



# HOLDER CONTINUITY OF SOLUTIONS TO QUASILINEAR HYPOELLIPTIC EQUATIONS

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# HÖLDER CONTINUITY OF SOLUTIONS TO QUASILINEAR HYPOELLIPTIC EQUATIONS

C. IMBERT & C. MOUHOT

ABSTRACT. We prove that  $L^2$  weak solutions to a quasilinear hypoelliptic equations with rough coefficients are Hölder continuous. The proof relies on classical techniques developed by De Giorgi and Moser together with the averaging lemma developed in kinetic theory. The latter tool is used in the proof of the local gain of integrability of sub-solutions and in the proof of an “hypoelliptic isoperimetric De Giorgi lemma”, obtained by combining the classical isoperimetric inequality on the diffusive variable with the structure of the integral curves of the first-order part of the operator.

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## 1. INTRODUCTION

**1.1. The question studied and its history.** We consider the following nonlinear kinetic Fokker-Planck equation

$$(1.1) \quad \partial_t f + v \cdot \nabla_x f = \rho \nabla_v \cdot (\nabla_v f + v f), \quad t \geq 0, \quad x \in \mathbb{T}^d, \quad v \in \mathbb{R}^d,$$

where  $d \in \mathbb{N}^*$ ,  $\mathbb{T}^d$  is the flat  $d$ -dimensional torus,  $f = f(t, x, v) \geq 0$  and  $\rho[f] = \int_{\mathbb{R}^d} f(t, x, v) dv$ . The construction of global smooth solutions for such a problem is one motivation for the present paper.

The linear kinetic Fokker-Planck equation  $\partial_t f + v \cdot \nabla_x f = \nabla_v \cdot (\nabla_v f + v f)$  is sometimes called the Kolmogorov-Fokker-Planck equation, as it was studied by Kolmogorov in the seminal paper [9], when  $x \in \mathbb{R}^d$ . In this note, Kolmogorov explicitly calculated the fundamental solution and deduced

regularisation in both variables  $x$  and  $v$ , even though the operator  $\nabla_v \cdot (\nabla_v + v) - v \cdot \nabla_x$  shows ellipticity in the  $v$  variable only. It inspired Hörmander and his theory of hypoellipticity [8], where the regularisation is recovered by more robust and more geometric commutator estimates (see also [13]).

Another question which has attracted a lot of attention in calculus of variations and partial differential equations along the 20th century is Hilbert's 19th problem about the analytic regularity of solutions to certain integral variational problems, when the quasilinear Euler-Lagrange equations satisfy ellipticity conditions. Several previous results had established the analyticity conditionally to some differentiability properties of the solution, but the full answer came with the landmark works of De Giorgi [2, 3] and Nash [11], where they prove that any solution to these variational problems with square integrable derivative is analytic. More precisely their key contribution is the following<sup>1</sup>: reformulate the quasilinear parabolic problem as

$$(1.2) \quad \partial_t f = \nabla_v \cdot (A(v, t) \nabla_v f), \quad t \geq 0, \quad v \in \mathbb{R}^d$$

with  $f = f(v, t) \geq 0$  and  $A = A(v, t)$  satisfies the ellipticity condition  $0 < \lambda I \leq A \leq \Lambda I$  for two constants  $\lambda, \Lambda > 0$  but is, besides that, merely measurable. Then the solution  $f$  is Hölder continuous.

In view of the nonlinear (quasilinear) equation (1.1) it is natural to ask whether a similar result as the one of De Giorgi-Nash holds for quasilinear hypoelliptic equations. In order to avoid unnecessary generality we shall focus on the following equation motivated by physics:

$$(1.3) \quad \partial_t f + v \cdot \nabla_x f = \nabla_v \cdot (A(x, t) \nabla_v f), \quad t \in (0, T), (x, v) \in \Omega,$$

where  $\Omega$  is an open set of  $\mathbb{R}^{2d}$ ,  $f = f(t, x, v) \geq 0$  and the  $d \times d$  symmetric matrix  $A$  satisfies the ellipticity condition

$$(1.4) \quad 0 < \lambda I \leq A \leq \Lambda I$$

for two constants  $\lambda, \Lambda$  but is, besides that, merely measurable. We want to establish the Hölder continuity of  $L^2$  solutions to this problem. In order to do so, we first prove that  $L^2$  sub-solutions are locally bounded; we refer to such a result as an  $L^2 - L^\infty$  estimate. We then prove that solutions are Hölder continuous by proving a lemma which is an hypoelliptic counterpart to De Giorgi's isoperimetric lemma. We denote by  $B_r(x_0)$  and  $B_r(v_0)$  the usual Euclidian balls in  $x$  and  $v$ .

**Theorem 1** (Hölder continuity). *Let  $f$  be a non-negative solution of (1.3) in a cube  $Q_0 := B_2(v_0) \times B_2(x_0) \times (-1, 0)$ . Then  $f$  is  $\alpha$ -Hölder continuous with respect to  $(x, v, t)$  in  $Q_1 = B_1(v_0) \times B_1(x_0) \times (-1/2, 0)$  and*

$$\|f\|_{C^\alpha(Q_1)} \lesssim \|f\|_{L^2(Q_0)}$$

*for some  $\alpha$  only depending on dimension and ellipticity constants.*

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<sup>1</sup>We give the parabolic version due to Nash here.

*Remark 2.* Cubes of general size are then obtained by scaling. Note also that the equation is translation invariant in  $x$  but not in  $v$  which explains why we keep track of the centers of the balls (we could however have reduced to  $x_0 = 0$  without loss of generality in the statement but preferred not to).

