 > t(1-x)(1 \mp \epsilon'_t) \right)$  and  $\epsilon'_t = e^{-c_2 h t}$ . For any  $t > 0$ , define also

$$\tilde{f}_\gamma(x) := E \left( \mathbb{P}_{\tilde{m}_1}^{W_\kappa} \left( \sup_{y \in \mathcal{D}_1} \mathcal{L}_{X'}(t(1-x), y) \leq \gamma t, H'(\tilde{L}_1) > t(1-x), H'(\tilde{L}_1) < H'(\tilde{L}_1^-) \right) \right),$$

with  $\mathcal{D}_1$  defined in (3.24), then  $\tilde{f}_\gamma(x)$  satisfies also (5.57).  $X'$  is an independent copy of  $X$  starting at  $\tilde{m}_1$ , definition of  $H'$  for  $X'$  is the same as definition of  $H$  for  $X$ .

**Lemma 5.4.** *For any  $0 < a < 1$ , we have for any  $t > 0$*

$$\sum_{k \leq n_t} \mathbb{P} \left[ \bar{\mathcal{H}}_k > 1 - a/2, 1 - 2a < \bar{\mathcal{H}}_{k-1} \leq 1 - a \right] \leq s(a, t), \quad (5.58)$$

with  $s(a, t)$  such that  $\lim_{t \rightarrow +\infty} s(a, t) = \text{const} \times a^\kappa$ . For any  $\epsilon > 0$  and  $t > 0$

$$\mathbb{P}(\epsilon t \leq H(m_{N_t}) \leq (1 - \epsilon)t) \geq 1 - \tilde{s}(\epsilon, t). \quad (5.59)$$

with  $\tilde{s}(\epsilon, t)$  such that  $\lim_{t \rightarrow +\infty} \tilde{s}(\epsilon, t) = \text{const} \times \epsilon^{(1-\kappa) \wedge \kappa}$ .

We postpone the proof of these Lemmata after the proof of the Proposition.

**Proof of Proposition 5.1 :** Recall that  $N_t$  is the largest index such that  $\sup_{s \leq t} X_s \geq m_{N_t}$ . The main idea is to use the fact that the supremum of the local time until the instant  $t$  is achieved in the neighborhood of the  $h_t$ -valleys.

**Starting with the upper bound** using (3.12), (3.13), (3.14), (5.59) and Lemma 2.2 we have for  $t$  large enough

$$\begin{aligned} \mathbb{P} \left( \sup_{x \in \mathbb{R}} \mathcal{L}(t, x) \leq \alpha t \right) &\leq E \mathbb{P}^{W_\kappa} \left[ \left( \max_{1 \leq j \leq N_t} \mathcal{L}(t, m_j) \leq \alpha t \right) \right] \\ &E \left[ \mathbb{P}^{W_\kappa} \left( \max_{1 \leq j \leq N_t-1} \mathcal{L}(H(L_j), m_j) \leq \alpha t, \mathcal{L}(t, m_{N_t}) \leq \alpha t, \mathcal{Q}, \mathcal{B}_1(n_t), \mathcal{B}_2(n_t), \mathcal{V}_t \right) \right] \\ &+ \bar{s}(\epsilon, t). \end{aligned} \quad (5.60)$$

with  $\mathcal{Q} := \{\epsilon t \leq H(m_{N_t}) \leq (1 - \epsilon)t, N_t \leq n_t\}$  and  $\bar{s}$  satisfying  $\lim_{t \rightarrow +\infty} \bar{s}(\epsilon, t) \leq C_+ \epsilon^{(1-\kappa) \wedge \kappa}$ . Define, for every  $0 < y < 1$  measures  $\nu_1^{W_\kappa}$  and  $\nu_2^{W_\kappa}$  as:

$$\begin{aligned} \nu_1^{W_\kappa}(y) &= \nu_1^{W_\kappa}([0, y]) := \mathbb{P}^{W_\kappa} \left( \max_{1 \leq j \leq k-1} \mathcal{L}(H(\tilde{L}_j), \tilde{m}_j) \leq \alpha t, H(\tilde{m}_k) - \sum_{i=1}^{k-1} U_i < \tilde{v}_t, H(\tilde{m}_k) \leq yt \right) \\ \nu_2^{W_\kappa}(y) &= \nu_2^{W_\kappa}([0, y]) := \mathbb{P}_{\tilde{m}_k}^{W_\kappa} \left( \mathcal{L}_{X'}(t(1-y), \tilde{m}_k) \leq \alpha t, H'(\tilde{m}_{k+1}) > t(1-y), H'(\tilde{m}_{k+1}) < H'(\tilde{L}_k^-), \right. \\ &\quad \left. H'(\tilde{m}_{k+1}) - H'(\tilde{L}_k) \leq \tilde{v}_t \right), \end{aligned}$$

with  $X'$  the diffusion starting from  $\tilde{m}_k$  independent of  $X$  (conditionally on  $W_\kappa$ ), and  $H'$  has the same definition than  $H$  (see (1.1)) but for  $X'$ . Also all or  $\nu$  above and in the sequel depend on  $k$  but we do not make appear this dependence for notational simplicity. Partitioning on the values of  $N_t$ , and  $H(\tilde{m}_k)$ , we obtain by the strong markov property (under  $\mathbb{P}^{W_\kappa}$ ), that probability below (5.60), is smaller than

$$\sum_{k \leq n_t} \int_\epsilon^{1-\epsilon} E \left( \nu_2^{W_\kappa}(x) d\nu_1^{W_\kappa}(x) \right) = \sum_{k \leq n_t} E \left[ \int_\epsilon^{1-\epsilon} \nu_2^{W_\kappa}(x) d\nu_1^{W_\kappa}(x) \right]. \quad (5.61)$$

The next step is to prove that the previous expectation can be approximated by a product of expectations. First notice that both  $y \rightarrow \nu_1^{W_\kappa}(y)$  and  $y \rightarrow \nu_2^{W_\kappa}(y)$  are positive increasing. So integrating by parts

$$\begin{aligned} \int_\epsilon^{1-\epsilon} \nu_2^{W_\kappa}(x) d\nu_1^{W_\kappa}(x) &= \left[ \nu_2^{W_\kappa}(x) \nu_1^{W_\kappa}(x) \right]_\epsilon^{1-\epsilon} - \int_\epsilon^{1-\epsilon} \nu_1^{W_\kappa}(x) d\nu_2^{W_\kappa}(x) \\ &\leq \left[ \nu_2^{W_\kappa}(x) \nu_1^{W_\kappa}(x) \right]_\epsilon^{1-\epsilon} - \int_\epsilon^{1-\epsilon} \tilde{\nu}_1^{W_\kappa}(x) d\nu_2^{W_\kappa}(x) \\ &= \left[ \nu_2^{W_\kappa}(x) (\nu_1^{W_\kappa}(x) - \tilde{\nu}_1^{W_\kappa}(x)) \right]_\epsilon^{1-\epsilon} + \mathcal{I}_1 \end{aligned} \quad (5.62)$$

with  $\tilde{\nu}_1^{W_\kappa}(x) := \mathbb{P}^{W_\kappa} \left( \mathcal{C}_k, H(\tilde{m}_k) - \sum_{i=1}^{k-1} U_i < \tilde{v}_t, \sum_{i=1}^{k-1} U_i + \tilde{v}_t \leq x \right)$ ,  
 $\mathcal{C}_k := \{ \max_{1 \leq j \leq k-1} \mathcal{L}(H(\tilde{L}_j), \tilde{m}_j) \leq \alpha t \}$  and

$$\mathcal{I}_1 := \int_\epsilon^{1-\epsilon} \nu_2^{W_\kappa}(x) d\tilde{\nu}_1(x) \leq \int_\epsilon^{1-\epsilon} \nu_2^{W_\kappa}(x) d\nu_3^{W_\kappa}(x) =: \mathcal{I}'_1,$$

$$\nu_3^{W_\kappa}(x) := \mathbb{P}^{W_\kappa} \left( \mathcal{C}_k, \sum_{i=1}^{k-1} U_i + \tilde{v}_t \leq xt \right).$$

**First** we deal with what is going to be a negligible part, that is to say the first term in (5.62).

As  $\nu_1^{W_\kappa}(x) \leq \mathbb{P}^{W_\kappa} \left( \mathcal{C}_k, H(\tilde{m}_k) - \sum_{i=1}^{k-1} U_i < \tilde{v}_t, \sum_{i=1}^{k-1} U_i \leq x \right)$ ,

$$|\nu_1^{W_\kappa}(x) - \tilde{\nu}_1^{W_\kappa}(x)| \leq \mathbb{P}^{W_\kappa} \left( xt - \tilde{v}_t < \sum_{i=1}^{k-1} U_i \leq xt \right) =: h(x).$$

so  $[\nu_2^{W_\kappa}(x)(\nu_1^{W_\kappa}(x) - \tilde{\nu}_1^{W_\kappa}(x))]_\epsilon^{1-\epsilon} \leq \nu_2^{W_\kappa}(1-\epsilon)h(1-\epsilon) + \nu_2^{W_\kappa}(\epsilon)h(\epsilon)$ .  $\sum_{i=1}^{k-1} U_i$  is measurable with respect to  $\sigma(X_s, 0 \leq s \leq H(\tilde{L}_{k-1}); W_\kappa(x), x \leq \tilde{L}_{k-1})$ , and the event in the definition of  $\nu_2^{W_\kappa}$  belongs to  $\sigma(X'_s, H'(\tilde{L}_k^-) \leq s \leq H'(\tilde{L}_k); W_\kappa(x) - W_\kappa(\tilde{m}_k), \tilde{L}_k^- \leq x \leq \tilde{L}_k)$ , and recall  $X'$  an independent copy of  $X$  starting at  $\tilde{m}_k$ .

