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HÖLDER CONTINUITY OF SOLUTIONS TO QUASILINEAR HYPOELLIPTIC EQUATIONS

C. IMBERT & C. MOUHOT

ABSTRACT. We prove that L^2 weak solutions to a quasilinear hypoelliptic equations with bounded measurable coefficients are Hölder continuous. The proof relies on classical techniques developed by De Giorgi and Moser together with the averaging lemma developed in kinetic theory. The latter tool is used in the proof of the local gain of integrability of sub-solutions and in the proof of an “hypoelliptic isoperimetric De Giorgi lemma”, obtained by combining the classical isoperimetric inequality on the diffusive variable with the structure of the integral curves of the first-order part of the operator.

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1. INTRODUCTION

1.1. **The question studied and its history.** We consider the following nonlinear kinetic Fokker-Planck equation

$$(1.1) \quad \partial_t f + v \cdot \nabla_x f = \rho \nabla_v \cdot (\nabla_v f + v f), \quad t \geq 0, \quad x \in \mathbb{T}^d, \quad v \in \mathbb{R}^d,$$

where $d \in \mathbb{N}^*$, \mathbb{T}^d is the flat d -dimensional torus, $f = f(t, x, v) \geq 0$ and $\rho[f] = \int_{\mathbb{R}^d} f(t, x, v) dv$. The construction of global smooth solutions for such a problem is one motivation for the present paper.

The linear kinetic Fokker-Planck equation $\partial_t f + v \cdot \nabla_x f = \nabla_v \cdot (\nabla_v f + v f)$ is sometimes called the Kolmogorov-Fokker-Planck equation, as it was studied by Kolmogorov in the seminal paper [9], when $x \in \mathbb{R}^d$. In this note, Kolmogorov explicitly calculated the fundamental solution and deduced

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regularisation in both variables x and v , even though the operator $\nabla_v \cdot (\nabla_v + v) - v \cdot \nabla_x$ shows ellipticity in the v variable only. It inspired Hörmander and his theory of hypoellipticity [8], where the regularisation is recovered by more robust and more geometric commutator estimates (see also [13]).

Another question which has attracted a lot of attention in calculus of variations and partial differential equations along the 20th century is Hilbert's 19th problem about the analytic regularity of solutions to certain integral variational problems, when the quasilinear Euler-Lagrange equations satisfy ellipticity conditions. Several previous results had established the analyticity conditionally to some differentiability properties of the solution, but the full answer came with the landmark works of De Giorgi [2, 3] and Nash [11], where they prove that any solution to these variational problems with square integrable derivative is analytic. More precisely their key contribution is the following¹: reformulate the quasilinear parabolic problem as

$$(1.2) \quad \partial_t f = \nabla_v \cdot (A(v, t) \nabla_v f), \quad t \geq 0, \quad v \in \mathbb{R}^d$$

with $f = f(v, t) \geq 0$ and $A = A(v, t)$ satisfies the ellipticity condition $0 < \lambda I \leq A \leq \Lambda I$ for two constants $\lambda, \Lambda > 0$ but is, besides that, merely measurable. Then the solution f is Hölder continuous.

In view of the nonlinear (quasilinear) equation (1.1) it is natural to ask whether a similar result as the one of De Giorgi-Nash holds for quasilinear hypoelliptic equations. More precisely, we consider

$$(1.3) \quad \partial_t f + v \cdot \nabla_x f = \nabla_v \cdot (A(x, v, t) \nabla_v f), \quad t \in (0, T), (x, v) \in \Omega,$$

where Ω is an open set of \mathbb{R}^{2d} , $f = f(t, x, v) \geq 0$ and the $d \times d$ symmetric matrix A satisfies the ellipticity condition

$$(1.4) \quad 0 < \lambda I \leq A \leq \Lambda I$$

for two constants λ, Λ but is, besides that, merely measurable. We want to establish the Hölder continuity of L^2 solutions to this problem. In order to do so, we first prove that L^2 sub-solutions are locally bounded; we refer to such a result as an $L^2 - L^\infty$ estimate. We then prove that solutions are Hölder continuous by proving a lemma which is an hypoelliptic counterpart to De Giorgi's isoperimetric lemma. We denote by $B_r(x_0)$ and $B_r(v_0)$ the usual Euclidian balls in x and v .

