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HÖLDER CONTINUITY OF SOLUTIONS TO HYPOELLIPTIC EQUATIONS WITH BOUNDED MEASURABLE COEFFICIENTS

C. IMBERT & C. MOUHOT

Abstract. We prove that L^2 weak solutions to hypoelliptic equations with bounded measurable coefficients are Hölder continuous. The proof relies on classical techniques developed by De Giorgi and Moser together with the averaging lemma and regularity transfers developed in kinetic theory. The latter tool is used repeatedly: first in the proof of the local gain of integrability of sub-solutions; second in proving that the gradient with respect to the velocity variable is $L^{2+\varepsilon}_{\mathrm{loc}}$; third, in the proof of an "hypoelliptic isoperimetric De Giorgi lemma". To get such a lemma, we develop a new method which combines the classical isoperimetric inequality on the diffusive variable with the structure of the integral curves of the first-order part of the operator. It also uses that the gradient of solutions w.r.t. v is $L^{2+\varepsilon}_{\mathrm{loc}}$.

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1. Introduction

1.1. **The question studied and its history.** We consider the following nonlinear kinetic Fokker-Planck equation

$$\partial_t f + v \cdot \nabla_x f = \rho \, \nabla_v \cdot (\nabla_v f + v f) \,, \quad t \geq 0, \; x \in \mathbb{R}^d, \; v \in \mathbb{R}^d, \tag{1.1}$$

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(with or without periodicity conditions with respect to the space variable) where $d \in \mathbb{N}^*$, $f = f(t,x,v) \geq 0$ and $\rho[f] = \int_{\mathbb{R}^d} f(t,x,v)\,\mathrm{d}v$. The construction of global smooth solutions for such a problem is one motivation for the present paper.

The linear kinetic Fokker-Planck equation $\partial_t f + v \cdot \nabla_x f = \nabla_v \cdot (\nabla_v f + v f)$ is sometimes called the Kolmogorov-Fokker-Planck equation, as it was studied by Kolmogorov in the seminal paper [11], when $x \in \mathbb{R}^d$. In this note, Kolmogorov explicitely calculated the fundamental solution and deduced regularisation in both variables x and v, even though the operator $\nabla_v \cdot (\nabla_v + v) - v \cdot \nabla_x$ shows ellipticity in the v variable only. It inspired Hörmander and his theory of hypoellipticity [10], where the regularisation is recovered by more robust and more geometric commutator estimates (see also [15]).

Another question which has attracted a lot of attention in calculus of variations and partial differential equations along the 20th century is Hilbert's 19th problem about the analytic regularity of solutions to certain integral variational problems, when the quasilinear Euler-Lagrange equations satisfy ellipticity conditions. Several previous results had established the analyticity conditionally to some differentiability properties of the solution, but the full answer came with the landmark works of De Giorgi [2, 3] and Nash [13], where they prove that any solution to these variational problems with square integrable derivative is analytic. More precisely their key contribution is the following[1]: reformulate the quasilinear parabolic problem as

$$\partial_t f = \nabla_v \left(A(v,t) \nabla_v f \right), \quad t \geq 0,\ v \in \mathbb{R}^d \tag{1.2}$$

with $f = f(v,t) \geq 0$ and $A = A(v,t)$ satisfies the ellipticity condition $0 < \lambda I \leq A \leq \Lambda I$ for two constants $\lambda, \Lambda > 0$ but is, besides that, merely measurable. Then the solution f is Hölder continuous.

In view of the nonlinear (quasilinear) equation (1.1) it is natural to ask whether a similar result as the one of De Giorgi-Nash holds for quasilinear hypoelliptic equations. More precisely, we consider the following Fokker-Planck equation

$$\partial_t f + v \cdot \nabla_x f = \nabla_v \cdot \left(A(x,v,t) \nabla_v f \right), \quad t \in (0,T), (x,v) \in \Omega, \tag{1.3}$$

where Ω is an open set of $\mathbb{R}^{2d}$, $f = f(t,x,v) \geq 0$ and the $d \times d$ symmetric matrix A satisfies the ellipticity condition

$$0 < \lambda I \leq A \leq \Lambda I \tag{1.4}$$

for two constants λ, Λ but is, besides that, merely measurable. We want to establish the Hölder continuity of L^2 solutions to this problem. In order to do so, we first prove that L^2 sub-solutions are locally bounded; we refer to such a result as an $L^2 - L^\infty$ estimate. We then prove that solutions are Hölder continuous by proving a lemma which is an hypoelliptic counterpart of De Giorgi's isoperimetric lemma.

[1] We give the parabolic version due to Nash here.

Given $z_0 = (x_0, v_0, t_0) \in \mathbb{R}^{2d+1}$, $Q = Q_r(z_0)$ denotes a cylinder centered at z_0 of "radius" r: it is defined as $Q = B_{r^3}(x_0) \times B_r(v_0) \times (t_0 - R^2, t_0]$ where $B_r(x_0)$ and $B_r(v_0)$ denote the usual Euclidian balls in x and v.

Theorem 1 (Hölder continuity)**.** *Let f be a solution of (1.3) in $Q_0 = Q(z_0, R_0)$ and $Q_1 = Q(z_0, R_1)$ with $R_1 < R_0$. Then f is α-Hölder continuous with respect to (x, v, t) in Q_1 and*

$$\|f\|_{C^\alpha(Q_1)} \leq C\|f\|_{L^2(Q_0)}$$

for some α universal, i.e. $\alpha = \alpha(d, \lambda, \Lambda)$, and $C = C(d, \lambda, \Lambda, Q_0, Q_1)$.

In [14], the authors obtain an $L^2 - L^\infty$ estimate with completely different techniques; however they cannot reach the Hölder continuity estimate. Our techniques rely on averaging lemmas [6, 7] in order to gain some regularity H^s_x, $s > 0$ small, in the space variable x from the natural H^1_v estimate. We emphasize that such H^s_x estimates do not hold for sub-solutions. From this Sobolev estimate, we can recover a gain of integrability for L^2 sub-solutions, and we then prove the Hölder continuity through a De Giorgi type argument on the decrease of oscillation for solutions.