So independence of  $X$  and  $X'$ , and independence of the two portions of the environment involved (Lemma 2.2) imply independence between  $\nu_2^{W_\kappa}$  and  $h$ , so

$$\begin{aligned} E[\nu_2^{W_\kappa}(x)(\nu_1^{W_\kappa}(x) - \tilde{\nu}_1^{W_\kappa}(x))]_\epsilon^{1-\epsilon} &= E \left[ [\tilde{\nu}_2^{W_\kappa}(x)(\nu_1^{W_\kappa}(x) - \tilde{\nu}_1^{W_\kappa}(x))]_\epsilon^{1-\epsilon} \right] \\ &\leq E[\tilde{\nu}_2^{W_\kappa}(1-\epsilon)]E[h(1-\epsilon)] + E[\tilde{\nu}_2^{W_\kappa}(\epsilon)]E[h(\epsilon)]. \end{aligned} \quad (5.63)$$

with  $\tilde{\nu}_2^{W_\kappa}(x) := \mathbb{P}_{\tilde{m}_1}^{W_\kappa} [\mathcal{L}_{X'}(t(1-x), \tilde{m}_1) \leq \alpha t, H'(\tilde{m}_2) > t(1-x), H'(\tilde{m}_2) < H'(\tilde{L}_1^-), H'(\tilde{m}_2) - H'(\tilde{L}_1) \leq \tilde{v}_t]$ . As  $E[\tilde{\nu}_2^{W_\kappa}(x)] \leq \mathbb{P}[U_1 > t(1-x) - \tilde{v}_t]$  and for every small  $\epsilon > 0$  and  $t$  large enough  $h(x) \leq \mathbb{P}^{W_\kappa} \left( (x-\epsilon)t < \sum_{i=1}^{k-1} U_i \leq xt \right)$  we can apply Proposition 3.5, we get

$$\begin{aligned} &E[\tilde{\nu}_2^{W_\kappa}(1-\epsilon)]E[h(1-\epsilon)] \\ &\leq \mathbb{P} \left( (x-\epsilon)/(1+\epsilon t) < \sum_{i=1}^{k-1} \mathcal{H}_i \leq x/(1-\epsilon t) \right) \mathbb{P}(\mathcal{H}_1 > (t(1-x) - \tilde{v}_t)/(1+\epsilon t)) + e^{-D_1 h t} \end{aligned}$$

Now with Lemma 4.1 and basic computations (see Feller [21] pages 470-472, or first part of Lemma 5.1 in [3]), we prove that

$$\sum_{k \leq n_t} E[\tilde{\nu}_2^{W_\kappa}(1-\epsilon)]E[h(1-\epsilon)] \leq C_+ \cdot u(t, \epsilon)$$

with  $u$  a positive function such that  $\lim_{t \rightarrow +\infty} u(t, \epsilon) = \epsilon^{1-\kappa}$ . A similar argument also works for the second term in (5.63), which yields

$$\sum_{k \leq n_t} E \left[ [\nu_2^{W_\kappa}(x)(\nu_1^{W_\kappa}(x) - \tilde{\nu}_1^{W_\kappa}(x))]_\epsilon^{1-\epsilon} \right] \leq 2C_+ \cdot u(t, \epsilon). \quad (5.64)$$

**We now deal with  $\mathcal{I}'_1$** , by independence between  $X$  and  $X'$ , and the independent parts of the potential  $W_\kappa$  involved in  $\nu_2^{W_\kappa}(x)$  and  $\nu_3^{W_\kappa}(x)$ ,

$$E(\mathcal{I}'_1) = \int_\epsilon^{1-\epsilon} \nu_2(x) d\nu_3(x), \quad (5.65)$$

with  $\nu_2(x) := E(\tilde{\nu}_2^{W_\kappa}(x))$ , and  $\nu_3(x) := E(\nu_3^{W_\kappa}(x))$ .

Again as  $y \rightarrow \nu_2(y)$  is positive increasing and  $\nu_3$  is a repartition function, integrating by parts together with the upper bound in Lemma 5.2 with  $F_\alpha(x - \tilde{v}_t/t) = \nu_3(x)$  gives

$$\int_\epsilon^{1-\epsilon} \nu_2(x) d\nu_3(x) \leq \int_\epsilon^{1-\epsilon} \nu_2(x) dF_\alpha^-(x) + [(\nu_3(x) - F_\alpha^-(x))\nu_2(x)]_\epsilon^{1-\epsilon} + e^{-D_1 h_t}. \quad (5.66)$$

We can prove in a similar way we have obtained (5.64) that:

$$\sum_{k \leq n_t} [(\nu_3(x) - F_\alpha^-(x))\nu_2(x)]_\epsilon^{1-\epsilon} \leq C_+ \cdot u(t, \epsilon),$$

with as usual a possibly enlarge  $C_+$ . So the important term in (5.66) comes from the integral, notice also that we do not make appear the little correction  $-\tilde{v}_t/t$  in this main contribution in order to simplify notations. We see, indeed, that due to the fact that  $\epsilon \leq x \leq 1 - \epsilon$ , with  $\epsilon > 0$  and  $\lim_{t \rightarrow +\infty} \tilde{v}_t/t = 0$  that, this kind of correction has actually no importance. We now have to work on  $\nu_2(x)$

$$\nu_2(x) \leq E(\mathbb{P}_{\tilde{m}_1}^{W_\kappa}[\mathcal{L}_{X'}(t(1-x), \tilde{m}_1) \leq \alpha t, H'(\tilde{L}_1) > t(1-x) - \tilde{v}_t, H'(\tilde{L}_1) < H'(\tilde{L}_1^-)])$$

Then, as  $F_\alpha^-(x)$  is positive and increasing in  $x$ , using Lemma 5.3, we obtain

$$\int_\epsilon^{1-\epsilon} \nu_2(x) dF_\alpha^-(x) \leq \int_\epsilon^{1-\epsilon} f_\alpha^+(x) dF_\alpha^-(x) + o(n_t^{-1}). \quad (5.67)$$

Again for the same reason as above and to simplify the expression, we do not make appear the dependence on the small  $\tilde{v}_t/t$  correction. Now, as  $f_\alpha^+(x)$  can be written

$$f_\alpha^+(x) = \mathbb{P}\left(\left(1-x\right)\frac{\ell_k}{\mathcal{H}_k} \leq \alpha(1+\epsilon_t), \mathcal{H}_k > t(1-x)(1-\epsilon'_t)\right),$$

by independence of the variables  $\{(\ell_j, \mathcal{H}_j), j \leq n_t\}$

$$\begin{aligned} & \int_\epsilon^{1-\epsilon} dF_\alpha^-(x) f_\alpha^+(x) \\ & \leq \mathbb{P}\left[\left(1-\bar{\mathcal{H}}_{k-1}\right)\frac{\bar{\ell}_k - \bar{\ell}_{k-1}}{\bar{\mathcal{H}}_k - \bar{\mathcal{H}}_{k-1}} \leq \alpha(1+\tilde{\epsilon}_t(k)), \bar{\mathcal{H}}_k \geq 1 - \delta'_t, \max_{1 \leq j \leq k-1} \frac{\ell_j}{t} \leq \alpha_t^-, \bar{\mathcal{H}}_{k-1} \leq 1 - \epsilon + \delta'_t\right]. \end{aligned}$$

with  $\delta'_t = 2\tilde{v}_t/t$ , and  $\tilde{\epsilon}_t(k) = \delta'_t\left(\alpha + \frac{\ell_k}{\mathcal{H}_k}\right)$ , and we have integrated back the correction  $\tilde{v}_t/t$ .

The idea now is to make appear the event  $\mathcal{N}_t^{2\epsilon} = k$  in the above probability (recall the definition of  $\mathcal{N}$  given in Proposition 5.1) and sum over  $k$ .