Theorem 1 (Hölder continuity). *Let f be a solution of (1.3) in $Q_0 := B_2(v_0) \times B_2(x_0) \times (-2, 0)$. Then f is α -Hölder continuous with respect to (x, v, t) in $Q_1 = B_1(v_0) \times B_1(x_0) \times (-1, 0)$ and*

$$\|f\|_{C^\alpha(Q_1)} \lesssim \|f\|_{L^2(Q_0)}$$

for some α only depending on dimension and ellipticity constants.

¹We give the parabolic version due to Nash here.

Remark 2. Cubes of general size are then obtained by scaling, see Remark 11. Note also that the equation is translation invariant in x but not in v which explains why we keep track of the centers of the balls (we could however have reduced to $x_0 = 0$ without loss of generality in the statement but preferred not to).

In [12], the authors obtain an $L^2 - L^\infty$ estimate with completely different techniques; however they cannot reach the Hölder continuity estimate. Our techniques rely on averaging lemmas [4, 5] in order to gain some regularity H_x^s , $s > 0$ small, in the space variable x from the natural H_v^1 estimate. We emphasize that such H_x^s estimates do not hold for sub-solutions. From this Sobolev estimate, we can recover a gain of integrability for L^2 sub-solutions, and we then prove the Hölder continuity through a De Giorgi type argument on the decrease of oscillation for solutions.

In [15, 16], the authors get a Hölder estimate for L^2 weak solutions of so-called ultraparabolic equations, including (1.3). Their proof relies on the construction of cut-off functions and a particular form of weak Poincaré inequality satisfied by non-negative weak sub-solutions. Our paper proposes a new, short and simple strategy, that, we hope, sheds new light on the regularizing effect for hypoelliptic equations with bounded measurable coefficients and provide tools for further applications. With this respect, we mention that we plan to use the main result of this paper in a future work to study (1.1) thoroughly.

We finally mention that Golse and Vasseur told us that they proved independently a similar result [6].

1.2. Plan of the paper. In Section 2, we explain how to get a universal gain of regularity for (signed) L^2 solutions; we also exhibit a universal gain of integrability for non-negative L^2 sub-solutions. Section 3 is dedicated to deriving a local upper bound of such non-negative L^2 sub-solutions by using Moser iteration procedure. In Section 4, the Hölder estimate is derived by proving a reduction of oscillation lemma.

2. LOCAL GAIN OF REGULARITY / INTEGRABILITY

We consider the equation (1.3) and we want to establish a local gain of integrability of solutions in order to apply Moser's iteration and get a local L^∞ bound. Since we will need to perform convex changes of unknown, it is necessary to obtain this gain even for (non-negative) *sub-solutions*.

Theorem 3 (Gain of integrability for non-negative sub-solutions). *Consider two cubes $Q_1 \subset\subset Q_0$, both centered at $(x_0, v_0, t_0) \in \mathbb{R}^{2d} \times (0, +\infty)$. There exists $q > 2$ (universal) such that for all non-negative L^2 sub-solution f of (1.3) in Q_0 , we have*

$$(2.1) \quad \|f\|_{L^q(Q_1)} \leq C \|f\|_{L^2(Q_0)}$$

where and $C = \bar{C}C_{0,1}$ with

$$C_{0,1} = (1 + |v_0|) \left(\frac{1}{\tau_0 - \tau_1} + \frac{1}{R_0 - R_1} + \frac{1}{(r_0 - r_1)^2} \right)$$

where $\bar{C} = C(d, \lambda, \Lambda)$ depends on d and upper bounds on Λ and λ^{-1} , and $Q_i = B_{R_i}(x_0) \times B_{r_i}(v_0) \times (t_0 - \tau_i, t_0]$, $i = 0, 1$ and $0 < r_1 < r_0 < 1$ and $0 < R_1 < R_0 < 1$ and $0 < \tau_1 < \tau_0$.

