Local large deviations principle for occupation measures of the damped nonlinear wave equation perturbed by a white noise

Davit Martirosyan, Vahagn Nersesyan

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Abstract

We consider the damped nonlinear wave (NLW) equation driven by a spatially regular white noise. Assuming that the noise is non-degenerate in all Fourier modes, we establish a large deviations principle (LDP) for the occupation measures of the trajectories. The lower bound in the LDP is of a local type, which is related to the weakly dissipative nature of the equation and seems to be new in the context of randomly forced PDE's. The proof is based on an extension of methods developed in [JNPS] and [JNPS14] in the case of kick forced dissipative PDE's with parabolic regularisation property such as, for example, the Navier–Stokes system and the complex Ginzburg–Landau equations. We also show that a high concentration towards the stationary measure is impossible, by proving that the rate function that governs the LDP cannot have the trivial form (i.e., vanish on the stationary measure and be infinite elsewhere).

AMS subject classifications: 35L70, 35R60, 60B12, 60F10

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Contents

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0 Introduction

This paper is devoted to the study of the large deviations principle (LDP) for the occupation measures of the stochastic nonlinear wave (NLW) equation in a bounded domain $D \subset \mathbb{R}^3$ with a smooth boundary ∂D:

$$\partial_t^2 u + \gamma \partial_t u - \Delta u + f(u) = h(x) + \vartheta(t,x), \quad u|_{\partial D} = 0, \tag{0.1}$$

$$[u(0), \dot{u}(0)] = [u_0, u_1]. \tag{0.2}$$

Here $\gamma > 0$ is a damping parameter, h is a function in $H_0^1(D)$, and f is a nonlinear term satisfying some standard dissipativity and growth conditions (see (1.1)-(1.3)). These conditions are satisfied for the classical examples $f(u) = \sin u$ and $f(u) = |u|^\rho u - \lambda u$, where $\lambda \in \mathbb{R}$ and $\rho \in (0,2)$, coming from the damped sine–Gordon and Klein–Gordon equations. We assume that $\vartheta(t,x)$ is a white noise of the form

$$\vartheta(t,x) = \partial_t \xi(t,x), \quad \xi(t,x) = \sum_{j=1}^{\infty} b_j \beta_j(t) e_j(x), \tag{0.3}$$

where $\{\beta_j\}$ is a sequence of independent standard Brownian motions, the set of functions $\{e_j\}$ is an orthonormal basis in $L^2(D)$ formed by eigenfunctions

of the Dirichlet Laplacian with eigenvalues $\{\lambda_j\}$, and $\{b_j\}$ is a sequence of real numbers satisfying

$$\mathfrak{B}_1 := \sum_{j=1}^{\infty} \lambda_j b_j^2 < \infty. \tag{0.4}$$

We denote by $(\mathfrak{u}_t, \mathbb{P}_{\mathfrak{u}}), \mathfrak{u}_t = [u_t, \dot{u}_t]$ the Markov family associated with this stochastic NLW equation and parametrised by the initial condition $\mathfrak{u} = [u_0, u_1]$. The exponential ergodicity for this family is established in [Mar14], this result is recalled below in Theorem 1.1.

The LDP for the occupation measures of randomly forced PDE's has been previously established in [Gou07b, Gou07a] in the case of the Burgers equation and the Navier–Stokes system, based on some abstract results from [Wu01]. In these papers, the force is assumed to be a rough white noise, i.e., it is of the form (0.3) with the following condition on the coefficients:

$$cj^{-\alpha} \le b_j \le Cj^{-\frac{1}{2}-\varepsilon}, \quad \frac{1}{2} < \alpha < 1, \ \varepsilon \in \left(0, \alpha - \frac{1}{2}\right].$$

In the case of a perturbation which is a regular random kick force, the LDP is proved in [JNPS, JNPS14] for a family of PDE's with parabolic regularisation (such as the Navier–Stokes system or the complex Ginzburg–Landau equation). See also [JNPS15] for the proof of the LDP and the Gallavotti–Cohen principle in the case of a rough kick force.

The aim of the present paper is to extend the results and the methods of these works under more general assumptions on both stochastic and deterministic parts of the equations. The random perturbation in our setting is a spatially regular white noise, and the NLW equation is only weakly dissipative and lacks a regularising property. In what follows, we shall denote by μ the stationary measure of the family $(\mathfrak{u}_t, \mathbb{P}_{\mathfrak{u}})$, and for any bounded continuous function $\psi : H_0^1(D) \times L^2(D) \to \mathbb{R}$, we shall write $\langle \psi, \mu \rangle$ for the integral of ψ with respect to μ. We prove the following level-1 LDP for the solutions of problem (0.1), (0.3).

Main Theorem. *Assume that conditions* (0.4) *and* (1.1)-(1.3) *are verified and* $b_j > 0$ *for all* $j \ge 1$. *Then for any non-constant bounded Hölder-continuous function* $\psi : H_0^1(D) \times L^2(D) \to \mathbb{R}$, *there is* $\varepsilon = \varepsilon(\psi) > 0$ *and a convex function* $I^\psi : \mathbb{R} \to \mathbb{R}_+$ *such that, for any* $\mathfrak{u} \in H^{\mathfrak{s}+1}(D) \times H^{\mathfrak{s}}(D)$ *and any open subset* O *of the interval* $(\langle \psi, \mu \rangle - \varepsilon, \langle \psi, \mu \rangle + \varepsilon)$, *we have*

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}_{\mathfrak{u}} \left\{ \frac{1}{t} \int_0^t \psi(\mathfrak{u}(\tau))\, \mathrm{d}\tau \in O \right\} = - \inf_{\alpha \in O} I^\psi(\alpha), \tag{0.5}$$

where $\mathfrak{s} > 0$ *is a small number. Moreover, limit* (0.5) *is uniform with respect to* $\mathfrak{u}$ *in a bounded set of* $H^{\mathfrak{s}+1}(D) \times H^{\mathfrak{s}}(D)$.

We also establish a more general result of level-2 type in Theorem 1.2. These two theorems are slightly different from the standard Donsker–Varadhan form

(e.g., see Theorem 3 in [DV75]), since here the LDP is proved to hold locally on some part of the phase space.

The proof of the Main Theorem is obtained by extending the techniques and results introduced in [JNPS, JNPS14]. According to a local version of the Gärtner–Ellis theorem, relation (0.5) will be established if we show that, for some $\beta_0 > 0$, the following limit exists

$$Q(\beta) = \lim_{t\to+\infty} \frac{1}{t} \log \mathbb{E}_{\mathfrak{u}} \exp\biggl(\int_0^t \beta\psi(\mathfrak{u}_\tau)\, \mathrm{d}\tau \biggr), \quad |\beta| < \beta_0$$

and it is differentiable in β on $(-\beta_0, \beta_0)$. We show that both properties can be derived from a multiplicative ergodic theorem, which is a convergence result for the Feynman–Kac semigroup of the stochastic NLW equation. A continuous-time version of a criterion established in [JNPS14] shows that a multiplicative ergodic theorem holds provided that the following four conditions are satisfied: *uniform irreducibility, exponential tightness, growth condition,* and *uniform Feller property*. The smoothness of the noise and the lack of a strong dissipation and of a regularising property in the equation result in substantial differences in the techniques used to verify these conditions. While in the case of kick-forced models the first two of them are checked directly, they have a rather non-trivial proof in our case, relying on a feedback stabilisation result and some subtle estimates for the Sobolev norms of the solutions. Nonetheless, the most involved and highly technical part of the paper remains the verification of the uniform Feller property. Based on the coupling method, its proof is more intricate here mainly due to a more complicated Foiaş–Prodi type estimate for the stochastic NLW equation. We get a uniform Feller property only for potentials that have a sufficiently small oscillation, and this is the main reason why the LDP established in this paper is of a local type.

The paper is organised as follows. We formulate in Section 1 the second main result of this paper on the level-2 LDP for the NLW equation and, by using a local version of Kifer's criterion, we reduce its proof to a multiplicative ergodic theorem. Section 2 is devoted to the derivation of the Main Theorem. In Sections 3 and 4, we are checking the conditions of an abstract result about the convergence of generalised Markov semigroups. In Section 5, we prove the exponential tightness property and provide some estimates for the growth of Sobolev norms of the solutions. The multiplicative ergodic theorem is established in Section 6. In the Appendix, we prove the local version of Kifer's criterion, the abstract convergence result for the semigroups, and some other technical results which are used throughout the paper.

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Notation

For a Banach space X, we denote by $B_X(a,R)$ the closed ball in X of radius R centred at a. In the case when $a=0$, we write $B_X(R)$. For any function $V: X \to \mathbb{R}$, we set $\mathrm{Osc}_X(V) := \sup_X V - \inf_X V$. We use the following spaces:

$L^\infty(X)$ is the space of bounded measurable functions $\psi: X \to \mathbb{R}$ endowed with the norm $\|\psi\|_\infty = \sup_{u\in X} |\psi(u)|$.

$C_b(X)$ is the space of continuous functions $\psi \in L^\infty(X)$, and $C_+(X)$ is the space of positive continuous functions $\psi: X \to \mathbb{R}$.

$C_b^q(X)$, $q \in (0,1]$ is the space of functions $f \in C_b(X)$ for which the following norm is finite

$$\|\psi\|_{C_b^q} = \|\psi\|_\infty + \sup_{u\neq v} \frac{|\psi(u)-\psi(v)|}{\|u-v\|^q}.$$

$\mathcal{M}(X)$ is the vector space of signed Borel measures on X with finite total mass endowed with the topology of the weak convergence. $\mathcal{M}_+(X) \subset \mathcal{M}(X)$ is the cone of non-negative measures.

$\mathcal{P}(X)$ is the set of probability Borel measures on X. For $\mu \in \mathcal{P}(X)$ and $\psi \in C_b(X)$, we denote $\langle \psi, \mu\rangle = \int_X \psi(u)\mu(\mathrm{d}u)$. If $\mu_1, \mu_2 \in \mathcal{P}(X)$, we set

$$|\mu_1 - \mu_2|_{var} = \sup\{|\mu_1(\Gamma) - \mu_2(\Gamma)| : \Gamma \in \mathcal{B}(X)\},$$

where $\mathcal{B}(X)$ is the Borel σ-algebra of X.

For any measurable function $\mathfrak{w}: X \to [1,+\infty]$, let $C_\mathfrak{w}(X)$ (respectively, $L^\infty_\mathfrak{w}(X)$) be the space of continuous (measurable) functions $\psi: X \to \mathbb{R}$ such that $|\psi(u)| \leq C\mathfrak{w}(u)$ for all $u \in X$. We endow $C_\mathfrak{w}(X)$ and $L^\infty_\mathfrak{w}(X)$ with the seminorm

$$\|\psi\|_{L^\infty_\mathfrak{w}} = \sup_{u\in X} \frac{|\psi(u)|}{\mathfrak{w}(u)}.$$

$\mathcal{P}_\mathfrak{w}(X)$ is the space of measures $\mu \in \mathcal{P}(X)$ such that $\langle \mathfrak{w}, \mu\rangle < \infty$.

For an open set D of $\mathbb{R}^3$, we introduce the following function spaces:

$L^p = L^p(D)$ is the Lebesgue space of measurable functions whose p^{th} power is integrable. In the case $p=2$ the corresponding norm is denoted by $\|\cdot\|$.

$H^\mathfrak{s} = H^\mathfrak{s}(D), \mathfrak{s} \geq 0$ is the domain of definition of the operator $(-\Delta)^{\mathfrak{s}/2}$ endowed with the norm $\|\cdot\|_\mathfrak{s}$:

$$H^\mathfrak{s} = \mathfrak{D}\left((-\Delta)^{\mathfrak{s}/2}\right) = \left\{ u = \sum_{j=1}^\infty u_j e_j \in L^2 : \|u\|_\mathfrak{s}^2 := \sum_{j=1}^\infty \lambda_j^s u_j^2 < \infty \right\}.$$

In particular, H^1 coincides with $H_0^1(D)$, the space of functions in the Sobolev space of order 1 that vanish at the boundary. We denote by $H^{-\mathfrak{s}}$ the dual of $H^\mathfrak{s}$.

1 Level-2 LDP for the NLW equation

1.1 Stochastic NLW equation and its mixing properties

In this subsection we give the precise hypotheses on the nonlinearity and recall a result on the property of exponential mixing for the Markov family associated with the flow of (0.1). We shall assume that f belongs to $C^2(\mathbb{R})$, vanishes at zero, satisfies the growth condition

$$|f''(u)| \le C(|u|^{\rho-1}+1), \quad u \in \mathbb{R}, \tag{1.1}$$

for some positive constants C and $\rho < 2$, and the dissipativity conditions

$$F(u) \ge C^{-1}|f'(u)|^{\frac{\rho+2}{\rho}} - \nu u^2 - C, \tag{1.2}$$

$$f(u)u - F(u) \ge -\nu u^2 - C, \tag{1.3}$$

where F is a primitive of f, ν is a positive number less than $(\lambda_1 \wedge \gamma)/8$. Let us note that inequality (1.2) is slightly more restrictive than the one used in [Mar14]; this hypothesis allows us to establish the exponential tightness property (see Section 5.1). We consider the NLW equation in the phase space $\mathcal{H} = H^1 \times L^2$ endowed with the norm

$$|\mathfrak{u}|^2_{\mathcal{H}} = \|u_1\|_1^2 + \|u_2 + \alpha u_1\|^2, \quad \mathfrak{u} = [u_1, u_2] \in \mathcal{H}, \tag{1.4}$$

where $\alpha = \alpha(\gamma) > 0$ is a small parameter. Under the above conditions, for any initial data $\mathfrak{u}_0 = [u_0, u_1] \in \mathcal{H}$, there is a unique solution (or *a flow*) $\mathfrak{u}_t = \mathfrak{u}(t; \mathfrak{u}_0) = [u_t, \dot{u}_t]$ of problem (0.1)-(0.3) in $\mathcal{H}$ (see Section 7.2 in [DZ92]). For any $\mathfrak{s} \in \mathbb{R}$, let $\mathcal{H}^{\mathfrak{s}}$ denote the space $H^{\mathfrak{s}+1} \times H^{\mathfrak{s}}$ endowed with the norm

$$|\mathfrak{u}|^2_{\mathcal{H}^{\mathfrak{s}}} = \|u_1\|^2_{\mathfrak{s}+1} + \|u_2 + \alpha u_1\|^2_{\mathfrak{s}}, \quad \mathfrak{u} = [u_1, u_2] \in \mathcal{H}^{\mathfrak{s}}$$

with the same α as in (1.4). If $\mathfrak{u}_0 \in \mathcal{H}^{\mathfrak{s}}$ and $0 < \mathfrak{s} < 1 - \rho/2$, the solution $\mathfrak{u}(t; \mathfrak{u}_0)$ belongs [1] to $\mathcal{H}^{\mathfrak{s}}$ almost surely. Let us define a function $\mathfrak{w} : \mathcal{H} \to [0, \infty]$ by

$$\mathfrak{w}(\mathfrak{u}) = 1 + |\mathfrak{u}|^2_{\mathcal{H}^{\mathfrak{s}}} + \mathcal{E}^4(\mathfrak{u}), \tag{1.5}$$

which will play the role of the *weight function*. Here

$$\mathcal{E}(\mathfrak{u}) = |\mathfrak{u}|^2_{\mathcal{H}} + 2\int_D F(u_1)\, \mathrm{d}x, \quad \mathfrak{u} = [u_1, u_2] \in \mathcal{H},$$

is the energy functional of the NLW equation.

We consider the Markov family $(\mathfrak{u}_t, \mathbb{P}_{\mathfrak{u}})$ associated with (0.1) and define the corresponding Markov operators

$$\mathfrak{P}_t : C_b(\mathcal{H}) \to C_b(\mathcal{H}), \qquad \mathfrak{P}_t\psi(\mathfrak{u}) = \int_{\mathcal{H}} \psi(\mathfrak{v}) P_t(\mathfrak{u}, \mathrm{d}\mathfrak{v}),$$

$$\mathfrak{P}^*_t : \mathcal{P}(\mathcal{H}) \to \mathcal{P}(\mathcal{H}), \qquad \mathfrak{P}^*_t\sigma(\Gamma) = \int_{\mathcal{H}} P_t(\mathfrak{v}, \Gamma)\sigma(\mathrm{d}\mathfrak{v}), \quad t \ge 0,$$

[1] Some estimates for the $\mathcal{H}^{\mathfrak{s}}$-norm of the solutions are given in Section 5.2.

where $P_t(\mathfrak{u},\Gamma) = \mathbb{P}_{\mathfrak{u}}\{\mathfrak{u}_t \in \Gamma\}$ is the transition function. Recall that a measure $\mu \in \mathcal{P}(\mathcal{H})$ is said to be stationary if $\mathfrak{P}_t^*\mu = \mu$ for any $t \geq 0$. The following result is Theorem 2.3 in [Mar14].

Theorem 1.1. *Let us assume that conditions (0.4) and (1.1)-(1.3) are verified and $b_j > 0$ for all $j \geq 1$. Then the family $(\mathfrak{u}_t, \mathbb{P}_{\mathfrak{u}})$ has a unique stationary measure $\mu \in \mathcal{P}(\mathcal{H})$. Moreover, there are positive constants C and $\varkappa$ such that, for any $\sigma \in \mathcal{P}(\mathcal{H})$, we have*

$$|\mathfrak{P}_t^*\sigma - \mu|_L^* \leq Ce^{-\varkappa t} \int_{\mathcal{H}} \exp\left(\varkappa |\mathfrak{u}|_{\mathcal{H}}^4\right) \sigma(\mathrm{d}\mathfrak{u}),$$

where we set

$$|\mu_1 - \mu_2|_L^* = \sup_{\|\psi\|_{C_b^1} \leq 1} |\langle \psi, \mu_1\rangle - \langle \psi, \mu_2\rangle|$$

for any $\mu_1, \mu_2 \in \mathcal{P}(\mathcal{H})$.

1.2 The statement of the result

Before giving the formulation of the main result of this section, let us introduce some notation and recall some basic definitions from the theory of LDP (see [DZ00]). For any $\mathfrak{u} \in \mathcal{H}$, we define the following family of *occupation measures*

$$\zeta_t = \frac{1}{t}\int_0^t \delta_{\mathfrak{u}_\tau}\, \mathrm{d}\tau, \quad t > 0, \tag{1.6}$$

where $\mathfrak{u}_\tau := \mathfrak{u}(\tau;\mathfrak{u})$ and $\delta_{\mathfrak{v}}$ is the Dirac measure concentrated at $\mathfrak{v} \in \mathcal{H}$. For any $V \in C_b(\mathcal{H})$ and $R > 0$, we set

$$Q_R(V) = \limsup_{t\to+\infty} \frac{1}{t} \log \sup_{\mathfrak{u}\in X_R} \mathbb{E}_{\mathfrak{u}} \exp\bigl(t\langle V, \zeta_t\rangle\bigr),$$

where $X_R := B_{\mathcal{H}^{\mathfrak{s}}}(R)$, $\mathfrak{s} \in (0, 1-\rho/2)$. Then $Q_R : C_b(\mathcal{H}) \to \mathbb{R}$ is a convex 1-Lipschitz function, and its *Legendre transform* is given by

$$I_R(\sigma) := \begin{cases} \sup_{V\in C_b(\mathcal{H})}\bigl(\langle V,\sigma\rangle - Q_R(V)\bigr) & \text{for } \sigma \in \mathcal{P}(\mathcal{H}), \\ +\infty & \text{for } \sigma \in \mathcal{M}(\mathcal{H}) \setminus \mathcal{P}(\mathcal{H}). \end{cases} \tag{1.7}$$

The function $I_R : \mathcal{M}(\mathcal{H}) \to [0, +\infty]$ is convex lower semicontinuous in the weak topology, and Q_R can be reconstructed from I_R by the formula

$$Q_R(V) = \sup_{\sigma\in\mathcal{P}(\mathcal{H})} \bigl(\langle V,\sigma\rangle - I_R(\sigma)\bigr) \quad \text{for any } V \in C_b(\mathcal{H}). \tag{1.8}$$

We denote by $\mathcal{V}$ the set of functions $V \in C_b(\mathcal{H})$ satisfying the following two properties.

Property 1. For any $R > 0$ and $\mathfrak{u} \in X_R$, the following limit exists (called pressure function)

$$Q(V) = \lim_{t\to+\infty} \frac{1}{t} \log \mathbb{E}_{\mathfrak{u}} \exp\biggl(\int_0^t V(\mathfrak{u}_\tau)\, \mathrm{d}\tau \biggr)$$

and does not depend on the initial condition $\mathfrak{u}$. Moreover, this limit is uniform with respect to $\mathfrak{u} \in X_R$.

Property 2. There is a unique measure $\sigma_V \in \mathcal{P}(\mathcal{H})$ (called equilibrium state) satisfying the equality

$$Q_R(V) = \langle V, \sigma_V \rangle - I_R(\sigma_V).$$

A mapping $I : \mathcal{P}(\mathcal{H}) \to [0, +\infty]$ is a *good rate function* if for any $a \geq 0$ the level set $\{\sigma \in \mathcal{P}(\mathcal{H}) : I(\sigma) \leq a\}$ is compact. A good rate function I is *non-trivial* if the effective domain $D_I := \{\sigma \in \mathcal{P}(\mathcal{H}) : I(\sigma) < \infty\}$ is not a singleton. Finally, we shall denote by $\mathcal{U}$ the set of functions $V \in C_b(\mathcal{H})$ for which there is a number $q \in (0, 1]$, an integer $N \geq 1$, and a function $F \in C_b^q(\mathcal{H}_N)$ such that

$$V(\mathfrak{u}) = F(P_N \mathfrak{u}), \quad \mathfrak{u} \in \mathcal{H}, \tag{1.9}$$

where $\mathcal{H}_N := H_N \times H_N$, $H_N := \operatorname{span}\{e_1, \dots, e_N\}$, and P_N is the orthogonal projection in $\mathcal{H}$ onto $\mathcal{H}_N$. Given a number $\delta > 0$, $\mathcal{U}_\delta$ is the subset of functions $V \in \mathcal{U}$ satisfying $\operatorname{Osc}(V) < \delta$.

Theorem 1.2. *Under the conditions of the Main Theorem, for any $R > 0$, the function $I_R : \mathcal{M}(\mathcal{H}) \to [0, +\infty]$ defined by (1.7) is a non-trivial good rate function, and the family $\{\zeta_t, t > 0\}$ satisfies the following local LDP.*

Upper bound. *For any closed set $F \subset \mathcal{P}(\mathcal{H})$, we have*

$$\limsup_{t\to\infty} \frac{1}{t} \log \sup_{\mathfrak{u}\in X_R} \mathbb{P}_{\mathfrak{u}}\{\zeta_t \in F\} \leq -I_R(F). \tag{1.10}$$

Lower bound. *For any open set $G \subset \mathcal{P}(\mathcal{H})$, we have*

$$\liminf_{t\to\infty} \frac{1}{t} \log \inf_{\mathfrak{u}\in X_R} \mathbb{P}_{\mathfrak{u}}\{\zeta_t \in G\} \geq -I_R(\mathcal{W} \cap G). \tag{1.11}$$

Here[2] *$I_R(\Gamma) := \inf_{\sigma\in\Gamma} I(\sigma)$ for $\Gamma \subset \mathcal{P}(\mathcal{H})$ and $\mathcal{W} := \{\sigma_V : V \in \mathcal{V}\}$, where σ_V is the equilibrium state*[3] *corresponding to V.*

Furthermore, there is a number $\delta > 0$ such that $\mathcal{U}_\delta \subset \mathcal{V}$ and for any $V \in \mathcal{U}_\delta$, the pressure function $Q_R(V)$ does not depend on R.