In [17, 18], the authors get a Hölder estimate for L^2 weak solutions of so-called ultraparabolic equations, including (1.3). Their proof relies on the construction of cut-off functions and a particular form of weak Poincaré inequality satisfied by non-negative weak sub-solutions. Our paper proposes a new, short and simple strategy, that, we hope, sheds new light on the regularizing effect for hypoelliptic equations with bounded measurable coefficients and provide tools for further applications.

We finally mention that Golse and Vasseur proved independently a similar result [8].

1.2. **Plan of the paper.** In Section 2, we first explain how to get a universal gain of regularity for (signed) L^2 solutions; we then exhibit a universal gain of integrability for non-negative L^2 sub-solutions; we finally explain how to derive from this gain of integrability a local upper bound of such non-negative L^2 sub-solutions by using Moser iteration procedure. In Section 3, we prove that the v-gradient of solutions is $L^{2+\varepsilon}_{\text{loc}}$. In Section 4, the Hölder estimate is derived by proving a reduction of oscillation lemma.

2. Local gain of regularity / integrability

We consider the equation (1.3) and we want to establish a local gain of integrability of solutions in order to apply Moser's iteration and get a local L^∞ bound. Since we will need to perform convex changes of unknown, it is necessary to obtain this gain even for (non-negative) *sub-solutions*.

In the two following theorems, we consider cylinders with a scaling corresponding to the hypoelliptic structure of the equation. For $z_0 = (x_0, v_0, t_0) \in \mathbb{R}^{2d+1}$,

$$Q_R(z_0) = B_{R^3}(x_0) \times B_R(v_0) \times (t_0 - R^2, t_0].$$

The next theorem is stated in cylinders centered at the origin.

Theorem 2 (Gain of integrability for non-negative sub-solutions)**.** *Consider two cylinders $Q_1 = Q_{R_1}(0)$ and $Q_0 = Q_{R_0}(0)$ with $R_1 < R_0$. There exists $q > 2$ (universal) such that for all non-negative L^2 sub-solution f of (1.3) in Q_0, we have*

$$\|f\|_{L^q(Q_1)} \leq C\|f\|_{L^2(Q_0)} \tag{2.1}$$

where

$$C = \bar{C}\left(\frac{1}{R_0^2 - R_1^2} + \frac{R_0}{R_0^3 - R_1^3} + \frac{1}{(R_0 - R_1)^2}\right)$$

and $\bar{C} = C(d, \lambda, \Lambda)$.

This result is a consequence of the comparison principle and the fact that, for weak signed solutions f, we can even get a gain of regularity. This gain of regularity will be important in the proof of the decrease of oscillation lemma to get compactness of sequences of equi-bounded solutions. This is the reason why it is necessary to state it in cylinders not necessarily centered at the origin.

Theorem 3 (Gain of regularity for signed solutions)**.** *Consider $z_0 \in \mathbb{R}^{2d+1}$ and two cylinders $Q_1 = Q_{R_1}(z_0)$ and $Q_0 = Q_{R_0}(z_0)$ with $R_1 < R_0$. There exists $s > 0$ (universal) such that for all (signed) L^2 weak solution f of (1.3) in Q_0, we have*

$$\|f\|_{H^s_{x,v,t}(Q_1)} \leq C\|f\|_{L^2(Q_0)} \tag{2.2}$$

where and $C = C(d, \lambda, \Lambda, Q_0, Q_1)$.

2.1. **Gain of integrability with respect to v and t.** The gain of integrability with respect to v and t is classical. It derives from the natural energy estimate, after truncation.

We follow here [12] in order to get the following lemma.

Lemma 4 (Gain of integrability w.r.t. v and t)**.** *Under the assumptions of Theorem 2, the function f satisfies*

$$\begin{aligned} \int_{Q_1} |\nabla_v f|^2 &\leq C \int_{Q_0} f^2 \\ \|f\|^2_{L^2_t L^2_x L^q_v(Q_1)} &\leq C \int_{Q_0} f^2 \\ \|f\|^2_{L^\infty_t L^2_x L^2_v(Q_1)} &\leq C \int_{Q_0} f^2 \end{aligned} \tag{2.3}$$

for some $q > 2$ and $C = \bar{C}\left(\frac{1}{R_0^2 - R_1^2} + \frac{R_0}{R_0^3 - R_1^3} + \frac{1}{(R_0 - R_1)^2}\right)$ and $\bar{C} = \bar{C}(d, \lambda, \Lambda)$.

Proof. Consider $\Psi \in C_c^\infty(\mathbb{R}^{2d} \times \mathbb{R})$ and integrate the inequation satisfied by f against $2f\Psi^2$ in $\mathbb{R}^{2d} \times [t_1, 0] = \mathcal{R}$ with $t_1 \in (-R_1^2, 0]$ and get

$$\int_{\mathcal{R}} \partial_t(f^2)\Psi^2 + \int_{\mathcal{R}} v \cdot \nabla_x(f^2)\Psi^2 \leq 2\int_{\mathcal{R}} \nabla_v(A\nabla_v f) f \Psi^2.$$

Add $\int_{\mathcal{R}} f^2\partial_t(\Psi^2)$, integrate by parts several times and use the upper bound on A in order to get

$$\begin{aligned}\int_{\mathcal{R}} \partial_t(f^2\Psi^2) + 2\int_{\mathcal{R}} (A\nabla_v f\cdot\nabla_v f)\Psi^2 \\ \leq \int_{\mathcal{R}} f^2(\partial_t + v\cdot\nabla_x)(\Psi^2) + 2\int_{\mathcal{R}} \Psi\sqrt{A}\nabla_v f\cdot f\sqrt{A}\nabla_v\Psi \\ \leq \int_{\mathcal{R}} f^2(\partial_t + v\cdot\nabla_x)(\Psi^2) + \int_{\mathcal{R}} (A\nabla_v f\cdot\nabla_v f)\Psi^2 + \int_{\mathcal{R}} f^2(A\nabla_v\Psi\cdot\nabla_v\Psi).\end{aligned}$$