This result is a consequence of the comparison principle and the fact that, for weak signed solutions f , we can even get a gain of regularity. This gain of regularity will be crucial in the proof of the decrease of oscillation lemma.

Theorem 4 (Gain of regularity for signed solutions). *Consider two cubes $Q_1 \subset\subset Q_0$, both centered at $(x_0, v_0, t_0) \in \mathbb{R}^{2d} \times (0, +\infty)$. There exists $s > 0$ (universal) such that for all (signed) L^2 weak solution f of (1.3) in Q_0 , we have*

$$(2.2) \quad \|f\|_{H_{x,v,t}^s(Q_1)} \leq C \|f\|_{L^2(Q_0)}$$

where and $C = C(d, \lambda, \Lambda, Q_0, Q_1)$.

2.1. Truncation. Given (x_0, v_0, t_0) and $0 < R < S$ and $0 < r < s$ and $0 < \tau < \sigma$, we consider

$$\chi(x, v, t) = \bar{\chi}_{R,S}(x - x_0) \bar{\chi}_{r,s}(v - v_0) \bar{\theta}_{\tau,\sigma}(t - t_0)$$

where

- for $0 < a < b$, $\bar{\chi}_{a,b}: \mathbb{R}^d \rightarrow [0, 1]$ is C^∞ , equals 1 in B_a and 0 outside B_b ;
- for $0 < \tau < \sigma$, $\bar{\theta}_{\tau,\sigma}: \mathbb{R} \rightarrow [0, 1]$ is C^∞ , equals 1 in $[-\tau, 0]$ and 0 outside $[-\sigma, 0]$.

Lemma 5 (Truncation of sub-solutions). *Given a cube Q , if $f: Q \rightarrow \mathbb{R}$ solves*

$$(\partial_t + v \cdot \nabla_x) f \leq \nabla_v \cdot (A \nabla_v f)$$

and $\chi: Q \rightarrow \mathbb{R}$ is smooth, non-negative and compactly supported in Q , then $\tilde{f} = f\chi$ satisfies

$$(\partial_t + v \cdot \nabla_x) \tilde{f} \leq \nabla_v (\chi A \nabla_v f) - \nabla_v \chi \cdot A \nabla_v f + \alpha f$$

where $\alpha = (\partial_t + v \cdot \nabla_x) \chi$.

2.2. Gain of integrability with respect to v and t . The gain of integrability with respect to v and t is classical. It derives from the natural energy estimate, after truncation.

We follow here [10] in order to get the following lemma.

Lemma 6 (Gain of integrability w.r.t. v and t). *Under the assumptions of Theorem 3, the function f satisfies*

$$(2.3) \quad \int_{Q_1} |\nabla_v f|^2 \leq CC_{0,1} \int_{Q_0} f^2.$$

and

$$\|f\|_{L_t^2 L_x^2 L_v^q(Q_1)}^2 \leq CC_{0,1} \int_{Q_0} f^2$$

for some $q > 2$ and

$$\|f\|_{L_t^\infty L_x^2 L_v^2(Q_1)}^2 \leq CC_{0,1} \int_{Q_0} f^2$$

with $C = C(\Lambda, d)$ and $C_{0,1} = (1 + |v_0| + r_0) \left(\frac{1}{R_0 - R_1} + \frac{1}{(r_0 - r_1)^2} + \frac{1}{\tau_0 - \tau_1} \right)$.