We first prove that the sum over  $k \leq n_t$ , of the above probability is small if we intersect its event with the event  $\mathcal{N}_t^{2\epsilon} \neq k$ . In other word let us prove that

$$\sum_1 := \sum_{k \leq n_t} \mathbb{P}\left[\bar{\mathcal{H}}_k \geq 1 - \delta'_t, \bar{\mathcal{H}}_{k-1} \leq 1 - \epsilon + \delta'_t, \mathcal{N}_t^{2\epsilon} \neq k\right]$$

is small. As  $\{\mathcal{N}_t^{2\epsilon} \neq k\} = \{\bar{\mathcal{H}}_k < 1 - 2\epsilon\} \cup \{\bar{\mathcal{H}}_{k-1} \geq 1 - 2\epsilon\}$ , and that for  $t$  large enough  $\{\bar{\mathcal{H}}_k \geq 1 - \delta'_t\} \cap \{\bar{\mathcal{H}}_k < 1 - 2\epsilon\} = \phi$ , we have that  $\sum_1 \leq \sum_{k \leq n_t} \mathbb{P}\left[\bar{\mathcal{H}}_k \geq 1 - \delta'_t, 1 - 2\epsilon \leq \bar{\mathcal{H}}_{k-1} \leq 1 - \epsilon + \delta'_t\right]$ , and therefore for  $t$  large enough  $\sum_1 \leq \sum_{k \leq n_t} \mathbb{P}\left[\bar{\mathcal{H}}_k \geq 1 - \epsilon/2, 1 - 2\epsilon \leq \bar{\mathcal{H}}_{k-1} \leq 1 - \epsilon\right] \leq s(\epsilon, t)$  by (5.58). Finally collecting all we know,

$$\begin{aligned} & \sum_{k \leq n_t} E(\mathcal{I}_1^k) \leq \\ & \mathbb{P}\left[\left(1-\bar{\mathcal{H}}_{\mathcal{N}_t^{2\epsilon}-1}\right)\frac{\bar{\ell}_{\mathcal{N}_t^{2\epsilon}} - \bar{\ell}_{\mathcal{N}_t^{2\epsilon}-1}}{\bar{\mathcal{H}}_{\mathcal{N}_t^{2\epsilon}} - \bar{\mathcal{H}}_{\mathcal{N}_t^{2\epsilon}-1}} \leq \alpha(1+\tilde{\epsilon}_t(\mathcal{N}_t^{2\epsilon})), \max_{1 \leq j \leq \mathcal{N}_t^{2\epsilon}-1} \frac{\ell_j}{t} \leq \alpha_t^-\right] + s(\epsilon, t). \quad (5.68) \end{aligned}$$

To finish we have to deal with  $\tilde{\epsilon}_t(\mathcal{N}_t^{2\epsilon})$ , a basic computation partitioning on the values of on  $\mathcal{N}_t^{2\epsilon}$ , shows that  $\mathbb{P}(\tilde{\epsilon}_t(\mathcal{N}_t^{2\epsilon})) \geq \sqrt{\delta_t'} \leq C_+ \mathbb{P}(R_1 \leq \sqrt{\delta_t'}) = o(1)$  as  $R_1 \geq 0$ . Collecting this last fact, (5.68), (5.64), (5.61) and (5.60) finishes the proof of the upper bound.

### The lower bound

The proof here follows the same line than the upper bound. The main difference comes from the fact that we can no longer use the obvious inequality  $\sup_{x \in \mathbb{R}} \mathcal{L}(t, x) \geq \sup_{1 \leq j \leq N_t} \mathcal{L}(t, m_j)$ . So for this part of the proof we stress on what is different here and refer to the previous computations when very few changes occur.

Assume for the moment that  $\mathbb{P}(\{\sup_{x \in \mathbb{R}} \mathcal{L}(t, x) \geq 2\tilde{w}_t\} =: \mathcal{E}_2) \geq 1 - o(1)$ , with  $\tilde{w}_t := te^{\kappa(1+3\delta)-1}$ , and recall that  $\delta$  is chosen small enough in such a way that  $\kappa(1+3\delta) < 1$  (see Lemma 3.2). This fact is a direct consequence of the upper-bound of  $\mathbb{P}(\sup_{x \in \mathbb{R}} \mathcal{L}(t, x) \leq \alpha t)$  [see the end of this section for the proof]. Recall (3.24), define for any  $l \geq 1$

$$\begin{aligned} \mathcal{E}_3(l) &:= \mathcal{E}_3^1(l) \cap \mathcal{E}_3^2(l), \text{ with} \\ \mathcal{E}_3^1(l) &:= \bigcap_{j=1}^{l-1} \left\{ \sup_{x \in \mathcal{D}_j} [\mathcal{L}(H(\tilde{L}_j), x) - \mathcal{L}(H(\tilde{m}_j), x)] \leq t\tilde{\alpha}_t \right\}, \\ \mathcal{E}_3^2(l) &:= \left\{ \sup_{x \in \mathcal{D}_l} [\mathcal{L}(t, x) - \mathcal{L}(H(\tilde{m}_l), x)] \leq t\tilde{\alpha}_t \right\}, \end{aligned}$$

with  $\tilde{\alpha}_t := (\alpha t - 2\tilde{w}_t)/t$ . Recall the definitions of the events  $\mathcal{B}$  in Sections 3.1 and 3.2, we have that  $\{\sup_{x \in \mathbb{R}} \mathcal{L}(t, x) \leq \alpha t\} \cap \{\mathcal{V}_t \cap \mathcal{E}_2 \cap \bigcap_{i=1}^4 \mathcal{B}_i(n_t), N_t \leq n_t\}$  contains

$\mathcal{E}_3(N_t) \cap \{\mathcal{V}_t \cap \mathcal{E}_2 \cap \bigcap_{i=1}^4 \mathcal{B}_i(n_t), N_t \leq n_t\}$ . Notice that by Lemmata 2.2, 3.1, 3.2, 3.4 and the above assumption  $\mathbb{P}(\mathcal{V}_t \cap \mathcal{E}_2 \cap \bigcap_{i=1}^4 \mathcal{B}_i(n_t), N_t \leq n_t) \geq 1 - o(1)$ . We now deal with  $\mathbb{P}(\mathcal{E}_3(N_t) \cap \mathcal{B}_1(N_t) \cap \mathcal{B}_2(n_t) \cap \mathcal{V}_t, N_t \leq n_t)$ , using Lemma 2.2, the fact that  $H(\tilde{L}_k) \leq H(\tilde{m}_{k+1})$  and the strong Markov property with respect to  $\mathbb{P}^{W_\kappa}$  we obtain

$$\begin{aligned} &\mathbb{P}(\mathcal{E}_3(N_t) \cap \mathcal{B}_1(N_t) \cap \mathcal{B}_2(n_t) \cap \mathcal{V}_t, \mathcal{Q}) \\ &\geq \sum_{k=1}^{n_t} \int_{\epsilon}^{1-\epsilon} E \left( \nu_4^{W_\kappa}(y) d\mathbb{P}^{W_\kappa}(\mathcal{E}_3^1(k), \mathcal{B}_1(k), \mathcal{B}_2(k-1), H(\tilde{m}_k)/t \leq y) \right) - o(1) \end{aligned}$$

with

$$\nu_4^{W_\kappa}(y) := \mathbb{P}_{\tilde{m}_k}^{W_\kappa} \left( \sup_{x \in \mathcal{D}_k} \mathcal{L}_{X'}(t(1-y), x) \leq t\tilde{\alpha}_t, H'(\tilde{L}_k) > t(1-y), H'(\tilde{L}_k) < H'(\tilde{L}_k^-) \right),$$

Now by similar computations than for the upper bound in particular what yields (5.64) and (5.65) we have

$$\begin{aligned} &\mathbb{P}(\mathcal{E}_3(N_t) \cap \mathcal{B}_1(N_t) \cap \mathcal{B}_2(n_t) \cap \mathcal{V}_t, \mathcal{Q}) \\ &\geq \sum_{k=1}^{n_t} \int_{\epsilon}^{1-\epsilon} E \left( \nu_4^{W_\kappa}(y) d\nu_5^{W_\kappa}(y) \right) - o(1) = \sum_{k=1}^{n_t} \int_{\epsilon}^{1-\epsilon} \nu_4(y) d\nu_5(y) - o(1). \end{aligned}$$

with  $\nu_5^{W_\kappa}(y) := \mathbb{P}^{W_\kappa} \left( \mathcal{E}_3^1(k), \mathcal{B}_1(k), \mathcal{B}_2(k-1), \sum_{i=1}^{k-1} U_i/t \leq y \right)$ ,  $\nu_4 := E(\nu_4^{W_\kappa})$ ,  $\nu_5 := E(\nu_5^{W_\kappa})$ . The next step, is to remove  $\mathcal{B}_1(k)$  in the above expression, for that we only have to prove that

$$\sum_{k=1}^{n_t} \int_{\epsilon}^{1-\epsilon} E \left( \nu_4^{W_\kappa}(y) d\mathbb{P}^{W_\kappa} \left( \mathcal{E}_3^1(k), \bar{\mathcal{B}}_1(k), \mathcal{B}_2(k-1), \sum_{i=1}^{k-1} U_i/t \leq y \right) \right)$$

is negligible, one can check that this quantity is smaller than

$$\begin{aligned} & \sum_{k=1}^{n_t} \int_{\epsilon}^{1-\epsilon} E \left[ \mathbb{P}_{\tilde{m}_k}^{W_\kappa} (H(\tilde{L}_k) < H(\tilde{L}_k^-), H(\tilde{L}_k) > t(1-y)) \right] d\mathbb{P} \left( \bar{\mathcal{B}}_1(k), \mathcal{B}_2(k-1), \sum_{i=1}^{k-1} U_i/t \leq y \right) \\ & \leq \sum_{k=1}^{n_t} \mathbb{P} \left( \sum_{i=1}^{k-1} U_i/t \leq 1, \sum_{i=1}^k U_i/t > 1, \bar{\mathcal{B}}_1(k) \right) \leq \mathbb{P}(\bar{\mathcal{B}}_1(n_t)) \leq C_2 v_t = o(1), \end{aligned}$$

where the last inequality comes from (3.12). Therefore collecting the above computations yields

$$\mathbb{P} \left( \sup_{x \in \mathbb{R}} \mathcal{L}(t, x) \leq \alpha \right) \geq \sum_{k=1}^{n_t} \int_{\epsilon}^{1-\epsilon} \nu_4(y) d\tilde{\nu}_5(y) - o(1)$$

with  $\tilde{\nu}_5(y) := e^{-\kappa\phi(t)} \sum_{k \leq n_t} \mathbb{P} \left( \mathcal{E}_3^1(k), \mathcal{B}_2(k-1), \sum_{i=1}^{k-1} U_i/t \leq y \right)$ .