We thus get

$$\begin{aligned}\int_{\mathcal{R}} \partial_t(f^2\Psi^2) + \lambda\int_{\mathcal{R}} |\nabla_v f|^2\Psi^2 \\ \leq \bar{C}\left(\|\partial_t\Psi\|_\infty + R_0\|\nabla_x\Psi\|_\infty + \|\nabla_v\Psi\|_\infty^2\right)\int_{\mathcal{R}\cap\operatorname{supp}\Psi} f^2\end{aligned}$$

with $\bar{C} = C(\Lambda, d)$. Choose next Ψ^2 such that $\Psi(t=0) = 0$ and $\operatorname{supp}\Psi \subset Q_0$ and get

$$\int_{x,v} f^2\Psi^2(t_1) + \lambda\int |\nabla_v f|^2\Psi^2 \leq CC_{0,1}\int_{Q_0} f^2.$$

If Ψ^2 additionally satisfies $\Psi^2 \equiv 1$ in Q_1, we get (2.3). The Sobolev inequality then implies the estimate for $\|f\|^2_{L^2_tL^2_xL^q_v(Q_1)}$. If now $t_1 \in [t_0 - r_1^2, t_0]$ is arbitrary, we get the estimate for $\|f\|^2_{L^\infty_tL^2_xL^2_v(Q_1)}$. The proof is now complete. □

2.2. Gain of regularity with respect to x for signed weak solutions.

Lemma 5 (Gain of regularity w.r.t. x)**.** *Under the assumptions of Theorem 2, if f is a signed weak solution to (1.3),*

$$\left\|D_x^{1/3}f\right\|_{L^2(Q_1)} \leq C\|f\|_{L^2(Q_0)} \tag{2.4}$$

with $C = \bar{C}\left(\frac{1}{R_0^2-R_1^2} + \frac{R_0}{R_0^3-R_1^3} + \frac{1}{(R_0-R_1)^2}\right)$ and $\bar{C} = \bar{C}(d,\lambda,\Lambda)$. In the case $q > 1$ with $\mathcal{Q}_1$ instead of Q_1, we have

$$\left\|D_x^{1/3}f\right\|_{L^q(\mathcal{Q}_1)} \leq C\|\nabla_v f\|_{L^q(Q_0)} \tag{2.5}$$

with $C = C(d,\lambda,\Lambda,Q_0,\mathcal{Q}_1)$.

Proof. Let $R_{\frac{1}{2}} = \frac{R_1+R_0}{2}$ and $Q_{\frac{1}{2}} = Q_{R_{\frac{1}{2}}}$. In particular,

$$Q_1 \subset Q_{\frac{1}{2}} \subset Q_0.$$

For $i = 1, \frac{1}{2}$, consider $f_i = f\chi_i$ where χ_1 and $\chi_{\frac{1}{2}}$ are two truncation functions such that

$$\chi_1 \equiv 1 \text{ in } Q_1 \quad \text{and} \quad \chi_1 \equiv 0 \text{ outside } Q_{\frac{1}{2}}$$

$$\chi_{\frac{1}{2}} \equiv 1 \text{ in } Q_{\frac{1}{2}} \quad \text{and} \quad \chi_{\frac{1}{2}} \equiv 0 \text{ outside } Q_0.$$

We get

$$(\partial_t + v \cdot \nabla_x) f_1 = \nabla_v \cdot H_1 + H_0 \text{ in } \mathbb{R}^{2d} \times (-\infty; 0]$$

$$\text{with} \quad \begin{cases} H_1 &= \chi_1 A \nabla_v f_{\frac{1}{2}} \\ H_0 &= -\nabla_v \chi_1 \cdot A \nabla_v f_{\frac{1}{2}} + \alpha_1 f_{\frac{1}{2}} \\ \alpha_1 &= (\partial_t + v \cdot \nabla_x) \chi_1. \end{cases}$$

The previous equation holds true in $\mathbb{R}^{2d} \times (-\infty; 0]$ since f_1, H_0 and H_1 are supported in Q_0. We remark that using (2.3),

$$\|H_0\|_{L^2} + \|H_1\|_{L^2} \leq C \|f\|_{L^2(Q_0)}$$

with C as in the statement. Applying [1, Theorem 1.3] with $p = 2$, $r = 0$, $\beta = 1$, $m = 1$, $\kappa = 1$ and $\Omega = 0$ yields (2.4). To get (2.5), we simply use a cut-off function such that $\alpha_1 \equiv 0$ and we apply [1, Theorem 1.3] with $p = q$, $r = 0$, $\beta = 1$, $m = 1$, $\kappa = 1$ and $\Omega = 0$. The proof is now complete. □

2.3. Gain of integrability with respect to x for non-negative sub-solutions.

Lemma 6 (Gain of integrability w.r.t. x)**.** *Under the assumptions of Theorem 2, there exists $p > 2$ such that*

$$\|f\|_{L^2_t L^p_x L^1_v (Q_1)} \leq C \|f\|_{L^2(Q_0)} \tag{2.6}$$

with $C = \bar{C} \left(\frac{1}{R_1 - R_0} + \frac{1}{(r_1 - r_0)^2} + \frac{1}{\tau_1 - \tau_0} \right)$ *and* $\bar{C} = \bar{C}(d, \lambda, \Lambda)$.

Proof. We follow the reasoning of Lemma 5. The function f_1 now satisfies the following inequation

$$(\partial_t + v \cdot \nabla_x) f_1 \leq \nabla_v \cdot H_1 + H_0 \text{ in } \mathbb{R}^{2d} \times \mathbb{R}.$$

If g solves

$$\begin{cases} (\partial_t + v \cdot \nabla_x) g = \nabla_v \cdot H_1 + H_0 \\ g(x, v, -R_1^2) = f_1(x, v, -R_1^2) \end{cases}$$

then the comparison principle implies that $f_1 \leq g$ in $\mathbb{R}^{2d} \times [-R_1^2, 0]$. Applying Lemma 5 and Sobolev inequality, we get

$$\|f_1\|_{L^2_t L^p_x L^1_v} \leq \|g\|_{L^2_t L^p_x L^1_v} \leq C C_{0,1} \|f_0\|_2$$

(where $p = 2d/(d - 2/3) > 2$) which yields the desired estimate. □

2.4. **Proof of Theorems 2 and 3.**

Proof of Theorem 2. Combine Lemmas 4 and 6, use interpolation to get the result through a covering argument. □

Proof of Theorem 3. We first prove the result when cylinders are centered at the origin. In this case, it is enough to combine Lemmas 4 and 5, Aubin's lemma and use interpolation to get the result.

