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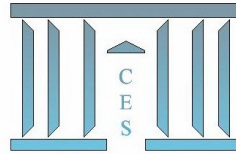
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**Credit segmentation in general equilibrium**

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# CREDIT SEGMENTATION IN GENERAL EQUILIBRIUM

SEBASTIÁN CEA-ECHENIQUE AND JUAN PABLO TORRES-MARTÍNEZ

ABSTRACT. We build a general equilibrium model with endogenous borrowing constraints compatible with credit segmentation. There are personalized trading restrictions connecting prices with both portfolio constraints and consumption possibilities, a setting which has not thoroughly been addressed by the literature. Our approach is general enough to be compatible with incomplete market economies where there exist wealth-dependent and/or investment-dependent credit access, borrowing constraints precluding bankruptcy, or assets backed by physical collateral.

To prove equilibrium existence, we assume that transfers implementable in segmented markets can be *super-replicated* by investments in non-segmented markets. We prove that equilibrium exists because of this super-replication property, which is satisfied if either (i) all individuals have access to borrow at a risk-free rate; or (ii) financial contracts make real promises in terms of non-perishable commodities; or (iii) promises are backed by physical collateral.

KEYWORDS. Incomplete Markets; General Equilibrium; Endogenous Trading Constraints.

JEL CLASSIFICATION. D52, D54.

## 1. INTRODUCTION

The differentiated access to commodity or asset markets endogenously emerges due to regulatory or institutional considerations. As a consequence, several kinds of trading restrictions are observed in financial markets: margin calls, collateral requirements, consumption quotas or income-based access to funding, among others. With the aim of understanding the effects of those restrictions in competitive markets, a vast literature of general equilibrium has been developed. That research has given consideration to models where financial trade is restricted by fixed, price-dependent, or

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Sebastián Cea-Echenique

Paris School of Economics - University Paris 1

e-mail: sebastian.cea@psemail.eu

Juan Pablo Torres-Martínez

Department of Economics, Faculty of Economics and Business, University of Chile

e-mail: juan.torres@fen.uchile.cl.

consumption-dependent portfolio constraints. Nevertheless, channels connecting prices with both portfolio constraints and consumption possibilities have not thoroughly been addressed by the literature. The objective of this paper is to contribute in this direction.

We analyze the existence of equilibria in a two period economy where agents are subject to price-dependent credit constraints that affect the access to physical and financial markets. Our approach is general enough to be compatible with incomplete market economies where there exist, for instance, wealth-dependent and/or investment-dependent credit access, borrowing constraints precluding bankruptcy, and assets backed by physical collateral.

Furthermore, we make credit segmentation compatible with the existence of a competitive equilibrium by assuming that assets payments can be *super-replicated* either by deliveries of non-perishable commodities or by the promises of assets that all agents can short-sell. For instance, this property holds when either (i) individuals have access to borrow resources through a portfolio making positive payments at all states of nature where remaining assets pay (e.g. risk-free asset); or (ii) all assets are real and promises are measured in units of non-perishable commodities; or (iii) debts are backed by physical collateral. In addition to our results of equilibrium existence, we provide examples clarifying the relevance of the super-replication assumption.

The related literature is described in the next section. In Sections 3 and 4 we state our model and introduce the basic assumptions over individuals' characteristics and trading constraints. Section 5 is devoted to discuss our main assumptions and results, whose proofs are given in the Appendix.

## 2. INSERTION IN THE LITERATURE

The existence of competitive equilibria was deeply studied in incomplete markets models where agents are subject to exogenous portfolio constraints. The case of portfolio restrictions determined by linear equality constraints is addressed by Balasko, Cass and Siconolfi (1990) for nominal assets, and by Polemarchakis and Siconolfi (1997) for real assets. When portfolio restrictions are determined by convex and closed sets containing zero, the case of nominal or numéraire assets is studied by Cass (1984, 2006), Siconolfi (1989), Cass, Siconolfi and Villanacci (2001), Martins-da-Rocha and Triki (2005), Won and Hahn (2007, 2012), Aouani and Cornet (2009, 2011), and Cornet and Gopalan (2010). In the same context, the case of real assets is analyzed by Radner (1972), Angeloni and Cornet (2006), and Aouani and Cornet (2011). In general terms, these authors prove equilibrium existence requiring non-redundancy hypotheses over financial structures and/or financial survival requirements. Under these assumptions, individuals' allocations and asset prices can be endogenously bounded without inducing frictions in the model.

There are several results that include price-dependent portfolio constraints in nominal or real assets markets. These models assume that financial constraints are determined by a finite number

of inequalities, and use differentiable techniques to ensure the existence of equilibrium and to analyze its stability and local-uniqueness. In this context, equilibrium existence is addressed by Carosi, Gori and Villanacci (2009) for numéraire asset markets with portfolio constraints, by Gori, Pireddu and Villanacci (2013) for numéraire and real asset markets with borrowing constraints, and by Hoelle, Pireddu, and Villanacci (2012) for real asset markets with wealth-dependent credit limits.

In addition to these approaches, Seghir and Torres-Martínez (2011) propose a model where trading constraints restrict the access to debt in terms of first-period consumption. Financial survival conditions are not required. However, they assume that preferences are such that individuals may compensate the losses on well-being generated by reductions of future consumption with increments in present demand.

We contribute to this literature with a model where borrowing constraints make the access to liquidity dependent on prices, investment, and consumption. Since we want to make trading constraints compatible with credit market segmentation, we do not impose financial survival conditions. Also, financial constraints are not necessarily determined by a finite number of inequalities and individual's preferences are not restricted by differentiable assumptions or by impatience conditions as in Seghir and Torres-Martínez (2011). Alternatively, we suppose that assets payments can be super-replicated either by deliveries of non-perishable commodities or by the promises of assets that all agents can short-sale.

### 3. MODEL

We consider a two-period economy with uncertainty about the realization of a state of nature in the second period, which belongs to a finite set  $S$ . Let  $\mathcal{S} = \{0\} \cup S$  be the set of states of nature in the economy, where  $s = 0$  denotes the unique state at the first period.

There is a finite set  $\mathcal{L}$  of perfectly divisible commodities, which are subject to transformation between periods and can be traded in spot markets at prices  $p = (p_s)_{s \in \mathcal{S}} \in \mathbb{R}_+^{\mathcal{L} \times \mathcal{S}}$ . We model the transformation of commodities between periods by linear technologies  $(Y_s)_{s \in S}$ . Thus, a bundle  $y \in \mathbb{R}_+^{\mathcal{L}}$  demanded at the first period is transformed, after its consumption and the realization of a state of nature  $s \in S$ , into the bundle  $Y_s y \in \mathbb{R}_+^{\mathcal{L}}$ .

There is a finite set  $\mathcal{J}$  of financial contracts available for trade at the first period that make promises contingent to the realization of uncertainty. Let  $q = (q_j)_{j \in \mathcal{J}} \in \mathbb{R}_+^{\mathcal{J}}$  be the vector of asset prices and denote by  $R_j(p) = (R_{s,j}(p))_{s \in S} \in \mathbb{R}_+^S$  the vector of payments associated to asset  $j \in \mathcal{J}$ .<sup>1</sup>

For notation convenience, let  $\mathbb{P} := \mathbb{R}_+^{\mathcal{L} \times \mathcal{S}} \times \mathbb{R}_+^{\mathcal{J}}$  be the space of commodity and asset prices, and let  $\mathbb{E} := \mathbb{R}_+^{\mathcal{L} \times \mathcal{S}} \times \mathbb{R}^{\mathcal{J}}$  be the space of consumption and portfolio allocations.

