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EXPLICIT FORMULATION FOR THE DIRICHLET PROBLEM FOR PARABOLIC-HYPERBOLIC CONSERVATION LAWS

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Abstract. We revisit the Cauchy-Dirichlet problem for degenerate parabolic scalar conservation laws. We suggest a new notion of strong entropy solution. It gives a straightforward explicit characterization of the boundary values of the solution and of the flux, and leads to a concise and natural uniqueness proof, compared to the one of the fundamental work [J. Carrillo, Arch. Ration. Mech. Anal., 1999]. Moreover, general dissipative boundary conditions can be studied in the same framework. The definition makes sense under the specific weak trace-regularity assumption. Despite the lack of evidence that generic solutions are trace-regular (especially in space dimension larger than one), the strong entropy formulation may be useful for modeling and numerical purposes.

1. **Introduction.** Consider a bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 1$, Lipschitz functions $f : \mathbb{R} \to \mathbb{R}^N$ and $\phi : \mathbb{R} \to \mathbb{R}$ such that in addition, ϕ is non-decreasing i.e. $\phi' \geq 0$ a.e. on $\mathbb{R}$. Given $u_0 \in L^\infty(\Omega)$, $T > 0$ and $u^D \in L^\infty(\Sigma)$, $\Sigma = (0,T) \times \partial\Omega$, we are interested in the appropriate definition of solution and well-posedness for the following Cauchy-Dirichlet problem for a degenerate parabolic conservation law in $(0,T) \times \Omega$:

$$u_t + \operatorname{div}(f(u) - \nabla\phi(u)) = 0, \quad u|_{t=0} = u_0, \quad u|_\Sigma = u^D. \tag{1}$$

By hyperbolic degeneracy we will understand the situation where $\phi' \equiv 0$ on certain nontrivial intervals; they are called hyperbolicity zones. Theory of such parabolic-hyperbolic problems was addressed in several contributions, first for the case $\phi \equiv 0$ (hyperbolic conservation law) and then for the general case. Without being exhaustive, let us cite the works [9] followed by [38] and [7, 8]; [28] followed by [29, 30, 37, 40]; [16] followed by [35] and [2]; and [20], related to the line of research initiated in [39].

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1.1. **Boundary-value degenerate parabolic-hyperbolic problems.** In the context of the Cauchy problem in the whole space, the definition of solutions is based on Kruzhkov entropy inequalities and doubling of variables method [25] adapted to the degenerate parabolic setting by Carrillo [16]. These ideas and techniques provide the foundations of the theory, however, their adaptation to boundary-value problems turned out to be particularly delicate. For instance, the reader can consult [13, 5] for definitions of solution and the associated results for the zero-flux boundary conditions $\big(f(u)-\nabla\phi(u)\big).\nu|_\Sigma = 0$, under the additional assumption that

$$u_0 \text{ takes values in an interval } [0, u_{max}] \text{ such that } f(0) = 0 = f(u_{max}). \tag{2}$$

The case of Robin boundary conditions is considered in [22], along the same guidelines. Under assumption (2), $[0, u_{max}]$ becomes an invariant domain for the parabolic-hyperbolic equation in (1) with zero-flux boundary condition. In this case uniqueness remains unjustified for the multi-dimensional degenerate parabolic problem, and even the one-dimensional problem requires a somewhat technical and rather tricky uniqueness proof (see our preceding work [5]). The proof is based upon a weak-strong comparison principle going back to [4]. We required existence of a dense set of solutions that are strongly trace-regular in the sense that the normal flux $\mathcal{F}[u] = \big(f(u)-\nabla\phi(u)\big).\nu$ admits a strong L^1 trace. Further, when (2) fails the question of what is the correct definition of solutions to the zero-flux problem remains open (cf. [7, 8] for the purely hyperbolic case); it is demonstrated numerically in [22, 6] that the formulation of [13, 5] is not appropriate in absence of (2).

Only the purely hyperbolic case is well understood, for a wide class of boundary conditions including zero-flux, Robin, Dirichlet and obstacle conditions. In [7, 8], a convenient formalism of maximal monotone graphs linking boundary values of the solution and of the normal flux component was exploited in order to express general boundary conditions of dissipative kind.

In this paper, we attempt to extend this formalism to the degenerate parabolic-hyperbolic problem. We focus only on the most classical Dirichlet conditions. For this case, the analysis of [9, 28, 16] and subsequent works implies that:

- the boundary conditions have to be relaxed within the hyperbolicity zones;
- they can be taken into account in an indirect way by means of well-chosen up-to-the-boundary entropy inequalities.

The work [35] attempted to give an explicit meaning to the homogeneous Dirichlet boundary conditions for the degenerate parabolic equation in (1) in the situation where solution regularity permits to express these conditions pointwise. This is also our aim, but we require less regularity from solutions and we provide a formulation of boundary condition in the language used in [7, 8]:

$$(u, \mathcal{F}[u].\nu) \in \widetilde{\mathcal{B}} \text{ pointwise on the boundary } \Sigma, \tag{3}$$

where we use the short-cut notation for the convection-diffusion flux:

$$\mathcal{F}[u] := f(u) - \nabla\phi(u), \tag{4}$$

and ν is the outer normal to $\partial\Omega$. The subset $\widetilde{\mathcal{B}} \subset \mathbb{R}^2$ is the maximal monotone graph that makes explicit the meaning to be given to the formal Dirichlet condition; it is described in Section 2. When u^D lies in hyperbolicity zones, the graph $\widetilde{\mathcal{B}}$ expresses both obstacle-like conditions enforced by the relation $\phi(u)|_\Sigma = \phi(u^D)$ (relation that can be understood literally in the sense of traces of Sobolev functions) and

the Bardos-LeRoux-Nédélec [9] relaxation of the formal condition $u|_\Sigma = u^D$, see Figure 1. Details and motivations for the definition of $\widetilde{\mathcal{B}}$ are given in Section 2.

Note that the boundary regularity of the flux needed in our study is less restrictive than the strong trace-regularity exploited in the zero-flux setting [5]. Indeed, it boils down to existence of a normal trace of the flux in the sense of the *weak* L^1 convergence (see Lemma 4.5), while in [5] *strong* L^1 convergence was needed. However, contrarily to [5] we are not able to compare a general solution to a trace-regular one: the weak trace-regularity of both solutions is required in the uniqueness proof we develop here.

1.2. **Aims of the paper.** In spite of the fact that we cannot justify in general the regularity of solutions needed to give sense to (3), our new formulation sheds light on the typical boundary behavior of solutions. Further, (3) is particularly convenient for understanding the arguments leading to uniqueness of solutions to (1). As a matter of fact, our uniqueness proof (under the suitable boundary regularity assumption) combines the relatively simple part of arguments of [16] leading to local contraction property (the Kato inequality, see (33) below) with a straightforward treatment of the boundary. In this way, the subtle and technically involved arguments of Carrillo ([16]) or of Otto ([28] and [29, 30, 37]) based upon up-to-the-boundary doubling of variables are avoided. Let us stress that the most general results on the Dirichlet problem (1) remain those of [28, 29, 30, 37].

The goals of this paper are the following:

- make explicit the suitable graph $\widetilde{\mathcal{B}}$ for (may be, somewhat heuristic) description of boundary behavior of solutions to (1) within the formalism (3);
- make apparent the weakest regularity of solutions required in order to give rigorous meaning to the pointwise boundary formulation (3) (Definition (4.3));
- put forward the direct and rather elementary arguments which permit to derive uniqueness of solutions satisfying local entropy inequalities and verifying (3);
- prepare grounds for forthcoming work on general initial-boundary value problems which naturally enter the formalism (3) (see [7, 8] for the hyperbolic case $\phi \equiv 0$).

Observe that formulations of kind (3) can be instrumental for scientific computing and modeling purposes. From the numerical perspective, implementing the effective boundary condition (3) may reduce the boundary layers that necessarily appear if one implements the formal boundary condition. From the modeling perspective, the explicit knowledge of the effective boundary condition may help to validate or to question the choice of the formal boundary condition proposed within a model.

1.3. **Assumptions on the domain, data and nonlinearities.** In order to leave aside technicalities that are not essential, we will concentrate on the one-dimensional case $N = 1$ with only one boundary point: $\Omega = (-\infty, 0)$, $\partial\Omega = \{0\}$ Notice that then, $\nu = 1$ is the outer normal to Σ thus $\mathcal{F}[u].\nu$ coincides with $\mathcal{F}[u]$. Since in this case the domain is unbounded, we will limit our attention to data

$$u_0 \in L^{1\cap\infty}((-\infty, 0)) := L^1((-\infty, 0)) \cap L^\infty((-\infty, 0)).$$

Remark 1. General L^∞ data can be dealt with by localizing the H^1 in space regularity requirement on $\phi(u)$ imposed in the definitions we exploit. General L^1 data can be dealt with in the context of renormalized solutions, cf. [12, 34], but this is beyond the goals of the present work. Abstract results for L^1 data, equivalent

to the renormalized solutions' setting, are easy to get in the nonlinear semigroup framework (see in particular [24] and Section 4.3).

