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# Finding stable flows in multi-agent networks under an equal-sharing policy

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## Abstract

A Multi-Agent network flow problem is addressed in this paper where a set of transportation-agents can control the capacities of a set of routes. Each agent incurs a cost proportional to the chosen capacity. A third-party agent, a customer, is interesting in maximizing the product flow transshipped from a source to a sink node through the network. He offers a reward proportional to the flow value, which is shared equally among the transportation-agents. We address the problem of finding a stable strategy (i.e., a Nash Equilibrium) that maximizes the network flow. In this paper, we present a Mixed Integer Linear Program (MILP) to model and solve this problem.

*Keywords:* Multi-Agent Network flow, Nash Equilibria, Equal-sharing Policy.

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## 1 Introduction

Multi-agent network games have become a promising interdisciplinary research area with important links to many application fields such as transportation networks, supply chain management, web services, production management, etc [1], [2]. This paper stands at the crossroad of two disciplines, namely multi-agent systems and social networks. To the best of our knowledge, the

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research presented here is an original way of presenting a transportation problem using multi-agent network flow with controllable arcs capacities. One important application is the expansion of transportation network capacity (railway, roads, pipelines, etc.) to meet current peak demand or to absorb future increase in the transportation demands. In this paper, we consider a particular multi-agent network flows problem involving a set of self-interested transportation-agents, each of them managing his proper set of network roads. Every agent is able to increase the capacity of his arcs by gathering extra resources, at a given cost. A customer-agent is interested in increasing the flow circulating in the network. He offers a reward, to be equally-shared among the agents, for each additional unit of flow delivered through the transportation network. The contribution of this paper is to propose a MILP to find a Nash equilibria maximizing the flow, assuming a given sharing policy of the reward. The paper is organized as follows: Section 2 defines formally the problem and recalls previous results. In Section 3 a MILP formulation is proposed. Thereafter, Section 4 provides experimental results.

## 2 Problem Statement

### 2.1 Notations and Definitions

The Multi-Agent Network-Flow with Controllable Capacities (**MA-NFCC**) can be defined as a tuple  $\langle G, \mathcal{A}, \underline{Q}, \overline{Q}, \mathcal{C}, \pi, \mathcal{W} \rangle$  where:

- $G = (\mathcal{V}, \mathcal{E})$  is a network flow.  $\mathcal{V}$  is the set of nodes and  $\mathcal{E}$  is the set of arcs, each one having its capacity and receiving a flow.
- $\mathcal{A} = \{A_1, \dots, A_u, \dots, A_m\}$  is a set of  $m$  transportation-agents. Arcs are distributed among the agents. An agent  $A_u$  owns a set of  $m_u$  arcs, denoted  $\mathcal{E}_u$ . Each arc  $(i, j)$  belongs to exactly one transportation-agent and  $\mathcal{E}_u$  represents the subset of arcs for agent  $A_u$ .
- $\underline{Q} = (q_{i,j})$  (resp.  $\overline{Q} = (\bar{q}_{i,j})$ ) represents the vector of normal capacity (resp. maximum capacity) for the agent  $A_u$  such that  $(i, j) \in \mathcal{E}_u$ .
- $\mathcal{C} = (c_{i,j})$  is the vector of unitary costs incurred by the agent  $A_u$  for increasing the capacity of  $(i, j) \in \mathcal{E}_u$  by one unit beyond the minimal capacity.
- $\pi$  is the reward given by the customer proportionally to the additional flow.
- $\mathcal{W} = \{w_u\}$  defines the sharing policy of rewards among the agents.

**Assumptions.** In this paper, for sake of simplicity, it is assumed that the minimal capacities are equal to 0 ( $q_{i,j} = 0$ ) and that the reward is shared

equally between the transportation-agents, i.e.,  $w_u = \frac{1}{m}$ .

Each transportation-agent has to decide the capacity  $q_{i,j}$  of his arcs, such that  $0 \leq q_{i,j} \leq \bar{q}_{i,j}$ . A *strategy*  $\mathcal{S}$  is the vector that gather the individual strategies of all agents, i.e., (their arc capacities):  $\mathcal{S} = \{Q_1, \dots, Q_m\}$  where  $Q_u = (q_{i,j})$ ,  $\forall (i,j) \in \mathcal{E}_u$ .  $F(\mathcal{S})$  denotes the *flow* that can circulate on the network given the current values of capacities. For each arc  $(i,j)$ , the circulating flow  $f_{i,j}$  is such that  $0 \leq f_{i,j} \leq q_{i,j}$ . When considering null minimal capacities, the maximum flow that can circulate on the network at zero cost is equal to 0. The total reward given by the customer-agent for a circulating flow  $F(\mathcal{S})$  is  $\pi \times F(\mathcal{S})$ . The profit  $Z_u(\mathcal{S})$  of transportation-agent  $A_u$  under strategy  $\mathcal{S}$  is equal to the difference between his reward and spending  $Z_u(\mathcal{S}) = \frac{1}{m} \times \pi \times F(\mathcal{S}) - \sum_{(i,j) \in \mathcal{E}_u} c_{i,j} \times q_{i,j}$ .