For cylinders that are not centered at the origin, we use *slanted* cylinders of the form:

$$\tilde{Q}_R(z_0) = \{(x, v, t) : |x - x_0 - (t - t_0)v_0| < R^3, |v - v_0| < R, t \in (t_0 - R^2, t_0]\}.$$

Now we cover Q_0 and Q_1 with such slanted cylinders, we get the gain of regularity (whose exponent remains universal) and we get the desired result. □

2.5. **Local upper bounds for non-negative sub-solutions.** In this subsection, we iterate the local gain of integrability to prove that non-negative L^2 sub-solutions are in fact locally bounded (with an estimate).

Theorem 7 (Upper bounds for non-negative L^2 sub-solutions)**.** *Given two cylinders $Q_0 = Q_{R_0}(z_0)$ and $Q_\infty = Q_{R_\infty}(z_0)$, let f be a non-negative L^2 sub-solution of*

$$(\partial_t + v\nabla_x)f \leq \nabla_v(A\nabla_v f) \quad \text{in } Q_0.$$

Then

$$\sup_{Q_\infty} f \leq C\|f\|_{L^2(Q_0)}$$

for some $C = C(d, \lambda, \Lambda, Q_0, Q_\infty)$.

Proof. We first prove the result for cylinders centered at the origin. To do so, we first remark that, for all $q > 1$, the function f^q satisfies

$$(\partial_t + v\nabla_x)f^q \leq \nabla_v \cdot (A\nabla_v f^q) \quad \text{in } Q_0.$$

We now rewrite (2.1) from $Q_q = Q_{R_q}(z_0)$ to Q_{q+1} with $R_{q+1} < R_q$ as follows:

$$\|(f^q)^\kappa\|^2_{L^2(Q_{q+1})} \leq C_{q+1}\|f^q\|^{2\kappa}_{L^2(Q_q)} \tag{2.7}$$

where $\kappa = p/2 > 1$ and

$$C_{q+1} = \bar{C}\left[\frac{1}{R_q^2 - R_{q+1}^2} + \frac{R_q}{R_q^3 - R_{q+1}^3} + \frac{1}{(R_q - R_{q+1})^2}\right]^\kappa$$

with $\bar{C} = \bar{C}(d, \lambda, \Lambda)$.

Choose now $q = q_n = 2\kappa^n$ for $n \in \mathbb{N}$, simply write Q_n for Q_{q_n} and C_n for C_{q_n} and get from (2.7)

$$\|f^{q_{n+1}}\|^2_{L^2(Q_{n+1})} \leq C_{n+1}\|f^{q_n}\|^{2\kappa}_{L^2(Q_n)}. \tag{2.8}$$

Moreover, we choose

$$R_{n+1} = R_n - \frac{1}{a(n+1)^2}$$

for some $a > 0$ so that

$$C_n \sim \bar{C}(a^2 n^4 + bn^2)^\kappa$$

with $b = \frac{5a}{6R_\infty}$. Applying iteratively (2.8), we get the result if

$$\prod_{n=0}^{+\infty} C_n^{\frac{1}{2\kappa^n}} < +\infty$$

which indeed holds true. This yields the desired result in the case of cylinders centered at the origin.

For cylinders that are not centered at the origin, we argue as in the proof of Theorem 3. The proof is now complete. □

3. Gain of integrability for the gradient w.r.t. the velocity variable

This subsection is devoted to the proof of the following theorem.

Theorem 8 (Gain of integrability for $\nabla_v f$)**.** *Let f be a solution of (1.3) in some cylinder $Q_0 = Q_{R_0}(z_0)$. There exists a universal $\varepsilon > 0$ such that for all $Q_i = Q_{R_i}(z_0)$, $i = 1, 2$ with $R_2 < R_1 < R_0$, $\nabla_v f \in L^{2+\varepsilon}(Q_2)$*

$$\int_{Q_2} |\nabla_v f|^{2+\varepsilon} dz \leq C \left(\int_{Q_1} |\nabla_v f|^2 dz \right)^{\frac{2+\varepsilon}{2}} \tag{3.1}$$

with $C = C(d, \lambda, \Lambda, Q_2, Q_1, Q_0)$.

The proof follows along the lines of the one of [5, Theorem 2.1]. It consists in deriving a reverse Hölder inequality which in turn implies the result thanks to the analogous of [5, Proposition 1.3].

Lemma 9 (A Gehring lemma)**.** *Let $g \geq 0$ in Q such that there exists $q > 1$ such that for all $z_0 \in Q$ and R such that $Q_{4R}(z_0) \subset Q$,*

$$\fint_{Q_R(z_0)} g^q \, dz \leq b \left(\fint_{Q_{8R}(z_0)} g \, dz \right)^q + \theta \fint_{Q_{8R}(z_0)} g^q \, dz$$

for some $\theta > 0$. There exists $\theta_0 = \theta_0(q, d)$ such that if $\theta < \theta_0$, then $g \in L^p_{\mathrm{loc}}(Q)$ for $p \in [q, q + \varepsilon)$ and

$$\left(\fint_{Q_R} g^p \, dz \right)^{\frac{1}{p}} \leq c \left(\fint_{Q_{4R}} g^q \, dz \right)^{\frac{1}{q}},$$

the constants c and $\varepsilon > 0$ depending only on b, q, θ and dimension.

The proof of Lemma 9 is an easy adaptation of the one of [4, Proposition 5.1], by changing Euclidian cubes with cylinders Q_R.