We start with an estimation of the repartition function  $\tilde{\nu}_5(y)$ . Recall that like in the proof of Lemma 3.6, by the strong Markov property, the occupation time formula (1.5) and (1.6) the sequence  $(U_j, \{\mathcal{L}(H(\tilde{L}_j), x), x \in \mathcal{D}_j\}, j \leq n_t)$  under  $\mathcal{B}_2(k-1)$  is equal in law to a sequence  $(H_j(\tilde{L}_j), \{\mathcal{L}_j(H_j(\tilde{L}_j), x), x \in \mathcal{D}_j\}, j \leq n_t)$ , with this time

$$\begin{aligned} H_j(\tilde{L}_j) &:= A^j(\tilde{L}_j) \int_{\tilde{L}_j^-}^{\tilde{L}_j} e^{-\tilde{V}^{(j)}(x)} \mathcal{L}_{B^j}[\tau^{B^j}(1), A^j(u)/A^j(\tilde{L}_j)] du, \\ \mathcal{L}_j(H_j(\tilde{L}_j), x) &:= A^j(\tilde{L}_j) e^{-\tilde{V}^{(j)}(x)} \mathcal{L}_{B^j}[\tau^{B^j}(1), A^j(x)/A^j(\tilde{L}_j)], \quad A^j(u) = \int_{\tilde{m}_j}^u e^{\tilde{V}^{(j)}(x)} dx. \end{aligned}$$

Using the fact that the sequence  $(\tilde{m}_j, j)$  is a subsequence of  $(m_j, j)$ , Fact 2.1 and then (6.87) for any  $j$ ,

$$\begin{aligned} P(\inf\{s > \tilde{m}_j, \tilde{V}^{(j)}(s) > r_t\} > \tilde{m}_j + r_t) &\leq e^{-c_- r_t}, \quad \text{and} \\ P(\sup\{s < \tilde{m}_j, \tilde{V}^{(j)}(s) > r_t\} < \tilde{m}_j - r_t) &\leq e^{-c_- r_t}. \end{aligned} \quad (5.69)$$

with  $c_- > 0$ . Therefore for any  $j$ ,  $P(D_j \subset [\tilde{\tau}_j^-(r_t), \tilde{\tau}_j(r_t)]) \geq 1 - 2e^{-c_- r_t}$ . Then on  $\{D_j \subset [\tilde{\tau}_j^-(r_t), \tilde{\tau}_j(r_t)]\}$  for any  $x \in D_j$ ,  $\mathcal{L}_j(H_j(\tilde{L}_j), x) \leq A^j(\tilde{L}_j) \mathcal{L}_{B^j}[\tau^{B^j}(1), A^j(x)/A^j(\tilde{L}_j)]$ . Also on  $\{D_j \subset [\tilde{\tau}_j^-(r_t), \tilde{\tau}_j(r_t)]\}$ , for any  $x \in D_j$ ,

$$A^j(\tilde{\tau}_j^-(r_t)) \leq A^j(x) \leq A^j(\tilde{\tau}_j(r_t)). \quad (5.70)$$

With Fact 2.1 and Lemma 6.3, we obtain with a probability larger than  $1 - e^{-c_- r_t}$

$$-e^{-ht/4} \leq -e^{r_t(1+1/2)} e^{-(1-1/2)ht} \leq \frac{A^j(\tilde{\tau}_j^-(r_t))}{A^j(\tilde{L}_j)} \leq \frac{A^j(\tilde{\tau}_j(r_t))}{A^j(\tilde{L}_j)} \leq e^{r_t(1+1/2)} e^{-(1-1/2)ht} \leq e^{-ht/4}. \quad (5.71)$$

Therefore taking  $\delta = e^{-ht/4}$  and  $\epsilon = \delta^{1/3}$  in (6.93) we obtain with a probability larger than  $1 - e^{-c_- r_t}$

$$\sup_{x \in \mathcal{D}_j} \mathcal{L}_{B^j}(\tau^{B^j}(1), A^j(x)/A^j(\tilde{L}_j)) \leq A^j(\tilde{L}_j) \mathcal{L}_{B^j}(\tau^{B^j}(1), 0)(1 + e^{-ht/12}). \quad (5.72)$$

Collecting the different estimates we then obtain,

$$\tilde{\nu}_5(y) \geq \mathbb{P} \left( \max_{1 \leq j \leq k-1} \mathcal{L}_j(H_j(\tilde{L}_j), \tilde{m}_j) \leq t\bar{\alpha}_t, \sum_{i=1}^{k-1} H_j(\tilde{L}_j)/t \leq y \right) - C_+ e^{-c_- r_t},$$

with  $\bar{\alpha}_t := \tilde{\alpha}_t(1 + e^{-ht/12})^{-1}$ . We can then inverse the equality in law we have used above, and then obtain

$$\tilde{\nu}_5(y) \geq F_{\bar{\alpha}_t}(y) - C_+ e^{-c-rt},$$

with  $F_{\bar{\alpha}_t}$  defined in Lemma 5.2. Then we can follow the same lines than for the upper bound (especially computations after (5.64)), and obtain via Lemma 5.2:

$$\int_{\epsilon}^{1-\epsilon} \nu_4(y) d\tilde{\nu}_5(y) \geq \int_{\epsilon}^{1-\epsilon} \nu_4(y) dF_{\bar{\alpha}_t}^+(y) - o(n_t^{-1}).$$

Remark also that (5.69), (5.70), (5.71) and 5.72 implies the concentration of the local time at the  $h_t$ -minima : with probability larger than  $1 - C_+ e^{-c-rt}$

$$\left| \sup_{y \in \mathcal{D}_j} \mathcal{L}(H(\tilde{L}_j), y) - \mathcal{L}(H(\tilde{L}_j), \tilde{m}_j) \right| \leq e^{-h_t/12} \mathcal{L}(H(\tilde{L}_j), \tilde{m}_j).$$

We now work on  $\nu_4(y)$ , by the second part of Lemma 2.2 it is equal to

$$E \left( \mathbb{P}_{\tilde{m}_1}^{W_\kappa} \left( \sup_{y \in \mathcal{D}_1} \mathcal{L}_{X'}(t(1-x), y) \leq t\tilde{\alpha}_t, H'(\tilde{L}_1) > t(1-x), H'(\tilde{L}_1) < H'(\tilde{L}_1^-) \right) \right) =: \tilde{\nu}_4(y),$$

and by Lemma 5.3,  $\tilde{\nu}_4(y) \geq f_{\bar{\alpha}_t}^-(y) - o(n_t^{-1})$ . Therefore

$$\int_{\epsilon}^{1-\epsilon} \nu_4(y) d\tilde{\nu}_5(y) \geq \int_{\epsilon}^{1-\epsilon} f_{\bar{\alpha}_t}^-(y) dF_{\bar{\alpha}_t}^+(y) - o(n_t^{-1}).$$

From now the computations are very closed from that of the upper bound (see (5.67) and below) and we do not give more details.  $\square$

## Proof of the Lemmata

**Proof of Lemma 5.2:** This is a direct consequence of Lemmata 2.2, 3.1 and Proposition 3.5.  $\square$

**Proof of Lemma 5.3:** To obtain the result we use a similar method than in [4]. That is to say we study the inverse of the local time at  $\tilde{m}_1$ , and use our knowledge about  $H(\tilde{L}_1)$ . Also for our purpose only the upper bound of  $f_\alpha$  and the lower bound of  $\tilde{f}_\alpha$  are needed, so we only present the proof for them. However few modifications of the proof we present below also lead to respectively the other lower and upper bounds.