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<sup>1</sup>Our financial structure is general enough to be compatible with several types of assets. For instance, to include a nominal asset  $j$  it is sufficient to assume that there is  $(N_{s,j})_{s \in S} \in \mathbb{R}_+^S$  such that  $R_{s,j} \equiv N_{s,j}, \forall s \in S$ . To include a real asset  $k$  we can define payments  $R_{s,k}(p) = p_s \cdot A_{s,k}, \forall s \in S$ , where  $(A_{s,k})_{s \in S} \in \mathbb{R}_+^{\mathcal{L} \times S}$ .

There is a finite set  $\mathcal{I}$  of consumers that may trade assets in order to smooth their consumption. Each agent  $i \in \mathcal{I}$  has a utility function  $V^i : \mathbb{R}_+^{\mathcal{L} \times \mathcal{S}} \rightarrow \mathbb{R}$  and endowments  $w^i = (w_s^i)_{s \in \mathcal{S}} \in \mathbb{R}_+^{\mathcal{L} \times \mathcal{S}}$ .

Each individual  $i$  is subject to personalized trading constraints, which are determined by a correspondences  $\Phi^i : \mathbb{P} \rightarrow \mathbb{E}$ . Notice that, agents may be subject to endogenous borrowing constraints, as the access to liquidity can depend on prices, investment and consumption. We assume that there are no restrictions on investment, i.e.,  $\Phi^i(p, q) + \mathbb{R}_+^{\mathcal{L} \times \mathcal{S}} \times \mathbb{R}_+^{\mathcal{J}} \subseteq \Phi^i(p, q)$ ,  $\forall (p, q) \in \mathbb{P}$ ,  $\forall i \in \mathcal{I}$ .<sup>2</sup>

Given prices  $(p, q) \in \mathbb{P}$ , each agent  $i$  chooses a consumption bundle  $x^i = (x_s^i)_{s \in \mathcal{S}}$  and a portfolio  $z^i = (z_j^i)_{j \in \mathcal{J}}$  in her choice set  $C^i(p, q)$ , which is characterized by the vectors  $(x^i, z^i) \in \Phi^i(p, q)$  satisfying the following budget restrictions:

$$p_0 \cdot x_0^i + q \cdot z^i \leq p_0 \cdot w_0^i; \quad p_s \cdot x_s^i \leq p_s \cdot (w_s^i + Y_s x_0^i) + \sum_{j \in \mathcal{J}} R_{s,j}(p) z_j^i, \quad \forall s \in \mathcal{S}.$$

DEFINITION 1. A vector  $((\bar{p}, \bar{q}), (\bar{x}^i, \bar{z}^i)_{i \in \mathcal{I}}) \in \mathbb{P} \times \mathbb{E}^{\mathcal{I}}$  is a competitive equilibrium for the economy with endogenous trading constraints when the following conditions hold:

- (i) Each agent  $i \in \mathcal{I}$  maximizes her preferences, i.e.,  $(\bar{x}^i, \bar{z}^i) \in \underset{(x^i, z^i) \in C^i(\bar{p}, \bar{q})}{\operatorname{argmax}} V^i(x^i)$ .
- (ii) Individuals' plans are market feasible,

$$\sum_{i \in \mathcal{I}} (\bar{x}_0^i, (\bar{x}_s^i)_{s \in \mathcal{S}}, \bar{z}^i) = \sum_{i \in \mathcal{I}} (w_0^i, (w_s^i + Y_s w_0^i)_{s \in \mathcal{S}}, 0).$$

Our objective is to determine conditions that make price-dependent trading constraints  $\{\Phi^i\}_{i \in \mathcal{I}}$  compatible with equilibrium existence, even in the presence of *credit market segmentation*, which means that there are contracts that not all agents can short-sale, i.e.,

$$\{j \in \mathcal{J} : \exists i \in \mathcal{I}, (x^i, z^i) \in \Phi^i(p, q) \implies z_j^i \geq 0, \forall (p, q) \in \mathbb{P}\} \neq \emptyset.$$

Notice that, the existence of credit market segmentation is not compatible with *financial survival*, which requires that independent of prices all agents have access to some amount of liquidity by selling endowments or financial contracts, i.e.,

$$\bigcap_{(p, q) \in \mathbb{P} \setminus \{0\}} \{i \in \mathcal{I} : \exists (x^i, z^i) \in \Phi^i(p, q), p_0 \cdot w_0^i - q \cdot z^i > 0\} = \mathcal{I}.$$

The following examples illustrate the generality of our approach to restricted participation.

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<sup>2</sup>At the cost of additional complexity, our model could be generalized to include price-dependent investment constraints (see Cea-Echenique and Torres-Martínez (2014)).

EXAMPLE 1. Suppose that  $\mathcal{J} = \{j_1, j_2, j_3\}$  and, for each  $(p, q) \in \mathbb{P}$  and  $i \in \mathcal{I}$ , we have that

$$(x^i, z^i) \in \Phi^i(p, q) \iff \begin{cases} z_{j_1}^i \in [\min\{p_0 \cdot (\tau_1 - w_0^i), 0\}, +\infty); \\ z_{j_2}^i \in [\min\{p_0 \cdot (w_0^i - \tau_2), 0\}, +\infty); \\ z_{j_3}^i \in [\min\{K - \sum_{k \in \mathcal{J}'} q_k z_k^i, 0\}, +\infty); \end{cases}$$

where  $\tau_1, \tau_2 \in \mathbb{R}_+^{\mathcal{L}}$ ,  $K > 0$  and  $\mathcal{J}' \subseteq \mathcal{J} \setminus \{j_3\}$ . It follows that  $j_1$  is a credit line available for high income agents, because agent  $i$  can short-sale it if and only if the value of her first period endowment is greater than the threshold  $p_0 \tau_1$ . Analogously, only low-income agents can short-sale asset  $j_2$ . In addition, the access to credit through asset  $j_3$  depends on the amount of investment in some financial contracts, i.e., only investors expending an amount greater than  $K$  in assets belonging to  $\mathcal{J}'$  have access to short-sale  $j_3$ .  $\square$

EXAMPLE 2. Since trading constraints may induce restrictions on consumption, we can allow for derivative contracts as *commodity options*. Indeed, let  $j \in \mathcal{J}$  be a financial contract such that, for every  $(p, q) \in \mathbb{P}$  and  $i \in \mathcal{I}$ ,

$$R_{s,j}(p) = \max\{Y_s y - K, 0\}, \quad \forall s \in S; \quad (x^i, z^i) \in \Phi^i(p, q) \implies x_0^i + \kappa y \min\{z_j^i, 0\} \geq 0,$$

where  $y \in \mathbb{R}_+^{\mathcal{L}}$ ,  $K > 0$  and  $\kappa \in [0, 1)$ . Then,  $j$  is a commodity option that gives the right to buy in the second period, at a strike price  $K$ , the bundle obtained by the transformation of  $y$  through time. To short-sell this option, agents are required to buy a portion  $\kappa$  of  $y$  as guarantee.  $\square$

