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NON-EQUILIBRIUM ISOTHERMAL TRANSFORMATIONS IN A TEMPERATURE GRADIENT FROM A MICROSCOPIC DYNAMICS

STEFANO OLLA AND VIVIANA LETIZIA

ABSTRACT. We consider a chain of anharmonic oscillators immersed in a heat bath with a temperature gradient and a time varying tension applied to one end of the chain while the other side is fixed to a point. We prove that under diffusive space-time rescaling the volume strain distribution of the chain evolves following a non-linear diffusive equation. The stationary states of the dynamics are of non-equilibrium and have a positive entropy production, so the classical relative entropy methods cannot be used. We develop new estimates based on entropic hypocoercivity, that allows to control the distribution of the positions configurations of the chain. The macroscopic limit can be used to model isothermal thermodynamic transformations between non-equilibrium stationary states.

1. INTRODUCTION

Macroscopic isothermal thermodynamic transformations can be modeled microscopically by putting a system in contact with Langevin heat bath at a given temperature β^{-1} . In [8] a chain of n anharmonic oscillators is *immersed* in a heat bath of Langevin thermostats acting independently on each particle. Macroscopically equivalent isothermal dynamics is obtained by elastic collision with an external gas of independent particles with maxwellian random velocities with variance β^{-1} . The effect is to renew fast the velocities distribution of the particles, so that at any given time it is very close to a maxwellian at given temperature. The chain is pinned only on one side, while at the opposite site a force (tension) τ is acting. The equilibrium distribution is characterized by the two control parameters β^{-1}, τ (temperature and tension). The total length and the energy of the system in equilibrium are in general non-linear functions of these parameters given by the standard thermodynamic relations.

By changing the tension τ applied to the system, a new equilibrium state, with the same temperature, will be eventually reached. For large n , the time scale in which the system converges to equilibrium is n^2 , much slower than the time scale the heat bath equilibrates the velocities at the corresponding temperature. In [8] it is proven that the length stretch of the system evolves in a diffusive space-time scale, i.e. after a scaling limit the empirical distribution of the interparticle distances converges to the solution of a non-linear diffusive equation governed by the local tension. Consequently

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this diffusive equation describe the non-reversible isothermal thermodynamic transformation from an equilibrium to another with a different tension. By a further rescaling of the time dependence of the changing tension, is obtained a so called *quasi-static* or *reversible* isothermal transformation. Corresponding Clausius equalities/inequalities relating work done and change in free energy can be proven.

The results of [8] resumed above concern isothermal transformations from an *equilibrium* state to another, by changing the applied tension. We are here interested in trasformations between *non-equilibrium* stationary states. We consider now the chain of oscillators immersed in a heat bath with a *macroscopic gradient* of temperatures: each particles is in contact with thermostats at different temperatures. These temperatures changes slowly from a particle to the neighbor one. A tension τ is again applied to the chain. In the stationary state, that is now characterized by the tension τ and the profile of temperatures $\beta_1^{-1}, \dots, \beta_n^{-1}$, there is a continuous flow of energy through the chain from the hot thermostats to the cold ones. Unlike the equilibrium case, the probability distribution of the configurations of the chain in the stationary state cannot be computed explicitly.

By changing the applied tension we can obtain transitions from a non-equilibrium stationary state to another, that will happen in a diffusive space-time scale as in the equilibrium case. The main result in the present article is that transformations are again governed by a diffusive equation that takes into account the local temperature profile. The free energy can be computed according to the local equilibrium rule and its changes in the transformation satisfy the Clausius inequality with respect to the work done. This provides a mathematically precise example for understanding non-equilibrium thermodynamics from microscopic dynamics.