## 2.2 The Nash-Stable Max-Flow Problem

In MA-NFCC setting, our goal is to find a Nash equilibrium between transportation-agents that maximizes the flow for the customer agent. A strategy is stable if there is no incentive for any agent to modify his capacities in order to improve his profit. Ideally, agents should choose a strategy which is both optimal, with respect to the flow value, and stable, with respect to the Nash equilibrium concept. Since such a strategy does not always exist, we search for a Nash equilibrium that is as efficient as possible with respect to the customer viewpoint. The problem of finding a Nash Equilibrium that maximizes  $F(\mathcal{S})$  was proved to be NP-Hard in [5] from a reduction of the 3-partition problem, which is known to be NP-Hard in the strong sense.

**Characterization of Nash Equilibria:** Finding a Nash-equilibria that maximizes the flow amounts to find profitable augmenting paths in residual graphs to increase flow, without generating decreasing paths that are profitable for some agent, hence preserving stability. Therefore, for a given non-poor strategy  $\mathcal{S}$  and given value  $w_u$ ,  $\mathcal{S}$  is a Nash Equilibrium if there is no path that can be profitable for any agent to increase or decrease the flow in the network. The proof of this statement, detailed in [5], is based on the fact that an agent can only improve his situation by increasing or decreasing the flow, which can only happen when there exists a profitable increasing / decreasing path.

We recall that, in the multi-agent context, there are as many residual graphs as agents. An augmenting (resp. decreasing) path, going from the source to the sink node (resp. *from the sink to the source node*), is composed by a set of forward and backward arcs, i.e.  $P = \{P^+, P^-\}$  and allows to increase (resp. decrease) the flow value by one unit at least.

### 2.3 Example of a MA-NFCC Problem

Let us consider a customer-agent willing to maximize a flow of products in the network, displayed in Fig. 1, from the source node  $A$  to the sink node  $D$ . He offers a reward  $\pi = 120$  which is shared equally between two agents  $A_1$  and  $A_2$ , i.e.  $w_1 = w_2 = \frac{1}{2}$ . The set of arcs of each transportation-agent are  $\mathcal{E}_1 = \{b, c, d\}$  and  $\mathcal{E}_2 = \{a, e\}$ , which are represented with plain and dotted arcs, respectively. Each arc of Fig. 1 is valued by the interval of normal and maximum capacities  $([0, \bar{q}_{i,j}])$ , and the unitary cost incurred when increasing the arc capacities  $(c_{i,j})$  beyond the minimal ones. Let consider the strategy

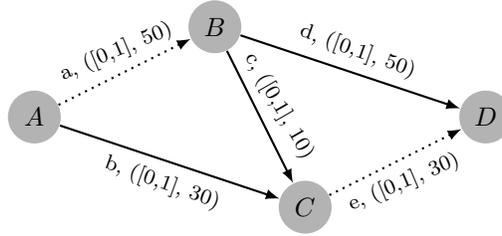


Fig. 1. Example of MA-NFCC problem

$\mathcal{S}$ , where both transportation-agents choose to open their respective routes  $(b, e)$  and  $(a, d)$  by one unit. The flow  $F(\mathcal{S})$  is then equal to 2, the profit for both agents is  $Z_1(\mathcal{S}) = Z_2(\mathcal{S}) = 40$ . From this strategy, there exists a profitable decreasing path  $(d, c, b)$  from sink node  $D$  to source node  $A$  which is profitable for agent  $A_1$  leading to the strategy  $\mathcal{S}'$  with  $F(\mathcal{S}') = 1$ . In fact,  $A_1$  can improve his own profit, by decreasing back the flow on  $b$  and  $d$  by one unit and increasing the flow on arc  $c$  by one unit ( $Z_1(\mathcal{S}') = 50$  and  $Z_2(\mathcal{S}') = -20$ ). Therefore, the strategy  $\mathcal{S}$  is not a stable strategy. Considering the strategy  $\mathcal{S}'$  there is no decreasing path for  $A_1$  and  $A_2$  and then  $\mathcal{S}'$  is a stable strategy.