We make a rather strong assumption on  $A$  by assuming that it does not depend on  $v$ . Indeed, in [12], the authors obtain an  $L^2 - L^\infty$  estimate with completely different techniques and for  $A$  depending on  $v$ ; however they cannot reach the Hölder continuity estimate in such a general setting. Our techniques rely on averaging lemma [4, 5] in order to gain some regularity  $H_x^s$ ,  $s > 0$  small, in the space variable  $x$  from the natural  $H_v^1$  estimate. We emphasize that such  $H_x^s$  estimates do not hold for sub-solutions. From this Sobolev estimate, we can recover a gain of integrability for  $L^2$  sub-solutions, and we then prove the Hölder continuity through a De Giorgi type argument on the decrease of oscillation for solutions.

In [16] (see also [15, 17]), the authors get a Hölder estimate for  $L^2$  weak solutions of so-called ultraparabolic equations, which include (1.3) with  $v$ -depending  $A$ 's. Their proof relies on the construction of cut-off functions and a particular form of weak Poincaré inequality satisfied by non-negative weak sub-solutions. However, we were not able to check their proof entirely and the meaning of the latter estimate is not clear to us. We hope that the method presented in the present article, in a more restrictive setting, sheds a new light on the regularizing effect for hypoelliptic equations with rough coefficients and provide tools for further applications. With this respect, we mention that we plan to use the main result of this paper in a future work to study (1.1) thoroughly.

We finally mention that Golse and Vasseur told us that they proved independently a similar result [6], also assuming that  $A$  is independent of  $v$ .

**1.2. Plan of the paper.** In Section 2, we exhibit the scaling of the equation, introduce some notation and recall an averaging lemma. In Section 3, we explain how to get a universal gain of integrability for non-negative  $L^2$  sub-solutions; we also explain how to get a gain of regularity for (signed)  $L^2$  solutions. Section 4 is dedicated to deriving a local upper bound of such non-negative  $L^2$  sub-solutions by using Moser iteration procedure. In the final section 5, the Hölder estimate is derived by proving a reduction of oscillation lemma.

## 2. PRELIMINARIES

**2.1. Scaling and neighbourhoods.** The scaling

$$(x, v, t) \mapsto (r^{-3}x, r^{-1}v, r^{-2}t)$$

only changes  $A(x, t)$  into  $A(r^{-3}x, r^{-2}t)$  which still satisfies (1.4). We use it to construct local neighborhoods around a point  $z := (x, v, t)$ :

$$(2.1) \quad Q_{z,r} = B_{r^3}(x) \times B_r(v) \times (t - r^2, t].$$

**2.2. Truncation.** For  $z_0 = (x_0, v_0, t_0)$  and  $0 < R < S$  and  $0 < r < s$  and  $0 < \tau < \sigma < \sqrt{\frac{3}{2}}\tau$ , we consider

$$\chi^{z_0}(x, v, t) = \bar{\chi}_{R,S}(x - x_0) \bar{\chi}_{r,s}(v - v_0) \bar{\theta}_{\tau,\sigma}(t - t_0)$$

where

- for  $0 < a < b$ ,  $\bar{\chi}_{a,b}: \mathbb{R}^d \rightarrow [0, 1]$  is  $C^\infty$ , equals 1 in  $B_a$  and 0 outside  $B_b$ ;
- for  $0 < \tau < \sigma$ ,  $\bar{\theta}_{\tau,\sigma}: \mathbb{R} \rightarrow [0, 1]$  is  $C^\infty$ , equals 1 in  $[-\tau, 0]$  and 0 outside  $[-\sigma, 0]$ .

**Lemma 3** (Truncation of sub-solutions). *Given a cube  $Q$ , if  $f: Q \rightarrow \mathbb{R}$  solves*

$$(\partial_t + v \cdot \nabla_x) f \leq \nabla_v \cdot (A \nabla_v f)$$

*and  $\chi: Q \rightarrow \mathbb{R}$  is smooth, non-negative and compactly supported in  $Q$ , then  $\tilde{f} = f\chi$  satisfies*

$$(\partial_t + v \cdot \nabla_x) \tilde{f} \leq \nabla_v \cdot (A \nabla_v \tilde{f}) + \nabla_v \cdot (f \Gamma) + \alpha f$$

*where*

$$\Gamma = 2A \nabla_v \chi \quad \text{and} \quad \alpha = (\partial_t + v \cdot \nabla_x) \chi - \nabla_v \cdot (A \nabla_v \chi).$$

*Remark 4.* The proof is a straightforward calculation. Note that we use here the fact that  $A$  does not depend on  $v$ . This is the only moment in the proof where it is used.