The proof of Theorem 8 is a consequence of some estimates involving weighted means of the solution. Given $z_0 \in \mathbb{R}^{2d+1}$, they are defined as follows of f are defined as follows:

$$\tilde{f}_{2R}(t) = (cR^{4d})^{-1} \int f(t,x,v)\chi_{2R}(x,v,t)dxdv$$

(for some c defined below) where χ_{2R} is a cut-off function such that

$$\chi_{2R}(x,v,t) = \phi_{R^3}((x-x_0)-(t-t_0)(v-v_0))\phi_R(v-v_0)$$

with $\phi_R(a) = \phi(a/R)$ for some ϕ such that $\sqrt{\phi} \in C^\infty(\mathbb{R}^d)$ and $\phi \equiv 1$ in B_1 and $\operatorname{supp}\phi \subset B_2$. In particular,

$$(\partial_t + v\cdot\nabla_x)\chi_R = 0 \quad \text{and} \quad \int \chi_R(x,v,t)\,dxdv = \int \phi_{R^3} \int \phi_R = cR^{4d}$$

with $c = (\int \phi)^2$. We now introduce "sheared" cylinders $\mathcal{Q}_R(z_0) = z_0 + \mathcal{Q}_R$ with

$$\mathcal{Q}_R = \{(x,v,t) : |x-tv| < R^3, |v| < R, t \in (-R^2, 0]\}.$$

Remark that

$$Q_{2^{-1/3}R} \subset \mathcal{Q}_R \subset Q_{2^{1/3}R}. \tag{3.2}$$

Remark also that $\chi_{2R} \equiv 1$ in $\mathcal{Q}_R$ and $\chi_{2R} \equiv 0$ outside $\mathcal{Q}_{2R}$.

Lemma 10 (Estimates)**.** *Let f be a solution of (1.3) in $\mathcal{Q}_0$. Then for $Q_{3R}(z_0) \subset \mathcal{Q}_0$,*

$$\int_{\mathcal{Q}_R(z_0)} |\nabla_v f|^2\,dz \le CR^{-2} \int_{\mathcal{Q}_{2R}(z_0)} |f - \tilde{f}_{2R}|^2\,dz \tag{3.3}$$

$$\sup_{t\in(t_0-R^2,t_0]} \int_{\mathcal{Q}_R^t(z_0)} |f(t) - \tilde{f}_R(t)|^2 \le CR^2 \int_{\mathcal{Q}_{3R}(z_0)} |\nabla_v f|^2\,dz \tag{3.4}$$

where $\mathcal{Q}_R^t(z_0) = z_0 + \{(x,v) : |x-tv| < R^3, |v| < R\}$.

Remark 11. This lemma corresponds to [5, Lemmas 2.1 & 2.2].

Proof. For the sake of clarity, we put $z_0 = 0$. Consider $\tau_{2R} \in C^\infty(\mathbb{R},\mathbb{R})$ such that $0 \le \tau_{2R} \le 1$, $\tau_{2R} \equiv 0$ in $(-\infty, -(2R)^2]$ and $\tau_{2R} \equiv 1$ in $[-R^2, 0]$. Use $2(f - \tilde{f}_{2R})\chi_{2R}\tau_{2R}$ as a test function for (1.3) and get

$$\begin{aligned}\int (f(0) - \tilde{f}_{2R}(0))^2 \chi_{2R}\,dxdv + 2\int (A\nabla_v f\cdot\nabla_v f)\chi_{2R}\tau_{2R}\,dz \\ = \int (f-\tilde{f}_{2R})^2\chi_{2R}(\partial_t\tau_{2R}) - \int v\cdot\nabla_x\left[(f-\tilde{f}_{2R})^2\right]\chi_{2R}\tau_{2R} \\ - 2\int (f-\tilde{f}_{2R})A\nabla_v f\cdot\nabla_v\chi_{2R}\tau_{2R}.\end{aligned}$$

Remark that the definition of $\tilde{f}_{2R}$ implies that the remaining term

$$-2\int(\partial_t\tilde{f}_{2R})(f-\tilde{f}_{2R})\chi_{2R}\tau_{2R}$$

vanishes. This equality yields

$$\int(f(0)-\tilde{f}_{2R}(0))^2\chi_{2R}\,dxdv+\lambda\int|\nabla_v f|^2\chi_{2R}\tau_{2R}\,dz$$
$$\leq\int(f-\tilde{f}_{2R})^2\left(\chi_{2R}|\partial_t\tau_{2R}|+|v\cdot\nabla_x\chi_{2R}|\tau_{2R}+\frac{\Lambda^2}{\lambda}|\nabla_v\sqrt{\chi_{2R}}|^2\tau_{2R}\right)$$

which yields (3.3). Changing the final time, we also get

$$\sup_{t\in(-R^2,0]}\int(f(t)-\tilde{f}_{2R}(t))^2\chi_{2R}(t)\,dxdv\leq CR^{-2}\int_{\mathcal{Q}_{2R}}|f-\tilde{f}_{2R}|^2\,dz.$$

Now the function $F=f-\tilde{f}_{2R}$ is such that $\int F(x,v,t)dxdv=0$. In particular, we have

$$\int_{\mathcal{Q}_{2R}}(f-\tilde{f}_{2R})^2\,dz\leq C\int_{\mathcal{Q}_{2R}}(R^2|\nabla_v f|^2+R^{2s}|D_x^s f|^2)\,dxdvdt.$$

Arguing as in the proof of Lemma 5 with a cut-off function $\chi_1=\chi_{2R}$ which satisfies $(\partial_t+v\cdot\nabla_x)\chi_1=0$, we get

$$\int_{\mathcal{Q}_{2R}}R^{2s}|D_x^s f|^2\,dxdvdt\leq C\int_{\mathcal{Q}_{3R}}R^2|\nabla_v f|^2\,dxdvdt.$$

Combining the three previous estimates yields

$$\sup_{t\in(-R^2,0]}\int(f(t)-\tilde{f}_{2R}(t))^2\chi_{2R}(t)\,dxdv\leq CR^2\int_{\mathcal{Q}_{3R}}|\nabla_v f|^2\,dxdvdt.$$

Finally, we write for $t\in(-R^2,0]$

$$\begin{aligned}\frac{1}{2}\int_{\mathcal{Q}_R^t}(f(t)-\tilde{f}_R(t))^2\chi_{2R}(t)\leq&\int_{\mathcal{Q}_R^t}(f(t)-\tilde{f}_{2R}(t))^2\chi_{2R}(t)\\&+\int_{\mathcal{Q}_R^t}(\tilde{f}_{2R}(t)-\tilde{f}_R(t))^2\chi_{2R}(t)\\\leq&\int(f(t)-\tilde{f}_{2R}(t))^2\chi_{2R}(t)\\&+|\mathcal{Q}_R^t|\left((cR^{4d})^{-1}\int(f-\tilde{f}_{2R}(t))\chi_R(x,v,t)\,dxdv\right)^2\\\leq&\,C\int_{\mathcal{Q}_R^t}(f(t)-\tilde{f}_{2R}(t))^2\chi_{2R}(t)\end{aligned}$$

and we get the second desired estimate since $\chi_{2R}\equiv 1$ in $\mathcal{Q}_R$. □

We now turn to the proof of Theorem 8. The use of (2.5) is the main difference with [5].