Proof. Consider $\Psi \in C_c^\infty(\mathbb{R}^{2d} \times \mathbb{R})$ and integrate the inequation satisfied by f against $2f\Psi^2$ in $\mathbb{R}^{2d} \times [t_1, t_0]$ with $t_1 \in [t_0 - r_1^2, t_0]$ and get

$$\int \partial_t(f^2)\Psi^2 + \int v \cdot \nabla_x(f^2)\Psi^2 \leq \int \nabla_v(A\nabla_v f)f\Psi^2.$$

Add $\int f^2\partial_t(\Psi^2)$, integrate by parts several times and use the upper bound on A in order to get

$$\begin{aligned} & \int \partial_t(f^2\Psi^2) + \int (A\nabla_v f \cdot \nabla_v f)\Psi^2 \\ & \leq \int f^2(\partial_t + v \cdot \nabla_x)(\Psi^2) + 2 \int \Psi\sqrt{A}\nabla_v f \cdot f\sqrt{A}\nabla_v \Psi \\ & \leq \int f^2(\partial_t + v \cdot \nabla_x)(\Psi^2) + \frac{1}{2} \int (A\nabla_v f \cdot \nabla_v f)\Psi^2 + 2 \int f^2(A\nabla_v \Psi \cdot \nabla_v \Psi). \end{aligned}$$

We thus get

$$\begin{aligned} & \int \partial_t(f^2\Psi^2) + \frac{\lambda}{2} \int |\nabla_v f|^2\Psi^2 \\ & \leq \tilde{C} \left(\|\partial_t \Psi\|_\infty + \|\nabla_x \Psi\|_\infty + \|\nabla_v \Psi\|_\infty^2 \right) \int_{\text{supp } \Psi} f^2 \end{aligned}$$

with $\tilde{C} = C(\Lambda, d) \times (1 + |v_0| + r_0)$. Choose next Ψ^2 such that $\Psi^2(t_0) = 0$ and $\text{supp } \Psi^2 \subset Q_0$ and get

$$\int_{x,v} f^2\Psi^2(t_1) + \frac{\lambda}{2} \int |\nabla_v f|^2\Psi^2 \leq CC_{0,1} \int_{Q_0} f^2.$$

If Ψ^2 additionally satisfies $\Psi^2 \equiv 1$ in Q_1 , we get (2.3). The Sobolev inequality then implies the estimate for $\|f\|_{L_t^2 L_x^2 L_v^q(Q_1)}^2$. If now $t_1 \in [t_0 - r_1^2, t_0]$ is arbitrary, we get the estimate for $\|f\|_{L_t^\infty L_x^2 L_v^2(Q_1)}^2$. The proof is now complete. \square

2.3. Gain of regularity with respect to x for signed weak solutions.

Lemma 7 (Gain of regularity w.r.t. x). *Under the assumptions of Theorem 3, if f is a signed weak solution to (1.3), then there exists $s > 0$ such that*

$$(2.4) \quad \|D_x^s f\|_{L^2(Q_1)} \leq CC_{0,1} \|f\|_{L^2(Q_0)}$$

with $C = C(d, \Lambda)$ and $C_{0,1} = (1 + |v_0| + r_0) \left(\frac{1}{R_1 - R_0} + \frac{1}{(r_1 - r_0)^2} + \frac{1}{\tau_1 - \tau_0} \right)$.

Proof. Let $R_{\frac{1}{2}} = \frac{R_1 + R_0}{2}$, $r_{\frac{1}{2}} = \frac{r_1 + r_0}{2}$, $\tau_{\frac{1}{2}} = \frac{\tau_1 + \tau_0}{2}$ and $Q_{\frac{1}{2}} = B_{R_{\frac{1}{2}}}(x_0) \times B_{r_{\frac{1}{2}}}(v_0) \times (t_0 - \tau_{\frac{1}{2}}, t_0]$. In particular,

$$Q_1 \subset Q_{\frac{1}{2}} \subset Q_0.$$

For $i = 1, \frac{1}{2}$, consider $f_i = f\chi_i$ where χ_1 and $\chi_{\frac{1}{2}}$ are two truncation functions introduced in Subsection 2.1 associated with parameters (R_1, r_1, τ_1) and $(R_{\frac{1}{2}}, r_{\frac{1}{2}}, \tau_{\frac{1}{2}})$ and centered at (x_0, v_0, t_0) . In particular,

$$\text{for } i = 1, \frac{1}{2}, \quad \chi_i \equiv 1 \text{ in } Q_i \quad \text{and} \quad \chi_i \equiv 0 \text{ outside } Q_{\frac{1}{2}}.$$