The results in [8] where obtained by using the relative entropy method, first developed by H.T.Yau in [16] for the Ginzburg-Landau dynamics, which is just the overdamped version of the bulk dynamics of the oscillators chain. The relative entropy method is very powerful and flexible, and was already applied to the interacting Ornstein-Uhlenbeck particles in the PhD thesis of Tremoulet [13] as well as many other cases, in particular in the hyperbolic scaling limit for Euler equation in the smooth regime [9, 4]. This method is based in looking at the time evolution of the relative entropy of the distribution of the particle with respect to the local Gibbs measure parametrized by the non-constant tension profile corresponding to the solution of the macroscopic diffusion equation. The point of the method is in proving that the time derivative of such relative entropy is small, so that the relative entropy itself remains small with respect to the size of the system and local equilibrium, in a weak but sufficient form, propagates in time. In the particular applications in [13, 8], the local Gibbs measure needs to be corrected by a small recentering of the damped velocities due to the local gradient of the tension.

The relative entropy method seems to fail when the stationary measures are not the equilibrium Gibbs measure, like in the present case. The reason is that when taking the time derivative of the relative entropy mentioned above, a big term, proportional to the gradient of the temperature, appears. This term is related to the *entropy production* of the stationary measure. Consequently we could not apply the relative entropy method in the present problem.

A previous method was developed by Guo, Papanicolaou and Varadhan in [6] for the overdamped dynamics. In this approach the main step in closing the macroscopic

equation is the direct comparison of the coarse grained empirical density in the microscopic and macroscopic space scale. They first obtain a bound of the Dirichlet form (more precisely called Fisher information) from the time derivative of the relative entropy with respect to the equilibrium stationary measures. This bound implies that the system is close to equilibrium on a local microscopic scale, and that the density on a large microscopic interval is close to the density in a small macroscopic interval (the so called one and two block estimates, see [7] chapter 5).

In the overdamped dynamics considered in [6], the Dirichlet form appearing in the time derivative of the relative entropy controls the gradients of the probability distributions with respects to the position of the particles. In the damped models the Dirichlet form appearing in the time derivative of the relative entropy controls only the gradients of the probability density with respect to the velocities of the particles. In order to deal with damped models a different approach for comparing densities on the different scales was developed in [10], after the overdamped case in [14], based on Young measures. Unfortunately this approach requires a control of higher moments of the density that are difficult to prove for lattice models. Consequently we could not apply this method either in the present situation.

The main mathematical novelty in the present article is the use of entropic hypocoercivity, inspired by [15]. We introduce a Fisher information form I_n associated to the vector fields $\{\partial_{p_i} + \partial_{q_i}\}_{i=1,\dots,n}$, defined by (2.25). By computing the time derivative of this Fisher information form on the distribution at time t of the configurations, we obtain a uniform bound $I_n \leq Cn^{-1}$. This implies that at the macroscopic diffusive time scale gradients of the distribution in the velocities are very close to gradients in the positions. This permits to obtain a bound on the Fisher information on the position from the bound on the Fisher information on the velocities. At this point we are essentially with the same information as in the overdamped model and proceed as in [6]. Observe that the form I_n we introduce in (2.25) is more specific and a bit different than the distorted Fisher information used by Villani in [15], in particular I_n is more degenerate. On the other hand the calculations, that are contained in appendix D are less *miraculous* than in [15], and they are stable enough to control the effect of the boundary tension and of the gradient of temperature. This also suggest that entropic hypocoercivity seems to be the right tool in order to obtain explicit estimates uniform in the dimension of the system.

Adiabatic thermodynamic transformations are certainly more difficult to be obtained from microscopic dynamics, for some preliminar results see [9, 4, 1, 11]. Equilibrium fluctuations for the dynamics with constant temperature can be treated as in [12]. The fluctuations in the case with a gradient of temperature are non-equilibrium fluctuation, we believe that can be treated with the techniques of the present article together with those developed in the overdamped case in [5].

Large deviations for the stationary measure also requires some further mathematical investigations, but we conjecture that the corresponding quasi-potential functional ([3]) is given by the free energy associated to the local Gibbs measure, without any non-local terms, unlike the case of the simple exclusion process.