This theorem is proved in the next subsection, using a multiplicative ergodic theorem and a local version of Kifer's criterion for LDP. Then in Section 2, we combine it with a local version of the Gärtner–Ellis theorem to establish the Main Theorem.

[2] The infimum over an empty set is equal to $+\infty$.

[3] By the fact that I_R is a good rate function, the set of equilibrium states is non-empty for any $V \in C_b(\mathcal{H})$. In Property 2, the important assumption is the uniqueness.

1.3 Reduction to a multiplicative ergodic theorem

In this subsection we reduce the proof of Theorem 1.2 to some properties related to the large-time behavior of the *Feynman–Kac semigroup* defined by

$$\mathfrak{P}_t^V \psi(\mathfrak{u}) = \mathbb{E}_{\mathfrak{u}} \left\{ \psi(\mathfrak{u}_t) \exp\left(\int_0^t V(\mathfrak{u}_\tau)\, \mathrm{d}\tau \right) \right\}.$$

For any $V \in C_b(\mathcal{H})$ and $t \geq 0$, the application $\mathfrak{P}_t^V$ maps $C_b(\mathcal{H})$ into itself. Let us denote by $\mathfrak{P}_t^{V*} : \mathcal{M}_+(\mathcal{H}) \to \mathcal{M}_+(\mathcal{H})$ its dual semigroup, and recall that a measure $\mu \in \mathcal{P}(\mathcal{H})$ is an *eigenvector* if there is $\lambda \in \mathbb{R}$ such that $\mathfrak{P}_t^{V*}\mu = \lambda^t \mu$ for any $t > 0$. Let $\mathfrak{w}$ be the function defined by (1.5). From (5.24) with $m = 1$ it follows that $\mathfrak{P}_t^V$ maps [4] $C_{\mathfrak{w}}(\mathcal{H}^{\mathfrak{s}})$ into itself (note that $\mathfrak{w}_1 = \mathfrak{w}$ in (5.24)). We shall say that a function $h \in C_{\mathfrak{w}}(\mathcal{H}^{\mathfrak{s}})$ is an eigenvector for the semigroup $\mathfrak{P}_t^V$ if $\mathfrak{P}_t^V h(\mathfrak{u}) = \lambda^t h(\mathfrak{u})$ for any $\mathfrak{u} \in \mathcal{H}^{\mathfrak{s}}$ and $t > 0$. Then we have the following theorem.

Theorem 1.3. *Under the conditions of the Main Theorem, there is $\delta > 0$ such that the following assertions hold for any $V \in \mathcal{U}_\delta$.*

Existence and uniqueness. *The semigroup $\mathfrak{P}_t^{V*}$ admits a unique eigenvector $\mu_V \in \mathcal{P}_{\mathfrak{w}}(\mathcal{H})$ corresponding to an eigenvalue $\lambda_V > 0$. Moreover, for any $m \geq 1$, we have*

$$\int_{\mathcal{H}} \left[|\mathfrak{u}|_{\mathcal{H}^{\mathfrak{s}}}^m + \exp(\varkappa \mathcal{E}(\mathfrak{u})) \right] \mu_V(\mathrm{d}\mathfrak{u}) < \infty, \tag{1.12}$$

where $\varkappa := (2\alpha)^{-1}\mathfrak{B}$ and $\mathfrak{B} := \sum b_j^2$. The semigroup $\mathfrak{P}_t^V$ admits a unique eigenvector $h_V \in C_{\mathfrak{w}}(\mathcal{H}^{\mathfrak{s}}) \cap C_+(\mathcal{H}^{\mathfrak{s}})$ corresponding to λ_V normalised by the condition $\langle h_V, \mu_V \rangle = 1$.

Convergence. *For any $\psi \in C_{\mathfrak{w}}(\mathcal{H}^{\mathfrak{s}})$, $\nu \in \mathcal{P}_{\mathfrak{w}}(\mathcal{H})$, and $R > 0$, we have*

$$\lambda_V^{-t} \mathfrak{P}_t^V \psi \to \langle \psi, \mu_V \rangle h_V \quad \text{in } C_b(X_R) \cap L^1(\mathcal{H}, \mu_V) \text{ as } t \to \infty, \tag{1.13}$$

$$\lambda_V^{-t} \mathfrak{P}_t^{V*} \nu \to \langle h_V, \nu \rangle \mu_V \quad \text{in } \mathcal{M}_+(\mathcal{H}) \text{ as } t \to \infty. \tag{1.14}$$

This result is proved in Section 6. Here we apply it to establish Theorem 1.2.

Proof of Theorem 1.2. Step 1: Upper and lower bounds. We apply Theorem 7.1 to prove estimates (1.10) and (1.11). Let us consider the following totally ordered set $(\Theta, \prec)$, where $\Theta = \mathbb{R}_+^* \times X_R$ and $\prec$ is a relation defined by $(t_1, \mathfrak{u}_1) \prec (t_2, \mathfrak{u}_2)$ if and only if $t_1 \leq t_2$. For any $\theta = (t, \mathfrak{u}) \in \Theta$, we set $r_\theta := t$ and $\zeta_\theta := \zeta_t$, where ζ_t is the random probability measure given by (1.6) defined on the probability space $(\Omega_\theta, \mathcal{F}_\theta, \mathbb{P}_\theta) := (\Omega, \mathcal{F}, \mathbb{P}_{\mathfrak{u}})$. The conditions of Theorem 7.1 are satisfied for the family $\{\zeta_\theta\}_{\theta \in \Theta}$. Indeed, a family $\{x_\theta \in \mathbb{R}, \theta \in \Theta\}$ converges if and only if it converges uniformly with respect to $\mathfrak{u} \in X_R$ as $t \to +\infty$.

[4] When we write $C_{\mathfrak{w}}(\mathcal{H}^{\mathfrak{s}})$ or $C(X_R)$, the sets $\mathcal{H}^{\mathfrak{s}}$ and X_R are assumed to be endowed with the topology induced by $\mathcal{H}$.

Hence (7.1) holds with $Q = Q_R$, and for any $V \in \mathcal{V}$, Properties 1 and 2 imply limit (7.3) and the uniqueness of the equilibrium state. It remains to check the following condition, which we postpone to Section 5.

Exponential tightness. There is a function $\Phi : \mathcal{H} \to [0, +\infty]$ whose level sets $\{\mathfrak{u} \in \mathcal{H} : \Phi(\mathfrak{u}) \le a\}$ are compact for any $a \ge 0$ and

$$\mathbb{E}_{\mathfrak{u}} \exp\left(\int_0^t \Phi(\mathfrak{u}_\tau)\, \mathrm{d}\tau \right) \le Ce^{ct}, \quad \mathfrak{u} \in X_R,\ t > 0$$

for some positive constants C and c.

Theorem 7.1 implies that I_R is a good rate function and the following two inequalities hold for any closed set $F \subset \mathcal{P}(\mathcal{H})$ and open set $G \subset \mathcal{P}(\mathcal{H})$

$$\limsup_{\theta \in \Theta} \frac{1}{r_\theta} \log \mathbb{P}_\theta\{\zeta_\theta \in F\} \le -I_R(F),$$
$$\liminf_{\theta \in \Theta} \frac{1}{r_\theta} \log \mathbb{P}_\theta\{\zeta_\theta \in G\} \ge -I_R(\mathcal{W} \cap G).$$

These inequalities imply (1.10) and (1.11), since we have the equalities

$$\limsup_{\theta \in \Theta} \frac{1}{r_\theta} \log \mathbb{P}_\theta\{\zeta_\theta \in F\} = \limsup_{t \to \infty} \frac{1}{t} \log \sup_{\mathfrak{u} \in X_R} \mathbb{P}_{\mathfrak{u}}\{\zeta_t \in F\},$$
$$\liminf_{\theta \in \Theta} \frac{1}{r_\theta} \log \mathbb{P}_\theta\{\zeta_\theta \in G\} = \liminf_{t \to \infty} \frac{1}{t} \log \inf_{\mathfrak{u} \in X_R} \mathbb{P}_{\mathfrak{u}}\{\zeta_t \in G\}.$$

Step 2: Proof of the inclusion $\mathcal{U}_\delta \subset \mathcal{V}$. Let $\delta > 0$ be the constant in Theorem 1.3. Taking $\psi = \mathbf{1}$ in (1.13), we get Property 1 with $Q_R(V) := \log \lambda_V$ for any $V \in \mathcal{U}_\delta$ (in particular, $Q(V) := Q_R(V)$ does not depend on R).

Property 2 is deduced from limit (1.13) in the same way as in [JNPS14]. Indeed, for any $V \in \mathcal{U}_\delta$, we introduce the semigroup

$$\mathfrak{S}_t^{V,F} \psi(\mathfrak{u}) = \lambda_V^{-t} h_V^{-1} \mathfrak{P}_t^{V+F}(h_V \psi)(\mathfrak{u}), \quad \psi, F \in C_b(\mathcal{H}),\ t \ge 0, \tag{1.15}$$

the function

$$Q_R^V(F) := \limsup_{t \to +\infty} \frac{1}{t} \log \sup_{\mathfrak{u} \in X_R} \log(\mathfrak{S}_t^{V,F} \mathbf{1})(\mathfrak{u}), \tag{1.16}$$

and the Legendre transform $I_R^V : \mathcal{M}(\mathcal{H}) \to [0, +\infty]$ of $Q_R^V(\cdot)$. The arguments of Section 5.7 of [JNPS14] show that $\sigma \in \mathcal{P}(\mathcal{H})$ is an equilibrium state for V if and only if $I_R^V(\sigma) = 0$. So the uniqueness follows from the following result which is a continuous-time version of Proposition 7.5 in [JNPS14]. Its proof is given in the Appendix.

Proposition 1.4. *For any $V \in \mathcal{U}_\delta$ and $R > 0$, the measure $\sigma_V = h_V \mu_V$ is the unique zero of I_R^V.*

Step 3: Non-triviality of I_R. We argue by contradiction. Let us assume that D_{I_R} is a singleton. By Proposition 1.4 with $V = \mathbf{0}$, we have that the stationary measure μ is the unique zero [5] of I_R, so $D_{I_R} = \{\mu\}$. Then (1.8) implies that $Q(V) = \langle V, \mu\rangle$ for any $V \in C_b(\mathcal{H})$. Let us choose any non-constant $V \in \mathcal{U}_\delta$ such that $\langle V, \mu\rangle = 0$. Then $Q(V) = 0$, and limit (1.13) with $\psi = \mathbf{1}$ implies that $\lambda_V = e^{Q(V)} = 1$ and

$$\sup_{t\geq 0} \mathbb{E}_0 \exp\left(\int_0^t V(\mathfrak{u}_\tau)\,\mathrm{d}\tau\right) < \infty, \tag{1.17}$$

where $\mathbb{E}_0$ means that we consider the trajectory issued from the origin. Combining this with the central limit theorem (see Theorem 2.5 in [Mar14] and Theorem 4.1.8 and Proposition 4.1.4 in [KS12]), we get $V = \mathbf{0}$. This contradicts the assumption that V is non-constant and completes the proof of Theorem 1.2. □

2 Proof of the Main Theorem

Step 1: Proof in the case $\psi \in \mathcal{U}$. For any $R > 0$ and non-constant $\psi \in \mathcal{U}$, we denote

$$I_R^\psi(p) = \inf\{I_R(\sigma) : \langle \psi, \sigma\rangle = p,\, \sigma \in \mathcal{P}(\mathcal{H})\}, \quad p \in \mathbb{R},$$

where I_R is given by (1.7). Then $Q_R(\beta\psi)$ is convex in $\beta \in \mathbb{R}$, and using (1.8), it is straightforward to check that

$$Q_R(\beta\psi) = \sup_{p\in\mathbb{R}} \Big(\beta p - I_R^\psi(p)\Big) \quad \text{for } \beta \in \mathbb{R}.$$

By well-known properties of convex functions of a real variable (e.g., see [RV73]), $Q_R(\beta\psi)$ is differentiable in $\beta \in \mathbb{R}$, except possibly on a countable set, the right and left derivatives $D^+Q_R(\beta\psi)$ and $D^-Q_R(\beta\psi)$ exist at any β and $D^-Q_R(\beta\psi) \leq D^+Q_R(\beta\psi)$. Moreover, the following equality holds for some $\beta, p \in \mathbb{R}$

$$Q_R(\beta\psi) = \beta p - I_R^\psi(p) \tag{2.1}$$

if and only if $p \in [D^-Q_R(\beta\psi), D^+Q_R(\beta\psi)]$. Let us set $\beta_0 := \delta/(4\|\psi\|_\infty)$, where $\delta > 0$ is the constant in Theorem 1.2. Then for any $|\beta| \leq \beta_0$, we have $\beta\psi \in \mathcal{U}_\delta \subset \mathcal{V}$ and $Q_R(\beta\psi)$ does not depend on $R > 0$; we set $Q(\beta\psi) := Q_R(\beta\psi)$. Let us show that $D^-Q(\beta\psi) = D^+Q(\beta\psi)$ for any $|\beta| < \beta_0$, i.e., $Q(\beta\psi)$ is differentiable at β. Indeed, assume that $p_1, p_2 \in [D^-Q(\beta\psi), D^+Q(\beta\psi)]$. Then equality (2.1) holds with $p = p_i, i = 1, 2$. As I_R is a good rate function, there are measures $\sigma_i \in \mathcal{P}(\mathcal{H})$ such that $\langle \psi, \sigma_i\rangle = p_i$ and $I_R(\sigma_i) = I_R^\psi(p_i), i = 1, 2$. Thus

$$Q(\beta\psi) = \beta p_i - I_R^\psi(p_i) = \langle \beta\psi, \sigma_i\rangle - I_R(\sigma_i),$$

i.e., σ_1 and σ_2 are equilibrium states corresponding to $V = \beta\psi$. As $\beta\psi \in \mathcal{V}$, from Property 2 we derive that $\sigma_1 = \sigma_2$, hence $p_1 = p_2$. Thus $Q(\beta\psi)$ is differentiable

[5] Note that when $V = \mathbf{0}$, we have $\lambda_V = 1$, $h_V = \mathbf{1}$, $I_R^V = I_R$, and $\mu_V = \mu$.

at β for any $|\beta| < \beta_0$. Let us define the convex function

$$Q^\psi(\beta) := \begin{cases} Q(\beta\psi), & \text{for } |\beta| \le \beta_0, \\ +\infty, & \text{for } |\beta| > \beta_0 \end{cases} \tag{2.2}$$

and its Legendre transform

$$I^\psi(p) := \sup_{\beta\in\mathbb{R}} \big(\beta p - Q^\psi(\beta)\big) \quad \text{for } p \in \mathbb{R}. \tag{2.3}$$

Then I^ψ is a finite convex function not depending on $R > 0$. As $Q^\psi(\beta)$ is differentiable at any $|\beta| < \beta_0$ and (7.3) holds with $Q = Q^\psi(\beta)$ (with respect to the directed set $(\Theta, \prec)$ defined in the proof of Theorem 1.2), we see that the conditions of Theorem A.5 in [JOPP12] are satisfied [6]. Hence, we have (0.5) for any open subset O of the interval $J^\psi := (D^+Q^\psi(-\beta_0), D^-Q^\psi(\beta_0))$.

Step 2: Proof in the case $\psi \in C_b(\mathcal{H})$. Let us first define the rate function $I^\psi : \mathbb{R} \to \mathbb{R}_+$ in the case of a general function $\psi \in C_b(\mathcal{H})$. To this end, we take a sequence $\psi_n \in \mathcal{U}$ such that $\|\psi_n\|_\infty \le \|\psi\|_\infty$ and $\psi_n \to \psi$ in $C(K)$ for any compact $K \subset \mathcal{H}$. The argument of the proof of property (a) in Section 5.6 in [JNPS14] implies that Property 1 holds with $V = \beta\psi$ for any $|\beta| \le \beta_0$, where β_0 is defined as in Step 1, and for any compact set $\mathcal{K} \subset \mathcal{P}(\mathcal{H})$, we have

$$\sup_{\sigma\in\mathcal{K}} |\langle \psi_n - \psi, \sigma\rangle| \to 0 \quad \text{as } n \to \infty. \tag{2.4}$$

Moreover, from the proof of Proposition 3.17 in [FK06] it follows that

$$Q_R(\beta\psi_n) \to Q_R(\beta\psi) \quad \text{for } |\beta| \le \beta_0. \tag{2.5}$$

This implies that $Q_R(\beta\psi)$ does not depend on R when $|\beta| \le \beta_0$, so we can define the functions Q^ψ and I^ψ by (2.2) and (2.3), respectively.

Let J^ψ be the interval defined in Step 1. To establish limit (0.5), it suffices to show that for any open subset $O \subset J^\psi$ the following two inequalities hold

$$\limsup_{t\to\infty} \frac{1}{t} \log \sup_{\mathrm{u}\in X_R} \mathbb{P}_\mathrm{u}\{\zeta_t^\psi \in O\} \le -I^\psi(O), \tag{2.6}$$

$$\liminf_{t\to\infty} \frac{1}{t} \log \inf_{\mathrm{u}\in X_R} \mathbb{P}_\mathrm{u}\{\zeta_t^\psi \in O\} \ge -I^\psi(O), \tag{2.7}$$

where $\zeta_t^\psi := \langle \psi, \zeta\rangle$. To prove (2.6), we first apply (1.10) for a closed subset $F \subset \mathcal{P}(\mathcal{H})$ defined by $F = \{\sigma \in \mathcal{P}(\mathcal{H}) : \langle \psi, \sigma\rangle \in \overline{O}\}$, where $\overline{O}$ is the closure of O in $\mathbb{R}$:

$$\begin{aligned} \limsup_{t\to\infty} \frac{1}{t} \log \sup_{\mathrm{u}\in X_R} \mathbb{P}_\mathrm{u}\{\zeta_t^\psi \in O\} &\le \limsup_{t\to\infty} \frac{1}{t} \log \sup_{\mathrm{u}\in X_R} \mathbb{P}_\mathrm{u}\{\zeta_t^\psi \in \overline{O}\} \\ &= \limsup_{t\to\infty} \frac{1}{t} \log \sup_{\mathrm{u}\in X_R} \mathbb{P}_\mathrm{u}\{\zeta_t \in F\} \\ &\le -I_R(F). \end{aligned} \tag{2.8}$$

[6] Theorem A.5 in [JOPP12] is stated in the case $\Theta = \mathbb{R}_+$. However, the proof presented there remains valid for random variables indexed by a directed set.

As $Q_R(\beta\psi) \le Q^\psi(\beta)$ for any $\beta \in \mathbb{R}$, we have

$$I^\psi(\overline{O}) \le I_R^\psi(\overline{O}). \tag{2.9}$$

It is straightforward to check that

$$I_R^\psi(\overline{O}) = I_R(F). \tag{2.10}$$

From the continuity of I^ψ it follows that $I^\psi(O) = I^\psi(\overline{O})$. Combining this with (2.8)-(2.10), we get (2.6).

To establish (2.7), we first recall that the exponential tightness property and Lemma 3.2 in [JNPS14] imply that for any $a > 0$ there is a compact $\mathcal{K}_a \subset \mathcal{P}(\mathcal{H})$ such that

$$\limsup_{t\to\infty} \frac{1}{t} \log \sup_{\mathfrak{u}\in X_R} \mathbb{P}_\mathfrak{u}\{\zeta_t \in \mathcal{K}_a^c\} \le -a. \tag{2.11}$$

Let us take any $p \in O$ and choose $\varepsilon > 0$ so small that that $(p-2\varepsilon, p+2\varepsilon) \subset O$. Then for any $a > 0$, we have

$$\mathbb{P}_\mathfrak{u}\{\zeta_t^\psi \in O\} \ge \mathbb{P}_\mathfrak{u}\{\zeta_t^\psi \in (p-2\varepsilon, p+2\varepsilon), \zeta_t \in \mathcal{K}_a\}. \tag{2.12}$$

By (2.4), we can choose $n \ge 1$ so large that

$$\sup_{\sigma\in\mathcal{K}_a} |\langle \psi_n - \psi, \sigma\rangle| \le \varepsilon.$$

Using (2.12), we get

$$\begin{aligned}\mathbb{P}_\mathfrak{u}\{\zeta_t^\psi \in O\} &\ge \mathbb{P}_\mathfrak{u}\{\zeta_t^{\psi_n} \in (p-\varepsilon, p+\varepsilon), \zeta_t \in \mathcal{K}_a\} \\ &\ge \mathbb{P}_\mathfrak{u}\{\zeta_t^{\psi_n} \in (p-\varepsilon, p+\varepsilon)\} - \mathbb{P}_\mathfrak{u}\{\zeta_t \in \mathcal{K}_a^c\}. \end{aligned} \tag{2.13}$$

We need the following elementary property of convex functions; see the Appendix for the proof.

Lemma 2.1. *Let $J \subset \mathbb{R}$ be an open interval and $f_n : J \to \mathbb{R}$ be a sequence of convex functions converging pointwise to a finite function f. Then we have*

$$\begin{aligned}\limsup_{n\to\infty} D^+ f_n(x) &\le D^+ f(x), \\ \liminf_{n\to\infty} D^- f_n(x) &\ge D^- f(x), \quad x \in J.\end{aligned}$$

This lemma implies that, for sufficiently large $n \ge 1$, we have

$$(p-\varepsilon, p+\varepsilon) \subset J^{\psi_n} = (D^+ Q^{\psi_n}(-\beta_0^n), D^- Q^{\psi_n}(\beta_0^n)),$$

where $\beta_0^n := \delta/(4\|\psi_n\|_\infty)$. Hence the result of Step 1 implies that

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}_\mathfrak{u}\{\zeta_t^{\psi_n} \in (p-\varepsilon, p+\varepsilon)\} = -I^{\psi_n}((p-\varepsilon, p+\varepsilon))$$

uniformly with respect to $\mathfrak{u} \in X_R$. As

$$\limsup_{n\to\infty} Q^{\psi_n}(\beta) \le Q^{\psi}(\beta), \quad \beta \in \mathbb{R},$$

we have

$$\liminf_{n\to\infty} I^{\psi_n}(q) \ge I^{\psi}(q), \quad q \in \mathbb{R}.$$

This implies that

$$\liminf_{n\to\infty} I^{\psi_n}((p-\varepsilon, p+\varepsilon)) \ge I^{\psi}((p-\varepsilon, p+\varepsilon)).$$

Thus we can choose $n \ge 1$ so large that

$$\liminf_{t\to\infty} \frac{1}{t} \log \inf_{\mathfrak{u}\in X_R} \mathbb{P}_{\mathfrak{u}}\{\zeta_t^{\psi_n} \in (p-\varepsilon, p+\varepsilon)\} \ge -I^{\psi}((p-\varepsilon, p+\varepsilon)) - \varepsilon.$$

Combining this with (2.13) and (2.11) and choosing $a > I^{\psi}((p-\varepsilon, p+\varepsilon)) + \varepsilon$, we obtain

$$\liminf_{t\to\infty} \frac{1}{t} \log \inf_{\mathfrak{u}\in X_R} \mathbb{P}_{\mathfrak{u}}\{\zeta_t^{\psi} \in O\} \ge -I^{\psi}((p-\varepsilon, p+\varepsilon)) - \varepsilon.$$

Since $p \in O$ is arbitrary and $\varepsilon > 0$ can be chosen arbitrarily small, we get (2.7).