**2.3. An averaging lemma.** This result is a consequence of [1] for instance.

**Lemma 5** (An averaging lemma). *There exists  $\eta > 0$  universal ( $\eta = \frac{1}{7}$ ) such that for all  $h \in L^2_{t,x,v}$  satisfying*

$$(\partial_t + v \cdot \nabla_x) h = \nabla_v^2 H_2 + \nabla_v \cdot H_1 + H_0$$

*where  $\nabla_v^2 H_2 = \partial_{v_i} \partial_{v_j} H_2^{ij}$  with  $H_0, H_1, H_2^{ij} \in L^2$ , the function*

$$\rho_\phi(t, x) = \int_{\mathbb{R}^d} h(t, x, v) \phi(v) dv$$

*(for  $\phi \in C_c^2(\mathbb{R}^d)$ ) satisfies*

$$\|\rho_\phi\|_{L_t^2 H_x^\eta} \leq C_d \|\phi\|_{2,\infty} \max \left\{ R_\phi^{\frac{1}{2}}, 1 \right\} R_\phi^{\frac{d-1}{2}} \left( \|h\|_2 + \|H_0\|_2 + \|H_1\|_2 + \|H_2\|_2 \right)$$

*where  $C_d$  only depends on dimension and  $R_\phi > 0$  is such that  $\text{supp } \phi \subset B_{R_\phi}$ .*

### 3. LOCAL GAIN OF REGULARITY / INTEGRABILITY

We consider the equation (1.3) and we want to establish a local gain of integrability in order to apply Moser's iteration. Since we will need to perform convex changes of unknown, it is crucial to obtain this gain on (non-negative) *sub-solutions*. We will see that we can in fact also obtain a gain of regularity for signed weak solutions.

**Theorem 6** (Gain of integrability for non-negative sub-solutions). *Consider two cubes  $Q_1 \subset\subset Q_0$ , both centered at  $(x_0, v_0, t_0) \in \mathbb{R}^{2d} \times (0, +\infty)$ . There exists  $q > 2$  (universal) such that for all non-negative  $L^2$  sub-solution  $f$  of (1.3) in  $Q_0$ , we have*

$$(3.1) \quad \|f\|_{L^q(Q_1)} \leq C \|f\|_{L^2(Q_0)}$$

where and  $C = \bar{C} C_0 C_{0,1}$  with

$$\begin{cases} C_0 &= \max \left\{ (|v_0| + r_0)^{\frac{1}{2}}, (|v_0| + r_0)^{\frac{d}{2}}, (|v_0| + r_0)^{\frac{d-1}{2}} \right\} \\ C_{0,1} &= [(\tau_0 - \tau_1)^{-1} + (R_0 - R_1)^{-1} + (r_0 - r_1)^{-2} + (r_0 - r_1)^{-4}]^{\frac{1}{2}} \end{cases}$$

where  $\bar{C} = C(d, \lambda, \Lambda)$  depends on  $d$  and upper bounds on  $\Lambda$  and  $\lambda^{-1}$ , and  $Q_i = B_{R_i}(x_0) \times B_{r_i}(v_0) \times (t_i - \tau_i, t_i]$ ,  $i = 0, 1$  and  $0 < r_0 < r_1$  and  $0 < R_0 < R_1$ .

Observe that the result extends to weak signed solutions  $f$  to (1.3) immediately by observing that then  $f_{\pm} = (|f| \pm f)/2$  are sub-solutions (1.3). As a matter of fact, for weak signed solutions  $f$ , we can even get a gain of regularity. This will be crucial in the proof of the decrease of oscillation lemma.

**Theorem 7** (Gain of regularity for signed solutions). *Consider two cubes  $Q_1 \subset\subset Q_0$ , both centered at  $(x_0, v_0, t_0) \in \mathbb{R}^{2d} \times (0, +\infty)$ . There exists  $s > 0$  (universal) such that for all (signed)  $L^2$  weak solution  $f$  of (1.3) in  $Q_0$ , we have*

$$(3.2) \quad \|f\|_{H_{x,v,t}^s(Q_1)} \leq C \|f\|_{L^2(Q_0)}$$

where and  $C = C(d, \lambda, \Lambda, Q_0, Q_1)$ .

**3.1. Gain of integrability with respect to  $x$ .** In order to prove Theorem 6, we first show how to improve the integrability with respect to the  $x$  variable.

**Lemma 8** (Gain of integrability w.r.t.  $x$ ). *Under the assumptions of Theorem 6, there exists  $p > 2$  such that*

$$(3.3) \quad \|f\|_{L_t^2 L_x^p L_v^1(Q_1)} \leq \bar{C} C_0 (r_0 - r_1)^{-2} \|f\|_{L^2(Q_0)}$$

with  $C_0 = \max \left\{ (|v_0| + r_0)^{\frac{d}{2}}, (|v_0| + r_0)^{\frac{d-1}{2}} \right\}$  and  $\bar{C} = C(d, \Lambda)$  depends on  $d$  and an upper bound on  $\Lambda$ .

*Remark 9.* We can choose from the proof:  $p = (2d)/(d - 2/7) > 2$ .

*Proof.* For  $i = 0, 1$ , consider  $f_i = f \chi_i$  where  $\chi_i$  are two truncation functions that we previously introduced associated with parameters

$$(R_1, R_0, r_1, r_0, \tau_1, \tau_0).$$

In particular,  $\chi_i$  equals 1 in  $Q_i$  for  $i = 0, 1$ . We know from Lemma 3 that

$$(\partial_t + v \cdot \nabla_x) f_1 \leq \nabla_v \cdot (A \nabla_v f_1) + \nabla_v (f_0 \Gamma_1) + \alpha_1(f_0) \text{ in } \mathbb{R}^{2d} \times \mathbb{R}$$

$$\text{with } \begin{cases} \alpha_1 = (\partial_t + v \cdot \nabla_x) \chi_1 - \nabla_v(A \nabla_v \chi_1) \\ \Gamma_1 = 2 \nabla_v \chi_1. \end{cases}$$

The previous inequation holds true in  $\mathbb{R}^{2d} \times (-\infty; t_0]$  since  $\alpha_1, \beta_1, \Gamma_1$  and  $f_1$  are supported in  $Q_0 := Q_{r_0}(z_0)$ .