Then the problem reads

$$(P)\begin{cases} u_t + (f(u) - \phi(u)_x)_x &= 0 \ \text{ in } \ Q = (0,T)\times(-\infty,0), \\ u(0,x) &= u_0 \ \text{ in } \ (-\infty,0), \\ u(t,x) &= u^D(x) \ \text{ on } \ \Sigma = (0,T)\times\partial\Omega. \end{cases}$$

Mainly because we link our work to the setting of [16], we limit our attention to constant in time boundary condition u^D; but we stress that the formulation (3) in the context of trace-regular solutions introduced here does not require this restriction.

Further, we will assume that ϕ degenerates only on the interval $(-\infty, u_c]$ for some $u_c \in \mathbb{R}$, namely

$$\phi|_{(-\infty,u_c]} \equiv 0 \text{ and } \phi|_{[u_c,+\infty)} \text{ is strictly increasing.}$$

This is the setting of typical degenerate parabolic-hyperbolic models of sedimentation, see, e.g., [21, 14]. Besides, we ask for genuine nonlinearity of the convective flux f in the hyperbolicity zones:

$$f \text{ is not affine on any nontrivial subinterval of } (-\infty, u_c]. \tag{5}$$

This assumption is not merely technical: it guarantees strong precompactness properties of (approximate) entropy solutions and the existence of strong initial and boundary traces of u, see Panov [31, 33, 32] and also [26] and references therein.

1.4. **Outline.** The paper is organized as follows. In Section 2 we describe the graph $\widetilde{\mathcal{B}}$ appropriate for taking into account the Dirichlet condition within the formulation (3). In Section 3 we accurately motivate our approach in the setting of the stationary problem associated with (P), namely, $u + (f(u) - \phi(u)_x)_x = g$ in $\Omega = (-\infty, 0)$, $u(0^-) = u^D$. Here, equivalence of the formulation (3) with the Carrillo formulation [16] is justified. Next, in Section 4 we introduce the notion of weakly trace-regular solutions and extend the whole theory to (P) by showing equivalence between weakly trace-regular Carrillo solutions and strong entropy solutions in the sense (3). We also recast the one-dimensional problem (P) into the abstract framework, exploiting the results of Section 3 and nonlinear semigroup methods ([11]). Conclusions and directions of extension of results based on the ideas of this paper are presented in Section 5.

2. **Adequate pointwise expression of the boundary condition for (P).** In the formalism used in [7, 8], the Dirichlet boundary-value condition $u = u^D$ on $(0,T)\times\{0\}$ in problem (P) is prescribed formally in terms of the maximal monotone graph $\beta := \{u^D\}\times\mathbb{R}$. This means that at the boundary, the couple $(u, \mathcal{F}[u].\nu)$ is supposed to belong to β, where ν the unit normal to $\partial\Omega$ outward to Ω (here $\Omega = (-\infty, 0)$, $\partial\Omega = \{0\}$ and $\nu = 1$). This is indeed the case at the level of approximate solutions obtained by the vanishing viscosity approximation of (P) or by a finite volume scheme (see [9, 40, 30, 3]; cf. [22, 23] for related analysis). However at the limit, the graph β in the above statement should be replaced by an appropriately projected graph that we denote by $\widetilde{\mathcal{B}}$ (cf. [7, 8] for the purely hyperbolic case and general graphs β). To illustrate this idea, let us first recall the known results for the purely hyperbolic case ($\phi(u) \equiv 0$) and for the non-degenerate parabolic case ($\phi' > 0$). The formulation suitable for (P) is given next.

2.1. **Case of the purely hyperbolic problem.** Consider the problem

$$(P^\epsilon)\begin{cases} u^\epsilon_t + (\mathcal{F}^\epsilon[u^\epsilon])_x = 0,\ \mathcal{F}^\epsilon[u^\epsilon] := f(u^\epsilon) + \epsilon u^\epsilon_x & \text{in } (0,T)\times(-\infty,0),\\ u^\epsilon(0,x) = u_0 & \text{in } (-\infty,0),\\ \Big(u^\epsilon, \mathcal{F}^\epsilon[u^\epsilon]\Big) \in \beta = \{u^D\}\times\mathbb{R} & \text{on } \Sigma = (0,T)\times\{0\}. \end{cases}$$

which is a natural vanishing viscosity approximate of the problem (P). In this case, in accordance with the general guidelines of the theory of hyperbolic conservation laws we expect that (P) is a formal limit of (P^ϵ). If we have enough compactness properties on sequence (u^ϵ), we can pass to the limit into the local weak and entropy formulations of (P), however we cannot hope that when passing to the limit $\epsilon \to 0$, the boundary condition be satisfied as the formal limit of "$\Big(u^\epsilon, \mathcal{F}^\epsilon[u^\epsilon]\Big) \in \beta$". Indeed, $L^1((0,T)\times(-\infty,0))$ compactness of (u^ϵ) is the strongest property we can prove, and it gives no information on the convergence of $\mathcal{F}^\epsilon[u^\epsilon]$ nor even of u^ϵ on the boundary. In general, the term ϵu_x becomes singular as $\epsilon \to 0$. This explains that boundary layers can appear in u^ϵ as $\epsilon \to 0$. As a matter of fact, in general the boundary condition $u = u^D$ is not the correct limit obtained from u^ϵ as ϵ tends to zero. In [9], Bardos, Leroux and Nédélec state that the Dirichlet boundary condition should be seen as a formal condition and that it must be interpreted by stating that the trace $\gamma u(t) = u(t,0^-)$ of u at the point $x = 0$ belongs to the subset $\mathcal{I} \subset \mathbb{R}$ defined in terms of u^D as

$$\mathcal{I} = \big\{ v \in \mathbb{R} \text{ with } sign(v - u^D)(f(v) - f(k)) \geq 0\ \forall k \in [\min(u^D, v), \max(u^D, v)] \big\}.$$

To sum up, the effective boundary condition expressing the formal Dirichlet condition reads:

$$\text{for a.e. } t \in (0,T) \quad \gamma u(t) \in \mathcal{I}. \tag{6}$$

This is the celebrated BLN (Bardos-LeRoux-Nédélec) interpretation of the Dirichlet condition. It is recognized as the correct one in the classical theory of hyperbolic conservation laws, and well-posedness in the BLN framework is well known, at least for the homogeneous boundary condition (see [18] for the general case). Let us stress that this effective boundary condition can also be expressed by (3) with the following definition:

$$\widetilde{\mathcal{B}} = \left\{ (k,\mathcal{K}) \in \mathbb{R}^2 \text{ s.t. if } k \in (-\infty, u^D) \text{ then } \mathcal{K} = \min_{[k,u^D]} f, \text{ else } \mathcal{K} = \max_{[u^D,k]} f \right\}. \tag{7}$$

Note in particular that $\widetilde{\mathcal{B}} \cap \{(k,\mathcal{K}) \,|\, \mathcal{K} = f(k)\}$ is a maximal monotone subgraph of the graph of the flux f. Let us briefly mention that existence of limits in (3) is straightforward for BV solutions ([9]) but it can also be justified, under a non-degeneracy assumption of the kind (5), for merely L^∞ solutions in the purely hyperbolic case we considered. We refer to [19] for this graphic interpretation of the BLN condition and to [7, 8] for rigorous statements and technical details.

2.2. **Case of the non-degenerate parabolic problem.** In the non-degenerate parabolic situation, the passage to the limit ($\epsilon \to 0$) gives the exact boundary condition $u(0) = u^D$ (see, e.g., [22, Lemma 3.4]). In this case, $\widetilde{\mathcal{B}} = \{u^D\}\times\mathbb{R} = \beta$.

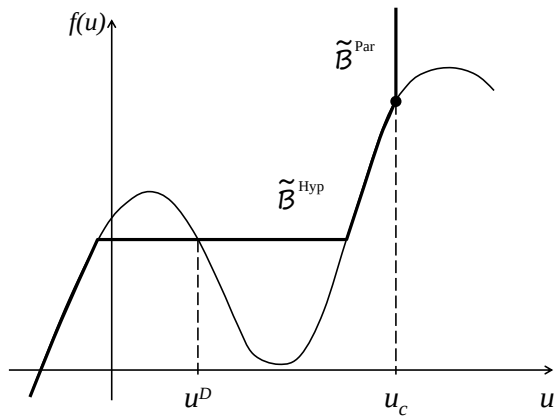

FIGURE 1. Construction of graph $\widetilde{\mathcal{B}} = \widetilde{\mathcal{B}}^{Hyp} \cup \widetilde{\mathcal{B}}^{Par}$ for formulation (3). Here the value u^D falls within the hyperbolic zone $(-\infty, u_c]$. For a general domain, f should be replaced by the normal flux $f.\nu$.