We know from Lemma 5 that

$$\begin{aligned} (\partial_t + v \cdot \nabla_x) f_1 &= \nabla_v \cdot H_1 + H_0 \text{ in } \mathbb{R}^{2d} \times \mathbb{R} \\ \text{with } \begin{cases} H_1 &= \chi_1 A \nabla_v f_{\frac{1}{2}} \\ H_0 &= -\nabla_v \chi_1 \cdot A \nabla_v f_{\frac{1}{2}} + \alpha_1 f_{\frac{1}{2}} \\ \alpha_1 &= (\partial_t + v \cdot \nabla_x) \chi_1. \end{cases} \end{aligned}$$

The previous equation holds true in $\mathbb{R}^{2d} \times (-\infty; t_0]$ since f_1, H_0 and H_1 are supported in $Q_{\frac{1}{2}}$. We remark that (2.3) implies

$$\|H_0\|_{L^2(Q_{\frac{1}{2}})} + \|H_1\|_{L^2(Q_{\frac{1}{2}})} \leq CC_{0,1} \|f\|_{L^2(Q_0)}.$$

Applying [1, Theorem 1.3] with $p = 2$, $r = 0$, $\beta = 1$, $m = 1$, $\kappa = 1$ and $\Omega = 0$ yields the desired estimate with $s = \frac{1}{4} > 0$. \square

2.4. Gain of integrability with respect to x for non-negative subsolutions.

Lemma 8 (Gain of integrability w.r.t. x). *Under the assumptions of Theorem 3, there exists $p > 2$ such that*

$$(2.5) \quad \|f\|_{L_t^2 L_x^p L_v^1(Q_1)} \leq CC_{0,1} \|f\|_{L^2(Q_0)}$$

with $C = C(d, \Lambda)$ and $C_{0,1} = (1 + |v_0| + r_0) \left(\frac{1}{R_1 - R_0} + \frac{1}{(r_1 - r_0)^2} + \frac{1}{\tau_1 - \tau_0} \right)$.

Proof. We follow the reasoning of Lemma 7. The function f_1 now satisfies

$$(\partial_t + v \cdot \nabla_x) f_1 \leq \nabla_v \cdot H_1 + H_0 \text{ in } \mathbb{R}^{2d} \times \mathbb{R}$$

If g solves

$$\begin{cases} (\partial_t + v \cdot \nabla_x) g = \nabla_v \cdot H_1 + H_0 \\ g(x, v, t_0 - \tau_1) = f_1(x, v, t_0 - \tau_1) \end{cases}$$

then the comparison principle implies that $f_1 \leq g$ in $\mathbb{R}^{2d} \times [t_0 - \tau_1, t_0]$. Applying Lemma 7 and Sobolev inequality, then we get

$$\|f_1\|_{L_t^2 L_x^p L_v^1} \leq \|g\|_{L_t^2 L_x^p L_v^1} \leq CC_{0,1} \|f_0\|_2$$

(where $p = 2d/(d - 2s) > 2$) which yields the desired estimate. \square

2.5. Proof of Theorems 3 and 4.

Proof of Theorem 3. Combine Lemmas 6 and 8 and use interpolation to get the result. \square

Proof of Theorem 4. Combine Lemmas 6 and 7, Aubin's lemma and use interpolation to get the result. \square

3. LOCAL UPPER BOUNDS FOR NON-NEGATIVE SUB-SOLUTIONS

In this subsection, we iterate the local gain of integrability to prove that non-negative L^2 sub-solutions are in fact locally bounded (with an estimate).