If  $g$  solves

$$\begin{cases} (\partial_t + v \cdot \nabla_x) g = \nabla_v(A \nabla_v f_1) + \nabla_v(f_0 \Gamma_1) + \alpha_1(f_0) \\ g(x, v, t_0 - \tau_1) = f_1(x, v, t_0 - \tau_1) \end{cases}$$

then the comparison principle implies that  $f_1 \leq g$  in  $\mathbb{R}^{2d} \times [t_0 - \tau_1, t_0]$ . We now write

$$\nabla_v(A \nabla_v f_1) + \nabla_v(f_0 \Gamma_1) + \alpha_1(f_0) = \nabla_v^2 H_2 + \nabla_v \cdot H_1 + H_0$$

with

$$H_2 = A f_1, \quad H_1 = f_0 \Gamma_1, \quad H_0 = \alpha_1 f_0.$$

We now apply the averaging lemma 5 with  $\phi(v) = \bar{\chi}_{r_1, r_0}(v - v_0)$  and use the Sobolev inequality to get

$$\begin{aligned} \|f_1\|_{L_t^2 L_x^p L_v^1} &\leq \|g\|_{L_t^2 L_x^p L_v^1} \\ &= \|\rho_\phi\|_{L_t^2 L_x^p} \\ &\leq \bar{C}(r_0 - r_1)^{-2} C_0 \|f_0\|_2 \end{aligned}$$

where  $p = 2d/(d - 2\eta) > 2$  and  $\bar{C} = C(d, \Lambda)$  and  $C_0 = \max\{(|v_0| + r_0)^{d/2}, (|v_0| + r_0)^{(d-1)/2}\}$ . The proof of Lemma 8 is now complete.  $\square$

In view of the previous proof, we notice that the following lemma also holds:

**Lemma 10** (Gain of regularity w.r.t.  $x$ ). *Under the assumptions of Theorem 6, if  $f$  is a signed weak solution to (1.3) (instead of a non-negative sub-solution), then there exists  $s > 0$  such that*

$$(3.4) \quad \|D_x^s f\|_{L^2(Q_1)} \leq \bar{C} C_0 (r_0 - r_1)^{-2} \|f\|_{L^2(Q_0)}$$

with  $C_0$  and  $\bar{C}$  as above in Lemma 8.

Indeed, instead of applying Lemma 5, one should apply for instance [1, Theorem 1.3] with  $p = 2$ ,  $r = 0$ ,  $\beta = 1$ ,  $m = 2$ ,  $\kappa = 1$  and  $\Omega = 0$ . This gives  $s = 1/5$ .

**3.2. Gain of integrability with respect to  $v$  and  $t$ .** The gain of integrability with respect to  $v$  and  $t$  is more classical. It derives from the natural energy estimate, after truncation.

We follow here [10] in order to get the following lemma.

**Lemma 11** (Gain of integrability w.r.t.  $v$  and  $t$ ). *Under the assumptions of Theorem 6, the function  $f$  satisfies*

$$\|f\|_{L_t^2 L_x^2 L_v^q(Q_1)}^2 \leq C \left[ (\tau_0 - \tau_1)^{-1} + (R_0 - R_1)^{-1} + (r_0 - r_1)^{-2} \right] \int_{Q_0} f^2$$

with  $q = (2d)/(d-2) > 2$  and

$$\|f\|_{L_t^\infty L_x^2 L_v^2(Q_1)}^2 \leq C \left[ (\tau_0 - \tau_1)^{-1} + (R_0 - R_1)^{-1} + (r_0 - r_1)^{-2} \right] \int_{Q_0} f^2$$

with  $C = C(\Lambda, d) \times (|v_0| + r_0)$ .

*Proof.* Consider  $\Psi \in C_c^\infty(\mathbb{R}^{2d} \times \mathbb{R})$  and integrate the inequation satisfied by  $f$  against  $2f\Psi^2$  in  $\mathbb{R}^{2d} \times [t_1, t_0]$  with  $t_1 \in [t_0 - r_1^2, t_0]$  and get

$$\int \partial_t(f^2)\Psi^2 + \int v \cdot \nabla_x(f^2)\Psi^2 \leq \int \nabla_v(A\nabla_v f)f\Psi^2.$$

Add  $\int f^2 \partial_t(\Psi^2)$ , integrate by parts several times and use the upper bound on  $A$  in order to get

$$\begin{aligned} & \int \partial_t(f^2\Psi^2) + \int (A\nabla_v f \cdot \nabla_v f)\Psi^2 \\ & \leq \int f^2(\partial_t + v \cdot \nabla_x)(\Psi^2) + 2 \int \Psi \sqrt{A} \nabla_v f \cdot f \sqrt{A} \nabla_v \Psi \\ & \leq \int f^2(\partial_t + v \cdot \nabla_x)(\Psi^2) + \frac{1}{2} \int (A\nabla_v f \cdot \nabla_v f)\Psi^2 + 2 \int f^2(A\nabla_v \Psi \cdot \nabla_v \Psi). \end{aligned}$$

We thus get

$$(3.5) \quad \int \partial_t(f^2\Psi^2) + \frac{\lambda}{2} \int |\nabla_v f|^2 \Psi^2 \leq C \left( \|\Psi\|_{t,x;1,\infty} + \|\Psi\|_{v;2,\infty} \right) \int_{\text{supp } \Psi} f^2$$

with  $C = C(\Lambda, d) \times (|v_0| + r_0)$ . Choose next  $\Psi^2$  such that  $\Psi^2(t_0) = 0$  and  $\text{supp } \Psi^2 \subset Q_0$  and get

$$\int_{x,v} f^2 \Psi^2(t_1) + \frac{\lambda}{2} \int |\nabla_v f|^2 \Psi^2 \leq C \left( \|\Psi\|_{t,x;1,\infty} + \|\Psi\|_{v;2,\infty} \right) \int_{Q_0} f^2.$$