EXAMPLE 3. If there is  $\kappa \in (0, 1)$  such that, for any  $(p, q) \in \mathbb{P}$  and for some  $i \in \mathcal{I}$ ,

$$(x^i, z^i) \in \Phi^i(p, q) \implies \kappa p_s \cdot (w_s^i + Y_s x_0^i) + \sum_{j \in \mathcal{J}} R_{s,j}(p) \min\{z_j^i, 0\} \geq 0, \quad \forall s \in S,$$

then agent  $i$ 's trading constraints ensure that her debt is not greater than an exogenously-fixed portion  $\kappa$  of physical-resources' value. If a portion  $\rho > \kappa$  of physical resources can be garnished in case of bankruptcy, the above restriction ensures that  $i$  honors her commitments.  $\square$

#### 4. BASIC ASSUMPTIONS

The following are the basic hypotheses over individual characteristics and the financial structure:

ASSUMPTION (A1)

(i) For any agent  $i \in \mathcal{I}$ ,  $V^i$  is continuous, strictly increasing and strictly quasi-concave.<sup>3</sup>

(ii) For any agent  $i \in \mathcal{I}$ ,  $(W_s^i)_{s \in S} := (w_0^i, (w_s^i + Y_s w_0^i)_{s \in S}) \gg 0$ .

(iii) Asset payments are continuous functions of prices satisfying  $R_j(p) \neq 0, \forall j \in \mathcal{J}, \forall p \gg 0$ .

<sup>3</sup>Strict quasi-concavity of  $V^i$  requires that  $V^i(\lambda x^i + (1 - \lambda)y^i) > \min\{V^i(x^i), V^i(y^i)\}$  when  $V^i(x^i) \neq V^i(y^i)$ .

ASSUMPTION (A2)

$\{\Phi^i\}_{i \in \mathcal{I}}$  are lower hemicontinuous correspondences with closed graph and convex values.

In addition, agents are not burden to trade assets, i.e.  $(0, 0) \in \bigcap_{(p,q) \in \mathbb{P}} \Phi^i(p, q)$ ,  $\forall i \in \mathcal{I}$ .

Under Assumptions (A1)-(A2) individuals' choice set correspondences vary continuously with prices and, therefore, they do not compromise the continuity of individual demands (see Lemma 1 in the Appendix).

Notice that, if agents are subject to *exogenous borrowing constraints*, i.e.,  $\Phi^i(p, q) = \mathbb{R}_+^{\mathcal{L} \times \mathcal{S}} \times Z^i$ , where  $Z^i + \mathbb{R}_+^{\mathcal{J}} \subseteq Z^i$ , then Assumption (A2) is satisfied if and only if  $\{Z^i\}_{i \in \mathcal{I}}$  are closed and convex sets containing zero. Furthermore, when individuals are restricted by *price-dependent borrowing constraints*, i.e.,  $\Phi^i(p, q) = \{(x^i, z^i) \in \mathbb{E} : z^i + g_k^i(p, q) \geq 0, \forall k \in \{1, \dots, m_i\}\}$ , then (A2) holds if and only if  $g_k^i : \mathbb{P} \rightarrow \mathbb{R}_+^{\mathcal{J}}$  is a continuous function for every  $i \in \mathcal{I}$  and  $k \in \{1, \dots, m_i\}$ .

Furthermore, as in Seghir and Torres-Martínez (2011), we can include trading constraints that determine restrictions on borrowing and first-period consumption. That is, given  $(p, q) \in \mathbb{P}$  and  $i \in \mathcal{I}$ , we may have  $\Phi^i(p, q) = \{(x^i, z^i) \in \mathbb{E} : \exists(\theta^i, \varphi^i) \in \mathbb{R}_+^{\mathcal{J}} \times \mathbb{R}_+^{\mathcal{J}}, \varphi^i \in \Psi^i(x_0^i) \wedge z^i = \theta^i - \varphi^i\}$ , where  $\Psi^i : \mathbb{R}_+^{\mathcal{L}} \rightarrow \mathbb{R}_+^{\mathcal{J}}$ . In this context, if  $\{\Psi^i\}_{i \in \mathcal{I}}$  have a closed and convex graph and  $0 \in \Psi^i(x_0^i)$  for each  $x_0^i \in \mathbb{R}_+^{\mathcal{L}}$ , then Assumption (A2) is satisfied. Furthermore, if  $\Psi^i(x_0^i) \subseteq \Psi^i(y_0^i)$  for each  $y_0^i \geq x_0^i$ , then we can ensure that  $\Phi^i(p, q) + \mathbb{R}_+^{\mathcal{L} \times \mathcal{S}} \times \mathbb{R}_+^{\mathcal{J}} \subseteq \Phi^i(p, q)$ ,  $\forall (p, q) \in \mathbb{P}$ . We remark that our approach is complementary to Seghir and Torres-Martínez (2001), because at the cost of a non-redundancy assumption and a super-replication property (see Assumptions (A3) and (A4) below) we neither restrict individual preferences nor require  $\{\Psi^i\}_{i \in \mathcal{I}}$  to have compact values.

## 5. EQUILIBRIUM EXISTENCE

We analyze the existence of a competitive equilibrium using standard fixed point techniques. Hence, we need to ensure that endogenous variables can be bounded without inducing frictions over individual demand correspondences.

To obtain upper bounds for individual allocations we impose restrictions over the correspondence of *attainable allocations*  $\Omega : \mathbb{P} \rightarrow \mathbb{E}^{\mathcal{I}}$ , which is defined as the set-valued mapping that associates prices with market feasible allocations satisfying individuals' budget and trading constraints, i.e.,

$$\Omega(p, q) := \left\{ ((x^i, z^i))_{i \in \mathcal{I}} \in \prod_{i \in \mathcal{I}} C^i(p, q) : \sum_{i \in \mathcal{I}} (x^i, z^i) = \sum_{i \in \mathcal{I}} ((W_s^i)_{s \in \mathcal{S}}, 0) \right\}.$$

ASSUMPTION (A3)

For every compact set  $\mathbb{P}' \subseteq \mathbb{P}$ ,  $\bigcup_{(p,q) \in \mathbb{P}': (p,q) \gg 0} \Omega(p, q)$  is bounded.



This assumption holds when  $\mathcal{J}$  is composed by non-redundant nominal assets, by collateralized assets, or when agents are subject to exogenous short-sale constraints—i.e., when for every  $i \in \mathcal{I}$  there exists  $m \in \mathbb{R}_+^{\mathcal{J}}$  such that  $(x^i, z^i) \in \Phi^i(p, q) \implies z^i \geq -m, \forall (p, q) \in \mathbb{P}$ .

To obtain upper bounds for prices, we restrict the heterogeneity of financial transfers through a *super-replication property*, requiring that wealth transfers that are implementable in segmented credit markets can be super-replicated by investments in non-segmented markets. Indeed, since prices of commodities and unsegmented assets can be normalized without induce frictions on individual behavior, the super-replication property allows us to find endogenous upper bounds for the non-arbitrage prices of segmented assets (see Lemma 3 in the Appendix).