• For  $f_\gamma$ , let  $\sigma(u, \tilde{m}_1) := \inf\{s > 0, \mathcal{L}(s, \tilde{m}_1) > u\}$ , just notice that

$$\begin{aligned} E & \left[ \mathbb{P}_{\tilde{m}_1}^{W_\kappa} (\mathcal{L}(t(1-x), \tilde{m}_1) \leq \gamma t, H(\tilde{L}_1) > t(1-x), H(\tilde{L}_1) < H(\tilde{L}_1^-)) \right] \\ & = E \left[ \mathbb{P}_{\tilde{m}_1}^{W_\kappa} (\sigma(\gamma t, \tilde{m}_1) \geq t(1-x), H(\tilde{L}_1) > t(1-x), H(\tilde{L}_1) < H(\tilde{L}_1^-)) \right]. \end{aligned} \quad (5.73)$$

On  $\{H(\tilde{L}_1) > t(1-x), H(\tilde{L}_1) < H(\tilde{L}_1^-)\}$ ,  $X$  remains between  $\tilde{L}_1^-$  and  $\tilde{L}_1$  until the instant  $t(1-x)$ . On this event considering (1.5) and (1.6) and the scaling invariance of the underlying Brownien motion  $B'$ , the inverse of the local time and the hitting time can be written for  $X$



starting at  $\tilde{m}_1$  as

$$\begin{aligned} (\sigma(\gamma t, \tilde{m}_1), H(\tilde{L}_1)) \stackrel{\mathfrak{A}_{W_\kappa}}{=} & \left( \gamma t \int_{\tilde{L}_1^-}^{\tilde{L}_1} e^{-\tilde{V}^{(1)}(x)} \mathcal{L}_{B'}(\sigma_{B'}(1, 0), \tilde{a}(x)) dx =: I, \right. \\ & \left. A^1(\tilde{L}_1) \int_{\tilde{L}_1^-}^{\tilde{L}_1} e^{-\tilde{V}^{(1)}(x)} \mathcal{L}_{B'}(\tau_{B'}(1), \tilde{a}'(x)) dx =: \tilde{h}_1 \right), \end{aligned}$$

with  $\tilde{a}(x) := (\gamma t)^{-1} A^1(x) = (\gamma t)^{-1} \int_{\tilde{m}_1}^x e^{V^{(1)}(y)} dy$ ,  $\tilde{a}'(x) := A^1(x)/A^1(\tilde{L}_1)$ , as usual  $B'$  is denoted  $B$  in the sequel. Also  $\sigma_U(r, y) := \inf\{s > 0, \mathcal{L}_U(s, y) > r\}$  for  $r > 0, y \in \mathbb{R}$  is the inverse of the local time of the process  $U$ . We have already talk about  $\tilde{h}_1$  in 3.30, in particular Proposition 3.5 gives

$$\mathbb{P}\left(\left|H(\tilde{L}_1) - H(\tilde{m}_1) - \mathbf{e}_1 S_1 R_1\right| \leq \epsilon_t \mathbf{e}_1 S_1 R_1\right) =: \mathcal{G}_1 = 1 - e^{-D_1 h_t}, \text{ with } \epsilon_t := e^{-d_1 h_t}, \quad (5.74)$$

so we only deal with  $I$ . Let us define  $\hat{L}_1^- := \sup\{s < \tilde{m}_1, \tilde{V}^{(1)}(s) > h_t/2\}$  and  $\hat{L}_1 := \inf\{s > \tilde{m}_1, \tilde{V}^{(1)}(s) > h_t/2\}$ .  $(\gamma t)^{-1} I$  can be splitted into two terms  $(\gamma t)^{-1} I := I_1 + I_2$  with

$$I_1 := \int_{\hat{L}_1^-}^{\hat{L}_1} e^{-\tilde{V}^{(1)}(x)} \mathcal{L}_B(\sigma_B(1, 0), \tilde{a}(x)) dx.$$

We now prove that the main contribution in  $(\gamma t)^{-1} I$  is given by  $I_1$  and obtain its approximation in probability. First using Fact 2.1, (6.89) and (6.90) we have for any  $\hat{L} \in \{\hat{L}_1^-, \hat{L}_1\}$  and  $\epsilon > 0$ ,

$$P(|A^1(\hat{L})| \leq e^{h_t(1+\epsilon)/2}) \geq 1 - C_+ e^{-\kappa \epsilon h_t/2}.$$

Therefore by monotony of  $\tilde{a}(\cdot)$  (depending on whether its argument is larger or smaller than  $\tilde{m}_1$ )  $P(\forall x \in [\hat{L}_1, \hat{L}_1^-], |\tilde{a}(x)| \leq e^{-(\log t)(1-3\epsilon)/2}) \geq 1 - C_+ e^{-\kappa \epsilon h_t/2}$ . Also, using (6.97) we have

$$\mathbb{P}\left(\sup_{0 \leq y \leq e^{-(\log t)(1-3\epsilon)/2}} |\mathcal{L}_B(\sigma_B(1, 0), y) - 1| \geq \hat{\epsilon}_t\right) \leq e^{-e^{(\log t)/20}}. \quad (5.75)$$

with  $\hat{\epsilon}_t := e^{-(\log t)(1-\epsilon)}$ . So we obtain

$$\mathbb{P}\left(|I_1 - \mathbb{R}_1| \leq e^{-(\log t)(1-\epsilon)} \mathbb{R}_1\right) \geq 1 - C_+ e^{-\kappa \epsilon h_t/2},$$

with  $\mathbb{R}_1 := \int_{\hat{L}_1^-}^{\hat{L}_1} e^{-V^{(1)}(x)} dx$ . Then we prove that  $I_2$  is negligible compared to the integral in the previous equation. First thanks to (6.98), we have  $P(\sup_{x \in [\tilde{L}_1^-, \tilde{L}_1]} \mathcal{L}_B(\sigma_B(1, 0), \tilde{a}(x)) > e^{\log t}) \leq 2e^{-\epsilon \log t}$ . So with probability larger than  $1 - 2e^{-\epsilon \log t}$  we have

$$I_2 \leq e^{\epsilon \log t} \left( \int_{\tilde{L}_1^-}^{\tilde{L}_1^-} e^{-V^{(1)}(x)} dx + \int_{\hat{L}_1}^{\tilde{L}_1} e^{-V^{(1)}(x)} dx \right).$$

We now want to prove that both integral in the upper bound of  $I_2$  is smaller than  $e^{-(1-2\epsilon) \log t/2} \mathbb{R}_1$ , and obtain a lower bound for  $\mathbb{R}_1$ . Both of these estimations can be found in [3] : in the proof of step 2 of Lemma 4.7 (estimation of what is called  $\mathcal{J}_2$ ), we find that with a probability larger than  $1 - e^{-c-\epsilon h_t}$

$$\int_{\hat{L}_1}^{\tilde{L}_1} e^{-V^{(1)}(x)} dx \leq C_+ h_t^2 e^{-(1-\epsilon)h_t/2}$$

and the same is true for the first integral. For  $\mathbb{R}_1$ , by Fact 2.1 it has the same law as  $R_1$  (which is the same  $R_1$  than in 5.74), which law is given by the sum of two independent copies of  $F^-(h_t/2)$ .

So using (6.91), with a probability larger than  $1 - e^{-c-\epsilon^2 h_t^2/2}$

$$\mathbb{R}_1 \stackrel{\mathfrak{I}}{=} R_1 \geq e^{-\epsilon h_t}.$$

We deduce from the last two inequalities that with a probability larger than  $1 - e^{-c-\epsilon h_t}$ ,

$$I_2 \leq R_1 e^{-(1-5\epsilon)h_t/2}.$$

Finally collecting what we have done for the inverse of the local time  $\sigma(\cdot, \cdot)$

$$\begin{aligned} E \left[ \mathbb{P}_{\tilde{m}_1}^{W_\kappa} \left( |\sigma(\gamma t, \tilde{m}_1) - \gamma t R_1| \leq 2e^{-(1-5\epsilon)h_t/2} (\gamma t) R_1, H(\tilde{L}_1) > t(1-x), H(\tilde{L}_1) < H(\tilde{L}_1^-) \right) \right] \\ \geq 1 - C_+ e^{-\epsilon c - h_t}. \end{aligned} \quad (5.76)$$

Also by (3.13)  $E(\mathbb{P}_{\tilde{m}_1}^{W_\kappa}(H(L_1^-) < H(L_1))) \leq C_3 n_t e^{-\delta \kappa h_t}$ . So collecting (5.73), (5.74) and the above inequality, for any  $0 < x < 1$  and  $t$  large enough

$$f_\gamma(x) \leq \mathbb{P} \left( R_1 \geq \frac{1-x}{\gamma} (1 - 2\tilde{\epsilon}_t), \mathbf{e}_1 S_1 R_1 > t(1-x)(1 - 2\epsilon_t) \right) + o(n_t^{-1})$$

with  $\tilde{\epsilon}_t := 2e^{-(1-5\epsilon)h_t/2}$ .