The article is structured in the following way. In section section 2 we define the dynamics and we state the main result (Theorem 2.1). In section 3 we discuss the consequences for the thermodynamic trasformations from a stationary state to another, the Clausius inequality and the quasi-static limit. In section 4 are obtained the

bounds on the entropy and the various Fisher informations needed in the proof of the hydrodynamic limit. In section 5 we show that any limit point of the distribution of the empirical density on strain of the volume is concentrated in the weak solutions of the macroscopic diffusion equation. The compactness, regularity and uniqueness of the corresponding weak solution, necessary to conclude the proof, are proven in the first three appendices. Appendix D contains the calculations and estimates for the time derivative of the Fisher information I_n .

2. THE DYNAMICS AND THE RESULTS

We consider a chain of n coupled oscillators in one dimension. Each particle has the same mass, equal to one. The configuration in the phase space is described by $\{q_i, p_i, i = 1, \dots, n\} \in \mathbb{R}^{2n}$. The interaction between two particles i and $i-1$ is described by the potential energy $V(q_i - q_{i-1})$ of an anharmonic spring. The chain is attached on the left to a fixed point, so we set $q_0 = 0, p_0 = 0$. We call $\{r_i = q_i - q_{i-1}, i = 1, \dots, n\}$ the interparticle distance. We assume V to be a positive smooth function, and that there exists a constant $C > 0$ such that:

$$\lim_{|r| \rightarrow \infty} \frac{V(r)}{|r|} = \infty, \quad \lim_{|r| \rightarrow \infty} V''(r) \leq C < \infty \quad (2.1)$$

Energy is defined by the following Hamiltonian:

$$\mathcal{H} := \sum_{i=1}^n \left(\frac{p_i^2}{2} + V(r_i) \right) \quad (2.2)$$

The particle dynamics is subject to an interaction with an environment given by Langevin heat bath at different temperatures β_i^{-1} . We choose β_i as slowly varying on a macroscopic scale, i.e. $\beta_i = \beta(i/n)$ for a given smooth strictly positive function $\beta(x)$, $x \in [0, 1]$.

The equations of motion are given by

$$\begin{cases} dr_i(t) = n^2(p_i(t) - p_{i-1}(t))dt \\ dp_i(t) = n^2(V'(r_{i+1}(t)) - V'(r_i(t))) dt - n^2\gamma p_i(t)dt + n\sqrt{\frac{2\gamma}{\beta_i}}dw_i(t), \quad i = 1, \dots, N-1 \\ dp_n(t) = n^2(\bar{\tau}(t) - V'(r_n(t))) dt - n^2\gamma p_n(t) dt + n\sqrt{\frac{2\gamma}{\beta_n}}dw_n(t). \end{cases} \quad (2.3)$$

Here $\{w_i(t)\}_i$ are n -independent Wiener processes, $\gamma > 0$ is the coupling parameter with the Langevin thermostats. The time is rescaled according to the diffusive space-time scaling, i.e. t is the macroscopic time. The tension $\bar{\tau} = \bar{\tau}(t)$ changes at the macroscopic time scale (i.e. very slowly in the microscopic time scale). The generator of the diffusion is given by

$$\mathcal{L}_n^{\bar{\tau}(t)} := n^2 \mathcal{A}_n^{\bar{\tau}(t)} + n^2 \gamma \mathcal{S}_n, \quad (2.4)$$

where $\mathcal{A}_n^{\bar{\tau}}$ is the Liouville generator

$$\mathcal{A}_n^{\bar{\tau}} = \sum_{i=1}^n (p_i - p_{i-1}) \partial_{r_i} + \sum_{i=1}^{n-1} (V'(r_{i+1}) - V'(r_i)) \partial_{p_i} + (\bar{\tau} - V'(r_n)) \partial_{p_n} \quad (2.5)$$

while \mathcal{S}_n is the operator

$$\mathcal{S}_n = \sum_{i=1}^n (\beta_i^{-1} \partial_{p_i}^2 - p_i \partial_{p_i}) \quad (2.6)$$