2.3. **Case of the general degenerate parabolic problem.** Our aim is to adapt formula (7) to degenerate parabolic equation; not surprisingly, the maximal monotone graph $\widetilde{\mathcal{B}}$ which expresses the effective boundary condition combines features of the two preceding cases. Remark that if $u^D > u_c$, the passage to the limit ($\epsilon \to 0$) still gives the exact boundary condition $u(0) = u^D$. On the other side, the case $u^D \le u_c$ is delicate. In Section 3 we will argue that the right choice of the maximal monotone graph $\widetilde{\mathcal{B}}$ constructed from the formal graph $\beta = \{u^D\} \times \mathbb{R}$ is as follows (see Figure 1):

$$\widetilde{\mathcal{B}} = \widetilde{\mathcal{B}}^{Par} \cup \widetilde{\mathcal{B}}^{Hyp}, \tag{8}$$

$$\widetilde{\mathcal{B}}^{Par} = \begin{cases} \{u^D\} \times \mathbb{R} & \text{if } u^D > u_c; \\ \{u_c\} \times [\max_{[u^D,u_c]} f, +\infty) & \text{if } u^D \le u_c; \end{cases}$$

$$\widetilde{\mathcal{B}}^{Hyp} = \left\{ (k, \mathcal{K}) \in \mathbb{R}^2 \,|\, \mathcal{K} = \begin{cases} \min_{[k,u^D]} f, & k \le u^D \\ \max_{[u^D,k]} f, & u^D \le k \le u_c \end{cases} \right\} \text{ if } u^D \le u_c, \text{ else } \widetilde{\mathcal{B}}^{Hyp} = \emptyset.$$

The part $\widetilde{\mathcal{B}}^{Hyp}$ of $\widetilde{\mathcal{B}}$ is the Bardos-LeRoux-Nédélec projection of β on the graph of f (see (7) in Section 2.1) restricted to $(-\infty, u_c]$. The part $\widetilde{\mathcal{B}}^{Par}$ expresses the obstacle condition induced by the fact that $\phi(u)(t, 0^-) = \phi(u^D)$.

Lemma 2.1. *The effective graph $\widetilde{\mathcal{B}}$ defined by (8) is maximal monotone.*

Proof. If $u^D > u_c$, $\widetilde{\mathcal{B}} = \widetilde{\mathcal{B}}^{Par} = \beta$ and the claim is evident. So we assume $u^D \le u_c$. The monotonicity of $\widetilde{\mathcal{B}}$ is obvious; let us justify its maximality. Let $(l, \mathcal{L})$ such that $\widetilde{\mathcal{B}} \cup (l, \mathcal{L})$ is monotone. Consider any point $k \in (-\infty, u_c)$, then there exists a unique $\mathcal{K}(k)$ such that $(k, \mathcal{K}(k)) \in \widetilde{\mathcal{B}}^{Hyp}$ and $\mathcal{K}(k)$ depends continuously on $k \in (-\infty, u_c)$ because f is continuous. Now, we have $\mathcal{L} = \mathcal{K}(l)$ if $l < u_c$. Indeed, if $k > l$ then $\mathcal{L} \le \mathcal{K}(k) \underset{k \downarrow l}{\longrightarrow} \mathcal{K}(l)$. If $k < l$ then $\mathcal{L} \ge \mathcal{K}(k) \underset{k \uparrow l}{\longrightarrow} \mathcal{K}(l)$. Finally, if $l = u_c$, then using only $k \uparrow l$ we find $\mathcal{L} \ge \lim_{k \uparrow u_c} \mathcal{K}(k) = \max_{[u^D,u_c]} f$. In this case $(l, \mathcal{L}) \in \widetilde{\mathcal{B}}^{Par} \subset \widetilde{\mathcal{B}}$. In all cases, $(l, \mathcal{L}) \in \widetilde{\mathcal{B}}$, which proves the lemma. $\square$

3. **Entropy solution of stationary problem.** In this section, we consider the stationary Dirichlet problem associated to the evolution problem (P):
for $g \in L^{1\cap\infty}((-\infty,0))$, find

$$(S)\left\{\begin{array}{rl} u+(f(u)-\phi(u)_x)_x & = g \text{ in } (-\infty,0), \\ u & = u^D \text{ on } \{x=0\}. \end{array}\right.$$

3.1. **Definitions of entropy solution of (S).** We will provide two definitions of entropy solution for (S); the subsequent analysis will ensure their equivalence. First, we recall the local definition not taking the boundary into account.

Definition 3.1. Let $g \in L^{1\cap\infty}((-\infty,0))$. A bounded measurable function u on $(-\infty,0)$ is called a local entropy solution of equation $u+(f(u)-\phi(u)_x)_x = g$ if $\phi(u) \in H^1(-\infty,0)$ and the following local entropy inequality is satisfied:

for all $k \in \mathbb{R}$, $\xi \in \mathcal{C}_0^\infty((-\infty,0))$, $\xi \geq 0$,

$$\int_{-\infty}^0 \Big\{ sign(u-k)(g-u)\xi + sign(u-k)\Big(\mathcal{F}[u]-f(k)\Big).\xi_x\Big\}dx \geq 0. \quad (9)$$

The following definition is (up to a translation) the definition of Carrillo [16] which is one of the established ways to take into account the Dirichlet boundary condition "$u = u^D$ on $\partial\Omega$".

Definition 3.2. A bounded measurable function u on $(-\infty,0)$ is called an entropy solution of the Dirichlet problem (S) in the sense of Carrillo if it is a local entropy solution of equation $u+(f(u)-\phi(u)_x)_x = g$, there holds

$$\phi(u)(0) = \phi(u^D) \quad (10)$$

and moreover, for all $\xi \in \mathcal{C}_0^\infty((-\infty,0])$, $\xi \geq 0$, the following up-to-the-boundary entropy inequalities are satisfied:

$$\forall k \geq u^D \quad \int_{-\infty}^0 \Big\{ sign^+(u-k)(g-u)\xi + sign^+(u-k)\Big(\mathcal{F}[u]-f(k)\Big)\xi_x\Big\}dx \geq 0, \quad (11)$$

$$\forall k \leq u^D \quad \int_{-\infty}^0 \Big\{ sign^-(u-k)(g-u)\xi + sign^-(u-k)\Big(\mathcal{F}[u]-f(k)\Big)\xi_x\Big\}dx \geq 0. \quad (12)$$

This is indeed the definition of [16] under the change of u into $u-u^D$ (the value u^D being a constant). Remark that $\phi(u) \in \mathcal{C}((-\infty,0])$ for every local entropy solution, giving sense to the requirement (10). The following result is essentially contained in [16], see also [27] for the analysis in an unbounded domain:

For all $g \in L^{1\cap\infty}((-\infty,0))$, for all $u^D \in \mathbb{R}$ there exists a unique entropy solution in the sense of Carrillo to the Dirichlet problem (S).

As explained in the introduction, our goal is to give an obvious meaning to the boundary condition contained in the above entropy formulation of [16], and to provide a simpler proof of uniqueness of the solution associated to a given datum g. To this end we will reformulate the boundary conditions and give them a pointwise sense, thanks to our assumptions (one space dimension, stationary setting) which guarantee existence of strong traces.

We start with the following observation.

Proposition 1. *Suppose that u is a local entropy solution of the equation in (S), moreover, (10) holds. Then u is an entropy solution in the sense of Carrillo of the Dirichlet problem (S) if and only if the two following inequalities hold:*

$$\forall k \geq u^D \quad \overline{\lim_{h\to 0}} \frac{1}{h}\int_{-h}^{0} sign^+(u-k)(\mathcal{F}[u]-f(k))dx \geq 0 \ , \tag{13}$$

$$\forall k \leq u^D \quad \overline{\lim_{h\to 0}} \frac{1}{h}\int_{-h}^{0} sign^-(u-k)(\mathcal{F}[u]-f(k))dx \geq 0 \ . \tag{14}$$

Proof. Taking $\xi_h = \max\{0, 1+\frac{x}{h}\}$ as a test function in (11), (12) and passing to the limit in h goes to zero one gets (13), (14) in their strengthened version (with $\overline{\lim}$ replaced by $\underline{\lim}$). Reciprocally, combining (13), (14) with local entropy inequalities of Definition 3.1 written for the test function $(1-\xi_h(x))\xi(x)$, one finds (11), (12). □

Now, we give a new definition which uses the maximal monotone graph $\widetilde{\mathcal{B}}$ defined by (8) to link the traces of the solution and of the flux.