• For  $\tilde{f}_\gamma$ , here again we work on the event  $\{H(\tilde{L}_1) > t(1-x), H(\tilde{L}_1) < H(\tilde{L}_1^-)\}$  and also use the equality in law like for  $f_\gamma$ : for any  $z \in \mathcal{D}_1$ ,

$$\mathcal{L}(\sigma(yt, \tilde{m}_1), z) \stackrel{\mathfrak{I}}{=} (yt) e^{-\tilde{V}^{(1)}(z)} \mathcal{L}_B(\sigma_B(1, 0), \tilde{a}(z)) \quad (5.77)$$

with  $\tilde{a}(x) = (yt)^{-1} \int_{\tilde{m}_1}^x e^{\tilde{V}^{(1)}(y)} dy$ . Remember also that thanks to Fact 2.1 and (5.69), we can obtain an upper bound of  $\int_{\tilde{m}_1}^z e^{\tilde{V}^{(1)}(y)} dy$  for  $z \in \mathcal{D}_1$  by estimating  $\int_0^z e^{Q(s)} ds$  for  $z \in [0, \tau_R(r_t)]$ . Also as by (6.90)  $P(\int_0^{\tau_R(r_t)} e^{Q(s)} ds \geq e^{\epsilon h_t}) \leq e^{-c-\epsilon h_t}$ , with the same probability  $\tilde{a}(z) \leq (yt)^{-1} e^{\epsilon h_t}$  and with the same method we have obtained (5.75),

$$E \left( \mathbb{P}_{\tilde{m}_1}^{W_\kappa} \left( \sup_{z \in \mathcal{D}_1} |\mathcal{L}_B(\sigma_B(1, 0), \tilde{a}(z)) - 1| \leq \hat{\epsilon}_t \right) \right) \geq 1 - 2e^{-c-\epsilon h_t}.$$

(5.77) together with the above estimate imply

$$\begin{aligned} E \left( \mathbb{P}_{\tilde{m}_1}^{W_\kappa} \left( \left\{ \sup_{z \in \mathcal{D}_1} \left| \mathcal{L}(\sigma(yt, \tilde{m}_1), z) - yte^{-\tilde{V}^{(1)}(z)} \right| \leq 2yte^{-\tilde{V}^{(1)}(z)} \hat{\epsilon}_t \right\} \right) \right) \\ \geq 1 - 2e^{-c-\epsilon h_t}. \end{aligned} \quad (5.78)$$

Let us take  $y = (1-x)/(R_1(1-\tilde{\epsilon}_t))$  then using (5.76) (with  $\gamma = y$ ) implies that with a probability larger than  $1 - C_+ e^{-\epsilon c - h_t}$ ,  $t(1-x) \leq \sigma(yt, \tilde{m}_1)$ . Then as the local time is increasing in time, with the same probability (eventually increasing  $C_+$ ), by (5.78), for any  $z \in \mathcal{D}_1$ ,  $\mathcal{L}(t(1-x), z) \leq \mathcal{L}(\sigma(yt, \tilde{m}_1), z) \leq yte^{-\tilde{V}^{(1)}(z)}(1 + \hat{\epsilon}_t) \leq yt(1 + \hat{\epsilon}_t)$ . Collecting all this gives

$$E \left( \mathbb{P}_{\tilde{m}_1}^{W_\kappa} \left( \left\{ \frac{\sup_{z \in \mathcal{D}_1} \mathcal{L}(t(1-x), z)}{t} \leq \frac{(1-x)}{R_1} \frac{1 + \hat{\epsilon}_t}{1 - \tilde{\epsilon}_t} \right\} =: \mathcal{G}_2 \right) \right) \geq 1 - o(n_t^{-1}). \quad (5.79)$$

Collecting (5.74), (5.79) and (3.13) we obtain,

$$\begin{aligned} \tilde{f}_\gamma(x) &\geq E \left( \mathbb{P}_{\tilde{m}_1}^{W_\kappa} \left( \sup_{z \in \mathcal{D}_1} \mathcal{L}(t(1-x), z) \leq t\gamma, H(\tilde{L}_1) > t(1-x), H(\tilde{L}_1) < H(\tilde{L}_1^-), \mathcal{G}_1, \mathcal{G}_2 \right) \right) \\ &\geq E \left( \mathbb{P}_{\tilde{m}_1}^{W_\kappa} \left( \left( \frac{(1-x)}{R_1} \frac{1 + \hat{\epsilon}_t}{1 - \tilde{\epsilon}_t} \leq \gamma, \mathbf{e}_1 S_1 R_1 > t \frac{1-x}{1 - \epsilon_t}, H(\tilde{L}_1) < H(\tilde{L}_1^-), \mathcal{G}_1, \mathcal{G}_2 \right) \right) \right) \\ &\geq \mathbb{P} \left( \left( \frac{(1-x)}{R_1} \frac{1 + \hat{\epsilon}_t}{1 - \tilde{\epsilon}_t} \leq \gamma, \mathbf{e}_1 S_1 R_1 > t \frac{1-x}{1 - \epsilon_t} \right) - o(n_t^{-1}). \right) \end{aligned}$$

This finishes the proof.  $\square$

**Proof of Lemma 5.4:**

For (5.58), independence between  $\bar{\mathcal{H}}_{k-1}$  and  $\bar{\mathcal{H}}_k - \bar{\mathcal{H}}_{k-1}$  yields

$$\sum_{k \leq n_t} \mathbb{P} [\bar{\mathcal{H}}_k > 1 - a/2, 1 - 2a < \bar{\mathcal{H}}_{k-1} \leq 1 - a] = \int_{1-2a}^{1-a} d\mu_t(x) e^{\kappa\phi(t)} \mathbb{P} [\bar{\mathcal{H}}_k - \bar{\mathcal{H}}_{k-1} > 1 - x - a/2] \quad (5.80)$$

where measure  $\mu_t$  is defined as  $\int_0^x d\mu_t(y) = e^{-\kappa\phi(t)} \sum_{k \leq n_t} \mathbb{P} [\bar{\mathcal{H}}_{k-1} \leq x]$ , and we know that  $\mu_t$  converge vaguely to the measure  $\mu$  which has a density with respect to the Lebesgue measure equal to  $(\Gamma(1 - \kappa)C_\kappa)^{-1} x^{\kappa-1} \mathbb{1}_{x>0}$  (see Lemma 5.1 in [3]). Also we know from Lemma 4.1 that  $e^{\kappa\phi(t)} P[(\mathcal{H}_k - \mathcal{H}_{k-1})/t > x]$  converge uniformly on every compact subset of  $(0, +\infty)$  to  $C_\kappa x^{-\kappa}/\Gamma(1 - \kappa)$ , therefore

$$\begin{aligned} & \lim_{t \rightarrow +\infty} \sum_{k \leq n_t} \mathbb{P} [\bar{\mathcal{H}}_k > 1 - a/2, 1 - 2a < \bar{\mathcal{H}}_{k-1} \leq 1 - a] \\ &= \frac{1}{\Gamma(\kappa)\Gamma(1 - \kappa)} \int_{1-2a}^{1-a} x^{\kappa-1} (1 - x - a/2)^{-\kappa} dx \\ &\leq \text{const} \times a^{1-\kappa}. \end{aligned}$$

Also (5.59) is a direct consequence of Corrolary 1.5, eq. (1.2) in [3].  $\square$

**Proof of Theorem 1.3**

The proof of this theorem is a direct consequence of Propositions 5.1, 1.4 and Lemmata 4.4 and 4.5. Notice that the proof of the upper bound is independent of the proof of the lower bound, but we use the upper bound for the proof of the lower bound, in particular from the upper bound we have that :  $\lim_{t \rightarrow +\infty} \mathbb{P}(\mathcal{L}^*(t) \leq 2\tilde{w}_t/t) \leq \mathbb{P}(\mathcal{Y}_1^\natural(\mathcal{Y}_2^{-1}(1)^-) \leq \varepsilon)$  for any  $\varepsilon > 0$  as  $\lim_{t \rightarrow +\infty} \tilde{w}_t/t = 0$ . From that as  $\mathcal{Y}_1^\natural(\mathcal{Y}_2^{-1}(1)^-)$  is positive we obtain that  $\lim_{t \rightarrow +\infty} \mathbb{P}(\mathcal{L}^*(t) \leq 2\tilde{w}_t) = 0$ , which proves the assertion at the beginning of the proof of the lower bound in Proposition 5.1.

**5.2. Favorite sites (proof of Theorem 1.5).**

The first two point of Theorem 1.5 can be obtained similarly as Theorem 1.3, they are in fact easier to prove due to the presence of stopping times.

To obtain the result for the favorite sites, we first argue that we essentially need to obtain the asymptotic behavior of  $N_t^*/N_t$ , where  $N_t^* := \min\{j \geq 1, \mathcal{L}(m_j, t) = \max_{k \leq N_t} \mathcal{L}^*(m_k, t)\}$ . Indeed, define for any  $\varepsilon > 0$

$$\mathcal{K}_1 := \{m_{N_t}(1 - \varepsilon) \leq X_t \leq m_{N_t}(1 + \varepsilon)\}, \quad \mathcal{K}_2 := \{m_{N_t^*}(1 - \varepsilon) \leq F_t^* \leq m_{N_t^*}(1 + \varepsilon)\}$$

then we have by the localization result Theorem 1.2  $\lim_{t \rightarrow +\infty} \mathbb{P}(\mathcal{K}_1) = 1$ , also we can prove that  $\lim_{t \rightarrow +\infty} \mathbb{P}(\mathcal{K}_2) = 1$  by using the same arguments than in the proof of the lower bound of Proposition 5.1. So

$$\begin{aligned} \mathbb{P} \left[ \frac{F_t^*}{X_t} \leq x \right] &= \mathbb{P} \left[ \frac{F_t^*}{X_t} \leq x, \mathcal{K}_1, \mathcal{K}_2 \right] + v(\varepsilon, t) \\ &\leq \mathbb{P} \left[ \frac{m_{N_t^*}}{m_{N_t}} \leq x \frac{1 + \varepsilon}{1 - \varepsilon} \right] + v(\varepsilon, t). \end{aligned} \quad (5.81)$$

where  $v(\varepsilon, t) \geq 0$ , satisfies  $\lim_{\varepsilon \rightarrow 0} \lim_{t \rightarrow +\infty} v(\varepsilon, t) = 0$ . We now use two fact, the first one is that  $N_t^*$  and  $N_t$  diverge when  $t$  goes to infinity : in particular  $\mathbb{P}(N_t \geq e^{(1-\varepsilon)\kappa\phi(t)}) = 1 - o(1)$  as it is proved in ([3], Proposition 1.6), and a similar analysis also works for  $N_t^*$  giving also  $\mathbb{P}(N_t^* \geq e^{(1-\varepsilon)\kappa\phi(t)}) = 1 - o(1)$ .