An equivalent dynamics is given by a different modelling of the heat bath: particle i undergoes to stochastic elastic collisions with *particles* of the environment at temperature β_i^{-1} , i.e. a independent exponentially distributed times of intensity γn^2 particle i change its velocity to a new random velocity normally distributed with variance β_i^{-1} . The evolution equations are given by:

$$\begin{cases} dr_i(t) = n^2(p_i(t) - p_{i-1}(t))dt \\ dp_i(t) = n^2(V'(r_{i+1}(t)) - V'(r_i(t)))dt + (\tilde{p}_{i,N_i(n^2t)} - p(t^-)) dN_i(\gamma n^2t) \\ dp_n(t) = n^2(\bar{\tau}(t) - V'(r_n))dt + (\tilde{p}_{n,N_n(n^2t)} - p_n(t^-)) dN_n(\gamma n^2t) \end{cases} \quad (2.7)$$

where $\tilde{p}_{i,k}$ are independent gaussian variables on mean zero and variance β_i^{-1} , and $\{N_i(t), i = 1, \dots, n\}$ are independent Poisson processes of intensity 1. All result presented in this article can be extended in a straightforward way to the dynamics (2.7).

2.1. Gibbs measures. For $\bar{\tau}(t) = \tau$ constant, and $\beta_i = \beta$ homogeneous, the system has a unique invariant measure given by a product of invariant Gibbs measures $\mu_{\tau,\beta}^n$:

$$d\mu_{\tau,\beta}^n = \prod_{i=1}^n e^{-\beta(\mathcal{E}_i - \tau r_i) - \mathcal{G}_{\tau,\beta}} dr_i dp_i \quad (2.8)$$

where \mathcal{E}_i is the energy of the particle i :

$$\mathcal{E}_i = \frac{p_i^2}{2} + V(r_i). \quad (2.9)$$

The function $\mathcal{G}_{\tau,\beta}$ is the Gibbs potential defined as:

$$\mathcal{G}_{\tau,\beta} = \log \left[\sqrt{2\pi\beta^{-1}} \int e^{-\beta(V(r) - \tau r)} dr \right]. \quad (2.10)$$

The free energy of the equilibrium state (r, β) is given by the Legendre transform of $\beta^{-1}\mathcal{G}_{\tau,\beta}$:

$$\mathcal{F}_{r,\beta} = \inf_{\tau} \{\tau r - \beta^{-1}\mathcal{G}_{\tau,\beta}\} \quad (2.11)$$

The corresponding convex conjugate variables are the lenght

$$\mathbf{r}(\tau, \beta) = \beta^{-1}\partial_{\tau}\mathcal{G}_{\tau,\beta} \quad (2.12)$$

and the tension

$$\boldsymbol{\tau}(r, \beta) = \partial_r\mathcal{F}_{r,\beta}. \quad (2.13)$$

Observe that

$$\mathbb{E}_{\mu_{\tau,\beta}^n}[r] = \mathbf{r}, \quad \mathbb{E}_{\mu_{\tau,\beta}^n}[V'(r)] = \boldsymbol{\tau}. \quad (2.14)$$

2.2. The hydrodynamic limit. We assume that for a given initial profile $r_0(x)$ the initial distribution satisfies:

$$\frac{1}{n} \sum_{i=1}^n G(i/n)r_i(0) \longrightarrow \int_0^1 G(x)r_0(x)dx \quad (2.15)$$

for any continuous test function $G \in \mathcal{C}_0([0, 1])$. We expect that this same convergence happens at the macroscopic time t :

$$\frac{1}{n} \sum_{i=1}^n G(i/n) r_i(t) \longrightarrow \int_0^1 G(x) r(x, t) dx \quad (2.16)$$

where $r(x, t)$ satisfies the following diffusive equation

$$\begin{cases} \partial_t r(x, t) = \frac{1}{\gamma} \partial_x^2 \tau(r(x, t), \beta(x)) & \text{for } x \in [0, 1] \\ \partial_x \tau(r(t, x), \beta(x))|_{x=0} = 0, \quad \tau(r(t, x), \beta(x))|_{x=1} = \bar{\tau}(t), & t > 0 \\ \tau(r(0, x), \beta(x)) = \tau_0(x), & x \in [0, 1] \end{cases} \quad (2.17)$$