Proof of Theorem 8. Pick $p > 2$ and let q denotes its conjugate exponent: $\frac{1}{q}+\frac{1}{p} = 1$. We follow [5] in writing (omitting the center of cylinders z_0),

$$\begin{aligned}
&\int_{\mathcal{Q}_{2R}} |f - \tilde{f}_{2R}|^2 \\
&\leq \sup_{t\in(t_0-(2R)^2,t_0]} \left(\int_{\mathcal{Q}^t_{2R}} |f - \tilde{f}_{2R}|^2\right)^{\frac{1}{2}} \int_{t_0-(2R)^2}^{t_0} dt \left(\int_{\mathcal{Q}^t_{2R}} |f - \tilde{f}_{2R}|^2\right)^{\frac{1}{2}} \\
&\lesssim R \left(\int_{\mathcal{Q}_{4R}} |\nabla_v f|^2\right)^{\frac{1}{2}} \int_{t_0-(2R)^2}^{t_0} dt \left(\int_{\mathcal{Q}^t_{2R}} |f - \tilde{f}_{2R}|^q\right)^{\frac{1}{2q}} \left(\int_{\mathcal{Q}^t_{2R}} |f - \tilde{f}_{2R}|^p\right)^{\frac{1}{2p}}
\end{aligned}$$

where (3.4) and Hölder inequality are used successively.

We now use Sobolev inequalities and Hölder inequality (twice) successively to get

$$\begin{aligned}
&\int_{\mathcal{Q}_{2R}} |f - \tilde{f}_{2R}|^2 \lesssim R \left(\int_{\mathcal{Q}_{4R}} |\nabla_v f|^2\right)^{\frac{1}{2}} \\
&\times \int_{t_0-(2R)^2}^{t_0} dt \left(\int_{\mathcal{Q}^t_{2R}} R^q|\nabla_v f|^q + R^{q/3}|D_x^{1/3} f|^q\right)^{\frac{1}{2q}} \left(\int_{\mathcal{Q}^t_{2R}} R^2|\nabla_v f|^2 + R^{2/3}|D_x^{1/3} f|^2\right)^{\frac{1}{4}} \\
&\lesssim R \left(\int_{\mathcal{Q}_{4R}} |\nabla_v f|^2\right)^{\frac{1}{2}} \\
&\times \left(\int_{\mathcal{Q}_{2R}} R^q|\nabla_v f|^q + R^{q/3}|D_x^{1/3} f|^q\right)^{\frac{1}{2q}} \left(\int_{t_0-(2R)^2}^{t_0} \left(\int_{\mathcal{Q}^t_{2R}} R^2|\nabla_v f|^2 + R^{2/3}|D_x^{1/3} f|^2\right)^{\frac{q}{2(2q-1)}}\right)^{\frac{2q-1}{2q}} \\
&\lesssim R \left(\int_{\mathcal{Q}_{4R}} |\nabla_v f|^2\right)^{\frac{1}{2}} \\
&\times \left(\int_{\mathcal{Q}_{2R}} R^q|\nabla_v f|^q + R^{q/3}|D_x^{1/3} f|^q\right)^{\frac{1}{2q}} \left(\int_{\mathcal{Q}_{2R}} R^2|\nabla_v f|^2 + R^{2/3}|D_x^{1/3} f|^2\right)^{\frac{1}{4}} R^{\frac{3}{2}q-1}.
\end{aligned}$$

We now use (2.5) and get

$$\begin{aligned}
\int_{\mathcal{Q}_{2R}} |f - \tilde{f}_{2R}|^2 &\lesssim R^{\frac{3}{2}q+1} \left(\int_{\mathcal{Q}_{4R}} |\nabla_v f|^2\right)^{\frac{1}{2}} \left(\int_{\mathcal{Q}_{2R}} |\nabla_v f|^q\right)^{\frac{1}{2q}} \left(\int_{\mathcal{Q}_{2R}} |\nabla_v f|^2\right)^{\frac{1}{4}} \\
&\lesssim R^{\frac{3}{2}q+1} \left(\int_{\mathcal{Q}_{4R}} |\nabla_v f|^2\right)^{\frac{3}{4}} \left(\int_{\mathcal{Q}_{2R}} |\nabla_v f|^q\right)^{\frac{1}{2q}}.
\end{aligned}$$

Now use (3.3) and get for all $\varepsilon > 0$,

$$\fint_{\mathcal{Q}_R} |\nabla_v f|^2 \lesssim R^{\frac{3}{2}q-1} |\mathcal{Q}_{2R}|^{\frac{1}{2q}-\frac{1}{4}} \left(\fint_{\mathcal{Q}_{4R}} |\nabla_v f|^2\right)^{\frac{3}{4}} \left(\fint_{\mathcal{Q}_{4R}} |\nabla_v f|^q\right)^{\frac{1}{2q}}$$

$$\lesssim R^{\gamma_d}\left(\fint_{\mathcal{Q}_{4R}}|\nabla_v f|^2\right)^{\frac{3}{4}}\left(\fint_{\mathcal{Q}_{4R}}|\nabla_v f|^q\right)^{\frac{1}{2q}}$$
$$\lesssim \varepsilon\fint_{\mathcal{Q}_{4R}}|\nabla_v f|^2+c_\varepsilon R^{4\gamma_d}\left(\fint_{\mathcal{Q}_{4R}}|\nabla_v f|^q\right)^{\frac{2}{q}}$$

where $\gamma_d=(4d+2)(\frac{1}{2q}-\frac{1}{4})+\frac{3}{2}q-1>0$. Using (3.2), we finally get

$$\fint_{Q_R}|\nabla_v f|^2\lesssim\varepsilon\fint_{Q_{8R}}|\nabla_v f|^2+c_\varepsilon R^{4\gamma_d}\left(\fint_{Q_{8R}}|\nabla_v f|^q\right)^{\frac{2}{q}}.$$

Apply now Proposition 9 in order to achieve the proof of Theorem 8. □

4. The decrease of oscillation lemma

It is classical that Hölder continuity is a consequence of the decrease of the oscillation of the solution "at unit scale".