In order to formalize these ideas, we need a previous definition: a contract  $j$  is *unsegmented* if for every vector of prices  $(p, q) \in \mathbb{P}$  there exists  $\delta > 0$  such that  $-\delta \vec{e}_j \in \bigcap_{i \in \mathcal{I}} \Phi^i(p, q)$ , where  $\vec{e}_j \in \mathbb{E}$  is the allocation composed by just one unit of  $j$ . Let  $\mathcal{J}_a$  be the set of unsegmented contracts.

ASSUMPTION (A4)

There exists an allocation  $(\hat{x}_0, \hat{z}) \in \mathbb{R}_+^{\mathcal{L}} \times \mathbb{R}_+^{\mathcal{J}_a}$  such that, at every state of nature  $s \in S$ ,

$$\sum_{j \in \mathcal{J} \setminus \mathcal{J}_a} R_{s,j}(p) \leq p_s \cdot Y_s \hat{x}_0 + \sum_{k \in \mathcal{J}_a} R_{s,k}(p) \hat{z}_k, \quad \forall p \in \mathbb{R}_+^{\mathcal{L}} \times (\mathbb{R}_+^{\mathcal{L}} \setminus \{0\})^S.$$

Notice that  $\mathcal{J} \setminus \mathcal{J}_a$  is the set of segmented credit contracts. Thus, Assumption (A4) requires that payments associated to segmented contracts can be super-replicated by positions on durable goods and/or contracts that all agents can short-sale. Furthermore, under Assumption (A4), the set of states of nature where inter-temporal transfers are available is the same for all agents.

Our main result ensures that credit segmentation is compatible with equilibria:

THEOREM. *Under Assumptions (A1)-(A4) there is a competitive equilibrium.*

We want to illustrate several situations where the super-replication property holds.

In the following result we assume that there is an asset that all agents can short-sale and whose payments are positive whenever one of the remaining contracts has non-trivial promises. Note that, this is satisfied when all agents have access to borrowing through a risk-free asset.

COROLLARY 1. *Under Assumptions (A1)-(A3), assume that there is an unsegmented contract  $j \in \mathcal{J}_a$  such that  $(R_{s,j}(p))_{s \in S_j} \gg 0, \forall p \in \mathbb{R}_+^{\mathcal{L}} \times (\mathbb{R}_+^{\mathcal{L}} \setminus \{0\})^S$ , where  $S_j := \{s \in S : \exists k \neq j, R_{s,k} \neq 0\}$ . Then, a competitive equilibrium exists.*

The super-replication property also holds when assets' promises are measure in units of non-perishable commodities. Indeed, if we define  $\mathcal{L}^* := \{l \in L : Y_s(l, l) > 0, \forall s \in \bigcup_j S_j\}$  as the set of commodities that not-fully depreciate at the states of nature where financial contracts make non-trivial promises, we have that:

**COROLLARY 2.** *Under Assumptions (A1)-(A3), there is a competitive equilibrium if assets are real and promises are given in terms of commodities in  $\mathcal{L}^*$ .*

Our model is general enough to be compatible with the inclusion of default and non-recourse collateralized assets. That is, assets whose promises are backed by non-perishable commodities such that the only payment enforcement mechanism in case of default is given by the seizure of these guarantees.<sup>4</sup> In this direction, we can extend the model of Geanakoplos and Zame (2013) to include financial market segmentation and price-dependent trading constraints.

More precisely, assume that each  $j \in \mathcal{J}$  is characterized by a pair  $(C_j, (D_{s,j}(p))_{s \in S})$ , where  $C_j = (C_{j,l})_{l \in \mathcal{L}} \in \mathbb{R}_+^{\mathcal{L}} \setminus \{0\}$  is the collateral guarantee, and  $(D_{s,j}(p))_{s \in S} \in \mathbb{R}_+^S$  are the state contingent promises. Borrowers are required to pledge the associated collateral, i.e., for any  $(x^i, z^i) \in \Phi^i(p, q)$  we have that  $x_0^i \geq \sum_{j \in \mathcal{J}} C_j \max\{-z_j^i, 0\}$ . In addition, as the only enforcement in case of default is the seizure of collateral guarantees, borrowers give strategic default, delivering the minimum between collateral value and promises, i.e.,  $R_{s,j}(p) := \min\{D_{s,j}(p), p_s Y_s C_j\}$ ,  $\forall s \in S$ .

Since payments associated to non-recourse collateralized loans can be super-replicated by the bundle used as guarantee, non-arbitrage asset prices are bounded from above by the collateral cost (see Appendix).

**COROLLARY 3.** *Under Assumptions (A1)-(A3) there is equilibria in collateralized asset markets.*

Since Corollaries 2 and 3 do not require  $\mathcal{J}_a \neq \emptyset$ , they are compatible with an extreme form of financial segmentation: the exclusion of some agents from credit markets.

The following examples highlight the difficulties to guarantee equilibrium existence without the super-replication property. In the first example, we present a collection of economies that have equilibria if and only if the super-replication property holds. In the second example, we show that without Assumption (A4) it may be difficult to find upper bounds for equilibrium asset prices.

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<sup>4</sup>In the absence of payment enforcement mechanisms over collateral repossession, the monotonicity of preferences guarantees that borrowers of a collateralized loan always deliver the minimum between promises and collateral values. Therefore, lenders that finance these loans perfectly foresight the payments that they will receive. Hence, as in Geanakoplos and Zame (2013), we can capture with a same financial contract both the collateralized line of credit and the collateralized loan obligation (CLO) that passthrough the payments made by borrowers.

EXAMPLE 4. Consider a two-period economy  $\mathcal{E}(\rho)$  without uncertainty at the second period, where the parameter  $\rho$  is non-negative. There is a perishable commodity and two assets, a segmented contract  $j$  with payment  $R_j(p) = 1$  and an unsegmented contract  $k$  with payment  $R_k(p) = \rho$ .

There are two agents,  $A$  and  $B$ . Each agent  $i \in \{A, B\}$  is characterized by a utility function  $V^i : \mathbb{R}_+^2 \rightarrow \mathbb{R}$  and by endowments  $w^i = (w_0^i, w_1^i) \neq 0$ . We assume that Assumption (A1)(i) holds,  $w^A \gg 0$ , and  $w_1^B = 0$ . Also,  $V^B(x_0, x_1) = f(x_0) + g(x_1)$ , where  $g$  satisfies Inada conditions.

Only agent  $B$  can short-sale asset  $j$ . That is,  $\Phi^A(p, q) = \mathbb{R}_+^2 \times (\mathbb{R}_+ \times \mathbb{R})$  and  $\Phi^B(p, q) = \mathbb{R}_+^2 \times \mathbb{R}^2$ .