We say that $r(x, t)$ is a weak solution of (2.17) if for any smooth function $G(x)$ on $[0, 1]$ such that $G(1) = 0$ and $G'(0) = 0$ we have

$$\int_0^1 G(x) (r(x, t) - r_0(x)) dx = \gamma^{-1} \int_0^t ds \left[\int_0^1 G''(x) \tau(r(x, s), \beta(x)) dx - G'(1) \bar{\tau}(s) \right]. \quad (2.18)$$

In appendix C we prove that the weak solution is unique in the class of functions such that:

$$\int_0^t ds \int_0^1 (\partial_x \tau(r(x, s), \beta(x)))^2 dx < +\infty \quad (2.19)$$

Let ν_β^n the inhomogeneous Gibbs measure

$$d\nu_\beta^n = \prod_{i=1}^n \frac{e^{-\beta_i \mathcal{E}_i}}{Z_{\beta_i}} \quad (2.20)$$

Observe that this is **not** the stationary measure for this dynamics for $\bar{\tau} = 0$.

Let f_t^n the density, with respect to ν_β^n , of the probability distribution of the system at time t , i.e. the solution of

$$\partial_t f_t^n = \mathcal{L}_n^{\bar{\tau}(t),*} f_t^n, \quad (2.21)$$

where $\mathcal{L}_n^{\bar{\tau}(t),*}$ is the adjoint of $\mathcal{L}_n^{\bar{\tau}(t)}$ with respect to ν_β^n , i.e. explicitly

$$(\mathcal{L}_n^{\bar{\tau}(t)})^* = -n^2 \mathcal{A}_n^{\bar{\tau}(t)} - n \sum_{i=1}^{n-1} \nabla_n \beta(i/n) p_i V'(r_{i+1}) + n^2 \beta(1) p_n \bar{\tau} + n^2 \gamma \mathcal{S}_n. \quad (2.22)$$

Define the relative entropy of $f_t^n d\nu_\beta^n$ with respect to $d\nu_\beta^n$:

$$H_n(t) = \int f_t^n \log f_t^n d\nu_\beta^n. \quad (2.23)$$

We assume that the initial density f_0^n satisfy the bound

$$H_n(t) \leq Cn. \quad (2.24)$$

We also need some regularity of f_0^n : define the hypercoercive Fisher information functional:

$$I_n(t) = \sum_{i=1}^{n-1} \beta_i^{-1} \int \frac{(\partial_{p_i} f_t^n + \partial_{q_i} f_t^n)^2}{f_t^n} d\nu_\beta. \quad (2.25)$$

where $\partial_{q_i} = \partial_{r_i} - \partial_{r_{i+1}}$, $i = 1, \dots, n-1$, and $\nu_\beta := \nu_\beta^n$. We assume that

$$I_n(0) \leq K_n \quad (2.26)$$

with K_n growing less than exponential in n . We will show in Appendix D that for any $t > 0$ we have $I_n(t) \leq Cn^{-1}$.

Furthermore we assume that

$$\lim_{n \rightarrow \infty} \int \left| \frac{1}{n} \sum_{i=1}^n G\left(\frac{i}{n}\right) r_i - \int_0^1 G(x) r_0(x) dx \right| f_0^n d\nu_\beta = 0 \quad (2.27)$$

for any continuous test function $G \in \mathcal{C}_0([0, 1])$.

Theorem 2.1. *Assume that the starting initial distribution satisfy the above conditions. Then*

$$\lim_{n \rightarrow \infty} \int \left| \frac{1}{n} \sum_{i=1}^n G\left(\frac{i}{n}\right) r_i - \int_0^1 G(x) r(x, t) dx \right| f_t^n d\nu_\beta = 0, \quad (2.28)$$

where $r(x, t)$ is the unique weak solution of (2.17) satisfying (2.19).