If  $\Psi^2$  additionally satisfies  $\Psi^2 \equiv 1$  in  $Q_1$ , we get

$$\int_{Q_1} |\nabla_v f|^2 \leq C \left[ (\tau_0 - \tau_1)^{-1} + (R_0 - R_1)^{-1} + (r_0 - r_1)^{-2} \right] \int_{Q_0} f^2$$

with  $C = C(d, \Lambda, \lambda^{-1}) \times (|v_0| + r_0)$ . The Sobolev inequality then implies

$$\|f\|_{L_t^\infty L_x^2 L_v^q(Q_1)}^2 \leq C \left[ (\tau_0 - \tau_1)^{-1} + (R_0 - R_1)^{-1} + (r_0 - r_1)^{-2} \right] \int_{Q_0} f^2$$

with  $q = (2d)/(d-2) > 2$ .

If now  $t_1 \in [t_0 - r_1^2, t_0]$  is arbitrary, we get the second estimate, that is

$$\|f\|_{L_t^\infty L_x^2 L_v^2(Q_1)}^2 \leq C \left[ (\tau_0 - \tau_1)^{-1} + (R_0 - R_1)^{-1} + (r_0 - r_1)^{-2} \right] \int_{Q_0} f^2. \quad \square$$



### 3.3. Proof of Theorems 6 and 7.

*Proof of Theorem 6.* Combine Lemmas 8 and 11 and use interpolation to get the result.  $\square$

*Proof of Theorem 7.* Combine Lemma 10 and (3.5), Aubin lemma and use interpolation to get the result.  $\square$

## 4. LOCAL UPPER BOUNDS FOR NON-NEGATIVE SUB-SOLUTIONS

In this subsection, we iterate the local gain of integrability to prove that non-negative  $L^2$  sub-solutions are in fact locally bounded (with an estimate).

**Theorem 12** (Upper bounds for non-negative  $L^2$  sub-solutions). *Let  $f$  be an non-negative  $L^2$  sub-solution of*

$$(\partial_t + v \nabla_x) f \leq \nabla_v \cdot (A \nabla_v f) \quad \text{in } Q_0$$

*for a given cube  $Q_0 = B_{R_0}(x_0) \times B_{r_0}(v_0) \times (t_0 - \tau_0, t_0]$ . Given another cube  $Q_\infty = B_{R_\infty}(x_0) \times B_{r_\infty}(v_0) \times (t_0 - \tau_\infty, t_0]$  with  $Q_\infty \subset \subset Q_0$ , we have*

$$\sup_{Q_\infty} f \leq C \|f\|_{L^2(Q_0)}$$

*where  $C$  depends on dimension, ellipticity constants,  $p'$ ,  $Q_0$  and  $Q_\infty$ .*

*Proof.* To do so, we first remark that, for all  $q > 1$ , the function  $f^q$  satisfies

$$(\partial_t + v \nabla_x) f^q \leq \nabla_v \cdot (A \nabla_v f^q) \quad \text{in } Q_0.$$

We now rewrite (3.1) from  $Q_q \subset Q_0$  to  $Q_{q+1} \subset \subset Q_q$  as follows:

$$(4.1) \quad \|(f^q)^\kappa\|_{L^2(Q_{q+1})}^2 \leq C_{q+1} \|f^q\|_{L^2(Q_q)}^2$$

where  $\kappa = p/2 > 1$  and

$$C_{q+1} = C_0(q+1) \left[ (\tau_q - \tau_{q+1})^{-1} + (R_q - R_{q+1})^{-1} + (r_q - r_{q+1})^{-4} \right]^\kappa$$

(with  $|r_q - r_{q+1}| \leq 1$ ) and  $C_0$  only depends on  $d$ , upper bounds on  $\Lambda$ ,  $\lambda^{-1}$  and  $Q_0$ .

Choose now  $q = q_n = 2\kappa^n$  for  $n \in \mathbb{N}$ , simply write  $Q_n$  for  $Q_{q_n}$  and  $C_n$  for  $C_{q_n}$  and get from (4.1)

$$(4.2) \quad \|f^{q_{n+1}}\|_{L^2(Q_{n+1})}^2 \leq C_{n+1} \|f^{q_n}\|_{L^2(Q_n)}^{2\kappa}.$$

Moreover, we choose

$$\begin{cases} R_{n+1} = R_n - \frac{1}{a(n+1)^2} \\ r_{n+1} = r_n - \frac{1}{b(n+1)^2} \\ \tau_{n+1} = \tau_n - \frac{1}{c(n+1)^2} \end{cases}$$

for some  $a, b, c > 0$  so that

$$C_n = C_0(2\kappa^n + 1)(dn^2 + en^8)^\kappa \quad \text{with} \quad \begin{cases} d = a + b \\ e = c^4. \end{cases}$$

Applying iteratively (4.2), we get the result if

$$\prod_{n=0}^{+\infty} C_n^{\frac{1}{2\kappa^n}} < +\infty$$

which indeed holds true. This achieves the proof in the case  $p' = 2$ .  $\square$

## 5. THE DECREASE OF OSCILLATION LEMMA

It is classical that Hölder continuity is a consequence of the decrease of the oscillation of the solution “at unit scale”.