Lemma 12 (Decrease of oscillation)**.** *Let f be a solution of (1.3) in $Q_2=B_2(x_0)\times B_2(v_0)\times(-2,0)$ with $|f|\le 1$. Then*

$$\mathrm{osc}_{Q_{\frac{1}{2}}}f\le 2-\lambda$$

with $Q_{\frac{1}{2}}=B_{\frac{1}{2}}(x_0)\times B_{\frac{1}{2}}(v_0)\times(-\frac{1}{2},0)$ for some $\lambda\in(0,2)$ only depending on dimension and ellipticity constants.

Remark 13. The equation is "invariant" under the following scaling

$$(x,v,t)\mapsto(r^{-3}x,r^{-1}v,r^{-2}t);$$

indeed, it changes $A(x,v,t)$ into $A(r^{-3}x,r^{-1}v,r^{-2}t)$ which still satisfies (1.4).

This lemma is an immediate consequence of the following one.

Lemma 14 (Decrease of the supremum bound)**.** *Let f be a solution of (1.3) in Q_2 with $|f|\le 1$. If*

$$|\{f\le 0\}\cap Q_1|\ge\frac{1}{2}|Q_1|$$

with $Q_1=B_1(x_0)\times B_1(v_0)\times(-1,0)$, then

$$\sup_{Q_{\frac{1}{2}}}f\le 1-\lambda$$

for some $\lambda\in(0,2)$ only depending on dimension and ellipticity constants.

As explained in [16] for instance, this lemma itself is a consequence of the following one. The details are given in Appendix for the reader's convenience.

Lemma 15 (A De Giorgi-type lemma)**.** *For all $\delta_1 > 0$ and $\delta_2 > 0$, there exists $\alpha > 0$ such that for all solution f of (1.3) in Q_2 with $|f| \leq 1$ and*

$$|\{f \geq \frac{1}{2}\} \cap Q_1| \geq \delta_1$$
$$|\{f \leq 0\} \cap Q_1| \geq \delta_2$$

we have

$$|\{0 < f < \frac{1}{2}\} \cap Q_1| \geq \alpha.$$

Remark 16. It is important to emphasize that the lemma is stated for solutions of (1.3), not sub-solutions.

Remark 17. The idea of proving such a generalization of the classical isoperimetric lemma of De Giorgi is reminiscent of an argument of Guo [9]. See also the very nice survey by Vasseur [16].

Proof. We argue by contradiction by assuming that there exists a sequence f_k of solutions of (1.3) for some diffusion matrix A_k such that $|f_k| \leq 1$ and

$$|\{f_k \geq \frac{1}{2}\} \cap Q_1| \geq \delta_1$$
$$|\{f_k \leq 0\} \cap Q_1| \geq \delta_2$$
$$|\{0 < f_k < \frac{1}{2}\} \cap Q_1| \to 0 \quad \text{as } k \to +\infty.$$

Compactness in L^2. Since the sequence f_k is bounded in $L^2(Q_2)$, Theorem 2 implies that it is relatively compact in $L^2(Q_1)$ for any $Q_1 \Subset Q_2$. With thus can assume that f_k converges in $L^2(Q_1)$ towards f as $k \to +\infty$. In particular, it satisfies

$$\begin{aligned} |\{f \geq \frac{1}{2}\} \cap Q_1| &\geq \tfrac{\delta_1}{2} \\ |\{f \leq 0\} \cap Q_1| &\geq \tfrac{\delta_2}{2} \\ |\{0 < f < \frac{1}{2}\} \cap Q_1| &= 0. \end{aligned} \tag{4.1}$$

Moreover, the natural energy estimate for solutions of (1.3) implies that $f \in L^2_{t,x}H^1_v$ by weak limit. Hence, by the classical de Giorgi isoperimetric inequality, for almost every $(t,x) \in B_1(x_0) \times (-1,0)$, we have

$$\begin{cases} \text{either} & \text{for almost every } v \in B_1(v_0), \quad f(t,x,v) \leq 0 \\ \text{or} & \text{for almost every } v \in B_1(v_0), \quad f(t,x,v) \geq \frac{1}{2}. \end{cases}$$

Truncation. Consider now a smooth non-decreasing function $T : [-1,1] \to \mathbb{R}$ such that $T \equiv 0$ in $[-1,0]$ and $T \equiv \frac{1}{2}$ in $[\frac{1}{2},1]$. We have that $\bar{f}_k = T(f_k)$ satisfies

$\bar f_k \to \bar f$ in $L^2(Q_1)$ such that

$$\begin{cases} \text{either} & \text{for almost every } v \in B_1(v_0), \quad \bar f(t,x,v) = 0 \\ \text{or} & \text{for almost every } v \in B_1(v_0), \quad \bar f(t,x,v) = \dfrac{1}{2}. \end{cases}$$

In particular,

$$\nabla_v \bar f = 0 \text{ in } L^2(Q_1)$$

i.e. the function is everywhere a *local equilibrium* in the terminology of kinetic theory. Hence,

$$\bar f(t,x,v) = \bar f(t,x) \in \{0, \frac{1}{2}\}$$

and

$$\begin{cases} |\{\bar f = \frac{1}{2}\} \cap B_1 \times (-1,0)| \geq \dfrac{\delta_1}{|B_1|} \\ |\{\bar f = 0\} \cap B_1 \times (-1,0)| \geq \dfrac{\delta_2}{|B_1|} \end{cases} \tag{4.2}$$

Passage to the limit. The function $\bar f_k$ satisfies in Q_1,

$$\partial_t \bar f_k + v \cdot \nabla_x \bar f_k = \nabla_v \cdot (A_k \nabla_v \bar f_k) - T''(f_k) A_k \nabla_v f_k \cdot \nabla_v f_k. \tag{4.3}$$