Step 3: The interval J^{ψ}. Let us show that if $\psi \in C_b^q(\mathcal{H})$, $q \in (0,1]$ is non-constant, then the interval $J^{\psi} = (D^+Q^{\psi}(-\beta_0), D^-Q^{\psi}(\beta_0))$ is non-empty and contains the point $\langle \psi, \mu\rangle$. Clearly we can assume that $\langle \psi, \mu\rangle = 0$. As $Q^{\psi}(0) = 0$, it is sufficient to show that $\beta = 0$ is the only point of the interval $[-\beta_0, \beta_0]$, where $Q^{\psi}(\beta)$ vanishes. Assume the opposite. Then, replacing ψ by $-\psi$ if needed, we can suppose that there is $\beta \in (0, \beta_0]$ such that $Q^{\psi}(\beta) = 0$. As in Step 3 of Theorem 1.2, this implies

$$\sup_{t\ge 0} \mathbb{E}_0 \exp\left(\beta \int_0^t \psi(\mathfrak{u}_\tau)\, \mathrm{d}\tau\right) < \infty$$

and $\psi \equiv 0$. This contradicts our assumption that ψ is non-constant and completes the proof of the Main Theorem.

3 Checking conditions of Theorem 7.4

The proof of Theorem 1.3 is based on an application of Theorem 7.4. In this section, we verify the growth condition, the uniform irreducibility property, and the existence of an eigenvector for the following generalised Markov family of transition kernels (see Definition 7.3)

$$P_t^V(\mathfrak{u}, \Gamma) = (\mathfrak{P}_t^{V*}\delta_{\mathfrak{u}})(\Gamma), \quad V \in C_b(\mathcal{H}),\ \Gamma \in \mathcal{B}(\mathcal{H}),\ \mathfrak{u} \in \mathcal{H},\ t \ge 0$$

in the phase space $X = \mathcal{H}$ endowed with a sequence of compacts $X_R = B_{\mathcal{H}^s}(R)$, $R \ge 1$ and a weight function $\mathfrak{w}$ defined by (1.5). The uniform Feller property is the most delicate condition to check in Theorem 7.4, it will be established in Section 4. In the rest of the paper, we shall always assume that the hypotheses of Theorem 1.2 are fulfilled.

3.1 Growth condition

Since we take $X_R = B_{\mathcal{H}^{\mathfrak{s}}}(R)$, the set X_∞ in the growth condition in Theorem 7.4 will be equal to $\mathcal{H}^{\mathfrak{s}}$ which is dense in $\mathcal{H}$. For any $\mathfrak{u} \in \mathcal{H}^{\mathfrak{s}}$ and $t \geq 0$, we have $\mathfrak{u}(t;\mathfrak{u}) \in \mathcal{H}^{\mathfrak{s}}$, so the measure $P_t^V(\mathfrak{u},\cdot)$ is concentrated on $\mathcal{H}^{\mathfrak{s}}$. As V is a bounded function, condition (7.12) is verified. Let us show that estimate (7.11) holds for any V with a sufficiently small oscillation.

Proposition 3.1. *There is a constant $\delta > 0$ and an integer $R_0 \geq 1$ such that, for any $V \in C_b(\mathcal{H})$ satisfying $\operatorname{Osc}(V) < \delta$, we have*

$$\sup_{t\geq 0} \frac{\|\mathfrak{P}_t^V \mathfrak{w}\|_{L^\infty_{\mathfrak{w}}}}{\|\mathfrak{P}_t^V \mathbf{1}\|_{R_0}} < \infty, \tag{3.1}$$

where $\mathbf{1}$ is the function on $\mathcal{H}$ identically equal to 1 and $\|\cdot\|_{R_0}$ is the L^∞ norm on X_{R_0}.

Proof. Without loss of generality, we can assume that $V \geq 0$ and $\operatorname{Osc}(V) = \|V\|_\infty$. Indeed, it suffices to replace V by $V - \inf_H V$. We split the proof of (3.1) into two steps.

Step 1. Let us show that there are $\delta_0 > 0$ and $R_0 \geq 1$ such that

$$\sup_{t\geq 0} \frac{\|\mathfrak{P}_t^V \mathbf{1}\|_{L^\infty_{\mathfrak{w}}}}{\|\mathfrak{P}_t^V \mathbf{1}\|_{R_0}} < \infty, \tag{3.2}$$

provided that $\|V\|_\infty < \delta_0$. To prove this, we introduce the stopping time

$$\tau(R) = \inf\{t \geq 0 : |\mathfrak{u}_t|_{\mathcal{H}^{\mathfrak{s}}} \leq R\}$$

and use the following result.

Lemma 3.2. *There are positive numbers δ_0, C, and R_0 such that*

$$\mathbb{E}_{\mathfrak{u}} e^{\delta_0 \tau(R_0)} \leq C\mathfrak{w}(\mathfrak{u}), \quad \mathfrak{u} \in \mathcal{H}^{\mathfrak{s}}. \tag{3.3}$$

We omit the proof of this lemma, since it is carried out by standard arguments, using the Lyapunov function $\mathfrak{w}$ and estimate (5.24) for $m = 1$ (see Lemma 3.6.1 in [KS12]). Setting $G_t := \{\tau(R_0) > t\}$ and

$$\Xi_V(t) := \exp\left(\int_0^t V(\mathfrak{u}_s)\,\mathrm{d}s\right), \tag{3.4}$$

we get

$$\mathfrak{P}_t^V \mathbf{1}(\mathfrak{u}) = \mathbb{E}_{\mathfrak{u}} \Xi_V(t) = \mathbb{E}_{\mathfrak{u}}\big\{\mathbb{I}_{G_t}\Xi_V(t)\big\} + \mathbb{E}_{\mathfrak{u}}\big\{\mathbb{I}_{G_t^c}\Xi_V(t)\big\} =: I_1 + I_2. \tag{3.5}$$

Since $V \geq 0$, we have $\mathfrak{P}_t^V \mathbf{1}(\mathfrak{u}) \geq 1$. Combining this with (3.3) and $\|V\|_\infty < \delta_0$, we obtain for any $\mathfrak{u} \in \mathcal{H}^{\mathfrak{s}}$

$$I_1 \leq \mathbb{E}_{\mathfrak{u}} \Xi_V\big(\tau(R_0)\big) \leq \mathbb{E}_{\mathfrak{u}} \exp\big(\delta_0 \tau(R_0)\big) \leq C\,\mathfrak{w}(\mathfrak{u}) \leq C\,\mathfrak{w}(\mathfrak{u})\,\|\mathfrak{P}_t^V \mathbf{1}\|_{R_0}.$$

The strong Markov property and (3.3) imply

$$\begin{aligned} I_2 &\le \mathbb{E}_{\mathfrak{u}}\big\{\mathbb{I}_{G_t}\Xi_V(\tau(R_0))\,\mathbb{E}_{\mathfrak{u}(\tau(R_0))}\Xi_V(t)\big\} \\ &\le \mathbb{E}_{\mathfrak{u}}\{e^{\delta_0\tau(R_0)}\}\,\|\mathfrak{P}_t^V\mathbf{1}\|_{R_0} \le C\,\mathfrak{w}(\mathfrak{u})\,\|\mathfrak{P}_t^V\mathbf{1}\|_{R_0}, \end{aligned}$$

where we write $\mathfrak{u}(\tau(R_0))$ instead of $\mathfrak{u}_{\tau(R_0)}$. Using (3.5) and the estimates for I_1 and I_2, we get (3.2).

Step 2. To prove (3.1), we set $\delta := \delta_0 \wedge (\alpha/2)$ and assume that $\|V\|_\infty < \delta$ and $t = Tk$, where $k \ge 1$ is an integer and $T > 0$ is so large that $q := 2e^{-T\frac{\alpha}{2}} < 1$. Then, using the Markov property and (5.24), we get

$$\begin{aligned} \mathfrak{P}_{Tk}^V\mathfrak{w}(\mathfrak{u}) &\le e^{T\delta}\mathbb{E}_{\mathfrak{u}}\left\{\Xi_V(T(k-1))\mathfrak{w}(\mathfrak{u}_{Tk})\right\} \\ &= e^{T\delta}\mathbb{E}_{\mathfrak{u}}\left\{\Xi_V(T(k-1))\mathbb{E}_{\mathfrak{u}(T(k-1))}\mathfrak{w}(\mathfrak{u}_T)\right\} \\ &\le e^{T\delta}\mathbb{E}_{\mathfrak{u}}\left\{\Xi_V(T(k-1))[2e^{-T\alpha}\mathfrak{w}(\mathfrak{u}_{T(k-1)}) + C_1]\right\} \\ &\le q\mathfrak{P}_{T(k-1)}^V\mathfrak{w}(\mathfrak{u}) + e^{T\delta}C_1\mathfrak{P}_{T(k-1)}^V\mathbf{1}(\mathfrak{u}). \end{aligned}$$

Iterating this and using fact that $V \ge 0$, we obtain

$$\mathfrak{P}_{Tk}^V\mathfrak{w}(\mathfrak{u}) \le q^k\mathfrak{w}(\mathfrak{u}) + (1-q)^{-1}e^{T\delta}C_1\mathfrak{P}_{Tk}^V\mathbf{1}(\mathfrak{u}).$$

Combining this with (3.2), we see that

$$A := \sup_{k\ge 0}\frac{\|\mathfrak{P}_{Tk}^V\mathfrak{w}\|_{L_{\mathfrak{w}}^\infty}}{\|\mathfrak{P}_{Tk}^V\mathbf{1}\|_{R_0}} < \infty.$$

To derive (3.1) from this, we use the semigroup property and the fact that V is non-negative and bounded:

$$\begin{aligned} \|\mathfrak{P}_t^V\mathfrak{w}\|_{L_{\mathfrak{w}}^\infty} &= \|\mathfrak{P}_{t-Tk}^V(\mathfrak{P}_{Tk}^V\mathfrak{w})\|_{L_{\mathfrak{w}}^\infty} \le C_2\|\mathfrak{P}_{Tk}^V\mathfrak{w}\|_{L_{\mathfrak{w}}^\infty}, \\ \|\mathfrak{P}_t^V\mathbf{1}\|_{R_0} &\ge \|\mathfrak{P}_{Tk}^V\mathbf{1}\|_{R_0}, \end{aligned}$$

where $k \ge 0$ is such that $Tk \le t < T(k+1)$ and

$$C_2 := \sup_{s\in[0,T]}\|\mathfrak{P}_s^V\mathfrak{w}\|_{L_{\mathfrak{w}}^\infty} \le e^{T\|V\|_\infty}\sup_{s\in[0,T]}\|\mathfrak{P}_s\mathfrak{w}\|_{L_{\mathfrak{w}}^\infty} < \infty.$$

So we get

$$\sup_{t\ge 0}\frac{\|\mathfrak{P}_t^V\mathfrak{w}\|_{L_{\mathfrak{w}}^\infty}}{\|\mathfrak{P}_t^V\mathbf{1}\|_{R_0}} \le C_2A < +\infty.$$

This completes the proof of the proposition. □

3.2 Uniform irreducibility

In this section, we show that the family $\{P_t^V\}$ satisfies the uniform irreducibility condition with respect to the sequence of compacts $\{X_R\}$. Since V is bounded, we have

$$P_t^V(\mathfrak{u}, \mathrm{d}\mathfrak{v}) \geq e^{-t\|V\|_\infty} P_t(\mathfrak{u}, \mathrm{d}\mathfrak{v}), \quad \mathfrak{u} \in \mathcal{H},$$

where $P_t(\mathfrak{u}, \cdot)$ stands for the transition function of $(\mathfrak{u}_t, \mathbb{P}_\mathfrak{u})$. So it suffices to establish the uniform irreducibility for $\{P_t\}$.

Proposition 3.3. *For any integers $\rho, R \geq 1$ and any $r > 0$, there are positive numbers $l = l(\rho, r, R)$ and $p = p(\rho, r)$ such that*

$$P_l(\mathfrak{u}, B_\mathcal{H}(\hat{\mathfrak{u}}, r)) \geq p \quad \textit{for all } \mathfrak{u} \in X_R,\ \hat{\mathfrak{u}} \in X_\rho. \tag{3.6}$$

Proof. Let us show that, for sufficiently large $d \geq 1$ and any $R \geq 1$, there is a time $k = k(R)$ such that

$$P_k(\mathfrak{u}, X_d) \geq \frac{1}{2}, \quad \mathfrak{u} \in X_R. \tag{3.7}$$

Indeed, by (5.24) for $m = 1$, we have

$$\mathbb{E}_\mathfrak{u} |\mathfrak{u}_t|^2_{\mathcal{H}^\mathfrak{s}} \leq \mathbb{E}_\mathfrak{u} \mathfrak{w}(\mathfrak{u}_t) \leq 2e^{-\alpha t} \mathfrak{w}(\mathfrak{u}) + C_1.$$

Combining this with the estimate

$$|\mathcal{E}(\mathfrak{u})| \leq C_2(1 + |\mathfrak{u}|^4_\mathcal{H}), \tag{3.8}$$

we get

$$\mathbb{E}_\mathfrak{u} |\mathfrak{u}_t|^2_{\mathcal{H}^\mathfrak{s}} \leq C_3 e^{-\alpha t} R^{16} + C_1, \quad \mathfrak{u} \in X_R.$$

The Chebyshev inequality implies that

$$P_t(\mathfrak{u}, X_d) \geq 1 - d^{-2}(C_3 e^{-\alpha t} R^{16} + C_1).$$

Choosing $t = k$ and d so large that $e^{-\alpha k} R^{16} \leq 1$ and $d^2 > 2(C_3 + C_1)$, we obtain (3.7).

Combining (3.7) with Lemma 3.4 and the Kolmogorov–Chapman relation, we get (3.6) for $l = k + m$ and $p = q/2$. □

Lemma 3.4. *For any integers $d, \rho \geq 1$ and any $r > 0$, there are positive numbers $m = m(d, \rho, r)$ and $q = q(d, \rho, r)$ such that*

$$P_m(\mathfrak{v}, B_\mathcal{H}(\hat{\mathfrak{u}}, r)) \geq q \quad \textit{for all } \mathfrak{v} \in X_d,\ \hat{\mathfrak{u}} \in X_\rho. \tag{3.9}$$

Proof. It is sufficient to prove that there is $m \geq 1$ such that

$$P_m(\mathfrak{v}, B_\mathcal{H}(\hat{\mathfrak{u}}, r/2)) > 0 \quad \text{for all } \mathfrak{v} \in X_d,\ \hat{\mathfrak{u}} \in \tilde{X}_\rho, \tag{3.10}$$

where $\tilde{X}_\rho = \{\mathfrak{u} = [u_1, u_2] \in X_\rho : u_1, u_2 \in C_0^\infty(D)\}$. Indeed, let us take this inequality for granted and assume that (3.9) is not true. Then there are sequences $\mathfrak{v}_j \in X_d$ and $\hat{\mathfrak{u}}_j \in X_\rho$ such that

$$P_m(\mathfrak{v}_j, B_{\mathcal{H}}(\hat{\mathfrak{u}}_j, r)) \to 0. \tag{3.11}$$

Moreover, up to extracting a subsequence, we can suppose that $\mathfrak{v}_j$ and $\hat{\mathfrak{u}}_j$ converge in $\mathcal{H}$. Let us denote by $\mathfrak{v}_*$ and $\hat{\mathfrak{u}}_*$ their limits. Clearly, $\mathfrak{v}_* \in X_d$ and $\hat{\mathfrak{u}}_* \in X_\rho$. Choosing $j \geq 1$ so large that $|\hat{\mathfrak{u}}_j - \hat{\mathfrak{u}}_*|_{\mathcal{H}} < r/2$ and applying the Chebyshev inequality, we get

$$\begin{aligned} P_m(\mathfrak{v}_*, B_{\mathcal{H}}(\hat{\mathfrak{u}}_*, r)) &\leq P_m(\mathfrak{v}_j, B_{\mathcal{H}}(\hat{\mathfrak{u}}_j, r/2)) + \mathbb{P}\{|\mathfrak{u}(m;\mathfrak{v}_j) - \mathfrak{u}(m;\mathfrak{v}_*)|_{\mathcal{H}} \geq r/2\} \\ &\leq P_m(\mathfrak{v}_j, B_{\mathcal{H}}(\hat{\mathfrak{u}}_j, r/2)) + 4/r^2\, \mathbb{E}|\mathfrak{u}(m;\mathfrak{u}_j) - \mathfrak{u}(m;\mathfrak{v}_*)|^2_{\mathcal{H}}. \end{aligned}$$

Combining this with (3.11) and using the convergence $\mathfrak{v}_j \to \mathfrak{v}_*$ and a density property, we arrive at a contradiction with (3.10). Thus, inequality (3.9) is reduced to the derivation of (3.10). We shall prove the latter in three steps.

Step 1: Exact controllability. In what follows, given any $\varphi \in C(0,T;H^1)$, we shall denote by $S_\varphi(t;\mathfrak{v})$ the solution at time t of the problem

$$\partial_t^2 u + \gamma\partial_t u - \Delta u + f(u) = h + \dot{\varphi}, \quad u|_{\partial D} = 0, \quad t \in [0,T]$$

issued from $\mathfrak{v}$. Let $\hat{\mathfrak{v}} = [\hat{v}, 0]$, where $\hat{v} \in H^1$ is a solution of

$$-\Delta\hat{v} + f(\hat{v}) = h(x).$$

In this step we prove that for any $\hat{\mathfrak{u}} = [\hat{u}_1, \hat{u}_2] \in \tilde{X}_\rho$, there is φ_* satisfying

$$\varphi_* \in C(0,1;H^1) \quad \text{and} \quad S_{\varphi_*}(1;\hat{\mathfrak{v}}) = \hat{\mathfrak{u}}. \tag{3.12}$$

First note that, since the function f is continuous from H^1 to L^2, we have

$$-\Delta\hat{v} = -f(\hat{v}) + h \in L^2,$$

so that $\hat{v} \in H^2$. Moreover, since f is also continuous from H^2 to H^1 (recall that f vanishes at the origin), we have $f(\hat{v}) \in H^1$. As $h \in H^1$, it follows that

$$-\Delta\hat{v} \in H^1. \tag{3.13}$$

Let us introduce the functions

$$\begin{aligned} u(t) &= a(t)\hat{v} + b(t)\hat{u}_1 + c(t)\hat{u}_2, \\ \varphi_*(t) &= \int_0^t (\partial_t^2 u + \gamma\partial_t u - \Delta u + f(u) - h)\,\mathrm{d}\tau, \end{aligned} \tag{3.14}$$

where $a, b, c \in C^\infty([0,1],\mathbb{R})$ satisfy

$$\begin{aligned} &a(0) = 1, \quad a(1) = \dot{a}(0) = \dot{a}(1) = 0, \quad b(1) = 1, \quad b(0) = \dot{b}(0) = \dot{b}(1) = 0, \\ &\dot{c}(1) = 1, \quad c(0) = c(1) = \dot{c}(0) = 0. \end{aligned}$$

Then, we have $[u(0),\dot u(0)]=\hat{\mathfrak{v}}$, $[u(1),\dot u(1)]=\hat{\mathfrak{u}}$, and $S_{\varphi_*}(1;\hat{\mathfrak{v}})=\hat{\mathfrak{u}}$. Let us show the first relation in (3.12). In view of (3.14) and the smoothness of the functions a,b and c, we have

$$\partial_t^2 u+\gamma\partial_t u-h\in C(0,1;H^1)$$

and thus it is sufficient to prove that

$$-\Delta u+f(u)\in C(0,1;H^1). \tag{3.15}$$

Since $u\in C(0,1;H^2)$, we have $f(u)\in C(0,1;H^1)$. Moreover, in view of (3.13) and the smoothness of $\hat u_1$ and $\hat u_2$, we have $-\Delta u\in C(0,1;H^1)$. Thus, inclusion (3.15) is established and we arrive at (3.12). Let us note that by continuity and compactness, there is $\varkappa=\varkappa(\hat{\mathfrak{v}},\rho,r)>0$, not depending on $\hat{\mathfrak{u}}\in\tilde X_\rho$, such that

$$S_{\varphi_*}(1;\mathfrak{v})\in B_{\mathcal H}(\hat{\mathfrak{u}},r/4)\quad\text{ for any }\mathfrak{v}\in B_{\mathcal H}(\hat{\mathfrak{v}},\varkappa). \tag{3.16}$$

Step 2: Feedback stabilisation. We now show that there is $\tilde m\ge 1$ depending only on d and $\varkappa$ such that for any $\mathfrak{v}\in X_d$ there is $\tilde\varphi_{\mathfrak{v}}$ satisfying

$$\tilde\varphi_{\mathfrak{v}}\in C(0,\tilde m;H^1)\quad\text{ and }\quad S_{\tilde\varphi_{\mathfrak{v}}}(\tilde m,\mathfrak{v})\in B(\hat{\mathfrak{v}},\varkappa). \tag{3.17}$$

To see this, let us consider the flow $\tilde{\mathfrak{v}}(t;\mathfrak{v})$ associated with the solution of the equation

$$\partial_t^2\tilde v+\gamma\partial_t\tilde v-\Delta\tilde v+f(\tilde v)=h+\mathsf{P}_N[f(\tilde v)-f(\hat v)],\quad t\in[0,\tilde m] \tag{3.18}$$

issued from $\mathfrak{v}\in X_d$, where P_N stands for the orthogonal projection in L^2 onto the subspace spanned by the functions $e_1,e_2,\dots,e_N$. Then, in view of Proposition 6.5 in [Mar15], for $N\ge N(|\hat{\mathfrak{v}}|_{\mathcal H},d)$, we have

$$|\tilde{\mathfrak{v}}(\tilde m;\mathfrak{v})-\hat{\mathfrak{v}}|_{\mathcal H}^2\le|\mathfrak{v}-\hat{\mathfrak{v}}|_{\mathcal H}^2\,e^{-\alpha\tilde m}\le C_d\,e^{-\alpha\tilde m}<\varkappa$$

for $\tilde m$ sufficiently large. It follows that (3.17) holds with the function

$$\tilde\varphi_{\mathfrak{v}}(t)=\int_0^t\mathsf{P}_N[f(\tilde v)-f(\hat v)]\,\mathrm{d}\tau.$$

Step 3: Proof of (3.10). Let us take $m=\tilde m+1$ and, for any $\mathfrak{v}\in X_d$, define a function $\varphi_{\mathfrak{v}}(t)$ on the interval $[0,m]$ by

$$\varphi_{\mathfrak{v}}(t)=\begin{cases}\tilde\varphi_{\mathfrak{v}}(t) & \text{for } t\in[0,m-1],\\ \tilde\varphi_{\mathfrak{v}}(m-1)+\varphi_*(t-m+1) & \text{for } t\in[m-1,m].\end{cases}$$

In view of (3.12), (3.16), and (3.17), we have $\varphi_{\mathfrak{v}}(t)\in C(0,m;H^1)$ and $S_{\varphi_{\mathfrak{v}}}(m;\mathfrak{v})\in B_{\mathcal H}(\hat{\mathfrak{u}},r/2)$. Hence there is $\delta>0$ such that $S_\varphi(m;\mathfrak{v})\in B_{\mathcal H}(\hat{\mathfrak{u}},r/2)$ provided $\|\varphi-\varphi_{\mathfrak{v}}\|_{C(0,m;H^1)}<\delta$. It follows that

$$P_m(\mathfrak{v},B_{\mathcal H}(\hat{\mathfrak{u}},r/2))\ge\mathbb{P}\{\|\xi-\varphi_{\mathfrak{v}}\|_{C(0,m;H^1)}<\delta\}.$$

To complete the proof, it remains to note that, due to the non-degeneracy of ξ, the term on the right-hand side of this inequality is positive. □

3.3 Existence of an eigenvector

For any $m \geq 1$, let us define functions $\mathfrak{w}_m, \tilde{\mathfrak{w}}_m : \mathcal{H} \to [1, +\infty]$ by

$$\mathfrak{w}_m(\mathfrak{u}) = 1 + |\mathfrak{u}|^{2m}_{\mathcal{H}^s} + \mathcal{E}^{4m}(\mathfrak{u}), \tag{3.19}$$

$$\tilde{\mathfrak{w}}_m(\mathfrak{u}) = \mathfrak{w}_m(\mathfrak{u}) + \exp(\varkappa \mathcal{E}(\mathfrak{u})), \quad \mathfrak{u} \in \mathcal{H}, \tag{3.20}$$

where $\varkappa$ is the constant in Theorem 1.3. The following proposition proves the existence of an eigenvector $\mu = \mu(t, V, m)$ for the operator $\mathfrak{P}_t^{V*}$ for any $t > 0$. We shall see in Section 6 that the measure μ actually does not depend on t and m.