**Lemma 13** (Decrease of oscillation). *Let  $f$  be a solution of (1.3) in  $Q_2 = B_2(x_0) \times B_2(v_0) \times (-2, 0)$  with  $|f| \leq 1$ . Then*

$$\text{osc}_{Q_{\frac{1}{2}}} f \leq 2 - \lambda$$

*with  $Q_{\frac{1}{2}} = B_{\frac{1}{2}}(x_0) \times B_{\frac{1}{2}}(v_0) \times (-\frac{1}{2}, 0)$  for some  $\lambda \in (0, 2)$  only depending on dimension and ellipticity constants.*

This lemma is an immediate consequence of the following one.

**Lemma 14** (Decrease of the supremum bound). *Let  $f$  be a solution of (1.3) in  $Q_2$  with  $|f| \leq 1$ . If*

$$|\{f \leq 0\} \cap Q_1| \geq \frac{1}{2}|Q_1|$$

*with  $Q_1 = B_1(x_0) \times B_1(v_0) \times (-1, 0)$ , then*

$$\sup_{Q_{\frac{1}{2}}} f \leq 1 - \lambda$$

*for some  $\lambda \in (0, 2)$  only depending on dimension and ellipticity constants.*

As explained in [14] for instance, this lemma itself is a consequence of the following one. The details are given in Appendix for the reader's convenience.

**Lemma 15** (A De Giorgi-type lemma). *For all  $\delta_1 > 0$  and  $\delta_2 > 0$ , there exists  $\alpha > 0$  such that for all solution  $f$  of (1.3) in  $Q_2$  with  $|f| \leq 1$  and*

$$\begin{aligned} |\{f \geq \frac{1}{2}\} \cap Q_1| &\geq \delta_1 \\ |\{f \leq 0\} \cap Q_1| &\geq \delta_2 \end{aligned}$$

*we have*

$$|\{0 < f < \frac{1}{2}\} \cap Q_1| \geq \alpha.$$

*Remark 16.* It is important to emphasize that the lemma is stated for solutions of (1.3), not sub-solutions.

*Remark 17.* The idea of proving such a generalization of the classical isoperimetric lemma of De Giorgi is reminiscent of an argument of Guo [7]. See also [14].

*Proof.* We argue by contradiction by assuming that there exists a sequence  $f_k$  of solutions of (1.3) such that  $|f_k| \leq 1$  and

$$\begin{aligned} |\{f_k \geq \frac{1}{2}\} \cap Q_1| &\geq \delta_1 \\ |\{f_k \leq 0\} \cap Q_1| &\geq \delta_2 \\ |\{0 < f_k < \frac{1}{2}\} \cap Q_1| &\rightarrow 0 \quad \text{as } k \rightarrow +\infty. \end{aligned}$$

In view of Theorem 7, we can get that  $f_k$  converges (up to a subsequence) locally towards  $f$  in the strong  $L^2_{t,x,v}$ -topology as  $k$  tends to  $+\infty$ . Remark that  $f$  still satisfies (1.3) and  $|f| \leq 1$ . Moreover, the strong  $L^2$  convergence implies convergence in probability, in particular,

$$\begin{aligned} |\{f \geq \frac{1}{2}\} \cap Q_1| &\geq \delta_1 \\ |\{f \leq 0\} \cap Q_1| &\geq \delta_2 \\ |\{0 < f < \frac{1}{2}\} \cap Q_1| &= 0. \end{aligned}$$

Consider now a smooth function  $T_\varepsilon : [-1, 1] \rightarrow \mathbb{R}$  such that  $T'_\varepsilon(r) = \varepsilon$  in  $[-1, 0] \cup [\frac{1}{2}, 1]$  and  $T_\varepsilon(0) = 0$  and  $T_\varepsilon(\frac{1}{2}) = \frac{1}{2}$ . Hence,  $f_\varepsilon = T_\varepsilon(f)$  satisfies (1.3) in  $Q_1$ . Arguing as above, we can also prove that  $f_\varepsilon \rightarrow \bar{f}$  as  $\varepsilon \rightarrow 0$  in  $L^2$ ,  $\bar{f}$  solves (1.3) and

$$\bar{f} \in \{0, \frac{1}{2}\} \quad \text{and} \quad \bar{f} = \begin{cases} 0 & \text{at points where } f \leq 0 \\ \frac{1}{2} & \text{at points where } f \geq \frac{1}{2}. \end{cases}$$

Moreover, the natural energy estimate for solutions of (1.3) implies that  $\bar{f} \in L^2_{t,x} H^1_v$ . Hence, by the classical de Giorgi isoperimetric inequality, for almost every  $(t, x) \in B_1(x_0) \times (-1, 0)$ , we have

$$\begin{aligned} \text{either} \quad & \text{for almost every } v \in B_1(v_0), \quad \bar{f}(t, x, v) \equiv 0 \\ \text{or} \quad & \text{for almost every } v \in B_1(v_0), \quad \bar{f}(t, x, v) \equiv \frac{1}{2}. \end{aligned}$$