Furthermore a local equilibrium result is valid in the following sense: consider a local function $\phi(\mathbf{r}, \mathbf{p})$ such that

$$|\phi(\mathbf{r}, \mathbf{p})| \leq C \sum_{i \in \Lambda_\phi} (p_i^2 + V(r_i))^\alpha, \quad \alpha < 1$$

where Λ_ϕ is the local support of ϕ . Let $\theta_i \phi$ be the shifted function (well defined for a local function for $i \ll n$) and define

$$\hat{\phi}(r, \beta) = \mathbb{E}_{\mu_{\tau, \beta}}(\phi), \quad \tau = \tau(r, \beta) \quad (2.29)$$

Corollary 2.2.

$$\lim_{n \rightarrow \infty} \int \left| \frac{1}{n} \sum_{i=1}^n G\left(\frac{i}{n}\right) \theta_i \phi(\mathbf{r}, \mathbf{p}) - \int_0^1 \mathbf{G}(\mathbf{x}) \hat{\phi}(\mathbf{r}(\mathbf{x}, \mathbf{t}), \beta(\mathbf{x})) d\mathbf{x} \right| f_t^n d\nu_\beta = 0, \quad (2.30)$$

3. NON-EQUILIBRIUM THERMODYNAMICS

We collect in this section some interesting consequences of the main theorem for the non-equilibrium thermodynamics of this system. All statement contained in this section can be proven rigorously, except one that will require more investigation in the future. The aim is to build a non equilibrium thermodynamics in the spirit of [2, 3]. The equilibrium version of these results had been already proven in [8].

As we already mentioned, stationary states of our dynamics are not given by Gibbs measure if a gradient in the temperature profile is present, but they are characterized by the tension $\bar{\tau}$ applied. Let us denote $f_{ss, \tau}^n$ the density of the stationary distribution with respect to ν_β .

It is easy to see that

$$\int V'(r_i) f_{ss, \tau}^n \nu_\beta = \tau, \quad i = 1, \dots, n. \quad (3.1)$$

Also by using the main results and the techniques used in proving it that there exists a stationary profile of stretch $r_{ss, \tau}(y) = \mathfrak{r}(\tau, \beta(y))$ (defined by (2.12)) such that for any continuous test function G :

$$\lim_{n \rightarrow \infty} \int \left| \frac{1}{n} \sum_{i=1}^n G\left(\frac{i}{n}\right) r_i - \int_0^1 G(x) r_{ss, \tau}(x) dx \right| f_{ss, \tau}^n d\nu_\beta = 0, \quad (3.2)$$

In order to study the transition from one stationary state to another with different tension, we start the system at time 0 with a stationary state with tension τ_0 , and we change tension with time, setting $\bar{\tau}(t) = \tau_1$ for $t \geq t_1$. The distribution of the system will eventually converge to a stationary state with tension τ_1 .

Consequently in this section $r(x, t)$ is the solution of the macroscopic equation starting with $r_0(x) = r_{ss, \tau_0}(x)$. Clearly $r(x, t) \rightarrow r_1(x) = r_{ss, \tau_1}(x)$, as $t \rightarrow \infty$.

3.1. Excess Heat. The (normalized) total internal energy of the system is defined by

$$U_n := \frac{1}{n} \sum_{i=1}^n \left(\frac{p_i^2}{2} + V(r_i) \right) \quad (3.3)$$

then internal energy evolves as:

$$U_n(t) - U_n(0) = \mathcal{W}_n(t) + Q_n(t)$$

where

$$\mathcal{W}_n(t) = \int_0^t \bar{\tau}(s) n p_n(s) ds = \int_0^t \bar{\tau}(s) \frac{dq_n(s)}{n}$$

is the (normalized) work done by the force $\bar{\tau}(s)$ up to time t , while

$$Q_n(t) = \gamma n \sum_{j=1}^n \int_0^t ds \left(p_j^2(s) - \beta_j^{-1} \right) + \sum_{j=1}^n \sqrt{2\gamma\beta_j^{-1}} \int_0^t p_j(s) dw_i(s). \quad (3.4)$$

is the total flux of energy between the system and the heat bath (divided by n). As a consequence of theorem 2.1 we have that

$$\lim_{n \rightarrow \infty} \mathcal{W}_n(t) = \int_0^t \bar{\tau}(s) d\mathcal{L}(s)$$

where $\mathcal{L}(t) = \int_0^1 r(x, t) dx$, the total macroscopic length at time t . While for the energy difference we expect that

$$\lim_{n \rightarrow \infty} (U_n(t) - U_n(0)) = \int_0^1 [u(\boldsymbol{\tau}(r(x, t), \beta(x)), \beta(x)) - u(\tau_0, \beta(x))] dx \quad (3.5)$$

where $u(\tau, \beta)$ is the average energy for $\mu_{\beta, \tau}$, i.e.