Since the commodity is perishable and asset  $j$  is segmented, the super-replication property holds if and only if  $\rho > 0$ . Furthermore, when  $\rho > 0$  the economy  $\mathcal{E}(\rho)$  has a non-empty set of competitive equilibria, because financial markets are complete and preferences are strictly monotonic. However,  $\mathcal{E}(0)$  does not have equilibria.<sup>5</sup>  $\square$

EXAMPLE 5. Consider a two-period economy  $\mathcal{E}(\nu)$  without uncertainty at the second period, where the parameter  $\nu \in (0, 0.25)$ . There exists only one commodity, which is perishable and we use as numeraire, and a risk-free bond  $j$  that promises one unit of commodity at the second period, i.e.,  $R_j(p) = 1$ . There are two agents,  $A$  and  $B$ , characterized by

$$\begin{aligned} V^A(x_0, x_1) &= \frac{x_0}{1+x_0} + 0.25\sqrt{x_1}, & (w_0^A, w_1^A) &= (1, 1); \\ V^B(x_0, x_1) &= \sqrt{x_0} + 0.25\sqrt{x_1}, & (w_0^B, w_1^B) &= (1, \nu). \end{aligned}$$

We assume that only  $B$  can short-sale the bond, i.e.,  $\Phi^A(p, q) = \mathbb{R}_+^2 \times \mathbb{R}_+$  and  $\Phi^B(p, q) = \mathbb{R}_+^2 \times \mathbb{R}$ . Notice that Assumptions (A1)-(A3) are satisfied. However, as the commodity is perishable and credit contract  $j$  is segmented, Assumption (A4) does not hold.

It is not difficult to verify that *autarchy* is the unique equilibrium allocation in  $\mathcal{E}(\nu)$ , and that the risk-free bond has a price  $1/(4\sqrt{\nu})$  (see Appendix). Thus, since we do not restrict endowments to be bounded away from zero, without Assumption (A4) it is impossible to find an upper bound for the asset price in the collection of economies  $\{\mathcal{E}(\nu) : \nu \in (0, 1)\}$ .<sup>6</sup>  $\square$

## 6. CONCLUDING REMARKS

In this paper we extend the theory of general equilibrium with incomplete financial markets to include price-dependent trading constraints that restrict both consumption alternatives and credit access. Our approach is general enough to incorporate several types of dependencies between prices,

<sup>5</sup>Given a price  $q > 0$  for asset  $j$ , agent  $B$ 's individual problem is equivalent to  $\max_{z \in [0, w_0^B q^{-1}]} f(w_0^B - qz) + g(z)$ . By Inada's condition, we know that the optimal portfolio of agent  $B$  belongs to  $(0, w_0^B q^{-1}]$ . This is incompatible with credit segmentation, because only agent  $B$  can short-sale asset  $j$ .

<sup>6</sup>This example does not satisfy the hypothesis on preferences imposed by Seghir and Torres-Martínez (2011, Assumption (A2)). Thus, their alternative approach to bound asset prices can not be implemented.

consumption, and credit access. For instance, the access to liquidity may depend on individuals income, the short-sale of derivatives may require the deposit of margins, or borrowers could be required to pledge physical collateral to protect lenders in case of default.

As we want to include financial segmentation, our results of equilibrium existence do not rely on financial survival conditions. Thus, based on the idea that asset prices can be bounded when their promises can be *super-replicated* at a finite cost, we show equilibrium existence assuming that investments in non-segmented assets can super-replicate the deliveries of segmented contracts.

In our model, investment restrictions can be introduced at the cost of more notation and technical hypotheses (see Cea-Echenique and Torres-Martínez (2014)). As a matter of future research, we want to analyze the characteristics of trading constraints that are essential to guarantee equilibrium existence in economies with more than two periods (cf., Iraola and Torres-Martínez (2014)).

#### APPENDIX

PROOF OF THE THEOREM. Given  $M \in \mathbb{N}$ , consider the set of normalized prices  $\mathbb{P}(M) := \mathcal{P}_0 \times [0, M]^{\mathcal{J}_b} \times \mathcal{P}_1^S$ , where  $\mathcal{P}_0 := \{y \in \mathbb{R}_+^{\mathcal{L} \cup \mathcal{J}_a} : \|y\|_{\Sigma} = 1\}$ ,  $\mathcal{P}_1 := \{y \in \mathbb{R}_+^{\mathcal{L}} : \|y\|_{\Sigma} = 1\}$ , and  $\mathcal{J}_b := \mathcal{J} \setminus \mathcal{J}_a$ .<sup>7</sup> Note that, a typical element of  $\mathbb{P}(M)$  is of the form  $(p, q) = ((p_0, (q_k)_{k \in \mathcal{J}_a}), (q_j)_{j \in \mathcal{J}_b}, (p_s)_{s \in \mathcal{S}})$ . When  $(p, q) \in \mathbb{P}(M)$ , the commodity price  $p = (p_0, (p_s)_{s \in \mathcal{S}})$  belongs to  $\mathcal{P} := \{y \in \mathbb{R}_+^{\mathcal{L}} : \|y\|_{\Sigma} \leq 1\} \times \mathcal{P}_1^S$ .

LEMMA 1. *Under Assumptions (A1)(iii) and (A2), for every agent  $i \in \mathcal{I}$  the choice set correspondence  $C^i : \mathbb{P}(M) \rightarrow \mathbb{E}$  is lower hemicontinuous with closed graph and non-empty and convex values.*

PROOF. Fix  $i \in \mathcal{I}$ . Since for every  $(p, q) \in \mathbb{P}$  the allocation  $((W_s^i)_{s \in \mathcal{S}}, 0) \in C^i(p, q)$ ,  $C^i$  is non-empty valued. Assumption (A2) implies that  $C^i$  has convex values and closed graph. To prove that  $C^i$  is lower hemicontinuous, let  $\tilde{C}^i : \mathbb{P}(M) \rightarrow \mathbb{E}$  be the correspondence that associates to each  $(p, q) \in \mathbb{P}(M)$  the set of allocations  $(x^i, z^i) \in C^i(p, q)$  satisfying budget constraints with strict inequalities. We affirm that  $\tilde{C}^i$  is lower hemicontinuous and has non-empty values. Since  $C^i$  is the closure of  $\tilde{C}^i$ , these properties imply that  $C^i$  is lower hemicontinuous (see Border (1985, 11.19(c))). Thus, to obtain the results it is sufficient to ensure the claimed properties for  $\tilde{C}^i$ .

*Claim A.  $\tilde{C}^i$  has non-empty values.* It follows from Assumption (A2) that  $((W_0^i, (0.5W_s^i)_{s \in \mathcal{S}}), 0) \in \Phi^i(p, q)$  for all  $(p, q) \in \mathbb{P}(M)$ . Notice that, when  $p_0 \neq 0$  we always have that  $((W_0^i, (0.5W_s^i)_{s \in \mathcal{S}}), 0) \in \tilde{C}^i(p, q)$ .

Thus, assume that  $\mathcal{J}_a \neq \emptyset$  and fix  $(p, q) \in \mathbb{P}(M)$  such that  $p_0 = 0$  and, therefore  $(q_j)_{j \in \mathcal{J}_a} \neq 0$ . By definition of unsegmented contracts, for every  $j \in \mathcal{J}_a$  there exists  $\delta_j(p, q) \in (0, 1)$  such that  $-\delta_j \vec{e}_j \in \Phi^i(p, q)$  for all  $\delta_j \in (0, \delta_j(p, q))$ . Since  $\Phi^i$  has convex values, we conclude that there exists  $\delta(p, q) > 0$  such that  $((W_0^i, (0.5W_s^i)_{s \in \mathcal{S}}), 0) - \delta \sum_{j \in \mathcal{J}_a} \vec{e}_j \in \Phi^i(p, q)$ , for every  $\delta \in (0, \delta(p, q))$ .<sup>8</sup> Furthermore, assume that  $\delta \in$

<sup>7</sup>Trading constraints are not necessarily homogeneous of degree zero in prices. Consequently, the normalization of prices may induce a selection of equilibria.