In particular,  $\nabla_v \bar{f} \equiv 0$  a.e. in  $Q_1$  (i.e. the solutions is everywhere a *local equilibrium* in the terminology of kinetic theory) and consequently,

$$\begin{aligned} \bar{f}(t, x, v) &= \bar{f}(t, x) \in \{0, \frac{1}{2}\} \\ |\{\bar{f} = \frac{1}{2}\} \cap (-1, 0) \times B_1| &\geq \frac{\delta_1}{|B_1|} \\ |\{\bar{f} = 0\} \cap B_1 \times (-1, 0)| &\geq \frac{\delta_2}{|B_1|} \\ \forall v \in B_1(0), \quad \partial_t \bar{f} + (v_0 + v) \cdot \nabla_x \bar{f} &= 0 \quad \text{in } B_1 \times (-1, 0). \end{aligned}$$

In particular, rewriting the equation for  $-v$ , summing and using all  $v \in B_1(0)$ , we get

$$\partial_t f + v_0 \cdot \nabla_x f \equiv 0, \nabla_x f \equiv 0$$

which, in turn, yields that  $f$  is constant (i.e. is a *global equilibrium* in the terminology of kinetic theory), which contradicts the lower bounds on the measure of the sets above. We thus get the desired contradiction. The proof is complete.  $\square$

#### APPENDIX A. PROOF OF THE AVERAGING LEMMA

*Proof of Lemma 5.* The Laplace-Fourier transform of  $h$  with respect to time and space:

$$\tilde{h}(\tau, \xi, v) = \int_0^{+\infty} \int_{\mathbb{R}^d} e^{-t\tau} e^{-iv \cdot \xi} h(t, x, v) dt dx$$

satisfies

$$(\tau + i\xi \cdot v)\tilde{h} = \Delta_v \tilde{H}_2 + \nabla_v \cdot \tilde{H}_1 + \tilde{H}_0.$$

The function

$$\tilde{\rho}_\phi = \int \tilde{h}(\tau, \xi, v) \phi(v) dv$$

can be split into two parts:

$$\tilde{\rho}_\phi = R_\alpha + R^\alpha$$

with

$$\begin{cases} R^\alpha = \int \tilde{h}(\tau, \xi, v) \phi(v) \psi_\alpha(\tau + i\xi \cdot v) dv \\ R_\alpha = \int \tilde{h}(\tau, \xi, v) \phi(v) (1 - \psi_\alpha)(\tau + i\xi \cdot v) dv \end{cases}$$

with the truncation function  $\psi_\alpha(r) = \Psi(r/\alpha)$  with  $\Psi \in C^\infty(\mathbb{C}, [0, 1])$ ,  $\Psi \equiv 1$  outside  $D_1$ ,  $\Psi \equiv 0$  in  $D_{\frac{1}{2}}$ .

If  $\langle \xi \rangle$  denotes  $\sqrt{1 + |\xi|^2}$ , then we first remark that for  $\eta > 0$ ,

$$\begin{aligned} \int |\xi|^{2\eta} R_\alpha^2 d\tau d\xi &\leq \int |\xi|^{2\eta} \left( \int_{|\tau + i\xi \cdot v| \leq \alpha} \tilde{h} \phi \right)^2 d\tau d\xi \\ &\leq \int |\xi|^{2\eta} \left( \int \tilde{h}^2 dv \right) \left( \int_{|\tau + i\xi \cdot v'| \leq \alpha} \phi^2(v') dv' \right) d\tau d\xi \\ &\lesssim \|\phi\|_\infty^2 R_\phi^{d-1} \int \alpha |\xi|^{2\eta} |\xi|^{-1} \tilde{h}^2(\tau, \xi, v) d\tau d\xi dv \end{aligned}$$

where  $R_\phi > 0$  is such that  $\text{supp } \phi \subset B_{R_\phi}$ .

Write next that

$$R^\alpha = \int (\Delta_v \tilde{H}_2 + \nabla_v \cdot \tilde{H}_1 + \tilde{H}_0) \frac{\psi_\alpha(\tau + i\xi \cdot v)}{\tau + i\xi \cdot v} \phi(v) dv.$$

It is convenient to write  $\Psi_\alpha(r) = \Psi(r/\alpha)$  with  $\Psi(r) = \psi(r)/r$  and

$$R^\alpha = \alpha^{-1} \int (\Delta_v \tilde{H}_2 + \nabla_v \cdot \tilde{H}_1 + \tilde{H}_0) \Psi_\alpha(\tau + i\xi \cdot v) \phi(v) dv.$$

We can thus write

$$R^\alpha = R_2^\alpha - R_1^\alpha + R_0^\alpha$$

with

$$\begin{cases} R_2^\alpha = \alpha^{-1} \int \tilde{H}_2 \Delta_v (\Psi_\alpha(\tau + i\xi \cdot v) \phi(v)) \, dv \\ R_1^\alpha = \alpha^{-1} \int \tilde{H}_1 \nabla_v \cdot (\Psi_\alpha(\tau + i\xi \cdot v) \phi(v)) \, dv \\ R_0^\alpha = \alpha^{-1} \int \tilde{H}_0 \Psi_\alpha(\tau + i\xi \cdot v) \phi(v) \, dv \end{cases}$$