Theorem 9 (Upper bounds for non-negative L^2 sub-solutions). *Let f be an non-negative L^2 sub-solution of*

$$(\partial_t + v\nabla_x)f \leq \nabla_v \cdot (A\nabla_v f) \quad \text{in } Q_0$$

for a given cube $Q_0 = B_{R_0}(x_0) \times B_{r_0}(v_0) \times (t_0 - \tau_0, t_0]$. Given another cube $Q_\infty = B_{R_\infty}(x_0) \times B_{r_\infty}(v_0) \times (t_0 - \tau_\infty, t_0]$ with $Q_\infty \subset\subset Q_0$, we have

$$\sup_{Q_\infty} f \leq C \|f\|_{L^2(Q_0)}$$

where C depends on dimension, ellipticity constants, p' , Q_0 and Q_∞ .

Proof. To do so, we first remark that, for all $q > 1$, the function f^q satisfies

$$(\partial_t + v\nabla_x)f^q \leq \nabla_v \cdot (A\nabla_v f^q) \quad \text{in } Q_0.$$

We now rewrite (2.1) from $Q_q \subset Q_0$ to $Q_{q+1} \subset\subset Q_q$ as follows:

$$(3.1) \quad \|(f^q)^\kappa\|_{L^2(Q_{q+1})}^2 \leq C_{q+1} \|f^q\|_{L^2(Q_q)}^{2\kappa}$$

where $\kappa = p/2 > 1$ and

$$C_{q+1} = C_0 \left[(\tau_q - \tau_{q+1})^{-1} + (R_q - R_{q+1})^{-1} + (r_q - r_{q+1})^{-2} \right]^\kappa$$

(with $|r_q - r_{q+1}| \leq 1$ and $|R_q - R_{q+1}| \leq 1$) and C_0 only depends on d, Λ, λ and Q_0 .

Choose now $q = q_n = 2\kappa^n$ for $n \in \mathbb{N}$, simply write Q_n for Q_{q_n} and C_n for C_{q_n} and get from (3.1)

$$(3.2) \quad \|f^{q_{n+1}}\|_{L^2(Q_{n+1})}^2 \leq C_{n+1} \|f^{q_n}\|_{L^2(Q_n)}^{2\kappa}.$$

Moreover, we choose

$$\begin{cases} R_{n+1} = R_n - \frac{1}{a(n+1)^2} \\ r_{n+1} = r_n - \frac{1}{b(n+1)^2} \\ \tau_{n+1} = \tau_n - \frac{1}{c(n+1)^2} \end{cases}$$

for some $a, b, c > 0$ so that

$$C_n = C_0 (2\kappa^n + 1) (dn^2 + en^4)^\kappa \quad \text{with } \begin{cases} d = a + b \\ e = c^2. \end{cases}$$

Applying iteratively (3.2), we get the result if

$$\prod_{n=0}^{+\infty} C_n^{\frac{1}{2^n}} < +\infty$$

which indeed holds true. This achieves the proof. \square

4. THE DECREASE OF OSCILLATION LEMMA

It is classical that Hölder continuity is a consequence of the decrease of the oscillation of the solution “at unit scale”.

Lemma 10 (Decrease of oscillation). *Let f be a solution of (1.3) in $Q_2 = B_2(x_0) \times B_2(v_0) \times (-2, 0)$ with $|f| \leq 1$. Then*

$$\text{osc}_{Q_{\frac{1}{2}}} f \leq 2 - \lambda$$

with $Q_{\frac{1}{2}} = B_{\frac{1}{2}}(x_0) \times B_{\frac{1}{2}}(v_0) \times (-\frac{1}{2}, 0)$ for some $\lambda \in (0, 2)$ only depending on dimension and ellipticity constants.

Remark 11. The equation is “invariant” under the following scaling

$$(x, v, t) \mapsto (r^{-3}x, r^{-1}v, r^{-2}t);$$

indeed, it changes $A(x, v, t)$ into $A(r^{-3}x, r^{-1}v, r^{-2}t)$ which still satisfies (1.4).

This lemma is an immediate consequence of the following one.