## 3 A MILP for the Nash-Stable Max-Flow Problem

### 3.1 Mathematical model

This problem is described by the non linear mathematical model (1), where constraints (i), (ii) and (iii) are the constraints of arcs capacities and flow conservation, respectively. The non-explicit constraints (iv) impose that no decreasing path  $\bar{P}$  can exist in solution  $S$  having a saving  $sav_u(\bar{P}) \geq w_u \times \pi$ . In other words, it enforces any solution to be Nash stable. Note that  $sav_u(\bar{P})$  is the saving of agent  $A_u$ , when decreasing the capacity of its arcs by one unit through a decreasing path going from the sink to the source node.

Max  $F$

s.c.

$$\begin{aligned}
(i) \quad & f_{i,j} \leq q_{i,j}, \forall (i,j) \in \mathcal{E} \\
(ii) \quad & 0 \leq q_{i,j} \leq \bar{q}_{i,j}, \forall (i,j) \in \mathcal{E} \\
(iii) \quad & \sum_{j \in P^+(i)} f_{i,j} = \sum_{j \in P^-(i)} f_{j,i}, \forall i \in \mathcal{V} \\
(iv) \quad & sav_u(\bar{P}) < w_u \times \pi, \forall \bar{P} \in \bar{G}_f \\
& f_{i,j} \geq 0, \forall (i,j) \in \mathcal{E}
\end{aligned} \tag{1}$$

We focus on the linearization of the non-linear constraints (iv) of the formulation (1) replacing it by a finite number of primal-dual constraints.

### 3.2 Characterization of Nash constraint

In order to explain how to increase or decrease the flow value, we need first to detail the concept of residual graphs in a multi-agent context.

Given a network  $G = (\mathcal{V}, \mathcal{E})$  and a strategy  $\mathcal{S}$ , there is a residual graph  $G_f^u(\mathcal{S}) = (\mathcal{V}, \mathcal{E}_r)$  for each agent  $A_u$ . To each edge  $(i, j) \in \mathcal{E}$ , we associate a forward and a backward edge. Each forward edge  $(i, j)$  of  $G_f^u$  has a residual capacity  $r_{i,j} = \bar{q}_{i,j} - q_{i,j}$  and cost  $\delta_F^{i,j,u}$ . Each backward edge  $(j, i)$  of  $G_f^u$  is with a residual capacity  $r_{j,i} = q_{i,j}$  and cost  $\delta_B^{j,i,u}$ .

$$\delta_F^{i,j,u} = \begin{cases} -\infty & \text{if } (i,j) \in \mathcal{E}_u \wedge q_{i,j} = \bar{q}_{i,j} \\ -c_{i,j} & \text{if } (i,j) \in \mathcal{E}_u \wedge q_{i,j} < \bar{q}_{i,j} \\ -\infty & \text{if } (i,j) \notin \mathcal{E}_u \wedge q_{i,j} = \bar{q}_{i,j} \\ -\infty & \text{if } (i,j) \notin \mathcal{E}_u \wedge q_{i,j} < \bar{q}_{i,j} \end{cases} \quad \delta_B^{j,i,u} = \begin{cases} -\infty & \text{if } (i,j) \in \mathcal{E}_u \wedge q_{i,j} = 0 \\ c_{i,j} & \text{if } (i,j) \in \mathcal{E}_u \wedge q_{i,j} > 0 \\ -\infty & \text{if } (i,j) \notin \mathcal{E}_u \wedge q_{i,j} = 0 \\ 0 & \text{if } (i,j) \notin \mathcal{E}_u \wedge q_{i,j} > 0 \end{cases} \tag{2}$$

A forward arc having the cost  $-c_{i,j}$  indicates that  $A_u$  can possibly make a profit by increasing the capacity of  $(i, j)$  since  $q_{i,j} < \bar{q}_{i,j}$ . A forward arc of cost  $-\infty$  cannot be used to increase the flow in the network (i.e., it is already at its maximum capacity or the arc belongs to another agent). In a similar way, a backward arc valued by 0 indicates that the arc does not matter in the computation of the cost of agent  $A_u$ . An arc valued by the cost  $c_{i,j}$  reflects the fact that the agent can make a saving by decreasing his arcs capacities (i.e.,  $q_{i,j} > 0$ ). Finally, a  $-\infty$  cost on a backward arc guarantees the impossibility of using the corresponding arc to decrease the flow (i.e.,  $q_{i,j} = 0$ ).