Proposition 3.5. *For any $t > 0$, $V \in C_b(\mathcal{H})$ and $m \geq 1$, the operator $\mathfrak{P}_t^{V*}$ admits an eigenvector $\mu = \mu(t, V, m) \in \mathcal{P}(\mathcal{H})$ with a positive eigenvalue $\lambda = \lambda(t, V, m)$:*

$$\mathfrak{P}_t^{V*}\mu = \lambda\mu.$$

Moreover, we have

$$\int_{\mathcal{H}} \tilde{\mathfrak{w}}_m(\mathfrak{u})\mu(\mathrm{d}\mathfrak{u}) < \infty, \tag{3.21}$$

$$\|\mathfrak{P}_t^V \mathfrak{w}_m\|_{X_R} \int_{X_R^c} \mathfrak{w}_m(\mathfrak{u})\mu(\mathrm{d}\mathfrak{u}) \to 0 \quad \text{as } R \to \infty. \tag{3.22}$$

Proof. Step 1. We first establish the existence of an eigenvector μ for $\mathfrak{P}_t^{V*}$ with a positive eigenvalue and satisfying (3.21). Let $t > 0$ and V be fixed. For any $A > 0$ and $m \geq 1$, let us introduce the convex set

$$D_{A,m} = \{\sigma \in \mathcal{P}(\mathcal{H}) : \langle \tilde{\mathfrak{w}}_m, \sigma \rangle \leq A\},$$

and consider the continuous mapping from $D_{A,m}$ to $\mathcal{P}(\mathcal{H})$ given by

$$G(\sigma) = \mathfrak{P}_t^{V*}\sigma / \mathfrak{P}_t^{V*}\sigma(\mathcal{H}).$$

Thanks to inequality (5.25), we have

$$\begin{aligned} \langle \tilde{\mathfrak{w}}_m, G(\sigma) \rangle &\leq \exp\left(t \operatorname{Osc}_{\mathcal{H}}(V)\right) \langle \tilde{\mathfrak{w}}_m, \mathfrak{P}_t^* \sigma \rangle \\ &\leq 2 \exp\left(t(\operatorname{Osc}_{\mathcal{H}}(V) - \alpha m)\right) \langle \tilde{\mathfrak{w}}_m, \sigma \rangle + C_m \exp\left(t \operatorname{Osc}_{\mathcal{H}}(V)\right). \end{aligned} \tag{3.23}$$

Assume that m is so large that

$$\operatorname{Osc}_{\mathcal{H}}(V) \leq \alpha m/2 \quad \text{and} \quad \exp(-\alpha m t/2) \leq 1/4,$$

and let $A := 2C_m e^{\alpha m t}$. Then, in view (3.23), we have $\langle \tilde{\mathfrak{w}}_m, G(\sigma) \rangle \leq A$ for any $\sigma \in D_{A,m}$, i.e., $G(D_{A,m}) \subset D_{A,m}$. Moreover, it is easy to see that the set $D_{A,m}$ is compact in $\mathcal{P}(\mathcal{H})$ (we use the Prokhorov compactness criterion to show that it is relatively compact and the Fatou lemma to prove that it is closed). Due to the Leray–Schauder theorem, the map G has a fixed point $\mu \in D_{A,m}$.

Note that, by the definitions of $D_{A,m}$ and G, the measure μ is an eigenvector of $\mathfrak{P}_t^{V*}$ with positive eigenvalue $\lambda := \mathfrak{P}_t^{V*}\mu(\mathcal{H})$ and satisfies (3.21).

Step 2. We now establish (3.22). Let us fix an integer $m \geq 1$ and let $n = 17m$. In view of the previous step, there is an eigenvector μ satisfying $\langle \mathfrak{w}_n, \mu\rangle < \infty$. From the Cauchy–Schwarz and Chebyshev inequalities it follows that

$$\int_{X_R^c} \mathfrak{w}_m(\mathfrak{u})\mu(\mathrm{d}\mathfrak{u}) \leq \langle \mathfrak{w}_m^2, \mu\rangle^{1/2}(\mu(X_R^c))^{1/2} \leq C_m\langle \mathfrak{w}_n, \mu_{t,V}\rangle R^{-n}. \tag{3.24}$$

On the other hand, using (5.24) and (3.8), we get

$$\|\mathfrak{P}_t^V \mathfrak{w}_m\|_{X_R} \leq \exp(t\|V\|_\infty) \sup_{\mathfrak{u}\in X_R} \mathbb{E}_{\mathfrak{u}}\mathfrak{w}_m(\mathfrak{u}_t) \leq C'_m \exp(t\|V\|_\infty)(R^{16m}+1).$$

Combining this with (3.24), we obtain (3.22). □

4 Uniform Feller property

4.1 Construction of coupling processes

As in the case of discrete-time models considered in [JNPS, JNPS14], the proof of the uniform Feller property is based on the coupling method. This method has proved to be an important tool for the study of the ergodicity of randomly forced PDE's (see Chapter 3 in [KS12] and the papers [KS02, Mat02, Oda08, Mar14]). In this section, we recall a construction of coupled trajectories from [Mar14], which was used to establish the exponential mixing for problem (0.1), (0.3). This construction will play a central role in the proof of the uniform Feller property in the next section.

For any $\mathfrak{z}, \mathfrak{z}' \in \mathcal{H}$, let us denote by $\mathfrak{u}_t$ and $\mathfrak{u}'_t$ the flows of (0.1), (0.3) issued from $\mathfrak{z}$ and $\mathfrak{z}'$, respectively. For any integer $N \geq 1$, let $\mathfrak{v} = [v, \partial_t v]$ be the flow of the problem

$$\partial_t^2 v + \gamma\partial_t v - \Delta v + f(v) + \mathsf{P}_N(f(u) - f(v)) = h + \vartheta(t,x), \quad v|_{\partial D} = 0, \quad \mathfrak{v}(0) = \mathfrak{z}'. \tag{4.1}$$

The laws of the processes $\{\mathfrak{v}_t, t \in [0,1]\}$ and $\{\mathfrak{u}'_t, t \in [0,1]\}$ are denoted by $\lambda(\mathfrak{z}, \mathfrak{z}')$ and $\lambda(\mathfrak{z}')$, respectively. We have the following estimate for the total variation distance between $\lambda(\mathfrak{z}, \mathfrak{z}')$ and $\lambda(\mathfrak{z}')$.

Proposition 4.1. *There is an integer $N_1 \geq 1$ such that, for any $N \geq N_1$, $\varepsilon > 0$, and $\mathfrak{z}, \mathfrak{z}' \in \mathcal{H}$, we have*

$$|\lambda(\mathfrak{z}, \mathfrak{z}') - \lambda(\mathfrak{z}')|_{var} \leq C_*\varepsilon^a + C_*\left[\exp\left(C_N\varepsilon^{a-2}|\mathfrak{z} - \mathfrak{z}'|_{\mathcal{H}}^2 e^{(|\mathcal{E}(\mathfrak{z})|+|\mathcal{E}(\mathfrak{z}')|)}\right) - 1\right]^{1/2}, \tag{4.2}$$

where $a < 2$, C_*, and C_N are positive numbers not depending on $\varepsilon, \mathfrak{z}$, and $\mathfrak{z}'$.

This proposition is essentially established in Section 4.2 in [Mar14] in a different form, and we shall omit the proof. By Proposition 1.2.28 in [KS12], there

is a probability space $(\hat\Omega, \hat{\mathcal{F}}, \hat{\mathbb{P}})$ and measurable functions $\mathcal{V}, \mathcal{V}' : \mathcal{H}\times\mathcal{H}\times\hat\Omega \to C([0,1],\mathcal{H})$ such that $(\mathcal{V}(\mathfrak{z},\mathfrak{z}'), \mathcal{V}'(\mathfrak{z},\mathfrak{z}'))$ is a maximal coupling for $(\lambda(\mathfrak{z},\mathfrak{z}'), \lambda(\mathfrak{z}'))$ for any $\mathfrak{z},\mathfrak{z}'\in\mathcal{H}$. We denote by $\tilde{\mathfrak{v}} = [\tilde v_t, \partial_t\tilde v]$ and $\tilde{\mathfrak{u}}'_t = [\tilde u'_t, \partial_t\tilde u']$ the restrictions of $\mathcal{V}$ and $\mathcal{V}'$ to time $t\in[0,1]$. Then $\tilde v_t$ is a solution of the problem

$$\partial_t^2\tilde v + \gamma\partial_t\tilde v - \Delta\tilde v + f(\tilde v) - \mathsf{P}_N f(\tilde v) = h + \psi(t), \quad \tilde v|_{\partial D} = 0, \quad \tilde{\mathfrak{v}}(0) = \mathfrak{z}',$$

where the process $\{\int_0^t \psi(\tau)\,\mathrm{d}\tau, t\in[0,1]\}$ has the same law as

$$\left\{\xi(t) - \int_0^t \mathsf{P}_N f(u_\tau)\,\mathrm{d}\tau, t\in[0,1]\right\}.$$

Let $\tilde{\mathfrak{u}}_t = [\tilde u, \partial_t\tilde u]$ be a solution of

$$\partial_t^2\tilde u + \gamma\partial_t\tilde u - \Delta\tilde u + f(\tilde u) - \mathsf{P}_N f(\tilde u) = h + \psi(t), \quad \tilde u|_{\partial D} = 0, \quad \tilde{\mathfrak{u}}(0) = \mathfrak{z}.$$

Then $\{\tilde{\mathfrak{u}}_t, t\in[0,1]\}$ has the same law as $\{\mathfrak{u}_t, t\in[0,1]\}$ (see Section 6.1 in [Mar14] for the proof). Now the coupling operators $\mathcal{R}$ and $\mathcal{R}'$ are defined by

$$\mathcal{R}_t(\mathfrak{z},\mathfrak{z}',\omega) = \tilde{\mathfrak{u}}_t, \quad \mathcal{R}'_t(\mathfrak{z},\mathfrak{z}',\omega) = \tilde{\mathfrak{u}}'_t, \quad \mathfrak{z},\mathfrak{z}'\in\mathcal{H}, \omega\in\hat\Omega.$$

By Proposition 4.1, if $N\ge N_1$, then for any $\varepsilon>0$, we have

$$\begin{aligned}&\hat{\mathbb{P}}\{\exists t\in[0,1] \text{ s.t. } \tilde{\mathfrak{v}}_t \ne \tilde{\mathfrak{u}}'_t\}\\ &\qquad\le C_*\varepsilon^a + C_*\left[\exp\left(C_N\varepsilon^{a-2}|\mathfrak{z}-\mathfrak{z}'|_{\mathcal{H}}^2 e^{(|\mathcal{E}(\mathfrak{z})|+|\mathcal{E}(\mathfrak{z}')|)}\right) - 1\right]^{1/2}. \quad (4.3)\end{aligned}$$

Let $(\Omega^k, \mathcal{F}^k, \mathbb{P}^k)$, $k\ge 0$ be a sequence of independent copies of the probability space $(\hat\Omega, \hat{\mathcal{F}}, \hat{\mathbb{P}})$. We denote by $(\Omega,\mathcal{F},\mathbb{P})$ the direct product of the spaces $(\Omega^k, \mathcal{F}^k, \mathbb{P}^k)$, and for any $\mathfrak{z},\mathfrak{z}'\in\mathcal{H}$, $\omega = (\omega^1,\omega^2,\ldots)\in\Omega$, and $k\ge 0$, we set $\tilde u_0 = u$, $\tilde u'_0 = u'$, and

$$\begin{aligned}&\tilde{\mathfrak{u}}_t(\omega) = \mathcal{R}_\tau(\tilde{\mathfrak{u}}_k(\omega), \tilde{\mathfrak{u}}'_k(\omega), \omega^k), && \tilde{\mathfrak{u}}'_t(\omega) = \mathcal{R}'_\tau(\tilde{\mathfrak{u}}_k(\omega), \tilde{\mathfrak{u}}'_k(\omega), \omega^k),\\ &\tilde{\mathfrak{v}}_t(\omega) = \mathcal{V}_\tau(\tilde{\mathfrak{u}}_k(\omega), \tilde{\mathfrak{u}}'_k(\omega), \omega^k),\end{aligned}$$

where $t = \tau + k, \tau\in[0,1)$. We shall say that $(\tilde{\mathfrak{u}}_t, \tilde{\mathfrak{u}}'_t)$ is a *coupled trajectory at level N* issued from $(\mathfrak{z},\mathfrak{z}')$.

4.2 The result and its proof

The following theorem establishes the uniform Feller property for the semigroup $\mathfrak{P}_t^V$ for any function $V\in\mathcal{U}_\delta$ with sufficiently small $\delta>0$. The property is proved with respect to the space $\mathcal{C} = \mathcal{U}$ which is a determining family for $\mathcal{P}(\mathcal{H})$ and contains the constant functions.

Theorem 4.2. *There are positive numbers δ and R_0 such that, for any function $V\in\mathcal{U}_\delta$, the family $\{\|\mathfrak{P}_t^V\mathbf{1}\|_R^{-1}\mathfrak{P}_t^V\psi, t\ge 1\}$ is uniformly equicontinuous on X_R for any $\psi\in\mathcal{U}$ and $R\ge R_0$.*

Proof. To prove this result, we develop the arguments of the proof of Theorem 6.2 in [JNPS14]. For any $\delta > 0$, $V \in \mathcal{U}_\delta$, and $\psi \in \mathcal{U}$, we have

$$\mathfrak{P}_t^V \psi(\mathfrak{u}) = \mathbb{E}_{\mathfrak{u}}\big\{(\Xi_V \psi)(\mathfrak{u}_t, t)\big\},$$

where

$$(\Xi_V \psi)(\mathfrak{u}_t, t) := \exp\bigg(\int_0^t V(\mathfrak{u}_\tau)\,\mathrm{d}\tau\bigg)\psi(\mathfrak{u}_t). \tag{4.4}$$

We prove the uniform equicontinuity of the family $\{g_t, t \geq 1\}$ on X_R, where

$$g_t(\mathfrak{u}) = \|\mathfrak{P}_t^V \mathbf{1}\|_R^{-1} \mathfrak{P}_t^V \psi(\mathfrak{u}).$$

Without loss of generality, we can assume that $0 \leq \psi \leq 1$ and $\inf_H V = 0$, so that $\mathrm{Osc}_H(V) = \|V\|_\infty$. We can assume also that the integer N entering representation (1.9) is the same for ψ and V and it is denoted by N_0.

Step 1: Stratification. Let us take any $N \geq N_0$ and $\mathfrak{z}, \mathfrak{z}' \in X_R$ such that $d := |\mathfrak{z} - \mathfrak{z}'|_{\mathcal{H}} \leq 1$, and denote by $(\Omega, \mathcal{F}, \mathbb{P})$ the probability space constructed in the previous subsection. Let us consider a coupled trajectory $(\mathfrak{u}_t, \mathfrak{u}'_t) := (\tilde{\mathfrak{u}}_t, \tilde{\mathfrak{u}}'_t)$ at level N issued from $(\mathfrak{z}, \mathfrak{z}')$ and the associated process $\mathfrak{v}_t := \tilde{\mathfrak{v}}_t$. For any integers $r \geq 0$ and $\rho \geq 1$, we set[7]

$$\begin{aligned}
&\bar{G}_r = \bigcap_{j=0}^{r} G_j, \;\; G_j = \{\mathfrak{v}_t = \mathfrak{u}'_t, \forall t \in (j, j+1]\}, \;\; F_{r,0} = \varnothing, \\
&F_{r,\rho} = \bigg\{ \sup_{\tau \in [0,r]} \bigg(\int_0^\tau \big(\|\nabla u_\tau\|^2 + \|\nabla u'_\tau\|^2 \big)\,\mathrm{d}\tau - L\tau \bigg) \leq |\mathcal{E}(\mathfrak{z})| + |\mathcal{E}(\mathfrak{z}')| + \rho; \\
&\qquad\qquad |\mathcal{E}(\mathfrak{u}_r)| + |\mathcal{E}(\mathfrak{u}'_r)| \leq \rho \bigg\},
\end{aligned}$$

where L is the constant in (4.11). We also define the pairwise disjoint events

$$A_0 = G_0^c, \quad A_{r,\rho} = \big(\bar{G}_{r-1} \cap G_r^c \cap F_{r,\rho}\big) \setminus F_{r,\rho-1}, \; r \geq 1, \rho \geq 1, \quad \tilde{A} = \bar{G}_{+\infty}.$$

Then, for any $t \geq 1$, we have

$$\begin{aligned}
\mathfrak{P}_t^V \psi(\mathfrak{z}) - \mathfrak{P}_t^V \psi(\mathfrak{z}') &= \mathbb{E}\big\{\mathbb{I}_{A_0}\big[(\Xi_V \psi)(\mathfrak{u}_t, t) - (\Xi_V \psi)(\mathfrak{u}'_t, t)\big]\big\} \\
&\quad + \sum_{r,\rho=1}^{\infty} \mathbb{E}\big\{\mathbb{I}_{A_{r,\rho}}\big[(\Xi_V \psi)(\mathfrak{u}_t, t) - (\Xi_V \psi)(\mathfrak{u}'_t, t)\big]\big\} \\
&\quad + \mathbb{E}\big\{\mathbb{I}_{\tilde{A}}\big[(\Xi_V \psi)(\mathfrak{u}_t, t) - (\Xi_V \psi)(\mathfrak{u}'_t, t)\big]\big\} \\
&= I_0^t(\mathfrak{z}, \mathfrak{z}') + \sum_{r,\rho=1}^{\infty} I_{r,\rho}^t(\mathfrak{z}, \mathfrak{z}') + \tilde{I}^t(\mathfrak{z}, \mathfrak{z}'),
\end{aligned} \tag{4.5}$$

[7] The event $\bar{G}_r$ is well defined also for $r = +\infty$.

where

$$I_0^t(\mathfrak{z},\mathfrak{z}') := \mathbb{E}\big\{\mathbb{I}_{A_0}\big[(\Xi_V\psi)(\mathfrak{u}_t,t)-(\Xi_V\psi)(\mathfrak{u}'_t,t)\big]\big\},$$
$$I_{r,\rho}^t(\mathfrak{z},\mathfrak{z}') := \mathbb{E}\big\{\mathbb{I}_{A_{r,\rho}}\big[(\Xi_V\psi)(\mathfrak{u}_t,t)-(\Xi_V\psi)(\mathfrak{u}'_t,t)\big]\big\},$$
$$\tilde{I}^t(\mathfrak{z},\mathfrak{z}') := \mathbb{E}\big\{\mathbb{I}_{\tilde{A}}\big[(\Xi_V\psi)(\mathfrak{u}_t,t)-(\Xi_V\psi)(\mathfrak{u}'_t,t)\big]\big\}.$$

To prove the uniform equicontinuity of $\{g_t, t\ge 1\}$, we first estimate these three quantities.

Step 2: Estimates for I_0^t and $I_{r,\rho}^t$. Let $\delta_1>0$ and $R_0\ge 1$ be the numbers in Proposition 3.1. Then, if $\mathrm{Osc}(V)<\delta_1$ and $R\ge R_0$, we have the following estimates

$$|I_0^t(\mathfrak{z},\mathfrak{z}')| \le C_1(R,V)\|\mathfrak{P}_t^V\mathbf{1}\|_R\,\mathbb{P}\{A_0\}^{1/2}, \tag{4.6}$$
$$|I_{r,\rho}^t(\mathfrak{z},\mathfrak{z}')| \le C_2(R,V)e^{r\|V\|_\infty}\|\mathfrak{P}_t^V\mathbf{1}\|_R\,\mathbb{P}\{A_{r,\rho}\}^{1/2} \tag{4.7}$$

for any integers $r,\rho\ge 1$. Let us prove (4.7), the other estimate is similar. First assume that $r\le t$. Using the inequalities $0\le\psi\le 1$, the positivity of $\Xi_V\psi$, and the Markov property, we derive

$$\begin{aligned} I_{r,\rho}^t(\mathfrak{z},\mathfrak{z}') &\le \mathbb{E}\big\{I_{A_{r,\rho}}(\Xi_V\psi)(\mathfrak{u}_t,t)\big\} \le \mathbb{E}\big\{I_{A_{r,\rho}}(\Xi_V\mathbf{1})(\mathfrak{u}_t,t)\big\}\\ &= \mathbb{E}\big\{I_{A_{r,\rho}}\mathbb{E}\big[(\Xi_V\mathbf{1})(\mathfrak{u}_t,t)\,\big|\,\mathcal{F}_r\big]\big\} \le e^{r\|V\|_\infty}\mathbb{E}\big\{I_{A_{r,\rho}}(\mathfrak{P}_{t-r}^V\mathbf{1})(\mathfrak{u}_r)\big\}, \end{aligned}$$

where $\{\mathcal{F}_t\}$ stands for the filtration generated by $(\mathfrak{u}_t,\mathfrak{u}'_t)$. Then from (3.1) it follows that

$$\mathfrak{P}_{t-r}^V\mathbf{1}(\mathfrak{z}) \le M\|\mathfrak{P}_{t-r}^V\mathbf{1}\|_{R_0}\mathfrak{w}(\mathfrak{z}),$$

so we have

$$\begin{aligned} I_{r,\rho}^t(\mathfrak{z},\mathfrak{z}') &\le C_3e^{r\|V\|_\infty}\|\mathfrak{P}_{t-r}^V\mathbf{1}\|_{R_0}\mathbb{E}\big\{I_{A_{r,\rho}}\mathfrak{w}(\mathfrak{u}_r)\big\}\\ &\le C_3e^{r\|V\|_\infty}\|\mathfrak{P}_{t-r}^V\mathbf{1}\|_{R_0}\big\{\mathbb{P}(A_{r,\rho})\,\mathbb{E}\,\mathfrak{w}^2(\mathfrak{u}_r)\big\}^{1/2}. \end{aligned}$$

Using this, (5.24), and the symmetry, we obtain (4.7). If $r>t$, then

$$I_{r,\rho}^t(\mathfrak{z},\mathfrak{z}') \le e^{r\|V\|_\infty}\mathbb{P}\{A_{r,\rho}\} \le e^{r\|V\|_\infty}\|\mathfrak{P}_t^V\mathbf{1}\|_R\,\mathbb{P}\{A_{r,\rho}\}^{1/2},$$

which implies (4.7) by symmetry.