<sup>8</sup>It is sufficient to consider  $\delta(p, q) := \min_{j \in \mathcal{J}_a} \delta_j(p, q) / \#\mathcal{J}_a$ .

$(0, \delta(p, q))$  satisfies

$$\delta \sum_{k \in \mathcal{J}_a} \max_{(\tilde{p}, s) \in \mathcal{P} \times \mathcal{S}} R_{s,k}(\tilde{p}) < 0.5 \min_{(s,l) \in \mathcal{S} \times \mathcal{L}} W_{s,l}^i.$$

Then, promises can be honored with the resources that became available after the consumption of  $0.5W_s^i$ . Therefore, under these requirements, we have that  $((W_0^i, (0.5W_s^i)_{s \in \mathcal{S}}), 0) - \delta \sum_{j \in \mathcal{J}_a} \tilde{e}_j \in \tilde{C}^i(p, q)$ .

*Claim B.  $\tilde{C}^i$  is lower hemicontinuous.* Fix  $(p, q) \in \mathbb{P}(M)$  and  $(x^i, z^i) \in \tilde{C}^i(p, q)$ . Given a sequence  $\{(p_n, q_n)\}_{n \in \mathbb{N}} \subset \mathbb{P}(M)$  converging to  $(p, q)$ , the lower hemicontinuity of  $\Phi^i$  (Assumption (A2)) ensures that there is  $\{(x^i(n), z^i(n))\}_{n \in \mathbb{N}} \subset \mathbb{E}$  converging to  $(x^i, z^i)$  such that  $(x^i(n), z^i(n)) \in \Phi^i(p_n, q_n)$ ,  $\forall n \in \mathbb{N}$ . Thus, for  $n \in \mathbb{N}$  large enough,  $(x^i(n), z^i(n)) \in \tilde{C}^i(p_n, q_n)$ . It follows from the sequential characterization of hemicontinuity that  $\tilde{C}^i$  is lower hemicontinuous (see Border (1985, 11.11(b))).  $\square$

Since Assumption (A4) holds, there exists  $(\hat{x}_0, \hat{z})$  that super-replicates the financial payments of segmented contracts. Define

$$\bar{Q} := \max \left\{ 1, 2 \left( \|\hat{x}_0\|_{\Sigma} + \max_{k \in \mathcal{J}_a} \hat{z}_k \right) \right\}; \quad \bar{\Omega} := 2 \sup_{(p,q) \in \mathbb{P}(\bar{Q}): (p,q) \gg 0} \sup_{(x^i, z^i)_{i \in \mathcal{I}} \in \Omega(p,q)} \sum_{i \in \mathcal{I}} \|z^i\|_{\Sigma}.$$

Notice that, Assumption (A3) guarantees that  $\bar{\Omega}$  is finite.

Given  $(p, q) \in \mathbb{P}(M)$ , for any  $i \in \mathcal{I}$  we consider the truncated choice set  $C^i(p, q) \cap \mathbb{K}$ , where

$$\mathbb{K} := [0, 2\bar{W}]^{\mathcal{L} \times \mathcal{S}} \times [-\bar{\Omega}, \#\mathcal{I} \bar{\Omega}]^{\mathcal{J}},$$

$$\bar{W} := \left( \#J \#\mathcal{I} \bar{\Omega} + \sum_{(s,l) \in \mathcal{S} \times \mathcal{L}} \sum_{i \in \mathcal{I}} W_{s,l}^i \right) \left( 1 + \max_{(p,s) \in \mathcal{P} \times \mathcal{S}} \sum_{j \in \mathcal{J}} R_{s,j}(p) \right).$$

Let  $\Psi_M : \mathbb{P}(M) \times \mathbb{K}^{\mathcal{I}} \rightarrow \mathbb{P}(M) \times \mathbb{K}^{\mathcal{I}}$  be the correspondence given by

$$\Psi_M(p, q, (x^i, z^i)_{i \in \mathcal{I}}) = \phi_{0,M}((x_0^i, z^i)_{i \in \mathcal{I}}) \times \prod_{s \in \mathcal{S}} \phi_s((x_s^i)_{i \in \mathcal{I}}) \times \prod_{i \in \mathcal{I}} \phi^i(p, q),$$

where

$$\begin{aligned} \phi_{0,M}((x_0^i, z^i)_{i \in \mathcal{I}}) &:= \operatorname{argmax}_{(p_0, q) \in \mathcal{P}_0 \times [0, M]^{\mathcal{J}_b}} p_0 \cdot \sum_{i \in \mathcal{I}} (x_0^i - w_0^i) + q \cdot \sum_{i \in \mathcal{I}} z^i; \\ \phi_s((x_s^i)_{i \in \mathcal{I}}) &:= \operatorname{argmax}_{p_s \in \mathcal{P}_1} p_s \cdot \sum_{i \in \mathcal{I}} (x_s^i - W_s^i), \quad \forall s \in \mathcal{S}; \\ \phi^i(p, q) &:= \operatorname{argmax}_{(x^i, z^i) \in C^i(p, q) \cap \mathbb{K}} V^i(x^i), \quad \forall i \in \mathcal{I}. \end{aligned}$$

LEMMA 2. *Under Assumptions (A1)-(A3),  $\Psi_M$  has a non-empty set of fixed points.*

PROOF. By Kakutani's Fixed Point Theorem, it is sufficient to prove that  $\Psi_M$  has a closed graph with non-empty and convex values. Since  $\mathbb{P}(M)$  is non-empty, convex and compact, Berge's Maximum Theorem establishes that  $\{\phi_{0,M}, \{\phi_s\}_{s \in \mathcal{S}}\}$  have a closed graph with non-empty and convex values.

It remains to prove that the same properties hold for  $\{\phi^i\}_{i \in \mathcal{I}}$ . Given  $i \in \mathcal{I}$ , Lemma 1 implies that  $C^i$  has a closed graph with non-empty and convex values. Since  $\mathbb{K}$  is compact and convex and  $((W_s^i)_{s \in \mathcal{S}}, 0) \in \mathbb{K}$ , it follows that  $(p, q) \in \mathbb{P}(M) \rightarrow C^i(p, q) \cap \mathbb{K}$  has a closed graph and non-empty, compact, and convex values. The proof of Lemma 1 also ensures that  $C^i$  is lower hemicontinuous and  $((W_s^i)_{s \in \mathcal{S}}, 0) \in C^i(p, q) \cap \operatorname{int}(\mathbb{K})$ .