Definition 3.3. A bounded measurable function u is called *strong entropy solution* of the Dirichlet problem (S) if the following conditions are satisfied:

1. The function u is a local entropy solution and (10) holds.
2. There exists $u(0) := \lim_{x\to 0^-} u(x)$.
3. There exists $\mathcal{F}[u](0) = \lim_{x\to 0^-} \mathcal{F}[u](x)$.
4. The couple $(u(0), \mathcal{F}[u](0))$ belongs to $\widetilde{\mathcal{B}}$.

Lemma 3.4. *Properties 2. and 3. of Definition 3.3 are not restrictive. Indeed,*

(i) The item 3. above is automatically fulfilled for every solution in $\mathcal{D}'((-\infty,0))$ of the equation $u + (f(u) - \phi(u)_x)_x = g$.

(ii) Assume that the couple (f,ϕ) is non degenerate in the sense of (5). Then the item 2. above is automatically fulfilled for every local entropy solution of the equation $u + (f(u) - \phi(u)_x)_x = g$.

Proof. The claim (i) is immediate. Indeed, the equation contained in (S) gives $(\mathcal{F}[u])_x = g - u \in L^\infty((-\infty,0))$ so that the total flux $\mathcal{F}[u]$ is absolutely continuous on $(-\infty,0)$, thus admitting a limit as $x \to 0$. The claim (ii) can be deduced from [26] or from [1]. □

The following observation is a first step towards establishing that the two definitions are equivalent.

Proposition 2. *Assume that the couple (f,ϕ) is non degenerate in the sense of (5). Assume that $u \in L^\infty((-\infty,0))$ is an entropy solution of the Dirichlet problem (S) in the sense of Carrillo. Then it is also a strong entropy solution of the same problem.*

Proof. The proof is based on Lemma 3.4. We know that there exist $U := u(0)$ and $F := \mathcal{F}[u](0)$. This permits to compute the limits in (13),(14) for all $k \neq U$. Then Definition 3.2 (via Proposition 1) implies the properties

$$\begin{cases} \text{if } k > u^D \text{ then } sign^+(U-k)(F-f(k)) \geq 0 \\ \text{if } k < u^D \text{ then } sign^-(U-k)(F-f(k)) \geq 0 \\ \qquad\qquad \text{and in all cases, } \phi(U) = \phi(u^D), \end{cases} \tag{15}$$

the case $k = U$ being trivial. Observe that reciprocally, if (15) holds and $U \neq k$ then we readily get (13) and (14); we do not pursue the equivalence analysis in this proof, because the case $U = k$ requires delicate technical arguments.

Since Definition 3.3 simply reads

$$(U, F) \in \widetilde{\mathcal{B}}, \tag{16}$$

it is enough for the proof of the proposition to establish that (15) implies (16); as a matter of fact, we prove that (15) and (16) are equivalent,

First, observe that

$$\text{whenever } U < u_c, \text{ one has } F = f(U). \tag{17}$$

Indeed, if $U < u_c$ then $u < u_c$ in a neighbourhood of $x = 0$, by the definition of U; so that $\phi(u) \equiv 0$ and thus $\nabla\phi(u) \equiv 0$ in this neighbourhood. Therefore

$$F = \lim_{x\to 0^-} (f(u) - \phi(u)_x) = \lim_{x\to 0^-} f(u) + 0 = f(\lim_{x\to 0^-} u) = f(U).$$

Now, the equivalence between (15) and (16) is established by a direct case study.

1. Case $u^D > u_c$. In this case, (16) means that $U = u^D$ and $F \in \mathbb{R}$ is arbitrary. Regarding (16), we also find $U \in \phi^{-1}(\phi(u^D)) = \{u^D\}$, thus $U = u^D$ and inequalities in (15) carry no restriction on F: indeed, there exist no value k between U and u^D, thus both inequalities in (15) read "$0 \leq 0$".

2a. Case $u^D \leq u_c$ and $U \geq u^D$. In this case, first, (15) yields $U \in \phi^{-1}(0) = (-\infty, u_c]$ and also (16) yields $U \in \mathrm{Dom}(\widetilde{\mathcal{B}}) = (-\infty, u_c]$. Second, the inequalities in (15) carry the information that $F \geq f(k)$ for $k \in [u^D, U)$, while in all other cases the inequalities reduce to "$0 \geq 0$". This is equivalent to the fact that $F \geq \sup_{k\in[u^D,U)} f(k)$.

Now we have two possible situations. Either $U < u_c$, in which case we use (17) and get $F = f(U)$. Along with the inequality $F \geq \sup_{k\in[u^D,U)} f(k)$, this is equivalent to $F = max_{k\in[u^D,U]} f(k) = \widetilde{\mathcal{B}}(U)$ (in this case, the right-hand side is single-valued). Or $U = u_c$, in which case we have proved that $F \in [max_{k\in[u^D,U]} f(k), +\infty) = \widetilde{\mathcal{B}}(u_c) = \widetilde{\mathcal{B}}(U)$.

2b. Case $u^D \leq u_c$ and $U \leq u^D$. This case is completely analogous to the previous one, with one simplification due to the fact that the case $U = u_c$ becomes impossible. We find that both (15) and (16) boil down to the restriction $U \in (-\infty, u_c] = \mathrm{Dom}(\widetilde{\mathcal{B}})$. Further, (15) means that $F \leq \inf_{k\in(U,u^D]} f(k)$. Since we also have $U < u_c$, we get $F = f(k)$ and finally, (15) means $F = \min_{k\in[U,u^D]} f(k) = \widetilde{\mathcal{B}}(U)$, which is equivalent to (16).

To sum up, in all possible cases (15) and (16) carry the same restrictions on the couple (U, F). According to the preceding analysis, Definition 3.2 therefore implies Definition 3.3. □

3.2. **Uniqueness of a strong entropy solution for** (S)**.** Due to the boundary regularity results of Lemma 3.4 and the formulation of the Dirichlet boundary condition in terms of the monotone graph $\widetilde{\mathcal{B}}$, our uniqueness proof is straightforward.

Theorem 3.5. *Let u be a strong entropy solution of the Dirichlet problem (S) (i.e., a solution in the sense of Definition 3.3) with source term $g \in L^{1\cap\infty}((-\infty, 0))$; let $\hat{u}$ be a strong entropy solution of the (S) with the same Dirichlet condition and with*

another source term $\hat g \in L^{1\cap\infty}((-\infty,0))$. *Then*

$$\int_{-\infty}^{0} |u-\hat u| dx \le \Big[u-\hat u, (g-\hat g)\Big]_{L^1((-\infty,0))}. \tag{18}$$

In particular, the strong entropy solution of the Dirichlet problem (S) *is unique.*

Here and in the sequel, $\Big[f,g\Big]_{L^1(-\infty,0)} := \int_{-\infty}^{0}\Big(sign(f)\,g + \mathbf{1}_{[f=0]}|g|\Big)$ is the bracket in $L^1((-\infty,0))$ (see [10, 11]). The proof follows by passage to the limit, as $\xi \to \mathbf{1}_{(-\infty,0)}$ in the local Kato inequality (19) that we state first.

Proposition 3. *Let* $u, \hat u$ *be local entropy solutions of* (S) *with source terms* $g, \hat g$, *respectively. Then for all* $\xi \in \mathcal{C}_0^\infty((-\infty,0)))$, $\xi \ge 0$

$$\int_{-\infty}^{0} |u-\hat u|\xi dx - \int_{-\infty}^{0} sign(u-\hat u)\Big(\mathcal{F}[u]-\mathcal{F}[\hat u]\Big).\xi_x dx \le \Big[u-\hat u, (g-\hat g)\xi\Big]_{L^1((-\infty,0))}. \tag{19}$$

The proof of this proposition is the simpler part of the arguments of [16]. One utilizes the Carrillo entropy dissipative information within the Kruzhkov-like doubling of variables technique in the interior of the domain.

Now, we are ready to address the uniqueness proof.

Proof of Theorem 3.5. Let us concentrate on the case $g = \hat g$: this amounts to neglect the right-hand sides of (18) and (19) which can be easily dealt with. Suppose that u and $\hat u$ are two strong entropy solutions of the Dirichlet problem (S). Take $\xi_h = \min\{1, -\frac{x}{h}\}$ in the local Kato inequality (19). Denote $k = u(0), \hat k = \hat u(0)$ and $\mathcal{K} = \mathcal{F}[u](0), \widehat{\mathcal{K}} = \mathcal{F}[\hat u](0)$. Using the continuity at $x = 0^-$ of $u, \hat u$ and of $\mathcal{F}[u], \mathcal{F}[\hat u]$, as $h \to 0$ we find

$$\int_{-\infty}^{0} |u-\hat u| dx \le -sign(k-\hat k)\Big(\mathcal{K}-\widehat{\mathcal{K}}\Big), \tag{20}$$

provided $k \ne \hat k$ holds[1]. Recall that both $(k,\mathcal{K})$ and $(\hat k, \widehat{\mathcal{K}})$ belong to the same maximal monotone graph $\widetilde{\mathcal{B}}$. Therefore, if $k \ne \hat k$, by (16), the second member of (20) is non-positive then we have $u = \hat u$ a.e. on $(-\infty,0)$.