Step 3: Estimates for $\mathbb{P}\{A_0\}$ and $\mathbb{P}\{A_{r,\rho}\}$. Let us show that, for sufficiently large $N\ge 1$, we have

$$\mathbb{P}\{A_0\} \le C_4(R,N)d^{a/2}, \tag{4.8}$$
$$\mathbb{P}\{A_{r,\rho}\} \le C_5(R)\left\{\left(d^ae^{-a\alpha r/2}+\Big[\exp\Big(C_6(R,N)d^ae^{2\rho-a\alpha r/2}\Big)-1\Big]^{1/2}\right)\wedge e^{-\beta\rho}\right\}, \tag{4.9}$$

where a, C_*, and β are the constants in (4.2) and (4.11). Indeed, taking $\varepsilon = d$ in (4.3), using (3.8), and recalling that $d \le 1$, we get

$$\mathbb{P}\{A_0\} \le C_* d^a + C_* \left[\exp\left(C_N d^a e^{C_7 R^4}\right) - 1\right]^{1/2} \le C_4(R,N) d^{a/2},$$

provided that N is larger that the number N_1 in Proposition 4.1. This gives (4.8). To show (4.9), we use the estimates

$$\mathbb{E}_{\mathfrak{u}} \exp\left(\beta |\mathcal{E}(\mathfrak{u}_t)|\right) \le C \exp(\beta |\mathcal{E}(\mathfrak{u})|), \quad \mathfrak{u} \in \mathcal{H}, \tag{4.10}$$

$$\mathbb{P}_{\mathfrak{u}} \left\{ \sup_{t \ge 0} \left(\int_0^t \|\nabla u_\tau\|^2 \, \mathrm{d}\tau - Lt \right) \ge |\mathcal{E}(\mathfrak{u})| + \rho \right\} \le C e^{-\beta \rho}, \quad \rho > 0, \tag{4.11}$$

where L, β, and C are some positive constants depending on $\gamma, \|h\|$, and $\mathfrak{B}$; they follow immediately from Propositions 3.1 and 3.2 in [Mar14]. From the inclusion $A_{r,\rho} \subset F^c_{r,\rho-1}$ and inequalities (4.10), (4.11), and (3.8) it follows that

$$\mathbb{P}\{A_{r,\rho}\} \le C_8(R) e^{-\beta\rho}. \tag{4.12}$$

By the Foiaş–Prodi type estimate (see (7.29) in Proposition 7.5), there is $N_2 \ge 1$ such that for any $N \ge N_2$ on the event $\bar{G}_{r-1} \cap F_{r,\rho}$ we have

$$|\mathfrak{u}_r - \mathfrak{u}'_r|^2_{\mathcal{H}} \le \exp(-\alpha r + \rho + |\mathcal{E}(\mathfrak{z})| + |\mathcal{E}(\mathfrak{z}')|) d^2 \le C_9(R) e^{-\alpha r + \rho} d^2, \tag{4.13}$$

where we used (3.8). Recall that on the same event we have also

$$|\mathcal{E}(\mathfrak{u}_r)| + |\mathcal{E}(\mathfrak{u}'_r)| \le \rho. \tag{4.14}$$

So using the Markov property, (4.3) with $\varepsilon = d e^{-\alpha r/2}$, (4.14) and (4.13), we obtain

$$\begin{aligned}
\mathbb{P}\{A_{r,\rho}\} &\le \mathbb{P}\big\{\bar{G}_{r-1}) \cap G^c_r \cap F_{r,\rho}\big\} = \mathbb{E}\big\{\mathbb{I}_{\bar{G}_{r-1} \cap F_{r,\rho}} \mathbb{E}\big(I_{G^c_r} \,\big|\, \mathcal{F}_r\big)\big\} \\
&\le C_* d^a e^{-a\alpha r/2} + C_* \mathbb{E}\Big\{ \mathbb{I}_{\bar{G}_{r-1} \cap F_{r,\rho}} \\
&\quad \times \left[\exp\left(C_N d^{a-2} e^{-(a-2)\alpha r/2} |\mathfrak{u}_r - \mathfrak{u}'_r|^2_{\mathcal{H}} e^{(|\mathcal{E}(\mathfrak{u}_r)| + |\mathcal{E}(\mathfrak{u}'_r)|)}\right) - 1\right]^{1/2} \Big\} \\
&\le C_* d^a e^{-a\alpha r/2} + C_* \left[\exp\left(C_6(R,N) d^a e^{2\rho - a\alpha r/2}\right) - 1\right]^{1/2}.
\end{aligned}$$

Combining this with (4.12) and choosing $N \ge N_1 \vee N_2$, we get the required inequality (4.9).

Step 4: Estimate for $\tilde{I}^t$. Let us show that, for any $N \ge N_0$, we have

$$|\tilde{I}^t_\rho(\mathfrak{z}, \mathfrak{z}')| \le C_{10}(\psi, V) \|\mathfrak{P}^V_t \mathbf{1}\|_R d^q. \tag{4.15}$$

Indeed, we write

$$\begin{aligned}
\tilde{I}^t(\mathfrak{z}, \mathfrak{z}') &= \mathbb{E}\big\{\mathbb{I}_{\tilde{A}} (\Xi_V \mathbf{1})(\mathfrak{u}_t, t)[\psi(\mathfrak{u}_t) - \psi(\mathfrak{u}'_t)]\big\} \\
&\quad + \mathbb{E}\big\{\mathbb{I}_{\tilde{A}}[(\Xi_V \mathbf{1})(\mathfrak{u}_t, t) - (\Xi_V \mathbf{1})(\mathfrak{u}'_t, t)]\psi(\mathfrak{u}'_t)\big\}.
\end{aligned} \tag{4.16}$$

Let us denote by $J^t_{1,\rho}$ and $J^t_{2,\rho}$ the expectations in the right-hand side of this equality. Then by estimate (7.27), on the event $\tilde{A}$ we have

$$|P_N(\mathfrak{u}_\tau - \mathfrak{u}'_\tau)|^2_{\mathcal{H}} \le e^{-\alpha\tau} d^2, \quad \tau \in [0,t]. \tag{4.17}$$

Since $\psi \in C^q_b(\mathcal{H})$, we derive from (4.17)

$$\begin{aligned} |J^t_{1,\rho}| &\le \mathbb{E}\big\{\mathbb{I}_{\tilde{A}}(\Xi_V \mathbf{1})(\mathfrak{u}_t,t)|\psi(\mathfrak{u}_t) - \psi(\mathfrak{u}'_t)|\big\} \le \|\psi\|_{C^q_b} e^{-\alpha t/2} d^q \|\mathfrak{P}^V_t \mathbf{1}\|_R \\ &\le \|\psi\|_{C^q_b} \|\mathfrak{P}^V_t \mathbf{1}\|_R d^q. \end{aligned}$$

Similarly, as $V \in C^q_b(\mathcal{H})$,

$$\begin{aligned} |J^t_{2,\rho}| &\le \mathbb{E}\big\{\mathbb{I}_{\tilde{A}}|(\Xi_V \mathbf{1})(\mathfrak{u}_t,t) - (\Xi_V \mathbf{1})(\mathfrak{u}'_t,t)|\big\} \\ &\le \mathbb{E}\left\{\mathbb{I}_{\tilde{A}}(\Xi_V \mathbf{1})(\mathfrak{u}_t,t)\left[\exp\left(\int_0^t |V(\mathfrak{u}_\tau) - V(\mathfrak{u}'_\tau)|\,\mathrm{d}\tau\right) - 1\right]\right\} \\ &\le \left[\exp\left(\|V\|_{C^q_b} d^q (1 - e^{-\alpha q t/2})\right) - 1\right] \|\mathfrak{P}^V_t \mathbf{1}\|_R \\ &\le \left[\exp\left(\|V\|_{C^q_b} d^q\right) - 1\right] \|\mathfrak{P}^V_t \mathbf{1}\|_R. \end{aligned}$$

Combining these estimates for $J^t_{1,\rho}$ and $J^t_{2,\rho}$ with (4.16), we get (4.15).

Step 5. From (4.5)–(4.9) and (4.15) it follows that, for any $\mathfrak{z}, \mathfrak{z}' \in X_R$, $t \ge 1$, and $R \ge R_0$, we have

$$\begin{aligned} \big|g_t(\mathfrak{z}) - g_t(\mathfrak{z}')\big| &\le C_{11}(R,V,N,\psi)\Bigg(d^{a/4} + d^q \\ &+ \sum_{r,\rho=1}^{\infty} e^{r\|V\|_\infty}\left\{\left(d^{a/2} e^{-a\alpha r/4} + \left[\exp\left(C_6 d^a e^{2\rho - a\alpha r/2}\right) - 1\right]^{1/4}\right) \wedge e^{-\beta\rho/2}\right\}\Bigg), \end{aligned}$$

provided that $N \ge N_0 \vee N_1 \vee N_2$. When $d = 0$, the series in the right-hand side vanishes. So to prove the uniform equicontinuity of $\{g_t\}$, it suffices to show that the series converges uniformly in $d \in [0,1]$. Since its terms are positive and monotone, it suffices to show the converge for $d = 1$:

$$\sum_{r,\rho=1}^{\infty} e^{r\|V\|_\infty}\left\{\left(e^{-a\alpha r/4} + \left[\exp\left(C_6 e^{2\rho - a\alpha r/2}\right) - 1\right]^{1/4}\right) \wedge e^{-\beta\rho/2}\right\} < \infty. \tag{4.18}$$

To prove this, we will assume that $\mathrm{Osc}(V)$ is sufficiently small. Let us consider the sets

$$S_1 = \{(r,\rho) \in \mathbb{N}^2 : \rho \le a\alpha r/8\}, \quad S_2 = \mathbb{N}^2 \setminus S_1.$$

Then taking $\delta < \delta_1 \vee (a\alpha/32)$ and $\mathrm{Osc}(V) < \delta$, we see that

$$\begin{aligned} &\sum_{(r,\rho)\in S_1} e^{r\|V\|_\infty}\left(e^{-a\alpha r/4} + \left[\exp\left(C_6 e^{2\rho - a\alpha r/2}\right) - 1\right]^{1/4}\right) \\ &\qquad \le C_{12}(R,N) \sum_{(r,\rho)\in S_1} e^{r\|V\|_\infty} e^{-a\alpha r/16} \le C_{13}(R,N) \sum_{r=1}^{\infty} e^{-a\alpha r/32} < \infty. \end{aligned}$$

Choosing $\delta < a\alpha\beta/32$, we get

$$\sum_{(r,\rho)\in S_2} e^{r\|V\|_\infty} e^{-\beta\rho/2} \le C_{14} \sum_{\rho=1}^{\infty} e^{-\beta\rho/4} < \infty.$$

These two inequalities show that (4.18) holds.

□

5 Estimates for regular solutions

In this section, we establish the exponential tightness property and obtain some higher order moment estimates for solutions in $\mathcal{H}^{\mathfrak{s}}$.

5.1 Exponential tightness

Here we show that the exponential tightness property in Section 1.3 is verified for the function $\Phi(\mathfrak{u}) = |\mathfrak{u}|^{\varkappa}_{\mathcal{H}^{\mathfrak{s}}}$, if we choose $\varkappa > 0$ sufficiently small. Clearly, the level sets of Φ are compact in $\mathcal{H}$.

Theorem 5.1. *For any $\mathfrak{s} < 1/2$, there is $\varkappa \in (0,1)$ such that, for any $R \ge 1$, we have*

$$\mathbb{E}_{\mathfrak{v}} \exp\left(\int_0^t |\mathfrak{u}_\tau|^{\varkappa}_{\mathcal{H}^{\mathfrak{s}}}\, \mathrm{d}\tau\right) \le c\, e^{ct} \quad \text{for any } \mathfrak{v} \in X_R, t \ge 0, \tag{5.1}$$

where c is a positive constant depending on R.

Proof. It is sufficient to prove that there is $\varkappa \in (0,1)$ such that, for any $R \ge 1$, we have

$$\mathbb{E}_{\mathfrak{v}} \exp\left(\delta \int_0^t |\mathfrak{u}_\tau|^{\varkappa}_{\mathcal{H}^{\mathfrak{s}}}\, \mathrm{d}\tau\right) \le \tilde{c}\, e^{\tilde{c}t} \quad \text{for any } \mathfrak{v} \in X_R, t \ge 0, \tag{5.2}$$

where δ and $\tilde{c}$ are positive constants depending on R. Indeed, once this is proved, we can use the inequality

$$|\mathfrak{u}|^{\frac{\varkappa}{2}}_{\mathcal{H}^{\mathfrak{s}}} \le \delta |\mathfrak{u}|^{\varkappa}_{\mathcal{H}^{\mathfrak{s}}} + \delta^{-1}$$

to derive (5.1), where $\varkappa$ should be replaced by $\varkappa/2$. We divide the proof of (5.2) into several steps.

Step 1: Reduction. Let us split the flow $\mathfrak{u}(t)$ to the sum $\mathfrak{u} = \mathfrak{v}_1 + \mathfrak{v}_2 + \mathfrak{z}$, where $\mathfrak{v}_1(t) = [v_1(t), \dot{v}_1(t)]$ corresponds to the flow of (0.1) with $f = h = \vartheta = 0$ issued from $\mathfrak{v}$ and $\mathfrak{v}_2(t) = [v_2(t), \dot{v}_2(t)]$ is the flow of (0.1) with $f = 0$ issued from the origin. Some standard arguments show that the following a priori estimates hold:

$$|\mathfrak{v}_1(t)|^2_{\mathcal{H}^{\mathfrak{s}}} \le |\mathfrak{v}|^2_{\mathcal{H}^{\mathfrak{s}}} e^{-\alpha t}, \tag{5.3}$$

$$\mathbb{E} \exp\left(\delta_1 \int_0^t |\mathfrak{v}_2(\tau)|^2_{\mathcal{H}^{\mathfrak{s}}}\, \mathrm{d}\tau\right) \le c_1\, e^{c_1 t} \quad \text{for any } t \ge 0, \tag{5.4}$$

where δ_1 and c_1 are positive constants depending only on $\alpha, \mathfrak{B}_1$, and $\|h\|_1$. Now using the Cauchy–Schwarz inequality and (5.3), we get, for any $\delta < \delta_1/2$,

$$\begin{aligned}\mathbb{E}_{\mathfrak{v}} \exp\left(\delta \int_0^t |\mathfrak{u}(\tau)|_{\mathcal{H}^{\mathfrak{s}}}^{\varkappa} \,\mathrm{d}\tau\right) &\le \exp\left(\delta \int_0^t |\mathfrak{v}_1(\tau)|_{\mathcal{H}^{\mathfrak{s}}}^{\varkappa} \,\mathrm{d}\tau\right) \mathbb{E} \exp\left(2\delta \int_0^t |\mathfrak{v}_2(\tau)|_{\mathcal{H}^{\mathfrak{s}}}^{\varkappa} \,\mathrm{d}\tau\right) \\ &\quad \times \mathbb{E} \exp\left(2\delta \int_0^t |\mathfrak{z}(\tau)|_{\mathcal{H}^{\mathfrak{s}}}^{\varkappa} \,\mathrm{d}\tau\right) \\ &\le \exp\bigl(2\delta R^{\varkappa} (\alpha\varkappa)^{-1}\bigr) \mathbb{E} \exp\left(2\delta \int_0^t (|\mathfrak{v}_2(\tau)|_{\mathcal{H}^{\mathfrak{s}}}^2 + 1) \,\mathrm{d}\tau\right) \\ &\quad \times \mathbb{E} \exp\left(2\delta \int_0^t |\mathfrak{z}(\tau)|_{\mathcal{H}^{\mathfrak{s}}}^{\varkappa} \,\mathrm{d}\tau\right).\end{aligned}$$

Combining this with (5.4), we see that inequality (5.2) will be established if we prove that

$$\mathbb{E} \exp\left(\delta \int_0^t |\mathfrak{z}(\tau)|_{\mathcal{H}^{\mathfrak{s}}}^{\varkappa} \,\mathrm{d}\tau\right) \le c\, e^{ct} \quad \text{for all } t \ge 0 \tag{5.5}$$

for some $\delta > 0$ and $c > 0$. The rest of the proof is devoted to the derivation of this inequality.

Step 2: Pointwise estimates. Let us note that, by construction, $\mathfrak{z}$ is the flow of equation

$$\partial_t^2 z + \gamma \partial_t z - \Delta z + f(u) = 0, \quad z|_{\partial D} = 0, \quad [z(0), \dot{z}(0)] = 0. \tag{5.6}$$

Let us differentiate this equation in time, and set $a = \dot{z}(t)$. Then a solves

$$\partial_t^2 a + \gamma \partial_t a - \Delta a + f'(u)\partial_t u = 0, \quad a|_{\partial D} = 0, \quad [a(0), \dot{a}(0)] = [0, -f(u(0))]. \tag{5.7}$$

We write $\mathfrak{a}(t) = [a(t), \dot{a}(t)]$. Multiplying equation (5.7) by $2(-\Delta)^{\mathfrak{s}-1}(\dot{a} + \alpha a)$ and integrating over D, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} |\mathfrak{a}|_{\mathcal{H}^{\mathfrak{s}-1}}^2 + \frac{3\alpha}{2} |\mathfrak{a}|_{\mathcal{H}^{\mathfrak{s}-1}}^2 \le 2 \int_D |f'(u)||\dot{u}||(-\Delta)^{\mathfrak{s}-1}(\dot{a} + \alpha a)| \,\mathrm{d}x = \mathcal{L}. \tag{5.8}$$

Let $\varkappa < 1$ be a positive constant that will be fixed later. Then, by the triangle inequality, we have

$$\begin{aligned}\frac{\mathcal{L}}{2} &\le \int_D |f'(u)||\dot{v}_1|^{1-\varkappa} |\dot{u}|^{\varkappa} |(-\Delta)^{\mathfrak{s}-1}(\dot{a} + \alpha a)| \,\mathrm{d}x \\ &\quad + \int_D |f'(u)||\dot{v}_2|^{1-\varkappa} |\dot{u}|^{\varkappa} |(-\Delta)^{\mathfrak{s}-1}(\dot{a} + \alpha a)| \,\mathrm{d}x \\ &\quad + \int_D |f'(u)||a|^{1-\varkappa} |\dot{u}|^{\varkappa} |(-\Delta)^{\mathfrak{s}-1}(\dot{a} + \alpha a)| \,\mathrm{d}x = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3.\end{aligned} \tag{5.9}$$

Using the Hölder inequality, we derive

$$\mathcal{L}_1 \le |f'(u)|_{L^{p_1}} |\dot{v}_1|_{L^{(1-\varkappa)p_2}}^{1-\varkappa} |\dot{u}|_{L^{\varkappa p_3}}^{\varkappa} |(-\Delta)^{\mathfrak{s}-1}(\dot{a} + \alpha a)|_{L^{p_4}}, \tag{5.10}$$

$$\mathcal{L}_2 \le |f'(u)|_{L^{q_1}} |\dot{v}_2|_{L^{(1-\varkappa)q_2}}^{1-\varkappa} |\dot{u}|_{L^{\varkappa q_3}}^{\varkappa} |(-\Delta)^{\mathfrak{s}-1}(\dot{a} + \alpha a)|_{L^{q_4}}, \tag{5.11}$$

$$\mathcal{L}_3 \le |f'(u)|_{L^{p_1}} |a|_{L^{(1-\varkappa)p_2}}^{1-\varkappa} |\dot{u}|_{L^{\varkappa p_3}}^{\varkappa} |(-\Delta)^{\mathfrak{s}-1}(\dot{a} + \alpha a)|_{L^{p_4}}, \tag{5.12}$$

where the exponents p_i, q_i are Hölder admissible. We now need the following lemma, which is established in the appendix.

Lemma 5.2. *Let us take $p_1 = 6/\rho, p_3 = 2/\varkappa, q_1 = (\rho+2)/\rho$ and $q_3 = 2/\varkappa$. Then, for $\varkappa > 0$ sufficiently small, the exponents p_2, p_4, q_2 and q_4 can be chosen in such a way that we have the following embeddings:*

$$H^{\mathfrak{s}} \hookrightarrow L^{(1-\varkappa)p_2}, \qquad H^{1-\mathfrak{s}} \hookrightarrow L^{p_4}, \tag{5.13}$$

$$H^{1} \hookrightarrow L^{(1-\varkappa)q_2}, \qquad H^{1-\mathfrak{s}} \hookrightarrow L^{q_4}. \tag{5.14}$$

Step 3: Estimation of $\mathcal{L}_1$ and $\mathcal{L}_3$. In view of Lemma 5.2 and inequalities (1.1) and (5.10), we have

$$\begin{aligned}\mathcal{L}_1 &\leq C_0 |f'(u)|_{L^{6/\rho}} \|\dot v_1\|_{\mathfrak{s}}^{1-\varkappa} \|\dot u\|^{\varkappa} \|(-\Delta)^{\mathfrak{s}-1}(\dot a + \alpha a)\|_{1-\mathfrak{s}} \\ &\leq C_1 \|\dot v_1\|_{\mathfrak{s}}^{1-\varkappa} (\|u\|_1^{\rho} + 1) \|\dot u\|^{\varkappa} \|\dot a + \alpha a\|_{\mathfrak{s}-1}.\end{aligned}$$

Now let us suppose that $\varkappa < 2 - \rho$. Then using (5.3) together with the Young inequality, we derive

$$\mathcal{L}_1 \leq C_2 |\mathfrak{v}|_{\mathcal{H}^{\mathfrak{s}}}^{1-\varkappa} (\|u\|_1^2 + \|\dot u\|^2 + C_{\varkappa}) \|\dot a + \alpha a\|_{\mathfrak{s}-1} \leq C_3\, R(\mathcal{E}(\mathfrak{u}) + C_3) |\mathfrak{a}|_{\mathcal{H}^{\mathfrak{s}-1}}. \tag{5.15}$$

To estimate $\mathcal{L}_3$, we again apply Lemma 5.2 and inequalities (1.1) and (5.12)

$$\mathcal{L}_3 \leq C_4 (\|u\|_1^{\rho} + 1) \|a\|_{\mathfrak{s}}^{1-\varkappa} \|\dot u\|^{\varkappa} \|\dot a + \alpha a\|_{\mathfrak{s}-1} \leq C_4 (\|u\|_1^{\rho} + 1) \|\dot u\|^{\varkappa} |\mathfrak{a}|_{\mathcal{H}^{\mathfrak{s}-1}}^{2-\varkappa}.$$

Applying the Young inequality, we get

$$\mathcal{L}_3 \leq C_5 (\mathcal{E}(\mathfrak{u}) + C_5) |\mathfrak{a}|_{\mathcal{H}^{\mathfrak{s}-1}}^{2-\varkappa}. \tag{5.16}$$

Step 4: Estimation of $\mathcal{L}_2$. It follows from Lemma 5.2 and inequalities (1.2) and (5.11) that

$$\begin{aligned}\mathcal{L}_2 &\leq C_6 |f'(u)|_{L^{(\rho+2)/\rho}} \|\dot v_2\|_1^{1-\varkappa} \|\dot u\|^{\varkappa} \|(-\Delta)^{\mathfrak{s}-1}(\dot a + \alpha a)\|_{1-\mathfrak{s}} \\ &\leq C_7 \|\dot v_2\|_1^{1-\varkappa} \left(\int_D (F(u) + \nu u^2 + C)\, \mathrm{d}x \right)^{\rho/\rho+2} \|\dot u\|^{\varkappa} \|\dot a + \alpha a\|_{\mathfrak{s}-1} \\ &\leq C_8 \|\dot v_2\|_1^{1-\varkappa} \left(\mathcal{E}(\mathfrak{u}) + C_8\right)^{\rho/\rho+2} \|\dot u\|^{\varkappa} |\mathfrak{a}|_{\mathcal{H}^{\mathfrak{s}-1}}.\end{aligned}$$

Finally, applying the Young inequality, we obtain

$$\mathcal{L}_2 \leq C_9 (\mathcal{E}(\mathfrak{u}) + |\mathfrak{v}_2|_{\mathcal{H}^{\mathfrak{s}}}^2 + C_9) |\mathfrak{a}|_{\mathcal{H}^{\mathfrak{s}-1}}. \tag{5.17}$$

Step 5: Estimation of $|\mathfrak{a}|_{\mathcal{H}^{\mathfrak{s}-1}}$. Combining inequalities (5.8), (5.9) and (5.15)-(5.17), we see that

$$\frac{\mathrm{d}}{\mathrm{d}t} |\mathfrak{a}(t)|_{\mathcal{H}^{\mathfrak{s}-1}}^2 + \alpha |\mathfrak{a}(t)|_{\mathcal{H}^{\mathfrak{s}-1}}^2 \leq C_{10}\, R \left(\mathcal{E}(\mathfrak{u}(t)) + |\mathfrak{v}_2(t)|_{\mathcal{H}^{\mathfrak{s}}}^2 + C_{10} \right) \left(|\mathfrak{a}(t)|_{\mathcal{H}^{\mathfrak{s}-1}}^{2-\varkappa} + 1 \right). \tag{5.18}$$

We now need an auxiliary result, whose proof is presented in the appendix.