Lemma 12 (Decrease of the supremum bound). *Let f be a solution of (1.3) in Q_2 with $|f| \leq 1$. If*

$$|\{f \leq 0\} \cap Q_1| \geq \frac{1}{2}|Q_1|$$

with $Q_1 = B_1(x_0) \times B_1(v_0) \times (-1, 0)$, then

$$\sup_{Q_{\frac{1}{2}}} f \leq 1 - \lambda$$

for some $\lambda \in (0, 2)$ only depending on dimension and ellipticity constants.

As explained in [14] for instance, this lemma itself is a consequence of the following one. The details are given in Appendix for the reader’s convenience.

Lemma 13 (A De Giorgi-type lemma). *For all $\delta_1 > 0$ and $\delta_2 > 0$, there exists $\alpha > 0$ such that for all solution f of (1.3) in Q_2 with $|f| \leq 1$ and*

$$\begin{aligned} |\{f \geq \frac{1}{2}\} \cap Q_1| &\geq \delta_1 \\ |\{f \leq 0\} \cap Q_1| &\geq \delta_2 \end{aligned}$$

we have

$$|\{0 < f < \frac{1}{2}\} \cap Q_1| \geq \alpha.$$

Remark 14. It is important to emphasize that the lemma is stated for solutions of (1.3), not sub-solutions.

Remark 15. The idea of proving such a generalization of the classical isoperimetric lemma of De Giorgi is reminiscent of an argument of Guo [7]. See also [14].

Proof. We argue by contradiction by assuming that there exists a sequence f_k of solutions of (1.3) for some diffusion matrix A_k such that $|f_k| \leq 1$ and

$$\begin{aligned} |\{f_k \geq \frac{1}{2}\} \cap Q_1| &\geq \delta_1 \\ |\{f_k \leq 0\} \cap Q_1| &\geq \delta_2 \\ |\{0 < f_k < \frac{1}{2}\} \cap Q_1| &\rightarrow 0 \quad \text{as } k \rightarrow +\infty. \end{aligned}$$

In view of Theorem 4, we can get that f_k converges (up to a subsequence) locally towards f in the strong $L^2_{t,x,v}$ -topology as k tends to $+\infty$. Remark that f still satisfies (1.3) for some A satisfying (1.4) and $|f| \leq 1$. Moreover, the strong L^2 convergence implies convergence in probability, in particular,

$$\begin{aligned} |\{f \geq \frac{1}{2}\} \cap Q_1| &\geq \delta_1 \\ |\{f \leq 0\} \cap Q_1| &\geq \delta_2 \\ |\{0 < f < \frac{1}{2}\} \cap Q_1| &= 0. \end{aligned}$$

Consider now a smooth function $T_\varepsilon : [-1, 1] \rightarrow \mathbb{R}$ such that $T'_\varepsilon(r) = \varepsilon$ in $[-1, 0] \cup [\frac{1}{2}, 1]$ and $T_\varepsilon(0) = 0$ and $T_\varepsilon(\frac{1}{2}) = \frac{1}{2}$. Hence, $f_\varepsilon = T_\varepsilon(f)$ satisfies (1.3) in Q_1 . Arguing as above, we can also prove that $f_\varepsilon \rightarrow \bar{f}$ as $\varepsilon \rightarrow 0$ in L^2 , \bar{f} solves (1.3) and

$$\bar{f} \in \{0, \frac{1}{2}\} \quad \text{and} \quad \bar{f} = \begin{cases} 0 & \text{at points where } f \leq 0 \\ \frac{1}{2} & \text{at points where } f \geq \frac{1}{2}. \end{cases}$$

Moreover, the natural energy estimate for solutions of (1.3) implies that $\bar{f} \in L^2_{t,x} H^1_v$. Hence, by the classical de Giorgi isoperimetric inequality, for almost every $(t, x) \in B_1(x_0) \times (-1, 0)$, we have

$$\begin{aligned} \text{either} \quad & \text{for almost every } v \in B_1(v_0), \quad \bar{f}(t, x, v) \equiv 0 \\ \text{or} \quad & \text{for almost every } v \in B_1(v_0), \quad \bar{f}(t, x, v) \equiv \frac{1}{2}. \end{aligned}$$