For the second argument, following the work of Faggionato [19], we know that  $(m_i - m_{i-1}, i \geq 2)$  are i.i.d. random variables with a known Laplace transform (given by (2.19) in [19]), this allows to compute the first and fourth moments of  $\Delta m_1 := m_2 - m_1$  and therefore obtain after an elementary but tedious computation that for large  $t$ ,  $\mathbb{E}(\Delta m_1) \sim C_7 e^{\kappa h t}$  ( $C_7 > 0$ ) and  $\mathbb{E}((\Delta m_1 - \mathbb{E}(\Delta m_1))^4) \sim C_8 e^{4\kappa h t}$  ( $C_8 > 0$ ), which yields for  $t$  large

$$\mathbb{E} \left[ (m_k/k - \mathbb{E}(\Delta m_1))^4 \right] \sim C_8 e^{4\kappa h t} / k^2.$$

These facts allow us to write by a Markov inequality that

$$\begin{aligned} & \mathbb{P} [|m_{N_t} - \mathbb{E}(\Delta m_1)N_t| > \epsilon \mathbb{E}(\Delta m_1)N_t] \\ & \leq \sum_{j \geq e^{(1-\epsilon)\kappa\phi(t)}} \mathbb{P} [|m_j - \mathbb{E}(\Delta m_1)j| > \epsilon \mathbb{E}(\Delta m_1)j] + o(1) \\ & \leq \sum_{j \geq e^{(1-\epsilon)\kappa\phi(t)}} \frac{C_8(C_7)^{-4}}{\epsilon^4 j^2} + o(1) \leq C_+ \epsilon^{-4} e^{-(1-\epsilon)\kappa\phi(t)} + o(1), \end{aligned}$$

this yields that  $\{|m_{N_t} - \mathbb{E}(\Delta m_1)N_t| \leq \epsilon \mathbb{E}(\Delta m_1)N_t\}$  as well as (with a similar computation)  $\{|m_{N_t^*} - \mathbb{E}(\Delta m_1)N_t^*| \leq \epsilon \mathbb{E}(\Delta m_1)N_t^*\}$  are realized with a probability close to one.

Now including these events in the probability in (5.81), eventually enlarging  $v(\epsilon, t)$  we get

$$\mathbb{P} \left[ \frac{F_t^*}{X_t} \leq x \right] \leq \mathbb{P} \left[ \frac{N_t^*}{N_t} \leq x \frac{(1+\epsilon)^2}{(1-\epsilon)^2} \right] + v(\epsilon, t).$$

Notice that the random variables involved now ( $N_t^*$  and  $N_t$ ) only depends of what happens in the bottom of the  $h_t$ -valleys, and we have to deal with

$$\mathbb{P} \left[ \frac{N_t^*}{N_t} \leq y \right] = \mathbb{P} [N_t^* = N_t] \mathbf{1}_{y=1} + \mathbb{P} \left[ \frac{N_t^*}{N_t} \leq y, N_t^* < N_t \right] \mathbf{1}_{y < 1}.$$

With very closed computations than in Section 5.1 together with Lemma 4.6, we can prove that

$$\begin{aligned} & \lim_{t \rightarrow +\infty} \mathbb{P} [N_t^* = N_t] = \mathbb{P} [\mathcal{I}_1 < \mathcal{I}_2], \text{ and} \\ & \lim_{t \rightarrow +\infty} \mathbb{P} \left[ \frac{N_t^* e^{-\kappa\phi(t)}}{N_t e^{-\kappa\phi(t)}} \leq y, N_t^* < N_t \right] = \mathbb{P} \left[ \frac{F^*(\mathcal{Y}_1, \mathcal{Y}_2)}{\mathcal{Y}_2^{-1}(1)} \leq y, \mathcal{I}_1 \geq \mathcal{I}_2 \right], \end{aligned}$$

where  $F^*$  is defined at the beginning of Section 4.3. To finish the proof we finally have to prove the following Lemma

**Lemma 5.5.** *The random variable  $\frac{F^*(\mathcal{Y}_1, \mathcal{Y}_2)}{\mathcal{Y}_2^{-1}(1)}$  follows a uniform law  $U_{[0,1]}$  independent of the couple  $(\mathcal{I}_1, \mathcal{I}_2)$ .*

*Proof.* For any  $s > 0$ , let  $\mathcal{G}_1(s) := \inf\{u \leq s, \mathcal{Y}_1(u) = \sup_{v \leq s} \mathcal{Y}_1^\sharp(v)\}$ . The fact that for every  $s > 0$ ,  $\mathcal{G}_1(s^-)/s$  follows a uniform law  $U_{[0,1]}$  is basic, as well as the fact that  $\mathcal{G}_1(s^-)$  is independent of  $\mathcal{Y}_1^\sharp(s^-)$ ,  $\mathcal{Y}_2(s^-)$ ,  $\mathcal{Y}_1(s^-)$ ,  $\mathcal{Y}_2(s) - \mathcal{Y}_2(s^-)$ , and  $\mathcal{Y}_1(s) - \mathcal{Y}_1(s^-)$ . To obtain the Lemma we only have to prove that this remains true when replacing  $s$  by  $\mathcal{Y}_2^{-1}(1)$ . For that we can consider for example the dyadic approximations of  $\mathcal{Y}_2^{-1}(1) : (t_n := \max\{k \in \mathbb{N}, \frac{k}{2^n} < \mathcal{Y}_2^{-1}(1)\}, n)$ . Then partitioning on values of  $t_n$ , using the independence we previously talked about and letting  $n$  goes to infinity we obtain the result.  $\square$

## 6. APPENDIX

We start this section with some known formulas for the processes we encounter in our study. The following Lemma is about Laplace transform of the exponential functional defined in (1.3), the proof can be found in ([3], Lemma 4.2). Recall that  $C_+$  (respectively  $c_-$ ) is a positive constant that is as large (resp. small) as needed.

**Lemma 6.1.** *There exists  $C_9 > 0$ ,  $M > 0$  and  $\eta_1 \in (0, 1)$  such that  $\forall y > M, \forall \gamma \in (0, \eta_1]$ ,*

$$\left| E \left( e^{-\gamma F^+(y)/e^y} \right) - [1 - 2\gamma/(\kappa + 1)] \right| \leq C_9 \max(e^{-\kappa y}, \gamma^{3/2}), \quad (6.82)$$

$$\left| E \left( e^{-\gamma G^+(y/2, y)/e^y} \right) - [1 - \Gamma(1 - \kappa)(2\gamma)^\kappa / \Gamma(1 + \kappa)] \right| \leq C_9 \max(\gamma^\kappa e^{-\kappa y/2}, \gamma). \quad (6.83)$$

Moreover, there exists  $C_{10} > 0$ , such that for all  $y > 0$ ,  $E(F^+(y)/e^y) \leq C_{10}$ .

Recall that  $Q$  is the  $(-\kappa/2)$ -drifted Brownian motion  $W_{-\kappa/2}$  Doob-conditioned to stay positive (see above (1.3)), we have

**Lemma 6.2.** *Let  $0 < \gamma < \alpha < \omega$ . For all  $h$  large enough, we have*

$$P^{\alpha h}(\tau^Q(\gamma h) < \tau^Q(\omega h)) \leq 2e^{-\kappa(\alpha - \gamma)h}, \quad (6.84)$$

$$P(\tau^Q(\omega h) - \tau^Q(\alpha h) \leq 1) \leq C_+ e^{-c_-[(\omega - \alpha)h]^2}, \quad (6.85)$$

$$P(\tau^Q(h) > 8h/\kappa) \leq C_+ e^{-\kappa h/(2\sqrt{2})}, \quad (6.86)$$

$$P(\tau^Q(h) \leq h) \leq C_+ e^{-c_- h}, \quad (6.87)$$

$$P(\tau^Q(\gamma h) \leq 1) \leq C_+ e^{-c_- [\gamma h]^2}, \quad (6.88)$$

$P^{\alpha h}$  denotes the law of  $Q$  starting from  $\alpha h$ .

**Proof:**

The first 3 points comes from Lemma 2.6 in [3]. We give a proof for (6.87), the proof of (6.88) is very similar. By formula (2.3) in [19] taking  $\mu = \kappa/2$  and  $\alpha = 1 - \kappa$ , we have  $E(e^{-\alpha \tau^Q(h)}) \sim_{h \rightarrow +\infty} \text{conste}^{h(\kappa/2 - \sqrt{2\alpha + \kappa^2/4})}$ . Then we apply a Markov inequality for  $P(e^{-\alpha \tau^Q(h)} > e^{-\alpha h})$ .  $\square$

We also need this second Lemma focusing only on some exponential functionals.