Lemma 5.3. *Let $x(t)$ be an absolutely continuous nonnegative function satisfying the differential inequality*

$$\dot{x}(t) + \alpha x(t) \le g(t)x^{1-\beta}(t) + b(t) \quad \text{for all } t \in [0,T], \tag{5.19}$$

where α, T, and $\beta < 1$ are positive constants and $g(t)$ and $b(t)$ are nonnegative functions integrable on $[0,T]$. Then we have

$$\frac{\alpha}{2}\int_0^t x^\beta(\tau)\,\mathrm{d}\tau \le \beta^{-1}(1+x(0))^\beta + \int_0^t (\alpha + g(\tau) + b(\tau))\,\mathrm{d}\tau \quad \text{for } t \in [0,T]. \tag{5.20}$$

Applying this lemma to inequality (5.18), we obtain

$$\frac{\alpha}{2}\int_0^t |\mathfrak{a}(\tau)|^\varkappa_{\mathcal{H}^{\mathfrak{s}-1}}\,\mathrm{d}\tau \le 2\varkappa^{-1}(1+|\mathfrak{a}(0)|^2_{\mathcal{H}^{\mathfrak{s}-1}})^{\varkappa/2} + \alpha t + 2\,C_{10}\,R\int_0^t \left(\mathcal{E}(\mathfrak{u}(\tau)) + |\mathfrak{v}_2(\tau)|^2_{\mathcal{H}^{\mathfrak{s}}} + C_{10}\right)\mathrm{d}\tau. \tag{5.21}$$

Step 6: Completion of the proof. Note that

$$|\mathfrak{z}|^2_{\mathcal{H}^{\mathfrak{s}}} = \|z\|^2_{\mathfrak{s}+1} + \|\dot{z} + \alpha z\|^2_{\mathfrak{s}} = \|\Delta z\|^2_{\mathfrak{s}-1} + \|a + \alpha z\|^2_{\mathfrak{s}}.$$

On the other, in view of (5.6), we have

$$\|\Delta z\|^2_{\mathfrak{s}-1} = \|\dot{a} + \gamma a + f(u)\|^2_{\mathfrak{s}-1} \le C_{11}(|\mathfrak{a}|^2_{\mathcal{H}^{\mathfrak{s}-1}} + \|f(u)\|^2),$$

whence we get

$$|\mathfrak{z}|^2_{\mathcal{H}^{\mathfrak{s}}} \le C_{12}\left(|\mathfrak{a}|^2_{\mathcal{H}^{\mathfrak{s}-1}} + \mathcal{E}^3(\mathfrak{u}) + C_{12}\right). \tag{5.22}$$

It follows that

$$|\mathfrak{z}|^\varkappa_{\mathcal{H}^{\mathfrak{s}}} \le C_{13}\left(|\mathfrak{a}|^\varkappa_{\mathcal{H}^{\mathfrak{s}-1}} + \mathcal{E}(\mathfrak{u}) + C_{13}\right),$$

provided $\varkappa < 2/3$. Multiplying this inequality by $\alpha/2$, integrating over $[0,t]$ and using (5.21) together with the fact that

$$|\mathfrak{a}(0)|^2_{\mathcal{H}^{\mathfrak{s}-1}} = \|f(u(0))\|^2_{\mathfrak{s}-1} \le \|f(u(0))\|^2 \le C_{14}(\|\mathfrak{v}\|^6_1 + 1), \tag{5.23}$$

we derive

$$\frac{\alpha}{2}\int_0^t |\mathfrak{z}(\tau)|^\varkappa_{\mathcal{H}^{\mathfrak{s}}}\,\mathrm{d}\tau \le C_{15}\left(1 + \int_0^t \left[\mathcal{E}(\mathfrak{u}(\tau)) + |\mathfrak{v}_2(\tau)|^2_{\mathcal{H}^{\mathfrak{s}}} + C_{15}\right]\mathrm{d}\tau\right),$$

where C_{15} depends on R. Multiplying this inequality by a small constant $\delta(R) > 0$, taking the exponent and then the expectation, and using (5.4) together with Proposition 3.2 in [Mar14], we derive (5.5). □

5.2 Higher moments of regular solutions

For any $m \geq 1$, let $\mathfrak{w}_m$ and $\tilde{\mathfrak{w}}_m$ be the functions given by (3.19) and (3.20). The following result shows that they are both Lyapunov functions for the trajectories of problem (0.1), (0.3).

Proposition 5.4. *For any $\mathfrak{v} \in \mathcal{H}^{\mathfrak{s}}$, $m \geq 1$, and $t \geq 0$, we have*

$$\mathbb{E}_{\mathfrak{v}} \mathfrak{w}_m(\mathfrak{u}_t) \leq 2e^{-\alpha m t} \mathfrak{w}_m(\mathfrak{v}) + C_m, \tag{5.24}$$

$$\mathbb{E}_{\mathfrak{v}} \tilde{\mathfrak{w}}_m(\mathfrak{u}_t) \leq 2e^{-\alpha m t} \tilde{\mathfrak{w}}_m(\mathfrak{v}) + C_m. \tag{5.25}$$

Proof. Step 1: Proof of (5.24). We split the flow $\mathfrak{u}(t;\mathfrak{v})$ to the sum $\mathfrak{u}(t;\mathfrak{v}) = \tilde{\mathfrak{u}}(t) + \mathfrak{z}(t)$, where $\tilde{\mathfrak{u}}$ is the flow issued from $\mathfrak{v}$ corresponding to the solution of (0.1) with $f = 0$. Let us note that here $\mathfrak{z} = [z, \dot{z}]$ is the same as in Section 5.1. A standard argument shows that

$$\mathbb{E}|\tilde{\mathfrak{u}}(t)|^{2m}_{\mathcal{H}^{\mathfrak{s}}} \leq e^{-\alpha m t} |\mathfrak{v}|^{2m}_{\mathcal{H}^{\mathfrak{s}}} + C(m, \|h\|_1, \mathfrak{B}_1). \tag{5.26}$$

As in Section 5.1, we set $a = \dot{z}$ and write $\mathfrak{a} = [a, \dot{a}]$. Notice that thanks to the Hölder inequality, the Sobolev embeddings $H^1 \hookrightarrow L^6$ and $H^{1-\mathfrak{s}} \hookrightarrow L^{6/(3-\rho)}$ for $\mathfrak{s} < 1 - \rho/2$, and inequality $|\mathfrak{u}|^2_{\mathcal{H}} \leq 2|\mathcal{E}(\mathfrak{u})| + 3C$, we can estimate the right-hand side of inequality (5.8) by

$$\begin{aligned}
\mathcal{L} &\leq C_1(|u|^{\rho}_{L^6} + 1)\|\dot{u}\| |(-\Delta)^{\mathfrak{s}-1}(\dot{a} + \alpha a)|_{L^{6/(3-\rho)}} \\
&\leq C_2(\|u\|^2_1 + 1)\|\dot{u}\| \|(-\Delta)^{\mathfrak{s}-1}(\dot{a} + \alpha a)\|_{1-\mathfrak{s}} \leq C_3(|\mathfrak{u}|^3_{\mathcal{H}} + 1)\|\dot{a} + \alpha a\|_{\mathfrak{s}-1} \\
&\leq \frac{\alpha}{4}|\mathfrak{a}|^2_{\mathcal{H}^{\mathfrak{s}-1}} + C_4\left(\mathcal{E}^3(\mathfrak{u}) + C_4\right).
\end{aligned}$$

Combining this with (5.8), we infer

$$\frac{\mathrm{d}}{\mathrm{d}t}|\mathfrak{a}|^2_{\mathcal{H}^{\mathfrak{s}-1}} \leq -\frac{5\alpha}{4}|\mathfrak{a}|^2_{\mathcal{H}^{\mathfrak{s}-1}} + C_4\left(\mathcal{E}^3(\mathfrak{u}) + C_4\right).$$

It follows that[8]

$$\frac{\mathrm{d}}{\mathrm{d}t}|\mathfrak{a}|^{2m}_{\mathcal{H}^{\mathfrak{s}-1}} = m|\mathfrak{a}|^{2m-2}_{\mathcal{H}^{\mathfrak{s}-1}} \frac{\mathrm{d}}{\mathrm{d}t}|\mathfrak{a}|^2_{\mathcal{H}^{\mathfrak{s}-1}} \leq -\alpha m|\mathfrak{a}|^{2m}_{\mathcal{H}^{\mathfrak{s}-1}} + C_5\left(\mathcal{E}^{3m}(\mathfrak{u}) + C_5\right),$$

where we used the Young inequality. Taking the mean value in this inequality and applying the comparison principle, we derive

$$\mathbb{E}|\mathfrak{a}(t)|^{2m}_{\mathcal{H}^{\mathfrak{s}-1}} \leq e^{-\alpha m t}|\mathfrak{a}(0)|^{2m}_{\mathcal{H}^{\mathfrak{s}-1}} + C_6 \int_0^t e^{\alpha m(\tau - t)}\left(\mathbb{E}\mathcal{E}^{3m}(\mathfrak{u}(\tau)) + C_6\right) \mathrm{d}\tau.$$

Combining this with (5.22) and (5.23), we get

$$\mathbb{E}|\mathfrak{z}(t)|^{2m}_{\mathcal{H}^{\mathfrak{s}}} \leq C_7\left(e^{-\alpha m t}\mathcal{E}^{3m}(\mathfrak{v}) + \int_0^t e^{\alpha m(\tau - t)}\mathbb{E}\mathcal{E}^{3m}(\mathfrak{u}(\tau))\,\mathrm{d}\tau + C_7\right).$$

[8] All the constants $C_i, i \geq 5$ depend on m.

Using the Itô formula, it is not difficult to show (cf. Proposition 3.1 in [Mar14]) that

$$\mathbb{E}\mathcal{E}^k(\mathfrak{u}(t)) \le \exp(-\alpha k t)\mathcal{E}^k(\mathfrak{v}) + C(k, \|h\|, \mathfrak{B}) \quad \text{for any } k \ge 1. \tag{5.27}$$

It follows from the last two inequalities that

$$\mathbb{E}|\mathfrak{z}(t)|^{2m}_{\mathcal{H}^s} \le C_8(e^{-\alpha m t}\mathcal{E}^{3m}(\mathfrak{v}) + C_8).$$

Combining this with the inequality

$$(A+B)^{2m} \le 2A^{2m} + C_9 B^{2m} \quad \text{for any } A, B \ge 0.$$

and (5.26), we infer

$$\begin{aligned}\mathbb{E}|\mathfrak{u}(t)|^{2m}_{\mathcal{H}^s} &\le \mathbb{E}(|\tilde{\mathfrak{u}}(t)|_{\mathcal{H}^s} + |\mathfrak{z}(t)|_{\mathcal{H}^s})^{2m} \le 2\mathbb{E}|\tilde{\mathfrak{u}}(t)|^{2m}_{\mathcal{H}^s} + C_9\mathbb{E}|\mathfrak{z}(t)|^{2m}_{\mathcal{H}^s} \\ &\le 2e^{-\alpha m t}|\mathfrak{v}|^{2m}_{\mathcal{H}^s} + C_{10}(e^{-\alpha m t}\mathcal{E}^{3m}(\mathfrak{v}) + C_{10}).\end{aligned}$$

So that we have

$$\begin{aligned}\mathbb{E}\mathfrak{w}_m(\mathfrak{u}(t)) &\le 2e^{-\alpha m t}|\mathfrak{v}|^{2m}_{\mathcal{H}^s} + C_{10}(e^{-\alpha m t}\mathcal{E}^{3m}(\mathfrak{v}) + C_{10}) + \mathbb{E}\mathcal{E}^{4m}(\mathfrak{u}(t)) \\ &\le 2e^{-\alpha m t}\left(|\mathfrak{v}|^{2m}_{\mathcal{H}^s} + \mathcal{E}^{4m}(\mathfrak{v})\right) + C_{11} = 2e^{-\alpha m t}\mathfrak{w}_m(\mathfrak{v}) + C_{11},\end{aligned}$$

where we used the Young inequality together with (5.27).

Step 2: Proof of (5.25). It was shown in Section 3.2 of [Mar14], that for any $\varkappa \le (2\alpha)^{-1}\mathfrak{B}$, we have

$$\begin{aligned}\mathbb{E}_{\mathfrak{v}} \exp[\varkappa\mathcal{E}(\mathfrak{u}(t))] &\le \exp(\varkappa\mathcal{E}(\mathfrak{v})) \\ &\quad + \varkappa\int_0^t \mathbb{E}_{\mathfrak{v}} \exp[\varkappa\mathcal{E}(\mathfrak{u}(\tau))](-\alpha\mathcal{E}(\mathfrak{u}(\tau)) + C(\mathfrak{B}, \|h\|))\,\mathrm{d}\tau.\end{aligned}$$

Using this with inequality

$$e^r(-\alpha r + C) \le -\alpha m e^r + C_{12} \quad \text{for any } r \ge -C$$

and applying the Gronwall lemma, we see that

$$\mathbb{E}_{\mathfrak{v}} \exp[\varkappa\mathcal{E}(\mathfrak{u}(t))] \le e^{-\alpha m t}\exp(\varkappa\mathcal{E}(\mathfrak{v})) + C_{13}.$$

Finally, combining this inequality with (5.24), we arrive at (5.25). □

6 Proof of Theorem 1.3

The results of Sections 3-5 imply that the growth conditions, the uniform irreducibility and uniform Feller properties in Theorem 7.4 are satisfied if we take

$$\begin{gathered}X = \mathcal{H},\ \ X_R = B_{\mathcal{H}^s}(R),\ \ P^V_t(\mathfrak{u}, \Gamma) = (\mathfrak{P}^{V*}_t\delta_{\mathfrak{u}})(\Gamma), \\ \mathfrak{w}(\mathfrak{u}) = 1 + |\mathfrak{u}|^2_{\mathcal{H}^s} + \mathcal{E}^4(\mathfrak{u}),\ \ \mathcal{C} = \mathcal{U},\ \ V \in \mathcal{U}_\delta\end{gathered}$$

for sufficiently large integer $R_0 \geq 1$, small $\delta > 0$, and any $\mathfrak{s} \in (0, 1 - \rho/2)$. Let us show that the time-continuity property is also verified.

Step 1: Time-continuity property. We need to show that the function $t \mapsto \mathfrak{P}_t^V g(\mathfrak{u})$ is continuous from $\mathbb{R}_+$ to $\mathbb{R}$ for any $g \in C_{\mathfrak{w}}(\mathcal{H}^{\mathfrak{s}})$ and $\mathfrak{u} \in \mathcal{H}^{\mathfrak{s}}$ (recall that $X_\infty = \mathcal{H}^{\mathfrak{s}}$). For any $T, t \geq 0$ and $\mathfrak{u} \in \mathcal{H}^{\mathfrak{s}}$, we have

$$\begin{aligned}\mathfrak{P}_T^V g(\mathfrak{u}) - \mathfrak{P}_t^V g(\mathfrak{u}) &= \mathbb{E}_{\mathfrak{u}} \left\{ [\Xi_V(T) - \Xi_V(t)]\, g(\mathfrak{u}_t) \right\} + \mathbb{E}_{\mathfrak{u}} \left\{ [g(\mathfrak{u}_T) - g(\mathfrak{u}_t)]\, \Xi_V(T) \right\} \\ &=: S_1 + S_2, \end{aligned} \tag{6.1}$$

where Ξ_V is defined by (3.4). As V is bounded and $g \in C_{\mathfrak{w}}(\mathcal{H}^{\mathfrak{s}})$, we see that

$$\begin{aligned}|S_1| &\leq \mathbb{E}_{\mathfrak{u}} \left\{ \left| \exp\left(\int_t^T V(\mathfrak{u}_\tau)\, \mathrm{d}\tau \right) - 1 \right| \Xi_V(t) |g(\mathfrak{u}_t)| \right\} \\ &\leq C_1 \left(e^{|T-t| \|V\|_\infty} - 1 \right) e^{T\|V\|_\infty} \mathbb{E}_{\mathfrak{u}} \mathfrak{w}(\mathfrak{u}_t). \end{aligned}$$

Combining this with (5.24), we get $S_1 \to 0$ as $t \to T$. To estimate S_2, let us take any $R > 0$ and write

$$\begin{aligned} e^{-T\|V\|_\infty} |S_2| &\leq \mathbb{E}_{\mathfrak{u}} \left| g(\mathfrak{u}_T) - g(\mathfrak{u}_t) \right| \\ &= \mathbb{E}_{\mathfrak{u}} \left\{ \mathbb{I}_{G_R^c} \left| g(\mathfrak{u}_T) - g(\mathfrak{u}_t) \right| \right\} + \mathbb{E}_{\mathfrak{u}} \left\{ \mathbb{I}_{G_R} \left| g(\mathfrak{u}_T) - g(\mathfrak{u}_t) \right| \right\} \\ &=: S_3 + S_4, \end{aligned}$$

where $G_R := \{u_t, u_T \in X_R\}$. From the Chebyshev inequality, the fact that $g \in C_{\mathfrak{w}}(\mathcal{H}^{\mathfrak{s}})$, and inequality (5.24) we derive

$$\begin{aligned} S_3 &\leq C_1 \mathbb{E}_{\mathfrak{u}} \left\{ \mathbb{I}_{G_R^c} (\mathfrak{w}(\mathfrak{u}_T) + \mathfrak{w}(\mathfrak{u}_t)) \right\} \\ &\leq C_1 R^{-2} \mathbb{E}_{\mathfrak{u}} \left\{ \mathfrak{w}^2(\mathfrak{u}_T) + \mathfrak{w}^2(\mathfrak{u}_t) \right\} \leq C_2 R^{-2} \mathfrak{w}^2(\mathfrak{u}). \end{aligned}$$

On the other hand, by the Lebesgue theorem on dominated convergence, for any $R > 0$, we have $S_4 \to 0$ as $t \to T$. Choosing $R > 0$ sufficiently large and t sufficiently close to T, we see that $S_3 + S_4$ can be made arbitrarily small. This shows that $S_2 \to 0$ as $t \to T$ and proves the time-continuity property.

Step 2: Application of Theorem 7.4. We conclude from Theorem 7.4 that there is an eigenvector $\mu_V \in \mathcal{P}(\mathcal{H})$ for the semigroup $\mathfrak{P}_t^{V*}$ corresponding to some positive eigenvalue λ_V, i.e., $\mathfrak{P}_t^{V*} \mu_V = \lambda_V^t \mu_V$ for any $t > 0$. Moreover, the semigroup $\mathfrak{P}_t^V$ has an eigenvector $h_V \in C_{\mathfrak{w}}(\mathcal{H}^{\mathfrak{s}}) \cap C_+(\mathcal{H}^{\mathfrak{s}})$ corresponding to λ_V such that $\langle h_V, \mu_V \rangle = 1$. The uniqueness of μ_V and h_V follows immediately from (1.13) and (1.14). The uniqueness of μ_V implies that it does not depend on m and (1.12) holds for any $m \geq 1$. It remains to prove limits (1.13) and (1.14).

Step 3: Proof of (1.13). By (7.16), we have (1.13) for any $\psi \in \mathcal{U}$. To establish the limit for any $\psi \in C_{\mathfrak{w}}(\mathcal{H}^{\mathfrak{s}})$, we apply an approximation argument similar to the one used in Step 4 of the proof of Theorem 5.5 in [JNPS14]. Let

us take a sequence $\psi_n \in \mathcal{U}$ such that $\|\psi_n\|_\infty \le \|\psi\|_\infty$ and $\psi_n \to \psi$ as $n \to \infty$, uniformly on bounded subsets of $\mathcal{H}^{\mathfrak{s}}$. If we define

$$\Delta_t(g) = \sup_{\mathfrak{u}\in X_R} \left|\lambda_V^{-t}\mathfrak{P}_t^V g(\mathfrak{u}) - \langle g, \mu_V\rangle h_V(\mathfrak{u})\right|, \quad \|g\|_R = \sup_{\mathfrak{u}\in X_R} |g(\mathfrak{u})|,$$

then

$$\Delta_t(\psi) \le \Delta_t(\psi_n) + \|h_V\|_R\, |\langle \psi - \psi_n, \mu_V\rangle| + \lambda_V^{-t}\|\mathfrak{P}_t^V(\psi - \psi_n)\|_R$$

for any $t \ge 0$ and $n \ge 1$. In view of (1.13) for ψ_n and the Lebesgue theorem on dominated convergence,

$$\Delta_t(\psi_n) \to 0 \quad \text{as } t \to \infty \text{ for any fixed } n \ge 1,$$
$$|\langle \psi - \psi_n, \mu_V\rangle| \to 0 \quad \text{as } n \to \infty.$$

Thus, it suffices to show that

$$\sup_{t\ge 0} \lambda_V^{-t}\|\mathfrak{P}_t^V(\psi - \psi_n)\|_R \to 0 \quad \text{as } n \to \infty. \tag{6.2}$$

To this end, for any $\rho > 0$, we write

$$\|\mathfrak{P}_t^V(\psi - \psi_n)\|_R \le J_1(t,n,\rho) + J_2(t,n,\rho),$$

where

$$J_1(t,n,\rho) = \|\mathfrak{P}_t^V\big((\psi - \psi_n)\mathbb{I}_{X_\rho}\big)\big\|_R, \quad J_2(t,n,\rho) = \|\mathfrak{P}_t^V\big((\psi - \psi_n)\mathbb{I}_{X_\rho^c}\big)\|_R.$$

Since $\psi_n \to \psi$ uniformly on X_ρ, we have

$$J_1(t,n,\rho) \le \varepsilon(n,\rho)\, \|\mathfrak{P}_t^V \mathbf{1}\|_R,$$

where $\varepsilon(n,\rho) \to 0$ as $n \to \infty$. Using convergence (1.13) for $\psi = \mathbf{1}$, we see that

$$\lambda_V^{-t}\|\mathfrak{P}_t^V \mathbf{1}\|_R \le C_3(R) \quad \text{for all } t \ge 0. \tag{6.3}$$

Hence,

$$\sup_{t\ge 0} \lambda_V^{-t} J_1(t,n,\rho) \le C_3(R)\, \varepsilon(n,\rho) \to 0 \quad \text{as } n \to \infty.$$

We use (3.1) and (6.3), to estimate J_2:

$$\begin{aligned}\lambda_V^{-t} J_2(t,n,\rho) &\le 2\|\psi\|_\infty \rho^{-2}\lambda_V^{-t}\|\mathfrak{P}_t^V \mathfrak{w}\|_R \le C_4(R)\|\psi\|_\infty\rho^{-2}\lambda_V^{-t}\|\mathfrak{P}_t^V\mathbf{1}\|_{R_0}\\ &\le C_4(R)\|\psi\|_\infty\rho^{-2}C_3(R_0).\end{aligned}$$

Taking first ρ and then n sufficiently large, we see that $\sup_{t\ge 0}\lambda_V^{-t}\|\mathfrak{P}_t^V(\psi - \psi_n)\|_R$ can be made arbitrarily small. This proves (6.2) and completes the proof of (1.13).