$$u(\tau, \beta) = \int \mathcal{E}_1 d\mu_{\tau, \beta}^1 = \frac{1}{2\beta} + \int V(r) e^{-\beta(V(r) - \tau r) - \tilde{\mathcal{G}}(\tau, \beta)} dr$$

with $\tilde{\mathcal{G}}(\tau, \beta) = \log \int e^{-\beta(V(r) - \tau r)} dr$. Unfortunately for lack of uniform integrability for the energy distribution, we do not have a rigorous proof of (3.5). In the equilibrium case, constant energy profile, treated in [8], this limit can be computed rigorously thanks to the use on the relative entropy method [16] that gives a better control on the local distribution of the energy.

After the limit $t \rightarrow \infty$ the macroscopic equation will reach the stationary state at tension τ_1 , so that $u(\boldsymbol{\tau}(r(x, t), \beta(x)), \beta(x)) \rightarrow u(\tau_1, \beta(x))$ and the difference of energies will become

$$\int_0^1 (u(\tau_1, \beta(x)) - u(\tau_0, \beta(x))) dx = \int_0^{+\infty} \bar{\tau}(s) d\mathcal{L}(s) ds + Q = \mathcal{W} + Q \quad (3.6)$$

where Q is the limit of (3.4), that is called *excess heat*. So equation (3.6) is the expression of the first principle of thermodynamics in this *isothermal* transformation

between non-equilibrium stationary states. Here *isothermal* means that the profile of temperature does not change during the transformation.

3.2. Free energy. Define

$$\tilde{\mathcal{F}}(t) = \int_0^1 \mathcal{F}(r(x, t), \beta(x)) dx \quad (3.7)$$

and

$$\tilde{\mathcal{F}}_{ss}(\tau) = \int_0^1 \mathcal{F}(r_{ss, \tau}(x), \beta(x)) dx \quad (3.8)$$

A straightforward calculation gives

$$\tilde{\mathcal{F}}(t) - \tilde{\mathcal{F}}_{ss}(\tau_0) = \mathcal{W}(t) - \gamma^{-1} \int_0^t ds \int_0^1 (\partial_x \tau(r(x, s), \beta(x)))^2 dx \quad (3.9)$$

and after the time limit $t \rightarrow \infty$

$$\begin{aligned} \tilde{\mathcal{F}}_{ss}(\tau_1) - \tilde{\mathcal{F}}_{ss}(\tau_0) &= \mathcal{W} - \gamma^{-1} \int_0^{+\infty} dt \int_0^1 (\partial_x \tau(r(x, t), \beta(x)))^2 dx \\ &\leq \mathcal{W} \end{aligned} \quad (3.10)$$

i.e. Clausius inequality for NESS.

3.3. Quasi-static limit and reversible transformations. In the spirit of [2] and [8], the quasi static limit is obtained by rescaling the time dependence of the driving tension $\bar{\tau}$ by a small parameter ε , i.e. by choosing $\bar{\tau}(\varepsilon t)$. Of course the right time scale at which the evolution appears is $\varepsilon^{-1}t$ and the rescaled solution $\tilde{r}^\varepsilon(x, t) = r(x, \varepsilon^{-1}t)$ satisfy the equation

$$\begin{cases} \partial_t \tilde{r}^\varepsilon(x, t) = \frac{1}{\varepsilon \gamma} \partial_x^2 \tau(\tilde{r}^\varepsilon(x, t), \beta(x)) & \text{for } x \in [0, 1] \\ \partial_x \tau(\tilde{r}^\varepsilon(t, x), \beta(x))|_{x=0} = 0, \quad \tau(\tilde{r}^\varepsilon(t, x), \beta(x))|_{x=1} = \bar{\tau}(t), & t > 0 \\ \tau(\tilde{r}^\varepsilon(0, x), \beta(x)) = \tau_0, & x \in [0, 1] \end{cases} \quad (3.11)$$