In particular, $\nabla_v \bar{f} \equiv 0$ a.e. in Q_1 (i.e. the solutions is everywhere a *local equilibrium* in the terminology of kinetic theory) and consequently,

$$\begin{aligned} \bar{f}(t, x, v) &= \bar{f}(t, x) \in \{0, \frac{1}{2}\} \\ |\{\bar{f} = \frac{1}{2}\} \cap (-1, 0) \times B_1| &\geq \frac{\delta_1}{|B_1|} \end{aligned}$$

$$|\{\bar{f} = 0\} \cap B_1 \times (-1, 0)| \geq \frac{\delta_2}{|B_1|}$$

For a.e. $v \in B_1(0)$, $\partial_t \bar{f} + (v_0 + v) \cdot \nabla_x \bar{f} = 0$ in $B_1 \times (-1, 0)$.

In particular, rewriting the equation for $-v$, summing and using all $v \in B_1(0)$, we get

$$\partial_t f + v_0 \cdot \nabla_x f \equiv 0, \nabla_x f \equiv 0$$

which, in turn, yields that f is constant (i.e. is a *global equilibrium* in the terminology of kinetic theory), which contradicts the lower bounds on the measure of the sets above. We thus get the desired contradiction. The proof is complete. \square

APPENDIX A. GETTING LEMMA 12 FROM LEMMA 13

Proof of Lemma 12. We follow the nice exposition of [14]. Let C_0 be the universal constant such that solutions f of (1.3) in Q_2 satisfy

$$\|f_+\|_{L^\infty(Q_{\frac{1}{2}})} \leq C_0 \|f_+\|_{L^2(Q_1)}.$$

We now define $f_1 = f$ and $f_{k+1} = 2f_k - 1$. Remark that

$$\begin{aligned} |\{f_1 \leq 0\} \cap Q_1| &\geq \delta_1 \\ \{f_{k+1} \leq 0\} &\supset \{f_k \leq 0\} \end{aligned}$$

with $\delta_1 = |Q_1|/2$ (remark it is universal). Our goal is to prove that there exists k_0 universal such that

$$|\{f_{k_0} \geq 0\} \cap Q_1| \leq \delta_2$$

with $\delta_2 = (4C_0^2)^{-1}$ (remark it is universal). Indeed, this implies

$$\|(f_{k_0})_+\|_{L^\infty(Q_{\frac{1}{2}})} \leq C_0 \|(f_{k_0})_+\|_{L^2(Q_1)} \leq C_0 \left[|\{f_{k_0} \geq 0\} \cap Q_1| \right]^{\frac{1}{2}} \leq \frac{1}{2}$$

which, in turn, yields

$$f \leq 1 - 2^{-k_0-1} \quad \text{in } Q_{\frac{1}{2}}.$$

Assume that for all $k \geq 1$,

$$|\{f_k \geq 0\} \cap Q_1| \geq \delta_2.$$

Since $f_{k+1} = 2f_k - 1$, this also implies

$$|\{f_k \geq \frac{1}{2}\} \cap Q_1| \geq \delta_2.$$

But we also have

$$|\{f_k \leq 0\} \cap Q_1| \geq |\{f \leq 0\} \cap Q_1| \geq \delta_1.$$

Hence Lemma 13 implies that

$$|\{0 \leq f_k \leq \frac{1}{2}\} \cap Q_1| \geq \alpha.$$

Now remark that

$$\begin{aligned} |Q_1| &\geq |\{f_{k+1} \leq 0\} \cap Q_1| = |\{f_k \leq 0\} \cap Q_1| + |\{0 \leq f_k \leq \frac{1}{2}\} \cap Q_1| \\ &\geq |\{f_k \leq 0\} \cap Q_1| + \alpha \\ &\geq k\alpha \end{aligned}$$

which is impossible for k large enough. □

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