As  $(p, q) \in \mathbb{P}(M) \rightarrow \text{int}(\mathbb{K})$  has open graph, it follows that  $(p, q) \in \mathbb{P}(M) \rightarrow C^i(p, q) \cap \text{int}(\mathbb{K})$  is lower hemicontinuous (see Border (1985, 11.21(c))). Therefore,  $(p, q) \in \mathbb{P}(M) \rightarrow C^i(p, q) \cap \mathbb{K}$  is lower hemicontinuous too (see Border (1985, 11.19(c))). Berge's Maximum Theorem and the continuity and quasi-concavity of  $V^i$  guarantees that  $\phi^i$  satisfies the required properties.  $\square$

LEMMA 3. *Under Assumptions (A1)-(A4). Let  $(\bar{p}, \bar{q}, (\bar{x}^i, \bar{z}^i)_{i \in \mathcal{I}})$  be a fixed point of  $\Psi_M$  such that  $\bar{p} \gg 0$  and*

$$\sum_{i \in \mathcal{I}} \bar{z}_k^i \leq 0, \quad \forall k \in \mathcal{J}_a; \quad \sum_{i \in \mathcal{I}} \bar{x}_{s,l}^i < 2\bar{W}, \quad \forall (s, l) \in \mathcal{S} \times \mathcal{L}.$$

Then, for any  $j \in \mathcal{J}_b$  we have that,

$$\bar{q}_j > 0 \quad \wedge \quad \sum_{i \in \mathcal{I}} \bar{z}_j^i > 0 \quad \implies \quad \bar{q}_j \leq \bar{Q}.$$

PROOF. Let  $j \in \mathcal{J}_b$  such that  $\bar{q}_j > 0$  and  $\sum_{i \in \mathcal{I}} \bar{z}_j^i > 0$ . Due to  $\bar{p} \gg 0$ , it follows from Assumption (A1)(iii) that  $R_j(\bar{p}) \neq 0$ . Hence, there is  $s' \in \mathcal{S}$  such that  $\sum_{r \in \mathcal{J}_b} R_{s',r}(\bar{p}) \geq R_{s',j}(\bar{p}) > 0$ .

Fix  $\delta \in (0, 1)$ . By Assumption (A4), it follows that

$$\sum_{r \in \mathcal{J}_b} R_{s',r}(\bar{p}) < (1 + \delta) \left( \bar{p}_{s'} \cdot Y_{s'} \hat{x}_0 + \sum_{k \in \mathcal{J}_a} R_{s',k}(\bar{p}) \hat{z}_k \right) \leq (1 + \delta) \left( \bar{p}_{s'} \cdot Y_{s'} \hat{x}_0 + \left( \max_{k \in \mathcal{J}_a} \hat{z}_k \right) \sum_{k \in \mathcal{J}_a} R_{s',k}(\bar{p}) \right).$$

We affirm that,

$$\bar{q}_j \leq (1 + \delta) \left( \bar{p}_0 \cdot \hat{x}_0 + \left( \max_{k \in \mathcal{J}_a} \hat{z}_k \right) \sum_{k \in \mathcal{J}_a} \bar{q}_k \right).$$

Let  $i$  be an agent that invests in asset  $j$ . If the inequality above does not hold, then there is  $\varepsilon > 0$  such that, agent  $i$  can reduce her long position on asset  $j$  in  $\varepsilon \bar{z}_j^i$  units, change her first-period consumption to  $\bar{x}_0^i + (1 + \delta) \varepsilon \bar{z}_j^i \hat{x}_0$ , and increase in  $(1 + \delta) (\max_{r \in \mathcal{J}_a} \hat{z}_r) \varepsilon \bar{z}_j^i$  units the investment in each  $k \in \mathcal{J}_a$ .<sup>9</sup>

With this strategy,  $i$  changes her wealth at state of nature  $s \in \mathcal{S}$  by

$$\left( (1 + \delta) \left( \bar{p}_s \cdot Y_s \hat{x}_0 + \left( \max_{k \in \mathcal{J}_a} \hat{z}_k \right) \sum_{k \in \mathcal{J}_a} R_{s,k}(\bar{p}) \right) - R_{s,j}(\bar{p}) \right) \varepsilon \bar{z}_j^i \geq 0,$$

where the last inequality is a consequence of the super-replication property and holds as strict inequality for  $s = s'$ . This contradicts the optimality of  $(\bar{x}^i, \bar{z}^i)$  on  $C^i(\bar{p}, \bar{q}) \cap \mathbb{K}$ . We conclude that  $\bar{q}_j \leq \bar{Q}$ .  $\square$

LEMMA 4. *Under Assumptions (A1)-(A4), for any  $M > \bar{Q}$  the fixed points of  $\Psi_M$  are competitive equilibria.*

PROOF. Given  $M > \bar{Q}$ , let  $(\bar{p}, \bar{q}, (\bar{x}^i, \bar{z}^i)_{i \in \mathcal{I}})$  be a fixed point of  $\Psi_M$ . Adding first period budget constraints across agents, the definition of  $\phi_{0,M}$  guarantees that,

$$p_0 \cdot \sum_{i \in \mathcal{I}} (\bar{x}_0^i - w_0^i) + q \cdot \sum_{i \in \mathcal{I}} \bar{z}^i \leq \bar{p}_0 \cdot \sum_{i \in \mathcal{I}} (\bar{x}_0^i - w_0^i) + \bar{q} \cdot \sum_{i \in \mathcal{I}} \bar{z}^i \leq 0, \quad \forall (p_0, q) \in \mathcal{P}_0 \times [0, M]^{\mathcal{J}_b}.$$

Hence,

$$\sum_{i \in \mathcal{I}} (\bar{x}_0^i - w_0^i) \leq 0, \quad \sum_{i \in \mathcal{I}} \bar{z}_k^i \leq 0, \quad \forall k \in \mathcal{J}_a,$$

<sup>9</sup>As the new strategy needs to be on  $\mathbb{K}$ , the value of  $\varepsilon$  may depend on  $(\bar{q}_j, \bar{x}_0^i, (\bar{z}_k^i)_{k \in \mathcal{J}_a}, \bar{z}_j^i)$ .

and  $\bar{q}_j = M$  for every  $j \in \mathcal{J}_b$  such that  $\sum_{i \in \mathcal{I}} \bar{z}_j^i > 0$ . Furthermore, adding individual budget constraints at any state of nature in the second period, the definition of  $\mathbb{K}$  guarantees that,

$$p_s \cdot \sum_{i \in \mathcal{I}} (\bar{x}_s^i - W_s^i) \leq \bar{p}_s \cdot \sum_{i \in \mathcal{I}} (\bar{x}_s^i - W_s^i) \leq \bar{W}, \quad \forall p_s \in \mathcal{P}_1, \forall s \in S.$$

We obtain that  $\sum_{i \in \mathcal{I}} \bar{x}_{s,l}^i < 2\bar{W}$ ,  $\forall (s,l) \in \mathcal{S} \times \mathcal{L}$ , which implies that  $\bar{p} \gg 0$ . In another case, Assumptions (A1) and (A2) guarantee that at least one agent can improve her utility by increasing her consumption without additional costs. A contradiction to the optimality of plans  $(\bar{x}^i, \bar{z}^i)_{i \in \mathcal{I}}$ .

The strict positivity of commodity prices has several consequences. First, by Assumption (A1)(iii) asset promises are non-trivial and the strict monotonicity of preferences (Assumption (A1)(i)) jointly with the absence of restrictions on investment ensure that asset prices are strictly positive. Second, as  $(\bar{p}, \bar{q}, (\bar{x}^i, \bar{z}^i)_{i \in \mathcal{I}})$  satisfies the hypotheses of Lemma 3 and  $M > \bar{Q}$ , we obtain that  $\sum_{i \in \mathcal{I}} \bar{z}^i \leq 0$ . Third, Assumption (A1)(i) guarantees that budget constraints are satisfied by equality.