**Arcs forming decreasing paths:** As stated in [5], a profitable decreasing path  $\bar{P}$ , is a decreasing path *from sink to the source node* in the residual graph  $G_f^u(\mathcal{S})$ , which is profitable at least to one agent, i.e.,  $sav_u(\bar{P}) > w_u \times \pi$ . According to the definition of the costs of forward and backward arcs in the residual graphs (cf. equations 2), the

saving of agent  $A_u$  when decreasing capacity by one unit through a decreasing path is  $sav_u(\bar{P}) = \sum_{(i,j) \in \bar{P}^+ \cap E_u} \delta_F^{i,j,u} - \sum_{(i,j) \in \bar{P}^- \cap E_u} \delta_B^{i,j,u}$ .

**Longest path in the residual Network:** We are interested in finding the path that maximize  $sav_u(\bar{P})$ . In other words, in computing the longest decreasing path in  $G_f^u(S)$ . This can be done using primal-dual constraints. For a given  $G_f^u(S)$ , equations (3)-(4) define the primal constraints of the longest path problem where  $t_i^u \geq 0$ . Equations (5) define the dual constraints of unitary flow variation using the binary variables  $\varphi_{i,j}^u = 1$  if one unit of flow circulates on arc  $(i, j)$ , 0 otherwise.

$$t_j^u - t_i^u \geq \delta_F^{i,j,u}, \forall (i, j) \in \mathcal{E}_u, \forall A_u \in \mathcal{A} \quad (3)$$

$$t_i^u - t_j^u \geq \delta_B^{j,i,u}, \forall (i, j) \in \mathcal{E}_u, \forall A_u \in \mathcal{A} \quad (4)$$

Expression  $t_j^u - t_i^u$  represents the potential associated with the arc  $(i, j)$ .

$$\sum_{(i,j) \in \mathcal{E} \cup \mathcal{E}_r} \varphi_{i,j}^u - \sum_{(i,j) \in \mathcal{E} \cup \mathcal{E}_r} \varphi_{j,i}^u = \begin{cases} 0 & \forall i \neq s, t \\ -1, & i = s, \forall i \in \mathcal{V}, \forall A_u \in \mathcal{A} \\ 1, & i = t \end{cases} \quad (5)$$

With respect to the Complementary Slackness Theorem, which imposes, at optimality, that the cost of a path in  $G_f^u(S)$  is equal to the saving of the units of flows circulating in the network, we obtain the constraints (6).

$$t_{n+1}^u \geq \sum_{(i,j) \in \mathcal{E}} \varphi_{i,j}^u \times \delta_F^{i,j,u} + \sum_{(i,j) \in \mathcal{E}_r} \varphi_{i,j}^u \times \delta_B^{j,i,u}, \forall A_u \in \mathcal{A} \quad (6)$$

The right-hand-side of (6) contains multiplication of two variables. Fortunately, since  $\varphi_{i,j}^u$  is binary, the multiplication can be easily linearized by introducing variables  $\psi_1^{i,j,u} = \varphi_{i,j}^u \times \delta_F^{i,j,u}$  and  $\psi_2^{i,j,u} = \varphi_{i,j}^u \times \delta_B^{j,i,u}$ . For instance, for the first one, the set of inequalities (i) – (iv) will ensure the linearization.

$$\begin{aligned} (i) \quad & \psi_1^{i,j,u} \geq 0, \forall (i, j) \in \mathcal{E}, \forall A_u \in \mathcal{A} \\ (ii) \quad & \psi_1^{i,j,u} \leq \varphi_{i,j}^u \times M, \forall (i, j) \in \mathcal{E}, \forall A_u \in \mathcal{A} \\ (iii) \quad & \psi_1^{i,j,u} \leq -\delta_F^{i,j,u}, \forall (i, j) \in \mathcal{E}, \forall A_u \in \mathcal{A} \\ (iv) \quad & \psi_1^{i,j,u} \geq M(\varphi_{i,j}^u - 1) - \delta_F^{i,j,u}, \forall (i, j) \in \mathcal{E}, \forall A_u \in \mathcal{A} \end{aligned} \quad (7)$$

Finally, since  $t_{n+1}^u$  is equal to  $sav_u(\bar{P})$ , constraints (iv) can be replaced in the MILP model by  $t_{n+1}^u < w_u \times \pi, \forall A_u \in \mathcal{A}$ .

## 4 First experiments

The previous MILP model, was implemented using the C++ API for GUROBI Optimization 6.0.0. The algorithm performance was evaluated on a PC with Linux



## 5 Conclusion

In this paper, we show that the problem of finding a Nash Equilibrium that maximizes the flow, already proved to be NP-hard, can be modeled and solved using a MILP and we presents preliminary results to assess the approach efficiency. Further works are ongoing to extend our approach to find an optimal sharing policy of the reward and to consider the case where minimum capacity different from zero.

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