Step 4: Proof of (1.14). Let us show that

$$\lambda_V^{-t}\langle \mathfrak{P}_t^V \psi, \nu\rangle \to \langle \psi, \mu_V\rangle\langle h_V, \nu\rangle \quad \text{as } t \to \infty$$

for any $\psi \in C_b(\mathcal{H})$. In view of (1.13), it suffices to show that

$$\sup_{t\geq 0}\left\{\int_H \mathbb{I}_{X_R^c}\big|\lambda_V^{-t}\mathfrak{P}_t^V\psi(\mathfrak{u}) - \langle\psi,\mu_V\rangle h_V(\mathfrak{u})\big|\,\nu(\mathrm{d}\mathfrak{u})\right\} \to 0 \ \text{ as } R\to\infty. \tag{6.4}$$

From (3.2) and (6.3) we derive that

$$\|\mathfrak{P}_t^V\psi\|_{L^\infty_{\mathfrak{w}}} \leq \|\psi\|_\infty \|\mathfrak{P}_t^V\mathbf{1}\|_{L^\infty_{\mathfrak{w}}} \leq C_5\|\mathfrak{P}_t^V\mathbf{1}\|_{R_0} \leq C_6(R_0)\lambda_V^t, \quad t\geq 0,$$

hence

$$\big|\lambda_V^{-k}\mathfrak{P}_t^V\psi(\mathfrak{u})\big| \leq C_6(R_0)\mathfrak{w}(\mathfrak{u}), \quad \mathfrak{u}\in\mathcal{H}^{\mathfrak{s}}, \quad t\geq 0.$$

Since $h_V \in C_{\mathfrak{w}}(\mathcal{H}^{\mathfrak{s}})$ and

$$\int_{\mathcal{H}} \mathbb{I}_{X_R^c}(\mathfrak{u})\,\mathfrak{w}(\mathfrak{u})\,\nu(\mathrm{d}\mathfrak{u}) \to 0 \quad \text{as } R\to\infty,$$

we obtain (6.4). This completes the proof of Theorem 1.3.

7 Appendix

7.1 Local version of Kifer's theorem

In [Kif90], Kifer established a sufficient condition for the validity of the LDP for a family of random probability measures on a compact metric space. This result was extended by Jakšić et al. [JNPS14] to the case of a general Polish space. In this section, we obtain a local version of these results. Roughly speaking, we assume the existence of a pressure function (i.e., limit (7.3)) and the uniqueness of the equilibrium state for functions V in a set $\mathcal{V}$, which is not necessarily dense in the space of bounded continuous functions. We prove the LDP with a lower bound in which the infimum of the rate function is taken over a subset of the equilibrium states. To give the exact formulation of the result, we first introduce some notation and definitions. Assume that X is a Polish space, and ζ_θ is a random probability measure on X defined on some probability space $(\Omega_\theta, \mathcal{F}_\theta, \mathbb{P}_\theta)$, where the index θ belongs to some directed set[9] Θ. Let $r : \Theta \to \mathbb{R}$ be a positive function such that $\lim_{\theta\in\Theta} r_\theta = +\infty$. For any $V \in C_b(X)$, let us set

$$Q(V) := \limsup_{\theta\in\Theta} \frac{1}{r_\theta}\log \mathbb{E}_\theta \exp\big(r_\theta\langle V,\zeta_\theta\rangle\big), \tag{7.1}$$

where $\mathbb{E}_\theta$ is the expectation with respect to $\mathbb{P}_\theta$. The function $Q : C_b(X) \to \mathbb{R}$ is convex, $Q(V) \geq 0$ for any $V \in C_+(X)$, and $Q(C) = C$ for any $C \in \mathbb{R}$. Moreover, Q is 1-Lipschitz. Indeed, for any $V_1, V_2 \in C_b(X)$ and $\theta \in \Theta$, we have

$$\frac{1}{r_\theta}\log \mathbb{E}_\theta \exp\big(r_\theta\langle V_1,\zeta_\theta\rangle\big) \leq \|V_1 - V_2\|_\infty + \frac{1}{r_\theta}\log \mathbb{E}_\theta \exp\big(r_\theta\langle V_2,\zeta_\theta\rangle\big),$$

[9] i.e., a partially ordered set whose every finite subset has an upper bound.

which implies that

$$Q(V_1) \le \|V_1 - V_2\|_\infty + Q(V_2).$$

By symmetry we get

$$|Q(V_1) - Q(V_2)| \le \|V_1 - V_2\|_\infty.$$

The *Legendre transform* of Q is given by

$$I(\sigma) = \begin{cases} \sup_{V \in C_b(X)} \big(\langle V, \sigma \rangle - Q(V) \big) & \text{for } \sigma \in \mathcal{P}(X), \\ +\infty & \text{for } \sigma \in \mathcal{M}(X) \setminus \mathcal{P}(X) \end{cases} \tag{7.2}$$

(see Lemma 2.2 in [BD99]). Then I is convex and lower semicontinuous function, and

$$Q(V) = \sup_{\sigma \in \mathcal{P}(X)} \big(\langle V, \sigma \rangle - I(\sigma) \big).$$

A measure $\sigma_V \in \mathcal{P}(X)$ is said to be an *equilibrium state* for V if

$$Q(V) = \langle V, \sigma_V \rangle - I(\sigma_V).$$

We shall denote by $\mathcal{V}$ the set of functions $V \in C_b(X)$ admitting a unique equilibrium state σ_V and for which the following limit exists

$$Q(V) = \lim_{\theta \in \Theta} \frac{1}{r_\theta} \log \mathbb{E}_\theta \exp\big(r_\theta \langle V, \zeta_\theta \rangle \big). \tag{7.3}$$

We have the following version of Theorem 2.1 in [Kif90] and Theorem 3.3 in [JNPS14].

Theorem 7.1. *Suppose that there is a function $\Phi : X \to [0, +\infty]$ whose level sets $\{u \in X : \Phi(u) \le a\}$ are compact for all $a \ge 0$ and*

$$\mathbb{E}_\theta \exp\big(r_\theta \langle \Phi, \zeta_\theta \rangle \big) \le C e^{c r_\theta} \quad \text{for } \theta \in \Theta, \tag{7.4}$$

for some positive constants C and c. Then I defined by (7.2) is a good rate function, for any closed set $F \subset \mathcal{P}(X)$,

$$\limsup_{\theta \in \Theta} \frac{1}{r_\theta} \log \mathbb{P}_\theta\{\zeta_\theta \in F\} \le -I(F), \tag{7.5}$$

and for any open set $G \subset \mathcal{P}(X)$,

$$\liminf_{\theta \in \Theta} \frac{1}{r_\theta} \log \mathbb{P}_\theta\{\zeta_\theta \in G\} \ge -I(\mathcal{W} \cap G), \tag{7.6}$$

where $\mathcal{W} := \{\sigma_V : V \in \mathcal{V}\}$ and $I(\Gamma) := \inf_{\sigma \in \Gamma} I(\sigma)$, $\Gamma \subset \mathcal{P}(X)$.

Proof. The fact that I is a good rate function is shown in Step 1 of the proof of Theorem 3.3 in [JNPS14]. In Step 2 of the same proof, the upper bound (7.5) is established, under the condition that the limit $Q(V)$ in (7.3) exists for any $V \in C_b(X)$. The latter condition can be removed, using literally the same proof, if one defines $Q(V)$ by (7.1) for any $V \in C_b(X)$ (see Theorem 2.1 in [dA85]).

To prove the lower bound, following the ideas of [Kif90], for any integer $n \geq 1$ and any functions $V_1, \ldots, V_n \in C_b(X)$, we define an auxiliary family of finite-dimensional random variables $\zeta_\theta^n := f_n(\zeta_\theta)$, where $f_n : \mathcal{P}(X) \to \mathbb{R}^n$ is given by

$$f_n(\mu) := \big(\langle V_1, \mu\rangle, \ldots, \langle V_n, \mu\rangle\big).$$

Let us set

$$\mathcal{W}_n := \{\sigma_V : V \in \mathcal{V} \cap \operatorname{span}\{V_1, \ldots, V_n\}\}.$$

The following result is a local version of Lemma 2.1 in [Kif90] and Proposition 3.4 in [JNPS14]; its proof is sketched at the end of this section.

Proposition 7.2. *Assume that the hypotheses of Theorem 7.1 are satisfied and set* $J_n(\Gamma) = \inf_{\sigma \in f_n^{-1}(\Gamma)} I(\sigma), \Gamma \subset \mathbb{R}^n$. *Then for any closed set* $M \subset \mathbb{R}^n$ *and open set* $U \subset \mathbb{R}^n$, *we have*

$$\limsup_{\theta \in \Theta} \frac{1}{r_\theta} \log \mathbb{P}\{\zeta_\theta^n \in M\} \leq -J_n(M), \tag{7.7}$$

$$\liminf_{\theta \in \Theta} \frac{1}{r_\theta} \log \mathbb{P}\{\zeta_\theta^n \in U\} \geq -J_n(f_n(\mathcal{W}_n) \cap U). \tag{7.8}$$

To derive (7.6) from Proposition 7.2, we follow the arguments of Step 4 of the proof of Theorem 3.3 in [JNPS14]. The case $I(\mathcal{W} \cap G) = +\infty$ is trivial, so we assume that $I(\mathcal{W} \cap G) < +\infty$. Then for any $\varepsilon > 0$, there is $\nu_\varepsilon \in \mathcal{W} \cap G$ such that

$$I(\nu_\varepsilon) \leq I(\mathcal{W} \cap G) + \varepsilon, \tag{7.9}$$

and there is a function $V_1 \in \mathcal{V}$ such that $\nu_\varepsilon = \sigma_{V_1}$. By Lemma 3.2 in [JNPS14], the family $\{\zeta_\theta\}$ is exponentially tight, hence there is a compact set $\mathcal{K} \subset \mathcal{P}(X)$ such that $\nu_\varepsilon \in \mathcal{K}$ and

$$\limsup_{\theta \in \Theta} \frac{1}{r_\theta} \log \mathbb{P}\{\zeta_\theta \in \mathcal{K}^c\} \leq -(I(\mathcal{W} \cap G) + 1 + \varepsilon). \tag{7.10}$$

We choose functions $V_k \in C_b(X), k \geq 2$, $\|V_k\|_\infty = 1$ such that

$$d(\mu, \nu) := \sum_{k=1}^{\infty} 2^{-k} |\langle V_k, \mu\rangle - \langle V_k, \nu\rangle|$$

defines a metric on $\mathcal{K}$ compatible with the weak topology. As G is open, there are $\delta > 0$ and $n \geq 1$ such that if

$$\sum_{k=1}^{n} 2^{-k} |\langle V_k, \nu\rangle - \langle V_k, \nu_\varepsilon\rangle| < \delta$$

for some $\nu \in \mathcal{K}$, then $\nu \in G$. Let $x_\varepsilon := f_n(\nu_\varepsilon)$, and denote by $\mathring{B}_{\mathbb{R}^n}(x_\varepsilon, \delta)$ the open ball in $\mathbb{R}^n$ of radius $\delta > 0$ centered at x_ε, with respect to the norm

$$\|x\|_n := \sum_{k=1}^{n} 2^{-k}|x_k|, \quad x = (x_1, \dots, x_n).$$

Then we have $f_n^{-1}\big(\mathring{B}_{\mathbb{R}^n}(x_\varepsilon, \delta)\big) \cap \mathcal{K} \subset G$, hence

$$\begin{aligned}\mathbb{P}\{\zeta_\theta \in G\} &\geq \mathbb{P}\{\zeta_\theta \in G \cap \mathcal{K}\} \geq \mathbb{P}\big\{\zeta_\theta \in f_n^{-1}\big(\mathring{B}_{\mathbb{R}^n}(x_\varepsilon, \delta)\big) \cap \mathcal{K}\big\} \\ &= \mathbb{P}\{\zeta_\theta^n \in \mathring{B}_{\mathbb{R}^n}(x_\varepsilon, \delta)\} - \mathbb{P}\{\zeta_\theta \in \mathcal{K}^c\}.\end{aligned}$$

Using the inequality

$$\log(u - v) \geq \log u - \log 2, \quad 0 < v \leq u/2$$

and inequalities (7.8)-(7.10), we obtain

$$\begin{aligned}\liminf_{\theta \in \Theta} \frac{1}{r_\theta} \log \mathbb{P}\{\zeta_\theta \in G\} &\geq \liminf_{\theta \in \Theta} \frac{1}{r_\theta} \big(\log \mathbb{P}\{\zeta_\theta^n \in \mathring{B}_{\mathbb{R}^n}(x_\varepsilon, \delta)\} - \log 2\big) \\ &\geq -J_n(f_n(\mathcal{W}_n) \cap \mathring{B}_{\mathbb{R}^n}(x_\varepsilon, \delta)) \geq -I_n(x_\varepsilon) \\ &\geq -I(\nu_\varepsilon) \geq -I(\mathcal{W} \cap G) - \varepsilon,\end{aligned}$$

which proves (7.6). □

Sketch of the proof of Proposition 7.2. Inequality (7.7) follows from (7.5). To show (7.8), for any $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{R}^n$ and $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$, we set $V_\beta := \sum_{j=1}^n \beta_j V_j$, $Q_n(\beta) := Q(V_\beta)$, and $I_n(\alpha) := \inf_{\sigma \in f_n^{-1}(\alpha)} I(\sigma)$. One can verify that

$$\begin{aligned}Q_n(\beta) &= \sup_{\alpha \in \mathbb{R}^n} \Big(\sum_{j=1}^{n} \beta_j \alpha_j - I_n(\alpha)\Big), \\ J_n(U) &= \inf_{\alpha \in U} I_n(\alpha).\end{aligned}$$

Assume that $J_n(f_n(\mathcal{W}_n) \cap U) < +\infty$, and for any $\varepsilon > 0$, choose $\alpha_\varepsilon \in f_n(\mathcal{W}_n) \cap U$ such that

$$I_n(\alpha_\varepsilon) < J_n(f_n(\mathcal{W}_n) \cap U) + \varepsilon.$$

Then $\alpha_\varepsilon = f_n(\sigma_{V_{\beta_\varepsilon}})$ for some $\beta_\varepsilon \in \mathbb{R}^n$ such that $V_{\beta_\varepsilon} \in \mathcal{V}$. It is easy to verify that the following equality holds

$$Q_n(\beta_\varepsilon) = \sum_{j=1}^{n} \beta_{\varepsilon j} \alpha_{\varepsilon j} - I_n(\alpha_\varepsilon).$$

Literally repeating the proof of Proposition 3.4 in [JNPS14] (starting from equality (3.16)) and using the uniqueness of the equilibrium state for $V = V_{\beta_\varepsilon}$ and the existence of limit (7.3), one obtains

$$-J_n(f_n(\mathcal{W}_n) \cap U) - \varepsilon \leq -I_n(\alpha_\varepsilon) \leq \liminf_{\theta \in \Theta} \frac{1}{r_\theta} \log \mathbb{P}\{\zeta_\theta^n \in U\}$$

for any $\varepsilon > 0$. This implies (7.8). □

7.2 Large-time asymptotics for generalised Markov semigroups

In this section, we give a continuous-time version of Theorem 4.1 in [JNPS14] with some modifications, due to the fact that the generalised Markov family associated with the stochastic NLW equation does not have a regularising property. See also [KS01, LS06, JNPS] for some related results.

We start by recalling some terminology from [JNPS14].

Definition 7.3. Let X be a Polish space. We shall say that $\{P_t(\mathfrak{u},\cdot), \mathfrak{u} \in X, t \geq 0\}$ is a *generalised Markov family of transition kernels* if the following two properties are satisfied.

Feller property. For any $t \geq 0$, the function $\mathfrak{u} \mapsto P_t(\mathfrak{u},\cdot)$ is continuous from X to $\mathcal{M}_+(X)$ and does not vanish.

Kolmogorov–Chapman relation. For any $t, s \geq 0, \mathfrak{u} \in X$, and Borel set $\Gamma \subset X$, the following relation holds

$$P_{t+s}(\mathfrak{u},\Gamma) = \int_X P_s(\mathfrak{v},\Gamma) P_t(\mathfrak{u},\mathrm{d}\mathfrak{v}).$$

To any such family we associate two semigroups by the following relations:

$$\mathfrak{P}_t : C_b(X) \to C_b(X), \qquad \mathfrak{P}_t\psi(\mathfrak{u}) = \int_X \psi(\mathfrak{v}) P_t(\mathfrak{u},\mathrm{d}\mathfrak{v}),$$
$$\mathfrak{P}_t^* : \mathcal{M}_+(X) \to \mathcal{M}_+(X), \qquad \mathfrak{P}_t^*\mu(\Gamma) = \int_X P_t(\mathfrak{v},\Gamma)\mu(\mathrm{d}\mathfrak{v}), \quad t \geq 0.$$

For a measurable function $\mathfrak{w} : X \to [1,+\infty]$ and a family $\mathcal{C} \subset C_b(X)$, we denote by $\mathcal{C}^{\mathfrak{w}}$ the set of functions $\psi \in L^\infty_{\mathfrak{w}}(X)$ that can be approximated with respect to $\|\cdot\|_{L^\infty_{\mathfrak{w}}}$ by finite linear combinations of functions from $\mathcal{C}$. We shall say that a family $\mathcal{C} \subset C_b(X)$ is *determining* if for any $\mu, \nu \in \mathcal{M}_+(X)$ satisfying $\langle \psi, \mu\rangle = \langle \psi, \nu\rangle$ for all $\psi \in \mathcal{C}$, we have $\mu = \nu$. Finally, a family of functions $\psi_t : X \to \mathbb{R}$ is *uniformly equicontinuous* on a subset $K \subset X$ if for any $\varepsilon > 0$ there is $\delta > 0$ such that $|\psi_t(\mathfrak{u}) - \psi_t(\mathfrak{v})| < \varepsilon$ for any $\mathfrak{u} \in K$, $\mathfrak{v} \in B_X(\mathfrak{u},\delta) \cap K$, and $t \geq 1$. We have the following version of Theorem 4.1 in [JNPS14].

Theorem 7.4. *Let $\{P_t(\mathfrak{u},\cdot), \mathfrak{u} \in X, t \geq 0\}$ be a generalised Markov family of transition kernels satisfying the following four properties.*

Growth conditions. *There is an increasing sequence $\{X_R\}_{R=1}^\infty$ of compact subsets of X such that $X_\infty := \cup_{R=1}^\infty X_R$ is dense in X. The measures $P_t(\mathfrak{u},\cdot)$ are concentrated on X_∞ for any $\mathfrak{u} \in X_\infty$ and $t > 0$, and there is a*

measurable function $\mathfrak{w} : X \to [1, +\infty]$ *and an integer* $R_0 \geq 1$ *such that*[10]

$$\sup_{t\geq 0} \frac{\|\mathfrak{P}_t \mathfrak{w}\|_{L^\infty_{\mathfrak{w}}}}{\|\mathfrak{P}_t \mathbf{1}\|_{R_0}} < \infty, \tag{7.11}$$

$$\sup_{t\in[0,1]} \|\mathfrak{P}_t \mathbf{1}\|_\infty < \infty, \tag{7.12}$$

where $\|\cdot\|_R$ *and* $\|\cdot\|_\infty$ *denote the* L^∞ *norm on* X_R *and* X*, respectively, and we set* $\infty/\infty = 0$*.*

Time-continuity. *For any function* $g \in L^\infty_{\mathfrak{w}}(X_\infty)$ *whose restriction to* X_R *belongs to* $C(X_R)$ *and any* $\mathfrak{u} \in X_\infty$*, the function* $t \mapsto \mathfrak{P}_t g(\mathfrak{u})$ *is continuous from* $\mathbb{R}_+$ *to* $\mathbb{R}$*.*

Uniform irreducibility. *For sufficiently large* $\rho \geq 1$*, any* $R \geq 1$ *and* $r > 0$*, there are positive numbers* $l = l(\rho, r, R)$ *and* $p = p(\rho, r)$ *such that*

$$P_l(\mathfrak{u}, B_X(\hat{\mathfrak{u}}, r)) \geq p \quad \text{for all } \mathfrak{u} \in X_R,\ \hat{\mathfrak{u}} \in X_\rho.$$

Uniform Feller property. *There is a number* $R_0 \geq 1$ *and a determining family* $\mathcal{C} \subset C_b(X)$ *such that* $\mathbf{1} \in \mathcal{C}$ *and the family* $\{\|\mathfrak{P}_t \mathbf{1}\|_R^{-1} \mathfrak{P}_t \psi, t \geq 1\}$ *is uniformly equicontinuous on* X_R *for any* $\psi \in \mathcal{C}$ *and* $R \geq R_0$*.*

Then for any $t > 0$*, there is at most one measure* $\mu_t \in \mathcal{P}_{\mathfrak{w}}(X)$ *such that* $\mu_t(X_\infty) = 1$ *and*

$$\mathfrak{P}_t^* \mu_t = \lambda(t)\mu_t \quad \text{for some } \lambda(t) \in \mathbb{R} \tag{7.13}$$

satisfying the following condition:

$$\|\mathfrak{P}_t \mathfrak{w}\|_R \int_{X\setminus X_R} \mathfrak{w}\, \mathrm{d}\mu_t \to 0 \quad \text{as } R \to \infty. \tag{7.14}$$

Moreover, if such a measure μ_t *exists for all* $t > 0$*, then it is independent of* t *(we set* $\mu := \mu_t$*), the corresponding eigenvalue is of the form* $\lambda(t) = \lambda^t$*,* $\lambda > 0$*,* $\operatorname{supp} \mu = X$*, and there is a non-negative function* $h \in L^\infty_{\mathfrak{w}}(X_\infty)$ *such that* $\langle h, \mu\rangle = 1$*,*

$$(\mathfrak{P}_t h)(\mathfrak{u}) = \lambda^t h(\mathfrak{u}) \quad \text{for } \mathfrak{u} \in X_\infty,\ t > 0, \tag{7.15}$$

the restriction of h *to* X_R *belongs to* $C_+(X_R)$*, and for any* $\psi \in \mathcal{C}^{\mathfrak{w}}$ *and* $R \geq 1$*, we have*

$$\lambda^{-t} \mathfrak{P}_t \psi \to \langle \psi, \mu\rangle h \quad \text{in } C(X_R) \cap L^1(X, \mu) \text{ as } t \to \infty. \tag{7.16}$$

Finally, if a Borel set $B \subset X$ *is such that*

$$\sup_{\mathfrak{u}\in B} \left(\int_{X\setminus X_R} \mathfrak{w}(\mathfrak{v})\, P_s(\mathfrak{u}, \mathrm{d}\mathfrak{v}) \right) \to 0 \quad \text{as } R \to \infty \tag{7.17}$$

for some $s > 0$*, then for any* $\psi \in \mathcal{C}^{\mathfrak{w}}$*, we have*

$$\lambda^{-t} \mathfrak{P}_t \psi \to \langle \psi, \mu\rangle h \quad \text{in } L^\infty(B) \text{ as } t \to \infty. \tag{7.18}$$

[10]The expression $(\mathfrak{P}_t \mathfrak{w})(\mathfrak{u})$ is understood as an integral of a positive function $\mathfrak{w}$ against a positive measure $P_t(\mathfrak{u}, \cdot)$.

Sketch of the proof. Step 1: Existence of eigenvectors μ and h. For any $t > 0$, the conditions of Theorem 4.1 in [JNPS14] are satisfied[11] for the discrete-time semigroup $\{\tilde{\mathfrak{P}}_k = \mathfrak{P}_{tk}, k \geq 1\}$ generated by $\tilde{P} = P_t$. So that theorem implies the existence of at most one measure $\mu_t \in \mathcal{P}_{\mathfrak{w}}(X)$ satisfying $\mu_t(X_\infty) = 1$, (7.13), and (7.14). Moreover, if such a measure μ_t exists for any $t > 0$, it follows from the Kolmogorov–Chapman relation that $\mu_t = \mu_1 =: \mu$ and $\lambda(t) = (\lambda(1))^t =: \lambda^t$ for any t in the set $\mathbb{Q}_+^*$ of positive rational numbers, i.e.,

$$\mathfrak{P}_t^* \mu = \lambda^t \mu \quad \text{for } t \in \mathbb{Q}_+^*. \tag{7.20}$$

Using the time-continuity property and density, we get that (7.20) holds for any $t > 0$. So we have $\mu_t = \mu$ and $\lambda(t) = \lambda^t$ for any $t > 0$, by uniqueness of the eigenvector.