It remains to study the case $k = \hat k$. We go back to the definition (4) of $\mathcal{F}[u], \mathcal{F}[\hat u]$ and separate the convection and diffusion fluxes

$$sign(u-\hat u)\Big(\mathcal{F}[u]-\mathcal{F}[\hat u]\Big) = q(u,\hat u) - |\phi(u)-\phi(\hat u)|_x \tag{21}$$

where $q(u,\hat u) = sign(u-\hat u)(f(u)-f(\hat u))$. The diffusion term is obtained as follows:

$$sign(u-\hat u)(\phi(u)_x - \phi(\hat u)_x) = sign(\phi(u)-\phi(\hat u))(\phi(u)_x - \phi(\hat u)_x), \tag{22}$$

because $(\phi(u)-\phi(\hat u))_x = 0$ a.e. on $[\phi(u) = \phi(\hat u)]$ and ϕ is non-decreasing. Then, using the chain rule for Sobolev functions,

$$sign(\phi(u)-\phi(\hat u))(\phi(u)_x - \phi(\hat u)_x) = |\phi(u)-\phi(\hat u)|_x.$$

[1] In the case $k = \hat k$ the right-hand side of (20) can be replaced by $|\mathcal{K}-\widehat{\mathcal{K}}|\mathbf{1}_{k=\hat k}$: this is a standard estimate coming from the technique of the L^1 bracket. However this estimate is too rough for proving uniqueness: a finer argument for the case $k = \hat k$ will be required.

In (19), replacing the second term by (21) we have

$$-\int_{-h}^{0} sign(u-\hat{u})\Big(\mathcal{F}[u]-\mathcal{F}[\hat{u}]\Big).(\xi_h)_x dx = \frac{1}{h}\Big(\int_{-h}^{0} q(u,\hat{u}) - \int_{-h}^{0} |\phi(u)-\phi(\hat{u})|_x\Big)$$
$$= \frac{1}{h}\Big(-|\phi(u)-\phi(\hat{u})|(0) + |\phi(u)-\phi(\hat{u})|(-h)\Big) + \frac{1}{h}\int_{-h}^{0} q(u,\hat{u}).$$

Since $u(0) = k = \hat{k} = \hat{u}(0)$, letting h to zero, we have that $q(u,\hat{u})$ tend to $q(u(0),\hat{u}(0)) = q(k,k) = 0$; in addition, $|\phi(u)-\phi(\hat{u})|(0) = 0$. Then

$$-\int_{-h}^{0} sign(u-\hat{u})\Big(\mathcal{F}[u]-\mathcal{F}[\hat{u}]\Big).(\xi_h)_x dx = \frac{1}{h}|\phi(u)-\phi(\hat{u})|(-h) \geq 0.$$

Whence from (19) with $\xi_h = \min\{1, -\frac{x}{h}\}$ and $h \to 0$ we get $u = \hat{u}$ a.e. on $(-\infty, 0)$ also in the case $k = \hat{k}$. This ends the proof. □

3.3. **Equivalence of Carrillo entropy solutions and strong entropy solutions. Well-posedness of the Dirichlet problem** (S)**.** In view of the facts we established, the following existence result permits to conclude the study of (S).

Proposition 4. *Assume that the couple (f,ϕ) is non degenerate in the sense of (5). Assume $g \in L^{1\cap\infty}((-\infty,0))$. There exists an entropy solution in the sense of Carrillo of the Dirichlet problem (S).*

The proof, which is by standard passage to the limit from the vanishing viscosity approximated problem, is contained in [16] and in many subsequent works, see, e.g., [3, 5]. Note that the assumption of bounded and integrable source term g guarantees uniform L^∞ and L^1 estimates on u_ϵ and also a uniform H^1 estimate for $\phi_\epsilon(u_\epsilon) = \phi(u_\epsilon) + \epsilon u_\epsilon$, where u_ϵ is the weak solution of the viscosity regularized problem

$$(S_\epsilon)\begin{cases} u_\epsilon + (f(u_\epsilon) - \phi_\epsilon(u_\epsilon)_x)_x &= g \text{ in } (-\infty,0), \\ u_\epsilon &= u^D \text{ on } \{x=0\}. \end{cases}$$

Now, the main result of this section is an immediate corollary of:

- the existence of a solution in the sense of Carrillo (Proposition 4);
- the fact that a solution in the sense of Carrillo is also a strong entropy solution (Proposition 2);
- the uniqueness of a strong entropy solution (Theorem 3.5).

Combining the preceding results, we readily obtain the main result of this section.

Theorem 3.6. *Assume that the couple (f,ϕ) is non degenerate in the sense of (5) and $g \in L^{1\cap\infty}((-\infty,0))$. There exists a unique solution of the Dirichlet problem (S) in the sense of Definition 3.2, which is also its unique solution in the sense of Definition 3.3.*

4. **Weakly trace-regular entropy solutions of evolution problem** (P)**.** Now we will address the evolution problem (P). Contrarily to the previous section where the continuity of $\mathcal{F}[u]$ was exploited, the choice of space dimension 1 here is not essential: it is a mere technical simplification in order to keep focused on the important details. Again, we restrict the space of data to $L^{1\cap\infty}((-\infty,0))$. Our main objective is to make precise the boundary regularity of solutions - regularity that we cannot guarantee for general solutions - which is needed in order to give sense to a strong entropy formulation for the Cauchy-Dirichlet problem (P). Indeed, under

the assumption of the weak trace-regularity of Carrillo entropy solutions, introduced below, we will justify the strong entropy formulation of the solution to the Dirichlet problem, and its uniqueness.

We follow the same stages as in the previous section. Let $u_0 \in L^{1\cap\infty}((-\infty,0))$.

Definition 4.1. A bounded measurable function u on $(0,T)\times(-\infty,0)$ is called a local weak solution of the Cauchy problem in (P) if $\phi(u)\in L^2(0,T;H^1(-\infty,0))$ and the following identity is satisfied: for all $\xi\in\mathcal{C}_0^\infty([0,T)\times(-\infty,0))$,

$$\int_0^T\int_{-\infty}^0\left\{u\xi_t+\Big(f(u)-\phi(u)_x\Big).\xi_x\right\}dxdt+\int_{-\infty}^0 u_0\xi(0,x)dx=0.$$

A bounded measurable function u is called a local entropy solution of the Cauchy problem in (P) if $\phi(u)\in L^2(0,T;H^1(-\infty,0))$ and for all $k\in\mathbb{R}$, for all $\xi\in\mathcal{C}_0^\infty([0,T)\times(-\infty,0))$ such that $\xi\geq 0$, there holds

$$\int_0^T\int_{-\infty}^0\left\{|u-k|\xi_t+sign(u-k)\Big(f(u)-f(k)-\phi(u)_x\Big).\xi_x\right\}dxdt$$
$$+\int_{-\infty}^0(u_0-k)\xi(0,x)dx\geq 0. \tag{23}$$

Obviously, taking $\pm k>\|u\|_\infty$, one finds that a local entropy solution is a local weak solution. Now, given a constant boundary datum u^D, one classical way to take the Dirichlet condition in the account is the following ([16]).

Definition 4.2. A bounded measurable function u on $(0,T)\times(-\infty,0)$ is called an entropy solution in the sense of Carrillo of the Cauchy-Dirichlet problem (P) if

1. the function u is a local entropy solution of the Cauchy problem;
2. there holds, in the sense of traces of Sobolev functions, the equality
$$\phi(u)(\cdot,0)=\phi(u^D). \tag{24}$$
3. finally, for all $\xi\in\mathcal{C}_0^\infty((0,T)\times(-\infty,0])$, $\xi\geq 0$ the following up-to-the-boundary entropy inequalities are satisfied:
   - $\forall k\geq u^D$,
$$\int_0^T\int_{-\infty}^0\left\{(u-k)^+\xi_t+sign^+(u-k)\Big(f(u)-f(k)-\phi(u)_x\Big).\xi_x\right\}dxdt\geq 0. \tag{25}$$
   - $\forall k\leq u^D$,
$$\int_0^T\int_{-\infty}^0\left\{(u-k)^-\xi_t+sign^-(u-k)\Big(f(u)-f(k)-\phi(u)_x\Big).\xi_x\right\}dxdt\geq 0. \tag{26}$$

Proposition 5. *Suppose that u is a local entropy solution of (P) and (24) holds. Then u is entropy solution of the Cauchy-Dirichlet problem (P) in the sense of Carrillo if and only if for all $\zeta\in\mathcal{C}_0^\infty((0,T))$ with $\zeta\geq 0$ there holds*

$$\forall k\geq u^D\quad \varliminf_{h\to 0}\frac{1}{h}\int_0^T\int_{-h}^0 sign^+(u-k)(\mathcal{F}[u]-f(k))\zeta(t)dxdt\geq 0\ , \tag{27}$$

$$\forall k\leq u^D\quad \varliminf_{h\to 0}\frac{1}{h}\int_0^T\int_{-h}^0 sign^-(u-k)(\mathcal{F}[u]-f(k))\zeta(t)dxdt\geq 0. \tag{28}$$

The proof is fully analogous to the one of Proposition 1.