For a test function ϕ supported in Q_1, we can write

$$\left| \int T''(f_k) A_k \nabla_v f_k \cdot \nabla_v \tilde f_k \phi \right| \leq \Lambda \|T''\|_\infty \|\phi\|_\infty \int_{B_k} |\nabla_v f_k|^2$$

where

$$B_k = \{0 < \tilde f_k < \frac{1}{2}\} \cap Q_1.$$

In view of (4.1), we know that $|B_k| \to 0$ as $k \to +\infty$. In view of Theorem 8, this implies that

$$\int T''(f_k) A_k \nabla_v f_k \cdot \nabla_v f_k \phi \to 0 \quad \text{ as } n \to +\infty. \tag{4.4}$$

We also know that $\nabla \bar f_k$ is bounded in $L^2(Q_1)$. Hence, we can assume that

$$\bar h_k := A_k \nabla_v \bar f_k \rightharpoonup \bar h \quad \text{ in } L^2(Q_1). \tag{4.5}$$

In view of (4.3), (4.4) and (4.5), we thus have

$$(\partial_t + v \cdot \nabla_x) \bar f = \nabla_v \bar h. \tag{4.6}$$

Identification of $\bar h$. Given $\phi \in \mathcal{D}(Q_1)$, we can on one hand use $\bar f \phi$ as a test function in (4.6) and get after integrating in all variables,

$$\frac{1}{2} \int (\bar f)^2 (\partial_t + v \cdot \nabla_x) \phi = \int \bar h \nabla_v (\bar f \phi).$$

On the other hand, we can use $\bar{f}_k\phi$ as a test function in (4.3) and get at the limit

$$\frac{1}{2}\int(\bar{f})^2(\partial_t + v\cdot\nabla_x)\phi = \lim_{k\to+\infty}\int \bar{h}_k\cdot\nabla_v(\bar{f}_k\phi).$$

In particular,

$$\int \bar{h}\nabla_v(\bar{f}\phi) = \lim_{k\to+\infty}\int \bar{h}_k\cdot\nabla_v(\bar{f}_k\phi).$$

Since $f_k \to f$ strongly in L^2 we have

$$\lim_{k\to+\infty}\int \bar{h}_k\cdot\bar{f}_k\nabla_v\phi = \int \bar{h}\cdot\bar{f}\nabla_v\phi.$$

and then since $\nabla_v\bar{f} = 0$, this implies

$$\lim_{k\to+\infty}\int \bar{h}_k\cdot\nabla_v\bar{f}_k\phi = 0.$$

Hence, for $\phi \geq 0$,

$$\begin{aligned}\int |\bar{h}|^2\phi &\leq \liminf_{k\to+\infty}\int |\tilde{A}_k\nabla_v\bar{f}_k|^2\phi\\ &\leq \Lambda\lim_{k\to+\infty}\int \tilde{A}_k\nabla_v\bar{f}_k\cdot\nabla_v\bar{f}_k\phi\\ &\leq \Lambda\lim_{k\to+\infty}\int \bar{h}_k\cdot\nabla_v\bar{f}_k\phi = 0.\end{aligned}$$

which implies that $\bar{h} = 0$.

Conclusion. We deduce that

$$\text{for a.e. } v\in B_1(0),\quad \partial_t\bar{f} + (v_0+v)\cdot\nabla_x\bar{f} = 0 \quad \text{in } B_1\times(-1,0).$$

In particular, rewriting the equation for $-v$, summing and using all $v \in B_1(0)$, we get

$$\partial_t f + v_0\cdot\nabla_x f \equiv 0, \nabla_x f \equiv 0$$

which, in turn, yields that f is constant (i.e. is a *global equilibrium* in the terminology of kinetic theory), which contradicts the lower bounds on the measure of the sets above. We thus get the desired contradiction. The proof is complete. □

Appendix A. Isoperimetric lemma implies decrease of the upper bound

Proof of Lemma 14. We follow the nice exposition of [16]. Let C_0 be the universal constant such that solutions f of (1.3) in Q_2 satisfy

$$\|f_+\|_{L^\infty(Q_{\frac{1}{2}})} \leq C_0\|f_+\|_{L^2(Q_1)}.$$

We now define $f_1 = f$ and $f_{k+1} = 2f_k - 1$. Remark that

$$\begin{aligned}|\{f_1 \leq 0\}\cap Q_1| &\geq \delta_1\\ \{f_{k+1}\leq 0\} &\supset \{f_k \leq 0\}\end{aligned}$$

with $\delta_1 = |Q_1|/2$ (remark it is universal). Our goal is to prove that there exists k_0 universal such that

$$|\{f_{k_0} \geq 0\} \cap Q_1| \leq \delta_2$$

with $\delta_2 = (4C_0^2)^{-1}$ (remark it is universal). Indeed, this implies

$$\|(f_{k_0})_+\|_{L^\infty(Q_{\frac{1}{2}})} \leq C_0 \|(f_{k_0})_+\|_{L^2(Q_1)} \leq C_0 \Big[|\{f_{k_0} \geq 0\} \cap Q_1| \Big]^{\frac{1}{2}} \leq \frac{1}{2}$$

which, in turn, yields

$$f \leq 1 - 2^{-k_0 - 1} \quad \text{in } Q_{\frac{1}{2}}.$$

Assume that for all $k \geq 1$,

$$|\{f_k \geq 0\} \cap Q_1| \geq \delta_2.$$

Since $f_{k+1} = 2f_k - 1$, this also implies

$$|\{f_k \geq \frac{1}{2}\} \cap Q_1| \geq \delta_2.$$

But we also have

$$|\{f_k \leq 0\} \cap Q_1| \geq |\{f \leq 0\} \cap Q_1| \geq \delta_1.$$

Hence Lemma 15 implies that

$$|\{0 \leq f_k \leq \frac{1}{2}\} \cap Q_1| \geq \alpha.$$

Now remark that

$$\begin{aligned} |Q_1| \geq |\{f_{k+1} \leq 0\} \cap Q_1| &= |\{f_k \leq 0\} \cap Q_1| + |\{0 \leq f_k \leq \frac{1}{2}\} \cap Q_1| \\ &\geq |\{f_k \leq 0\} \cap Q_1| + \alpha \\ &\geq k\alpha \end{aligned}$$

which is impossible for k large enough. □

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