We conclude that,

$$(\bar{p}, \bar{q}) \in \mathbb{P}(\bar{Q}), \quad (\bar{p}, \bar{q}) \gg 0, \quad \sum_{i \in \mathcal{I}} (\bar{x}^i - (W_s^i)_{s \in S}) = 0, \quad \sum_{i \in \mathcal{I}} \bar{z}^i = 0,$$

and Assumption (A3) implies that  $(\bar{x}^i, \bar{z}^i)_{i \in \mathcal{I}} \in \Omega(\bar{p}, \bar{q}) \cap \text{int}(\mathbb{K})$ .<sup>10</sup>

As for any  $i \in \mathcal{I}$  the allocation  $(\bar{x}^i, \bar{z}^i)$  belongs to  $C^i(\bar{p}, \bar{q}) \cap \text{int}(\mathbb{K})$ , given  $(x^i, z^i) \in C^i(\bar{p}, \bar{q})$  with  $x^i \neq \bar{x}^i$  there exists  $\lambda \in (0, 1)$  such that  $\lambda(\bar{x}^i, \bar{z}^i) + (1 - \lambda)(x^i, z^i) \in C^i(\bar{p}, \bar{q}) \cap \mathbb{K}$ . The strict quasi-concavity of utility functions (Assumption (A1)(i)) implies that,  $V^i(\lambda(\bar{x}^i, \bar{z}^i) + (1 - \lambda)(x^i, z^i)) > \min\{V^i(\bar{x}^i), V^i(x^i)\}$ . Since  $(\bar{x}^i, \bar{z}^i) \in \phi^i(\bar{p}, \bar{q})$ , we obtain that  $V^i(x^i) < V^i(\bar{x}^i)$ .

Therefore,  $(\bar{x}^i, \bar{z}^i)$  is an optimal choice for agent  $i$  in  $C^i(\bar{p}, \bar{q})$ .  $\square$

PROOF OF COROLLARY 1. We need to guarantee that (A4) is satisfied. Let  $\mathcal{J}_a^* \subseteq \mathcal{J}_a$  be the set of contracts that satisfies the requirements imposed in the statement of Corollary 1. Then, Assumption (A4) holds by choosing  $\hat{x}_0 = 0$ ;  $\hat{z}_k = 0$ ,  $\forall k \in \mathcal{J}_a \setminus \mathcal{J}_a^*$ ; and  $\hat{z}_k = \max_{(p,s) \in \mathcal{P} \times S_k} \left( \sum_{j \in \mathcal{J}_b} R_{s,j}(p) / R_{s,k}(p) \right)$ ,  $\forall k \in \mathcal{J}_a^*$ .  $\square$

PROOF OF COROLLARY 2. As in the previous result, it is sufficient to ensure that Assumption (A4) holds. Since assets are real and payments are given in terms of commodities in  $\mathcal{L}^*$ , for any  $j \in \mathcal{J} \setminus \mathcal{J}_a$  there is  $A_{s,j} : \mathbb{R}_+^{\mathcal{L} \times S} \rightarrow \mathbb{R}_+^{\mathcal{L}}$  such that  $R_{s,j}(p) = p_s \cdot A_{s,j}(p)$  and  $(A_{s,j})_l = 0$ ,  $\forall l \notin \mathcal{L}^*$ . Thus, we can super-replicate financial payments of asset in  $\mathcal{J} \setminus \mathcal{J}_a$  by choosing  $(\hat{z}_k)_{k \in \mathcal{J}_a} = 0$  and  $\hat{x}_0 = a(1, \dots, 1)$ , where  $a > 0$  satisfies  $\max_{l \in \mathcal{L}^*} \max_{(p,s) \in \mathcal{P} \times \hat{S}} \sum_{j \in \mathcal{J} \setminus \mathcal{J}_a} (A_{s,j}(p))_l < a \min_{(s,l) \in \hat{S} \times \mathcal{L}^*} Y_s(l, l)$  and  $\hat{S} := \bigcup_{j \in \mathcal{J}} S_j$  are the states of nature in which assets make promises.  $\square$

<sup>10</sup>Assumption (A3) is only required to ensure that  $[(\bar{p}, \bar{q}) \gg 0 \wedge (\bar{x}^i, \bar{z}^i)_{i \in \mathcal{I}} \in \Omega(\bar{p}, \bar{q})] \implies (\bar{x}^i, \bar{z}^i)_{i \in \mathcal{I}} \in \text{int}(\mathbb{K})$ . In fact, if we change the upper bound  $\bar{\Omega}$  for an arbitrary positive number in the definition of  $\mathbb{K}$ , then all the other arguments in the proof of Theorem 1 still hold.

PROOF OF COROLLARY 3. Since assets are backed by physical collateral, it is possible to super-replicate financial payments by choosing  $(\widehat{z}_k)_{k \in \mathcal{J}_a} = 0$  and  $\widehat{x}_0 = \sum_{j \in \mathcal{J}} C_j$ . Therefore, Assumption (A4) holds.  $\square$

DETAILS OF EXAMPLE 5. Since there is only one commodity, financial allocations determine consumption. Hence, given a price  $q > 0$  for the asset, individual problems are equivalent to

$$\max_{z^A \in [0, q^{-1}]} 4 \frac{1 - qz^A}{2 - qz^A} + \sqrt{1 + z^A}, \quad \max_{z^B \in [-\nu, q^{-1}]} 4\sqrt{1 - qz^B} + \sqrt{\nu + z^B}.$$

Let  $(\bar{z}^A(q), \bar{z}^B(q))$  be the vector of optimal portfolios at prices  $q$ . By Inada's condition, we know that  $\bar{z}^B(q) \in (-\nu, q^{-1})$ . If we suppose that  $\bar{z}^A(q) \in (0, q^{-1})$ , then first order conditions of individual's problems implies that

$$\frac{16q^2}{(2 - q\bar{z}^A(q))^4} = \frac{1}{4(1 + \bar{z}^A(q))}, \quad \frac{4q^2}{1 - q\bar{z}^B(q)} = \frac{1}{4(\nu + \bar{z}^B(q))}.$$

Since  $\bar{z}^A(q) > 0$  and, in equilibrium,  $\bar{z}^B(q) = -\bar{z}^A(q)$ , we obtain that

$$\frac{\nu}{4} > \frac{(\nu - \bar{z}^A(q))}{4(1 + \bar{z}^A(q))} = \frac{1 + q\bar{z}^A(q)}{(2 - q\bar{z}^A(q))^4} > \frac{1}{16},$$

which is a contradiction with  $\nu \in (0, 0.25)$ . Thus, for every  $q > 0$ ,  $\bar{z}^A(q) \in \{0, q^{-1}\}$ . Notice that, agent A prefers to trade the asset if and only if  $3 \leq \sqrt{1 + q^{-1}}$ . Also, if there is an equilibrium with financial trade, then  $\bar{z}^A(q) = q^{-1}$ ,  $\bar{z}^B(q) = -q^{-1}$ . Therefore,  $q^{-1} \leq \nu \leq 0.25$ , which implies that  $\sqrt{1 + q^{-1}} \leq \sqrt{1.25} < 3$ , a contradiction.

Therefore, there is no trade in equilibrium and the asset price is determined by the first order condition of agent B:  $4q^2 = 1/(4\nu)$ . That is, the equilibrium asset price is equal to  $1/(4\sqrt{\nu})$ .  $\square$

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