Further, we would like to give a pointwise sense to the boundary values of the flux $\mathcal{F}[u]$; this is possible only under the appropriate regularity assumption. For this purpose, we put forward the notion of a weakly trace-regular solution.

Definition 4.3. We say that a local weak solution to the Cauchy problem in (P) is weakly trace-regular if the family $(\mathcal{F}[u](.,x))_{x<0}$ of $L^1((0,T))$ functions is equi-integrable in some neighborhood $(-\epsilon,0)$ of the boundary.

Recall that, given $(f_n)_{n>0}$ a sequence in $L^1((0,T))$, it is weakly precompact (i.e., it admits a weakly convergent subsequence) if and only if it is equi-integrable.

Remark 2. If u is weakly trace-regular, then the family $\left(\frac{1}{h}\int_{-h}^{0}\mathcal{F}[u](.,x)dx\right)_{h>0}$ is also equi-integrable on $(0,T)$ and thus it is weakly precompact.

Now, we are ready to give the rather non-standard notion of weak normal boundary trace of the flux that we have found appropriate in order to give a sense to the relation " $\Big(u(t,0^-),\mathcal{F}[u](t,0^-)\Big)\in\widetilde{\mathcal{B}}$ ".

Definition 4.4. We say that $t\mapsto\gamma_w\mathcal{F}[u](t)$ is the L^1-weak trace of $\mathcal{F}[u]$ at $x=0^-$ if

$$\frac{1}{h}\int_{-h}^{0}\mathcal{F}[u](.,x)dx \rightharpoonup_{h\to 0}(\gamma_w\mathcal{F}[u])(.) \quad \text{weakly in } L^1((0,T)). \tag{29}$$

Lemma 4.5. *Assume that u is a weakly trace-regular local weak solution to the Cauchy problem in (P). Then $\mathcal{F}[u]$ has L^1-weak trace $\gamma_w\mathcal{F}[u]$ at $x=0^-$.*

Proof. It is clear from Remark 2 that in the assumptions of the lemma, there exists a subsequence $(h_n)_n$, $h_n\to 0^+$, such that (29) is verified with the limit taken along the subsequence $(h_n)_n$. Further, one circumvents the use of the subsequence $(h_n)_n$ using the theory of normal traces of divergence-measure fields ([17]) for the field $\Big(u,\mathcal{F}[u]\Big)$. Indeed, the definition of a local weak solution says in particular that $\Big(u,\mathcal{F}[u]\Big)\in L^2_{loc}([0,T]\times(-\infty,0])^2$ and that $\mathrm{div}_{(t,x)}\Big(u,\mathcal{F}[u]\Big)=0$ in the sense of distributions. Thus by the results of [17] there exists the weak limit in the sense of $H^{-1/2}(0,T)$ of the left-hand side of (29). It follows, first, that the trace in the Chen-Frid sense is an $L^1(0,T)$ function $F(.)$; and second, that every accumulation point of the left-hand side of (29) - now in the weak L^1 sense - is equal to $F(.)$. Then, equi-integrability in Remark 2 and the uniqueness of the accumulation point guarantee the existence of the limit (29). □

Now we are ready to give a new definition for the Cauchy-Dirichlet problem (P), analogous to Definition 3.3 of the stationary Dirichlet problem (S).

Definition 4.6. A bounded measurable function u on $(0,T)\times(-\infty,0)$ is called *strong entropy solution* of the Cauchy-Dirichlet problem (P) if the following conditions are satisfied:

1. the function u is a local entropy solution and (24) holds;
2. there exists an L^∞ function γu representing the trace of u on the boundary $(0,T)\times\{0\}$ in the strong L^1 sense, i.e.,

$$u(.,x)\xrightarrow[x\to 0^-]{}(\gamma u)(.) \text{ strongly in } L^1((0,T));$$

3. there exists an L^1 function $\gamma_w\mathcal{F}[u]$ representing the trace of $\mathcal{F}[u]$ on the boundary $(0,T)\times\{0\}$ in the weak L^1 sense (29);
4. for a.e. $t\in(0,T)$ the couple $\Big((\gamma u)(t),(\gamma_w\mathcal{F}[u])(t)\Big)$ belongs to the maximal monotone graph $\widetilde{\mathcal{B}}$ defined by (8) in Section 2.

Remark 3. Let us discuss the important issue of existence of traces in 2. and 3.

(i) Assume that the couple (f,ϕ) is non degenerate in the sense (5). Let u be a local entropy solution of (P). Then existence of a strong L^1 trace γu of u on the boundary is guaranteed by the results of [33, 26].

(ii) For a weakly trace-regular solution u of (P), existence of a weak L^1 normal trace $\gamma_w\mathcal{F}[u]$ of $\mathcal{F}[u]$ on the boundary is guaranteed by Lemma 4.5.

We have the following relation between the notions of solution introduced above (a more precise relation will be obtained at the end of the section).

Proposition 6. *Assume that the couple (f,ϕ) is non degenerate in the sense of (5). Assume that $u\in L^\infty((-\infty,0))$ is a weakly trace-regular entropy solution in the sense of Carrillo of the Cauchy-Dirichlet problem (P). Then it is also a strong entropy solution of the same problem.*

Before turning our attention to the proof, we make the following observation which relies on the assumption of weak trace-regularity. It is the essential ingredient of the localization procedure needed to formulate boundary conditions pointwise.

Lemma 4.7. *Let u be a weakly trace-regular solution to the Cauchy problem in (P) and assume that for some $(k,\mathcal{K})\in\mathbb{R}^2$, for all $\zeta\in\mathcal{C}_0^\infty((0,T))$ with $\zeta\geq 0$ there holds*

$$I^\pm(\zeta)=\lim_{h\to 0}\frac{1}{h}\int_{-h}^0 sign^\pm(u-k)\Big(\mathcal{F}[u]-\mathcal{K}\Big)\zeta(t)dxdt\geq 0. \tag{30}$$

Then inequalities (30) still hold for all $\zeta\in L^\infty((0,T))$, $\zeta\geq 0$.

Proof. Take $\zeta\in L^\infty((0,T))$ with $\zeta\geq 0$. For a first step of approximation, take $(\zeta_n)_n$ a sequence of $\mathcal{C}((0,T))$ functions such that $O_n=\{t\mid \zeta_n(t)\neq\zeta(t)\}$ verifies $meas(O_n)\leq\frac{1}{2n}$, moreover $||\zeta_n||_{L^\infty}\leq||\zeta||_{L^\infty}$. Such sequence is given by the Lusin theorem. We modify ζ_n in a neighbourhood of $t=0$ and $t=T$ to get $meas(O_n)\leq\frac{1}{n}$ and ζ_n continuous compactly supported in $(0,T)$, for all n. Then we have

$$\frac{1}{h}\left|\int_0^T\int_{-h}^0 sign^\pm(u-k)(\mathcal{F}[u]-\mathcal{K})\zeta(t)dxdt-\int_0^T\int_{-h}^0 sign^\pm(u-k)(\mathcal{F}[u]-\mathcal{K})\zeta_n(t)dxdt\right|$$
$$\leq 2\|\zeta\|_{L^\infty}\frac{1}{h}\int_{-h}^0\int_{O_n}(|\mathcal{F}[u]|+|\mathcal{K}|)dtdx\quad\longrightarrow_{n\to\infty}\ 0$$

uniformly in h due to the equi-integrability, indeed, we have $meas(O_n)\to_{n\to\infty}0$. Further, it is easy to approximate a compactly supported $\mathcal{C}((0,T))$ function ζ_n by a sequence of $\mathcal{C}_0^\infty((0,T))$ functions ζ_n^m in L^∞ norm. Observe that

$$\left|\left|\frac{1}{h}\int_{-h}^0(|\mathcal{F}[u]|+|\mathcal{K}|)dx\right|\right|_{L^1((0,T))}$$

is bounded uniformly in h; therefore the convergence $||\zeta_n-\zeta_n^m||_{L^\infty}\to 0$, as $m\to\infty$, is enough to pass to the limit on $I(\zeta_n^m)$ and get $I(\zeta_n)$. Thus, for any fixed $\epsilon>0$, we can choose $\zeta_n\in\mathcal{C}((0,T))$ such that $|I(\zeta_n)-I(\zeta_n^m)|\leq\frac{\epsilon}{2}$ then $\zeta_n^m\in\mathcal{C}_0^\infty((0,T))$ such that $|I(\zeta)-I(\zeta_n^m)|\leq\frac{\epsilon}{2}$. Moreover, at all steps of the approximation, we can choose nonnegative functions. Since $I^\pm(\zeta_n^m)\geq 0$, this proves the lemma. □

Proof of Proposition 6. The only point that has to be justified is the last item of Definition 4.6.