We now estimate successively  $R_0^\alpha$ ,  $R_1^\alpha$  and  $R_2^\alpha$ . In order to do so, remark that for  $r \geq \frac{\alpha}{2}$ ,

$$\begin{cases} |\Psi_\alpha(r)| \lesssim 1 \\ |\Psi'_\alpha(r)| \lesssim \frac{1}{\alpha} \\ |\Psi''_\alpha(r)| \lesssim \frac{1}{\alpha^2}. \end{cases}$$

$$\begin{aligned} |R_0^\alpha| &\lesssim \frac{\|\phi\|_\infty}{\alpha} \int_{|v| \leq R_\phi} |\tilde{H}_0(\tau, \xi, v)| \, dv \\ |R_1^\alpha| &\lesssim \frac{1}{\alpha} \left( \|\phi\|_\infty \frac{|\xi|}{\alpha} + \|\nabla \phi\|_\infty \right) \int_{|v| \leq R_\phi} |\tilde{H}_1| \, dv \\ |R_2^\alpha| &\lesssim \frac{1}{\alpha} \left( \|\phi\|_\infty \frac{|\xi|^2}{\alpha^2} + \|\nabla \phi\|_\infty \frac{|\xi|}{\alpha} + \|\Delta \phi\|_\infty \right) \int_{|v| \leq R_\phi} |\tilde{H}_2| \, dv. \end{aligned}$$

The previous inequalities imply

$$\begin{aligned} \int |\xi|^{2\eta} |R_0^\alpha|^2 &\lesssim \|\phi\|_\infty^2 R_\phi^d \int |\xi|^{2\eta} \frac{1}{\alpha^2} |\tilde{H}_0(\tau, \xi, v)|^2 \, d\tau \, d\xi \, dv \\ \int |\xi|^{2\eta} |R_1^\alpha|^2 &\lesssim \|\phi\|_{1,\infty}^2 R_\phi^d \int |\xi|^{2\eta} \left( \frac{|\xi|^2}{\alpha^4} + \frac{1}{\alpha^2} \right) |\tilde{H}_1(\tau, \xi, v)|^2 \, d\tau \, d\xi \, dv \\ \int |\xi|^{2\eta} |R_2^\alpha|^2 &\lesssim \|\phi\|_{2,\infty}^2 R_\phi^d \int |\xi|^{2\eta} \left( \frac{|\xi|^4}{\alpha^6} + \frac{|\xi|^2}{\alpha^4} + \frac{1}{\alpha^2} \right) |\tilde{H}_2(\tau, \xi, v)|^2 \, d\tau \, d\xi \, dv. \end{aligned}$$

Choosing  $\alpha = \langle \xi \rangle^{1-2\eta}$  and  $\eta = \frac{1}{7}$ , we get

$$\begin{aligned} \int |\xi|^{2\eta} |R_\alpha|^2 &\lesssim \|\phi\|_\infty^2 \|h\|_2^2 R_\phi^{d-1} \\ \int |\xi|^{2\eta} |R^\alpha|^2 &\lesssim \|\phi\|_{2,\infty}^2 R_\phi^d (\|H_0\|_2^2 + \|H_1\|_2^2 + \|H_2\|_2^2). \end{aligned}$$

Recalling that  $\tilde{\rho}_\phi = R_\alpha + R^\alpha$ , the two previous inequalities yield the desired result for the Homogeneous Sobolev space. But estimating the  $L^2$ -norm yields a similar estimate. The proof is now complete.  $\square$

## APPENDIX B. GETTING LEMMA 14 FROM LEMMA 15

*Proof of Lemma 14.* We follow the nice exposition of [14]. Let  $C_0$  be the universal constant such that solutions  $f$  of (1.3) in  $Q_2$  satisfy

$$\|f_+\|_{L^\infty(Q_{\frac{1}{2}})} \leq C_0 \|f_+\|_{L^2(Q_1)}.$$

We now define  $f_1 = f$  and  $f_{k+1} = 2f_k - 1$ . Remark that

$$\begin{aligned} |\{f_1 \leq 0\} \cap Q_1| &\geq \delta_1 \\ \{f_{k+1} \leq 0\} &\supset \{f_k \leq 0\} \end{aligned}$$

with  $\delta_1 = |Q_1|/2$  (remark it is universal). Our goal is to prove that there exists  $k_0$  universal such that

$$|\{f_{k_0} \geq 0\} \cap Q_1| \leq \delta_2$$

with  $\delta_2 = (4C_0^2)^{-1}$  (remark it is universal). Indeed, this implies

$$\|(f_{k_0})_+\|_{L^\infty(Q_{\frac{1}{2}})} \leq C_0 \|(f_{k_0})_+\|_{L^2(Q_1)} \leq C_0 \left[ |\{f_{k_0} \geq 0\} \cap Q_1| \right]^{\frac{1}{2}} \leq \frac{1}{2}$$

which, in turn, yields

$$f \leq 1 - 2^{-k_0-1} \quad \text{in } Q_{\frac{1}{2}}.$$

Assume that for all  $k \geq 1$ ,

$$|\{f_k \geq 0\} \cap Q_1| \geq \delta_2.$$

Since  $f_{k+1} = 2f_k - 1$ , this also implies

$$|\{f_k \geq \frac{1}{2}\} \cap Q_1| \geq \delta_2.$$

But we also have

$$|\{f_k \leq 0\} \cap Q_1| \geq |\{f \leq 0\} \cap Q_1| \geq \delta_1.$$

Hence Lemma 15 implies that

$$|\{0 \leq f_k \leq \frac{1}{2}\} \cap Q_1| \geq \alpha.$$

Now remark that

$$\begin{aligned} |Q_1| &\geq |\{f_{k+1} \leq 0\} \cap Q_1| = |\{f_k \leq 0\} \cap Q_1| + |\{0 \leq f_k \leq \frac{1}{2}\} \cap Q_1| \\ &\geq |\{f_k \leq 0\} \cap Q_1| + \alpha \\ &\geq k\alpha \end{aligned}$$

which is impossible for  $k$  large enough.  $\square$

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