Theorem 4.1 in [JNPS14] also implies that $\operatorname{supp} \mu = X, \lambda > 0$, and there is a non-negative function $h_t \in L^\infty_{\mathfrak{w}}(X_\infty)$ such that $\langle h_t, \mu \rangle = 1$, the restriction of h_t to X_R belongs to $C_+(X_R)$, and

$$(\mathfrak{P}_t h_t)(\mathfrak{u}) = \lambda^t h_t(\mathfrak{u}) \quad \text{for } \mathfrak{u} \in X_\infty, \tag{7.21}$$

$$\lambda^{-tk} \mathfrak{P}_{tk} \psi \to \langle \psi, \mu \rangle h_t \quad \text{in } C(X_R) \cap L^1(X, \mu) \text{ as } k \to \infty \tag{7.22}$$

for any $\psi \in \mathcal{C}^{\mathfrak{w}}, R \geq 1$, and $t > 0$. Taking $\psi = \mathbf{1}$ in (7.22), we see that $h_t = h_1 =: h$ for any $t \in \mathbb{Q}_+^*$. The continuity of the function $t \mapsto \mathfrak{P}_t h(\mathfrak{u})$ and (7.21) imply that $h_t = h$ for any $t > 0$ and

$$\lambda^{-tk} \mathfrak{P}_{tk} \psi \to \langle \psi, \mu \rangle h \quad \text{in } C(X_R) \cap L^1(X, \mu) \text{ as } k \to \infty. \tag{7.23}$$

Step 2: Proof of (7.16). First let us prove (7.16) for any $\psi \in \mathcal{C}$. Replacing $P_t(\mathfrak{u}, \Gamma)$ by $\lambda^{-t} P_t(\mathfrak{u}, \Gamma)$, we may assume that $\lambda = 1$. Taking $\psi = \mathbf{1}$ and $t = 1$ in (7.23), we obtain $\sup_{k \geq 0} \|\mathfrak{P}_k \mathbf{1}\|_R < \infty$. So using (7.12), we get $\sup_{t \geq 0} \|\mathfrak{P}_t \mathbf{1}\|_R < \infty$. This implies that $\{\mathfrak{P}_t \psi, t \geq 1\}$ is uniformly equicontinuous on X_R for any $R \geq R_0$. Setting $g = \psi - \langle \psi, \mu \rangle h$, we need to prove that $\mathfrak{P}_t g \to 0$ in $C(X_R)$ for any $R \geq 1$. Since $\{\mathfrak{P}_t g, t \geq 1\}$ is uniformly equicontinuous on X_R, the required assertion will be established if we prove that

$$|\mathfrak{P}_t g|_\mu := \langle |\mathfrak{P}_t g|, \mu \rangle \to 0 \quad \text{as } t \to \infty. \tag{7.24}$$

[11] Let us note that in Theorem 4.1 in [JNPS14] it is assumed that the measures $P_t(\mathfrak{u}, \cdot)$ are concentrated on X_∞ for any $\mathfrak{u} \in X$. Here this is replaced by the condition that the measures $P_t(\mathfrak{u}, \cdot)$ and μ_t are concentrated on X_∞ for any $u \in X_\infty$. The uniform irreducibility property is slightly different from the one assumed in [JNPS14]. Both modifications are due to the lack of a regularising property for the stochastic NLW equation. These changes do not affect the proof given in [JNPS14], one only needs to replace inequality (4.16) in the proof by the inequality

$$\sup_{k \geq 0} \|\mathfrak{P}_k \psi\|_{L^\infty_{\mathfrak{w}}(X)} \leq M_1 \|\psi\|_{L^\infty_{\mathfrak{w}}(X)} \quad \text{for any } \psi \in L^\infty_{\mathfrak{w}}(X), \tag{7.19}$$

and literally repeat all the arguments. The proof of (7.19) is similar to the one of (4.16). Under these modified conditions, the concept of eigenfunction for $\mathfrak{P}_t$ is understood in a weaker sense; namely, relation (7.15) needs to hold only for $\mathfrak{u} \in X_\infty$.

For any $\varphi \in L^\infty_{\mathfrak{w}}(X)$, we have

$$|\mathfrak{P}_t\varphi|_\mu \le \langle \mathfrak{P}_t|\varphi|, \mu\rangle = \langle |\varphi|, \mu\rangle = |\varphi|_\mu,$$

thus $|\mathfrak{P}_t g|_\mu$ is a non-increasing function in t. By (7.23), we have $|\mathfrak{P}_{tk} g|_\mu \to 0$ as $k \to \infty$. This proves (7.24), hence also (7.16) for any $\psi \in \mathcal{C}$.

An easy approximation argument shows that (7.16) holds for any $\psi \in \mathcal{C}^{\mathfrak{w}}$ (see Step 4 of the proof of Theorem 4.1 in [JNPS14]). Finally, the proof of (7.18) under condition (7.17) is exactly the same as in Step 7 of the proof of the discrete-time case. □

7.3 Proofs of some auxiliary assertions

The Foiaş-Prodi estimate

Here we briefly recall an a priori estimate established in Proposition 4.1 in [Mar14]. Let $\mathfrak{u}_t = [u, \dot u]$ and $\mathfrak{v}_t = [v, \dot v]$ be some flows of the equations

$$\partial_t^2 u + \gamma \partial_t u - \Delta u + f(u) = h(x) + \partial_t \varphi(t,x), \tag{7.25}$$

$$\partial_t^2 v + \gamma \partial_t v - \Delta v + f(v) + \mathsf{P}_N[f(u) - f(v)] = h(x) + \partial_t \varphi(t,x), \tag{7.26}$$

where φ is a function belonging to $L^2_{loc}(\mathbb{R}_+, L^2(D))$. We recall that P_N stands for the orthogonal projection in $L^2(D)$ onto the vector span H_N of the functions $e_1, e_2, \dots, e_N$ and P_N is the projection in $\mathcal{H}$ onto $\mathcal{H}_N := H_N \times H_N$.

Proposition 7.5. *Assume that, for some non-negative numbers s and T, we have $\mathfrak{u}, \mathfrak{v} \in C(s, s+T; \mathcal{H})$. Then*

$$|P_N(\mathfrak{v}_t - \mathfrak{u}_t)|^2_{\mathcal{H}} \le e^{-\alpha(t-s)} |\mathfrak{v}_s - \mathfrak{u}_s|^2_{\mathcal{H}} \quad \text{for } s \le t \le s+T, \tag{7.27}$$

where $\alpha > 0$ is the constant entering (1.4). If we suppose that the inequality holds

$$\int_s^t \|\nabla z\|^2 \, d\tau \le l + K(t-s) \quad \text{for } s \le t \le s+T \tag{7.28}$$

for $z = u$ and $z = v$ and some positive numbers K and l, then, for any $\varepsilon > 0$, there is an integer $N_* = N_*(\varepsilon, K) \ge 1$ such that

$$|\mathfrak{v}_t - \mathfrak{u}_t|^2_{\mathcal{H}} \le e^{-\alpha(t-s)+\varepsilon l} |\mathfrak{v}_s - \mathfrak{u}_s|^2_{\mathcal{H}} \quad \text{for } s \le t \le s+T \tag{7.29}$$

for all $N \ge N_*$ and $s \le t \le s+T$.

Proof. Estimate (7.29) is proved in Proposition 4.1 in [Mar14]. To prove (7.27), let us note that $\mathfrak{z} = [z, \dot z] = P_N(\mathfrak{v} - \mathfrak{u})$ is a solution of the linear equation

$$\partial_t^2 z + \gamma \partial_t z - \Delta z = 0.$$

So we have

$$|P_N(\mathfrak{v}_t - \mathfrak{u}_t)|^2_{\mathcal{H}} = |\mathfrak{z}_t|^2_{\mathcal{H}} \le e^{-\alpha(t-s)} |\mathfrak{z}_s|^2_{\mathcal{H}} \le e^{-\alpha(t-s)} |\mathfrak{v}_s - \mathfrak{u}_s|^2_{\mathcal{H}}.$$

□

Proof of Proposition 1.4

Step 1: Preliminaries. We denote by $\mathfrak{S}_t^{V,F}$ the semigroup defined by (1.15), and write $\mathfrak{S}_t^V$ instead of $\mathfrak{S}_t^{V,0}$ (i.e., $F = \mathbf{0}$). Let $\mathfrak{D}(\mathbb{L}_V)$ be the space of functions $\psi \in C_b(\mathcal{H}^{\mathfrak{s}})$ such that

$$\mathfrak{S}_t^V \psi(\mathfrak{u}) = \psi(\mathfrak{u}) + \int_0^t \mathfrak{S}_\tau^V g(\mathfrak{u})\, \mathrm{d}\tau, \quad t \geq 0,\, \mathfrak{u} \in \mathcal{H}^{\mathfrak{s}} \tag{7.30}$$

for some $g \in C_b(\mathcal{H}^{\mathfrak{s}})$. Then the continuity of the mapping $t \mapsto \mathfrak{S}_t^V g(\mathfrak{u})$ from $\mathbb{R}_+$ to $\mathbb{R}$ implies the following limit

$$g(\mathfrak{u}) = \lim_{t \to 0} \frac{\mathfrak{S}_t^V \psi(\mathfrak{u}) - \psi(\mathfrak{u})}{t},$$

and proves the uniqueness of g in representation (7.30). We set $\mathbb{L}_V \psi := g$. The proof is based on the following two lemmas.

Lemma 7.6. *For any $F \in C_b(\mathcal{H}^{\mathfrak{s}})$, the following properties hold*

i) For any $\psi \in \mathfrak{D}(\mathbb{L}_V)$, we have $\varphi_t := \mathfrak{S}_t^{V,F} \psi \in \mathfrak{D}(\mathbb{L}_V)$ and

$$\partial_t \varphi_t = (\mathbb{L}_V + F)\varphi_t, \quad t > 0.$$

ii) The set $\mathfrak{D}_+ := \{\psi \in \mathfrak{D}(\mathbb{L}_V) : \inf_{\mathfrak{u} \in \mathcal{H}^{\mathfrak{s}}} \psi(\mathfrak{u}) > 0\}$ is determining for $\mathcal{P}(\mathcal{H}^{\mathfrak{s}})$, i.e., if $\langle \psi, \sigma_1 \rangle = \langle \psi, \sigma_2 \rangle$ for some $\sigma_1, \sigma_2 \in \mathcal{P}(\mathcal{H}^{\mathfrak{s}})$ and any $\psi \in \mathfrak{D}_+$, then $\sigma_1 = \sigma_2$.

This lemma is proved at the end of this subsection. The next result is established exactly in the same way as Lemma 5.9 in [JNPS14], by using limit (1.13); we omit its proof.

Lemma 7.7. *The Markov semigroup $\mathfrak{S}_t^V$ has a unique stationary measure, which is given by $\nu_V = h_V \mu_V$.*

Step 2. Let us show that, for any $\psi \in \mathfrak{D}_+$, we have

$$Q_R^V(F_\psi) = 0, \tag{7.31}$$

where $F_\psi := -\mathbb{L}_V \psi / \psi \in C_b(\mathcal{H}^s)$ and $Q_R^V(F_\psi)$ is defined by (1.16). Indeed, by property i) in Lemma 7.6, the function $\varphi_t = \mathfrak{S}_t^{V,F_\psi} \psi$ satisfies

$$\partial_t \varphi_t = \left(\mathbb{L}_V - \frac{\mathbb{L}_V \psi}{\psi} \right) \varphi_t, \quad \varphi_0 = \psi.$$

From the uniqueness of the solution we derive that $\psi = \varphi_t$ for any $t \geq 0$, hence

$$\lim_{t \to +\infty} \frac{1}{t} \log \sup_{\mathfrak{u} \in X_R} \log(\mathfrak{S}_t^{V,F_\psi} \psi)(\mathfrak{u}) = 0. \tag{7.32}$$

As $c \le \psi(\mathfrak{u}) \le C$ for any $\mathfrak{u} \in \mathcal{H}^{\mathfrak{s}}$ and some constants $C, c > 0$, we have

$$Q_R^V(F_\psi) \le \limsup_{t\to+\infty} \frac{1}{t} \log \sup_{\mathfrak{u}\in X_R} \log(\mathfrak{S}_t^{V,F_\psi}\psi)(\mathfrak{u}) \le Q_R^V(F_\psi).$$

Combining this with (7.32), we obtain (7.31).

Step 3. Let us assume [12] that $I_R^V(\sigma) = 0$. Then $\sigma \in \mathcal{P}(\mathcal{H}^{\mathfrak{s}})$ and

$$0 = I_R^V(\sigma) = \sup_{F\in C_b(\mathcal{H}^{\mathfrak{s}})} \big(\langle F, \sigma\rangle - Q_R^V(F)\big).$$

So taking here $F = F_\psi$ for any $\psi \in \mathfrak{D}_+$ and using the result of Step 2, we get

$$0 \le \inf_{\psi\in\mathfrak{D}_+} \int_{\mathcal{H}^{\mathfrak{s}}} \frac{\mathbb{L}_V\psi}{\psi}\sigma(\mathrm{d}\mathfrak{u}).$$

Since $\mathfrak{S}_t^V$ is a Markov semigroup, we have $\mathbb{L}_V\mathbf{1} = \mathbf{0}$. We see that $\theta = 0$ is a local minimum of the function

$$f(\theta) := \int_{\mathcal{H}^{\mathfrak{s}}} \frac{\mathbb{L}_V(1+\theta\psi)}{1+\theta\psi}\sigma(\mathrm{d}\mathfrak{u})$$

for any $\psi \in \mathfrak{D}_+$, so

$$0 = f'(0) = \int_{\mathcal{H}^{\mathfrak{s}}} \mathbb{L}_V\psi\,\sigma(\mathrm{d}\mathfrak{u}).$$

Combining this with property i) in Lemma 7.6, we obtain

$$\int_{\mathcal{H}^{\mathfrak{s}}} \mathfrak{S}_t^V\psi\,\sigma(\mathrm{d}\mathfrak{u}) = \int_{\mathcal{H}^{\mathfrak{s}}} \psi\,\sigma(\mathrm{d}\mathfrak{u}), \quad t > 0.$$

From ii) in Lemma 7.6, we derive that σ is a stationary measure for $\mathfrak{S}_t^V$, and Lemma 7.7 implies that $\sigma = h_V\mu_V$. This completes the proof of Proposition 1.4.

Proof of Lemma 7.6. Step 1: Property i). Let us show that, for any $\psi \in C_b(\mathcal{H}^{\mathfrak{s}})$, the function $\varphi_t = \mathfrak{S}_t^{V,F}\psi$ satisfies the equation in the Duhamel form

$$\varphi_t = \mathfrak{S}_t^V\psi + \int_0^t \mathfrak{S}_{t-s}^V(F\varphi_s)\,\mathrm{d}s. \tag{7.33}$$

Indeed, we have

$$\varphi_t - \mathfrak{S}_t^V\psi = \lambda_V^{-t}h_V^{-1} \times \mathbb{E}_{\mathfrak{u}}\left\{\exp\left(\int_0^t V(\mathfrak{u}_\tau)\,\mathrm{d}\tau\right)\left[\exp\left(\int_0^t F(\mathfrak{u}_\tau)\,\mathrm{d}\tau\right) - 1\right]h_V(\mathfrak{u}_t)\psi(\mathfrak{u}_t)\right\}.$$

[12] As I_R defined by (1.7) is a good rate function, the set of equilibrium measures for V is non-empty. So the set of zeros of I_R^V is also non-empty, by the remark made at the end of Step 2 of the proof of Theorem 1.2.

Integrating by parts and using the the Markov property, we get

$$\begin{aligned}
&\varphi_t - \mathfrak{S}_t^V \psi = \lambda_V^{-t} h_V^{-1} \\
&\quad \times \int_0^t \mathbb{E}_{\mathfrak{u}} \left\{ \exp\left(\int_0^t V(\mathfrak{u}_\tau)\, \mathrm{d}\tau \right) \left[F(\mathfrak{u}_s) \exp\left(\int_s^t F(\mathfrak{u}_\tau)\, \mathrm{d}\tau \right) \right] h_V(\mathfrak{u}_t) \psi(\mathfrak{u}_t) \right\} \mathrm{d}s \\
&= \int_0^t \lambda_V^{-s} h_V^{-1} \mathbb{E}_{\mathfrak{u}} \left\{ \exp\left(\int_0^s V(\mathfrak{u}_\tau)\, \mathrm{d}\tau \right) h_V(\mathfrak{u}_s) F(\mathfrak{u}_s) \varphi_{t-s}(\mathfrak{u}_s) \right\} \mathrm{d}s \\
&= \int_0^t \mathfrak{S}_s^V (F \varphi_{t-s})\, \mathrm{d}s = \int_0^t \mathfrak{S}_{t-s}^V (F \varphi_s)\, \mathrm{d}s.
\end{aligned}$$

This proves (7.33). The identity

$$\mathfrak{S}_t^V(\varphi_r)(\mathfrak{u}) = \varphi_{r+t}(\mathfrak{u}) = \varphi_r(\mathfrak{u}) + \int_0^t \mathfrak{S}_\tau^V (\mathfrak{S}_r^V g)(\mathfrak{u})\, \mathrm{d}\tau, \quad t \geq 0,\, \mathfrak{u} \in \mathcal{H}^{\mathfrak{s}}$$

shows that $\varphi_r \in \mathfrak{D}(\mathbb{L}_V)$ for $\psi \in \mathfrak{D}(\mathbb{L}_V)$ and $r > 0$.

Step 2: Property ii). Assume that, for some $\sigma_1, \sigma_2 \in \mathcal{P}(\mathcal{H}^{\mathfrak{s}})$, we have

$$\langle \psi, \sigma_1 \rangle = \langle \psi, \sigma_2 \rangle, \quad \psi \in \mathfrak{D}_+. \tag{7.34}$$

Let us take any $\psi \in C_b(\mathcal{H}^{\mathfrak{s}})$ such that $c \leq \psi(\mathfrak{u}) \leq C$ for any $\mathfrak{u} \in \mathcal{H}^{\mathfrak{s}}$ and some constants $c, C > 0$. Then $\tilde{\varphi}_r := \frac{1}{r} \int_0^r \mathfrak{S}_\tau^V \psi\, \mathrm{d}\tau$ belongs to $\mathfrak{D}_+$ for any $r > 0$. Indeed, the inequality $c \leq \tilde{\varphi}_r(\mathfrak{u}) \leq C$ follows immediately from the definition of $\mathfrak{S}_r^V$, and the fact that $\tilde{\varphi}_r \in \mathfrak{D}(\mathbb{L}_V)$ follows from the identity

$$\begin{aligned}
\mathfrak{S}_t^V \tilde{\varphi}_r - \tilde{\varphi}_r &= \frac{1}{r} \int_0^r (\mathfrak{S}_{\tau+t}^V \psi - \mathfrak{S}_\tau^V \psi)\, \mathrm{d}\tau = \frac{1}{r} \int_r^{r+t} \mathfrak{S}_\tau^V \psi\, \mathrm{d}\tau - \frac{1}{r} \int_0^t \mathfrak{S}_\tau^V \psi\, \mathrm{d}\tau \\
&= \int_0^t \mathfrak{S}_\tau^V \left(\frac{\mathfrak{S}_r^V \psi - \psi}{r} \right) \mathrm{d}\tau.
\end{aligned}$$

Then, by (7.34), we have

$$\langle \tilde{\varphi}_r, \sigma_1 \rangle = \langle \tilde{\varphi}_r, \sigma_2 \rangle, \quad r > 0. \tag{7.35}$$

Using the continuity of the mapping $r \mapsto \mathfrak{S}_r^V \psi(\mathfrak{u})$ from $\mathbb{R}_+$ to $\mathbb{R}$, we see that $\tilde{\varphi}_r(\mathfrak{u}) \to \psi(\mathfrak{u})$ as $r \to 0$. Passing to the limit in (7.35) and using the Lebesgue theorem on dominated convergence, we obtain $\langle \psi, \sigma_1 \rangle = \langle \psi, \sigma_2 \rangle$. It is easy to verify that the set $\{\psi \in C_b(\mathcal{H}^{\mathfrak{s}}) : \inf_{\mathfrak{u} \in \mathcal{H}^{\mathfrak{s}}} \psi(\mathfrak{u}) > 0\}$ is determining, so we get $\sigma_1 = \sigma_2$.

□

Proof of Lemma 2.1

The function $f : J \to \mathbb{R}$ is convex, so the derivatives $D^{\pm} f(x)$ exist for any $x \in J$. We confine ourselves to the derivation of the first inequality in the lemma. Assume the opposite, and let $x_0 \in J$, $(n_k) \subset \mathbb{N}$, and $\eta > 0$ be such that

$$D^+ f_{n_k}(x_0) \geq D^+ f(x_0) + \eta \quad \text{for } k \geq 1. \tag{7.36}$$

Let us fix $x_1 \in J$, $x_1 > x_0$ such that

$$D^+ f(x_0) \geq \frac{f(x_1) - f(x_0)}{x_1 - x_0} - \eta/4.$$

Since f_{n_k} is a convex function, we have

$$D^+ f_{n_k}(x_0) \leq \frac{f_{n_k}(x_1) - f_{n_k}(x_0)}{x_1 - x_0}.$$

Assume that $k \geq 1$ is so large that we have

$$|f_{n_k}(x_1) - f(x_1)| + |f_{n_k}(x_0) - f(x_0)| \leq \eta(x_1 - x_0)/4.$$

Then, combining last three inequalities, we derive

$$D^+ f_{n_k}(x_0) \leq D^+ f(x_0) + \eta/2,$$

which contradicts (7.36) and proves the lemma.

Proof of Lemma 5.2

Let us first prove (5.13). We take $p_4 = 6/(1+2\mathfrak{s})$ the maximal exponent for which the Sobolev embedding $H^{1-\mathfrak{s}} \hookrightarrow L^{p_4}$ holds. We choose p_2 in such a way that exponents (p_i) are Hölder admissible. It follows that $p_2 = 6/(5-\rho-2\mathfrak{s}-3\varkappa)$. Now let $\varkappa > 0$ be so small that $\rho + 2\mathfrak{s}\varkappa \leq 2$. Then a simple calculation shows that $(1-\varkappa)p_2 \leq 6/(3-2\mathfrak{s})$, so the Sobolev embedding implies the first inclusion in (5.13).

We now prove (5.14). Proceeding as above, we take $q_4 = 6/(1+2\mathfrak{s})$ and choose q_2 such that the exponents (q_i) are Hölder admissible, i.e., $q_2 = 6(\rho+2)/(12-(\rho+2)(1+2\mathfrak{s}+3\varkappa))$. It is easy to check that for $\varkappa < 1/2 - \mathfrak{s}$, we have $(1-\varkappa)q_2 \leq 6$. The Sobolev embedding allows to conclude.

Proof of Lemma 5.3

In view of inequality (5.19), we have

$$\begin{aligned}\beta^{-1}\frac{\mathrm{d}}{\mathrm{d}t}(1+x)^\beta = (1+x)^{\beta-1}\dot{x} &\leq (1+x)^{\beta-1}(-\alpha x + g x^{1-\beta} + b) \\ &\leq -\alpha\, x(1+x)^{\beta-1} + g + b \leq -\frac{\alpha}{2}x^\beta + \alpha + g + b.\end{aligned}$$

Fixing $t \in [0,T]$ and integrating this inequality over $[0,t]$, we obtain

$$\beta^{-1}(1+x(t))^\beta + \frac{\alpha}{2}\int_0^t x^\beta(\tau)\,\mathrm{d}\tau \leq \beta^{-1}(1+x(0))^\beta + \int_0^t (\alpha + g(\tau) + b(\tau))\,\mathrm{d}s,$$

which implies (5.20).

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