Denote by $U(.)$ the strong L^1 trace of u and by $F(.)$, the weak L^1 trace of $\frac{1}{h}\int_{-h}^{0}\mathcal{F}[u](.,x)\,dx$ (by Remark 3 and due to assumption (5), both traces do exist for a trace-regular entropy solution in the sense of Carrillo of problem (P)). Given $k\in\mathbb{R}$ and fixing an everywhere defined representative of U, we introduce the sets

$$E_k^0=\{t\ \ |U(t)=k\}\,;\ \ E_k^+=\{t\ \ |U(t)>k\};\ \ E_k^-=\{t\ \ |U(t)<k\}\,.$$

Because $u(.,x)$ converges to $U(.)$ as $x\to 0^-$ a.e. on $(0,T)$, we have:

$$\forall t\in E_k^{\pm}\ \exists h(t)>0\ \text{ such that for a.e. } x\in(-h(t),0),\ \ \pm(u(t,x)-k)>0.$$

Therefore, we can represent the sets $E_k^{\pm}$ as $\bigcup_{m\in\mathbb{N}^\star}E_{k,m}^{\pm}$ respectively, where

$$E_{k,m}^{\pm}=\Big\{t\in E_k^{\pm}\ \Big|\ \forall x\in(-\frac{1}{m},0)\ \ \pm(u(t,x)-k)>0\Big\}.$$

Due to Lemma 4.7, we can choose $\zeta=\theta\mathbf{1}_{E_{k,m}^+}$ in (27), with some $\theta\geq 0$, $\theta\in L^\infty(E_k^+)$. Then for all $h<\frac{1}{m}$, thanks to the definition of $E_{k,m}^+$ we can simply compute

$$\begin{aligned}I^+(\zeta)&=\frac{1}{h}\int_{-h}^{0}\int_0^T sign^+(u-k)(\mathcal{F}[u]-f(k))\,\zeta(t)\,dtdx\\&=\int_{E_{k,m}^+}\frac{1}{h}\int_{-h}^{0}(+1)(\mathcal{F}[u]-f(k))\,\theta(t)\,dxdt\longrightarrow\int_{E_{k,m}^+}(F(t)-f(k))\,\theta(t)\,dt,\end{aligned}\tag{31}$$

where the limit, as $h\to 0^+$, is due to the definition of $F(.)$. Thus by (27) of Proposition 5 we see that the right-hand side of (31) is nonnegative for all $k>u^D$, for all L^∞ function $\theta\geq 0$. Recalling that $U(t)>k$ on $E_{k,m}^+$, we conclude that $sign^+(U(.)-k)(F(.)-f(k))\geq 0$ pointwise on $E_{k,m}^+$. Since m is arbitrary, the inequality extends to E_k^+; moreover, this inequality is obviously true for $t\in E_k^0\cup E_k^-$. We conclude that the first inequality in (15) holds for the couple $(U(t),F(t))$ for almost all $t\in(0,T)$. The proof of the second inequality in (15) for the same couples is fully analogous. Finally, the last line of (15) is the pointwise expression of (24). To sum up, given a weakly trace-regular entropy solution in the sense of Carrillo of (P), we have proved (15) pointwise on $(0,T)$.

Recall that from the proof of Proposition 2 we know that, given a couple $(U,F)\in\mathbb{R}^2$, it fulfills the properties (15) if and only if it fulfills $(U,F)\in\widetilde{\mathcal{B}}$. We apply the above equivalence pointwise to $(U(t),F(t))$ for a.e. $t\in(0,T)$, and deduce that u is a strong entropy solution of (P). □

4.1. **Uniqueness of a strong entropy solution to (P).** As in the stationary case, monotonicity of $\widetilde{\mathcal{B}}$ ensures uniqueness of a strong entropy solution of (P).

Theorem 4.8. *Assume $u_0,\hat{u}_0\in L^{1\cap\infty}((-\infty,0))$. Assume that u and $\hat{u}$ are strong entropy solutions of the Cauchy-Dirichlet problem (P) with the corresponding initial data u_0 and $\hat{u}_0$, respectively. Then for a.e. $t>0$,*

$$||\hat{u}-u||_{L^1((-\infty,0))}(t)\leq||\hat{u}_0-u_0||_{L^1((-\infty,0))}.\tag{32}$$

The inequality (32) also holds whenever u, $\hat{u}$ are weakly trace-regular entropy solutions in the sense of Carrillo of (P) with initial data u_0, $\hat{u}_0$, respectively. In particular, there is uniqueness of a strong entropy solution to (P) and uniqueness of a weakly trace-regular Carrillo entropy solution to (P), for a given initial datum.

Proof. As in the proof of Theorem 3.5, using the Carrillo entropy dissipative information and doubling of variables technique, one gets the local Kato inequality: for all $\xi \in \mathcal{C}^\infty([0,T)\times(-\infty,0))$, $\xi \geq 0$,

$$-\int_0^T\int_{-\infty}^0 |u-\hat{u}|\xi_t dxdt \leq \int_0^T\int_{-\infty}^0 sign(u-\hat{u})\Big(\mathcal{F}[u]-\mathcal{F}[\hat{u}]\Big)\xi_x dxdt + \int_{-\infty}^0 |u_0-\hat{u}_0|\xi(0,x)dx. \tag{33}$$

By a classical density argument, we can take $\xi(t,x)=\xi_h(x)\mathbf{1}_{[0,t)}$ in the inequality (33) where $\xi_h = \min\{1,-\frac{x}{h}\}$ and $\zeta \in \mathcal{C}^\infty((0,T))$. For all Lebesgue point of the map $t \mapsto (u-\hat{u})(t) \in L^1((-\infty,0))$, we obtain after passing to the limit $h\to 0$

$$\int_{-\infty}^0 |u-\hat{u}|(t)dxdt \leq \int_{-\infty}^0 |u_0-\hat{u}|dx - \lim_{h\to 0}\frac{1}{h}\int_0^T\int_{-h}^0 sign(u-\hat{u})\Big(\mathcal{F}[u]-\mathcal{F}[\hat{u}]\Big)dxdt.$$

For $t\in(0,T)$, we introduce the notation $U := \gamma u$, $\hat{U}=\gamma\hat{u}$ and $F=\gamma_w\mathcal{F}[u]$, $\hat{F}=\gamma_w\mathcal{F}[\hat{u}]$. Recall that both $(U(t),F(t))$ and $(\hat{U}(t),\hat{F}(t))$ (for a.e. t) belong to the same maximal monotone graph $\widetilde{\mathcal{B}}$. To obtain (32), it is enough to prove that

$$\liminf_{h\to 0}\frac{1}{h}\int_0^T\int_{-h}^0 sign(u-\hat{u})\Big(\mathcal{F}[u]-\mathcal{F}[\hat{u}]\Big)dxdt \geq 0. \tag{34}$$

To prove (34), we fix $m\in\mathbb{N}$ and for all $h<\frac{1}{m}$, we split the integrals over $(0,T)$ into the integrals over four disjoint subsets:

$$E^0, \quad E_m^-, \quad E_m^+ \text{ and } E_m^r,$$

where $E^0 = \big\{t\in(0,T)\,\big|\,U(t)=\hat{U}(t)\big\}$,

$$E_m^\pm = \Big\{t\in(0,T)\,\Big|\,\text{for a.e. } x\in(-\frac{1}{m},0) \quad \pm(u(t,x)-\hat{u}(t,x))>0\Big\},$$

and the residual set E_m^r is the complementary in $(0,T)$ of the union of the three other sets. As in the proof of Proposition 6, we have

$$(0,T) = \Big(\bigcup_{m\in\mathbb{N}} E_m^+\Big)\cup E^0 \cup \Big(\bigcup_{m\in\mathbb{N}} E_m^-\Big).$$

Therefore, E_m^r is of vanishing measure, as $m\to\infty$.