**Lemma 6.3.** *Recall the definition of  $F^\pm$ , and  $G^+$  in (1.3). For all  $0 < \zeta \leq 1$  and  $0 < \varepsilon < 1/2$ , for  $t$  large enough,*

$$P \left[ e^{\zeta h t (1 - \varepsilon)} \leq F^+(\zeta h t) \leq e^{\zeta h t (1 + \varepsilon)} \right] \geq 1 - 4e^{-\kappa \varepsilon \zeta h t / 2}, \quad (6.89)$$

$$P \left[ F^+(\zeta h) \geq e^{(1 - \alpha)h} \right] \geq 1 - 3 \exp[-\kappa \alpha h / 2], \quad 0 < \alpha < 1, \quad (6.90)$$

$$P(F^-(h) \geq e^{-\varepsilon h}) \geq 1 - e^{-c_- \varepsilon^2 h^2}, \quad (6.91)$$

$$P(G^+(\alpha h, h) \leq b(h)e^h) \geq 1 - C_+(b(h))^{-\kappa} \quad 0 < \alpha < 1, \quad b > 0. \quad (6.92)$$

**Proof:** By Markov inequality and Lemma 6.1  $P[F^+(\zeta h t) > e^{\zeta h t (1 + \varepsilon)}] \leq c_1 e^{-\varepsilon \zeta h t}$ . For the lower bound, taking  $\gamma = \zeta(1 - \varepsilon/2)$ ,  $\alpha = \zeta(1 - \varepsilon)$  and  $\omega = \zeta$  in (6.84) we get with a probability larger than  $1 - C_+ e^{-\varepsilon \kappa h t}$ ,

$$F^+(\zeta h t) \geq e^{\zeta h t (1 - \varepsilon)} [\tau^Q(\zeta h t) - \tau^Q(\zeta h t (1 - \varepsilon/2))].$$

Finally using (6.85) we get the result. Proof of (6.90) is very similar. For (6.91), first  $F^-(h) \geq e^{-\epsilon h} \tau(\epsilon h_t)$ , and using (6.88), with a probability larger than  $1 - e^{-c - \epsilon^2 h^2}$ ,  $\tau(\epsilon h_t) \geq 1$ . For the last estimation, we notice that in law  $G^+(\alpha h, h) \leq \int_0^{+\infty} e^{-W_\kappa(x)} dx = A_\infty$ , by [16] r.v.  $2/A_\infty$  is a gamma variable of parameter  $(\kappa, 1)$ , and so has a density equal to  $e^{-x} x^{\kappa-1} \mathbb{1}_{\mathbb{R}_+}(x) / \Gamma(\kappa)$  which yields the estimate.  $\square$

The following Lemma is exactly Lemma 4.3 in [3] which proof can be found in that paper.

**Lemma 6.4.** *Let  $B$  a standard two-sided Brownian motion, for every  $0 < \varepsilon < 1$ ,  $0 < \delta < 1$  and  $x > 0$ ,*

$$\mathbb{P}\left(\sup_{u \in [-\delta, \delta]} |\mathcal{L}_B(\tau^B(1), u) - \mathcal{L}_B(\tau^B(1), 0)| > \varepsilon \mathcal{L}_B(\tau^B(1), 0)\right) \leq C_+ \frac{\delta^{1/6}}{\varepsilon^{2/5}}, \quad (6.93)$$

$$\mathbb{P}\left(\sup_{u \in [0, 1]} \mathcal{L}_B(\tau^B(1), u) \geq x\right) \leq 4e^{-x/2}, \quad (6.94)$$

$$\mathbb{P}\left(\sup_{u \leq 0} \mathcal{L}_B(\tau^B(1), u) \geq x\right) \leq 1/x. \quad (6.95)$$

The next Lemma deals with fluctuations for 2-dimension Bessel Processes :

**Lemma 6.5.** *Let  $(Q_2(u), u \geq 0)$  be a Bessel process of dimension 2, starting from 0, and two functions  $a(\cdot)$  and  $k(\cdot)$  from  $(0, +\infty)$  to  $(0, +\infty)$ , having limit  $+\infty$  on  $+\infty$ . We have for large  $t$ ,*

$$P(\forall u \in (0, k(t)], Q_2^2(u) \leq 2e[a(t) + 4 \log \log[ek(t)/u]]u) \geq 1 - C_+ \exp[-a(t)/2].$$

**Proof:** We consider for  $t > 0$  and  $i \in \mathbb{N}$ ,

$$\mathcal{A}_{1,i} := \left\{ \sup_{[k(t)/e^{i+1}, k(t)/e^i]} Q_2^2 \leq 2 \frac{k(t)}{e^i} [a(t) + 4 \log(i+1)] \right\}, \quad \mathcal{A}_2 := \bigcap_{i=0}^{\infty} \mathcal{A}_{1,i}.$$

We recall that there exist two standard independent Brownian motions  $(B_1(u), u \geq 0)$  and  $(B_2(u), u \geq 0)$  such that  $(Q_2^2(u), u \geq 0)$  is equal in law to  $(B_1^2(u) + B_2^2(u), u \geq 0)$ . So for  $i \in \mathbb{N}$ ,

$$\begin{aligned} P(\overline{\mathcal{A}}_{1,i}) &\leq 2P(\sup_{[k(t)/e^{i+1}, k(t)/e^i]} B_1^2 > k(t)e^{-i}[a(t) + 4 \log(i+1)]) \\ &\leq 4P\left(\sup_{[0, k(t)/e^i]} B_1 > \sqrt{k(t)e^{-i}[a(t) + 4 \log(i+1)]}\right) \\ &= 4P\left(|B_1(1)| > \sqrt{a(t) + 4 \log(i+1)}\right) \\ &\leq 8 \exp[-a(t)/2 - 2 \log(i+1)] \end{aligned}$$

for large  $t$  so that  $a(t) \geq 1$ , by scaling, and since  $B_1 \stackrel{\mathcal{L}}{=} -B_1$ ,  $\sup_{[0, 1]} B_1 \stackrel{\mathcal{L}}{=} |B_1(1)|$  and  $P(B_1(1) \geq x) \leq e^{-x^2/2}$  for  $x \geq 1$ . Consequently for large  $t$ ,

$$P(\overline{\mathcal{A}}_2) \leq \sum_{i=0}^{\infty} P(\overline{\mathcal{A}}_{1,i}) \leq 8 \exp[-a(t)/2] \sum_{i=0}^{\infty} \frac{1}{(i+1)^2} = C_+ \exp[-a(t)/2]. \quad (6.96)$$

Now, let  $0 < u \leq k(t)$ . There exists  $i \in \mathbb{N}$  such that  $k(t)/e^{i+1} < u \leq k(t)/e^i$ . We have,  $e^i \leq k(t)/u$ , so  $e^{i+1} \leq ek(t)/u$  and then  $\log(i+1) \leq \log \log[ek(t)/u]$ . Consequently on  $\mathcal{A}_2$ ,

$$Q_2^2(u) \leq 2(k(t)/e^i)[a(t) + 4 \log(i+1)] \leq 2eu[a(t) + 4 \log \log[ek(t)/u]].$$

This, combined with (6.96), proves the lemma.  $\square$

We also need some estimations on the local time of  $B$  at a given coordinate  $y \in \mathbb{R}$  within the inverse of the local time of  $B$  at 0. Recall  $\sigma_B(r, y) = \inf\{s > 0, \mathcal{L}_B(s, y) > r\}$  for  $r >$

0,  $y \in \mathbb{R}$ . By the second Ray-Knight Theorem the processes  $(\mathcal{L}_B(\sigma(r, 0), y), y \in \mathbb{R}^+)$  and  $(\mathcal{L}_B(\sigma(r, 0), -y), y \in \mathbb{R}^+)$  are two independent square of 0-dimensional Bessel process starting at  $r$ . We have the following Lemma which proof can be found in [15] Lemma 2.3 page 2307.

**Lemma 6.6.** *We denote by  $Q_0$  a square of 0-dimensional Bessel process starting at 1. Let  $M > 0$ ,  $u > 0$  and  $v > 0$  then*

$$P\left(\sup_{0 \leq x \leq u} |Q_0(t) - 1| \geq v\right) \leq 4 \frac{\sqrt{(1+u)v}}{u} \exp(-u^2/(8(1+u)v)), \quad (6.97)$$

$$P\left(\sup_{x \geq 0} Q_0(x) \geq M\right) = 1/M. \quad (6.98)$$

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UNIVERSITÉ D'ORLÉANS, LABORATOIRE MAPMO - FÉDÉRATION DENIS POISSON, BÂTIMENT DE MATHÉMATIQUES - RUE DE CHARTRES B.P. 6759 - 45067 ORLÉANS CEDEX 2, FRANCE

*E-mail address:* Pierre.Andreoletti@univ-orleans.fr

UNIVERSITÉ DE VERSAILLES SAINT-QUENTIN-EN-YVELINES, LABORATOIRE DE MATHÉMATIQUES DE VERSAILLES, CNRS UMR 8100, B,T. FERMAT, 45 AVENUE DES ETATS-UNIS, 78035 VERSAILLES CEDEX, FRANCE.

*E-mail address:* devulder@math.uvsq.fr

UNIVERSITÉ D'ORLÉANS, LABORATOIRE MAPMO - FÉDÉRATION DENIS POISSON, BÂTIMENT DE MATHÉMATIQUES - RUE DE CHARTRES B.P. 6759 - 45067 ORLÉANS CEDEX 2, FRANCE

*E-mail address:* Gregoire.Vechambre@ens-rennes.fr