Due to the weak trace-regularity of u, the contribution of the integral over E_m^r into (34) can be made as small as desired, as $m\to\infty$. Further, for h small enough, due to the definition of $E_m^\pm$ we can simply calculate the contributions of the integrals over $E_m^\pm$ into (34). Indeed, we have $sign(u(t,x)-\hat{u}(t,x))=\pm 1$ for a.e. $(t,x)\in E_m^\pm\times(-h,0)$, therefore we can pass to the weak L^1 limit for the fluxes $\mathcal{F}[u]$, $\mathcal{F}[\hat{u}]$. We get the following contribution to the limit (34):

$$\int_{E_m^-}(-1)(F(t)-\hat{F}(t))dt + \int_{E_m^+}(+1)(F(t)-\hat{F}(t))dt = \int_{E_{k,m}^-\cup E_{k,m}^+} sign(U(t)-\hat{U}(t))(F(t)-\hat{F}(t))dt.$$

This term is nonnegative because $(U(t),F(t)),(\hat{U}(t),\hat{F}(t))\in\widetilde{B}$ where $\widetilde{B}$ is a monotone graph. Finally, although the contribution of the set E^0 into (34) seems difficult to estimate directly, we can separate the convection and the diffusion fluxes as in

(21) in the proof of Theorem 3.5. Then for a.e. t in E_k^0, following the lines of the end of the proof Theorem 3.5 we find

$$\liminf_{h\to 0} \frac{1}{h}\int_{-h}^{0}\int_{E^0} sign(u-\hat{u})(F[u]-F[\hat{u}])dtdx \geq 0.$$

This concludes the proof of positivity of (34) and proves the result of the theorem. □

4.2. **Equivalence of trace-regular Carrillo solutions and strong entropy solutions to (P).** It is immediate to derive, from the preceding results, the following fact which is the main result of the paper.

Theorem 4.9. *Assume for given data $u_0 \in L^{1\cap\infty}((-\infty,0))$, $u^D \in \mathbb{R}$ there exists a trace-regular entropy solution in the sense of Carrillo to the Cauchy-Dirichlet problem (P). Then the solution is unique in this class, moreover, it is the unique strong entropy solution of the same problem.*

We recall that existence of a Carrillo solution is a relatively straightforward result ([16]), while the proof of its uniqueness - without any trace-regularity assumptions - requires a particularly involved analysis. Thus, the interest of the result of Theorem 4.9 depends on the possibility to prove trace-regularity, at least for a restricted class of data. This issue requires deeper analysis than we postpone for future work. Here, let us stress that this kind of results is sometimes available in one space dimension. In particular, in [15] the concept of bounded-flux solutions was put forward; it is clear that the L^∞ bound on $\mathcal{F}[u]$ investigated in [15] implies equi-integrability of $\big(\mathcal{F}[u](\cdot,x)\big)_{x<0}$, so that bounded-flux solutions are in particular weakly trace-regular. Some one-dimensional regularity results for the flux $\mathcal{F}[u]$ can also be found in [35, 36, 20]. The techniques of these works are limited to the one-dimensional situation, and the justification of weak trace-regularity in the general multi-dimensional case requires new ideas.

4.3. **Integral solutions and well-posedness of the abstract evolution problem associated with (P).** For the one-dimensional case of (P), let us point out the abstract well-posedness result that follows readily from the well-posedness result for the stationary problem in Section 3 in the strong entropy framework.

To this end, given f,ϕ verifying the assumptions of the introduction including the non-degeneracy assumption (5), given $u^D \in \mathbb{R}$ we define the (may be, multi-valued) operator $\mathcal{A}$ from $X := L^1((-\infty,0))$ into itself by

$$(v,h)\in\mathcal{A} \iff v \text{ is the strong entropy solution of } (S) \text{ for the datum } g = v+h.$$

Because $L^{1\cap\infty}((-\infty,0))$ is dense in X, it readily follows from the results of Section 3 that $\mathcal{A}$ is an m-accretive operator on X (see [11]). Moreover, it is not difficult to show that $\mathcal{A}$ is densely defined, see e.g. [5] for an analogous proof. Therefore, the general theory of nonlinear semigroups ([11]) yields existence and uniqueness of a mild solution to the evolution problem

$$u' + \mathcal{A}u \ni 0, \quad u(0) = u_0 \in L^1((-\infty,0)) \tag{35}$$

which is the abstract counterpart of the Cauchy-Dirichlet problem (P).

Definition 4.10. We say that $u \in C([0,T];L^1((-\infty,0)))$ is an integral solution of (35) if $u(0)=u_0$ and for all v local entropy solution of (S) with source g and with

$\Big(v(0^-), \mathcal{F}[v](0^-)\Big) \in \widetilde{\mathcal{B}}$ there holds

$$\frac{d}{dt}\|u(t)-v\|_{L^1((-\infty,0))} \leq \Big[u(t)-v, v-g\Big]_{L^1((-\infty,0))} \quad \text{in } \mathcal{D}'((0,T)). \tag{36}$$

Somewhat abusively, we will also say in this case that the function $(t,x) \mapsto u(t)(x)$ is an integral solution of the Cauchy-Dirichlet problem (P).

Restricting again to $L^{1\cap\infty}$ data for the sake of simplicity, we find that strong entropy solutions, whenever they exist, also coincide with the natural abstract notion of solution described hereabove.

Theorem 4.11. *For all datum $u_0 \in L^1((-\infty,0))$ there exists a unique integral solution of (P). Moreover, assume that $u_0 \in L^{1\cap\infty}((-\infty,0))$ and u is a trace-regular solution of (P) in the sense of Carrillo (or, equivalently, u a strong entropy solution of (P)) with initial datum u_0. Then u coincides with the unique integral solution of the same problem.*

Proof. The first statement is immediate due to the identification of mild solutions and integral solutions ([10, 11]). To justify the second statement, one follows the lines of the proof of Theorem 4.8 with the stationary solution $x \mapsto v(x)$ of S replacing $(t,x) \mapsto \hat{u}(t,x)$. Note that the weak trace-regularity of so defined $\hat{u}$ is obvious. □

For details on this section (including the existence result for mild and integral solutions), we refer to [24, Chap. 2].

5. **Conclusions, extensions and remarks.** Despite the fact that existence of weakly trace-regular solution remains a widely open question, let us stress the appealing aspects of the analysis presented in this section.

First, it is easy to generalize the definition of strong entropy solutions and the uniqueness result in several complementary directions. The specific configuration of the convective and the diffusive fluxes considered here can be replaced by a general configuration where ϕ can have several flatness regions. If $[a,b]$ is one of these regions, i.e., it is the maximal interval where ϕ is equal to a constant, then for all $u^D \in [a,b]$ the corresponding graph $\widetilde{\mathcal{B}}$ is Bardos-LeRoux-Nédélec graph (see [8]) combined with the obstacle that forces $\gamma_w u$ to stay within $[a,b]$; this can be expressed as the sum of the Bardos-LeRoux-Nédélec graph with the maximal monotone graph

$$\Big(\{a\} \times (-\infty,0]\Big) \cup \Big([a,b] \times \{0\}\Big) \cup \Big(\{b\} \times [0,+\infty)\Big).$$

General L^∞ initial data can be dealt with easily; see Remark 1 for the case of L^1 data. More importantly, problem (1) with variable boundary Dirichlet data $u^D(\cdot)$ in general Lipschitz domains in any space dimension can be considered within the definition of strong entropy solutions. It is enough to consider graphs $\widetilde{\mathcal{B}}$ that depend on the point of the boundary and work with the weak L^1 trace of the normal component of the flux $\gamma_w \mathcal{F}[u].\nu(x)$. Note that extension of the Carrillo definition [16] to variable boundary data is, on the contrary, quite delicate because of the necessity to localize the properties "$k \geq u^D(\cdot)$", "$k \leq u^D(\cdot)$" (for an extension to continuous data u^D - in the purely hyperbolic setting only - see [2]). We stress that the uniqueness arguments developed for (P) apply in the general context (1). Existence of strong entropy solutions remains a difficult open problem.

Second, formulation 3 gives a clear intuitive meaning to the sense in which the Dirichlet boundary conditions are relaxed. For this reason, it should be useful in applications: contrarily to the definitions of Carrillo [16] or Otto [28] (which have the advantage of being apparently more general but which rely on implicit ways to describe boundary layers), the definition of a strong entropy solution provides explicit information for the needs of scientific computation and engineering. In particular, in numerical analysis of problem (P) the graph $\widetilde{\mathcal{B}}$ can be exploited in order to avoid the appearance of boundary layers. Note that in the case of the pure hyperbolic problem, the direct use of $\widetilde{\mathcal{B}}$ to prescribe boundary conditions is equivalent to the use of the Godunov numerical flux for taking into account the boundary, see [8].

Our third observation develops the previous one: the use of maximal monotone relations $(\gamma u, \gamma_w \mathcal{F}[u].\nu) \in \widetilde{\mathcal{B}}$ provides a natural framework to encompass the formulation of general dissipative boundary conditions (zero-flux conditions, Robin conditions, obstacle conditions). The hyperbolic case has been studied in [7, 8] in such setting.

We will address these issues in a forthcoming work.

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