By repeating the argument above one obtains:

$$\tilde{\mathcal{F}}_{ss}(\tau_1) - \tilde{\mathcal{F}}_{ss}(\tau_0) = \mathcal{W}^\varepsilon - \frac{1}{\varepsilon \gamma} \int_0^{+\infty} dt \int_0^1 (\partial_x \tau(\tilde{r}^\varepsilon(x, t), \beta(x)))^2 dx \quad (3.12)$$

By the same argument used in [8] for β constant, it can be proven that the last term on the right hand side of (3.12) converges to 0 as $\varepsilon \rightarrow 0$, and that $\tau(\tilde{r}^\varepsilon(x, t), \beta(x)) \rightarrow \bar{\tau}(t)$ for almost any $x \in [0, 1]$ and $t \geq 0$. Consequently in the quasistatic limit we have the Clausius equality

$$\tilde{\mathcal{F}}_{ss}(\tau_1) - \tilde{\mathcal{F}}_{ss}(\tau_0) = \mathcal{W}$$

This implies the following equality in the heat in the quasi-static limit:

$$Q = \int_0^1 \beta^{-1}(x) (S(r_{ss}(x, \tau_1), u_{ss}(x, \tau_1)) - S(r_{ss}(x, \tau_0), u_{ss}(x, \tau_0))) dx \quad (3.13)$$

analogous of the equilibrium equality $Q = T \Delta S$.

4. ENTROPY AND HYPERCOERCIVE BOUNDS

Taking the time derivative of the entropy we obtain:

$$\frac{d}{dt}H_n(t) = \int (\mathcal{L}_n^{\bar{\tau}(t)})^* f_t^n \log f_t^n d\nu_\beta. \quad (4.1)$$

So that

$$\begin{aligned} \frac{d}{dt}H_n(t) &= \int f_t^n \mathcal{L}_n^{\bar{\tau}(t)} \log f_t^n d\nu_\beta. \\ &= \int n^2 \mathcal{A}_n^{\bar{\tau}(t)} f d\nu_\beta. - \gamma n^2 \sum_{i=1}^n \beta_i^{-1} \int \frac{(\partial_{p_i} f_t^n)^2}{f_t^n} d\nu_\beta. \\ &= -n \sum_{i=1}^{n-1} \nabla_n \beta(i/n) \int V'(r_{i+1}) p_i f_t^n d\nu_\beta. + n^2 \beta_n \bar{\tau}(t) \int p_n f_t^n d\nu_\beta. - \gamma n^2 \sum_{i=1}^n \mathcal{D}_{i,n}^p(f) \end{aligned} \quad (4.2)$$

where we have denoted

$$\mathcal{D}_{i,n}^p(f) = \beta_i^{-1} \int \frac{(\partial_{p_i} f)^2}{f} d\nu_\beta.$$

The second term on the right hand side of (4.2) is proportional, after time integration, to the work done by the tension, so it can be easily estimate in the following way. Recall that $q_n = \sum_{i=1}^n r_i$,

$$\begin{aligned} n^2 \beta_n \int_0^t ds \bar{\tau}(s) \int p_n f_s^n d\nu_\beta. &= \beta_n \int_0^t ds \bar{\tau}(s) \int \mathcal{L}_n^{\bar{\tau}(s)} q_n f_s^n d\nu_\beta. \\ &= \beta_n \bar{\tau}(t) \int q_n f_t^n d\nu_\beta. - \beta_n \bar{\tau}(0) \int q_n f_0^n d\nu_\beta. + \beta_n \int_0^t ds \bar{\tau}'(s) \int q_n f_s^n d\nu_\beta. \end{aligned} \quad (4.3)$$

By the entropy inequality, for any $a_1 > 0$

$$\int |q_n| f_s^n d\nu_\beta. \leq \frac{1}{a_1} \log \int e^{a_1 |q_n|} d\nu_\beta. + \frac{1}{a_1} H_n(s) \quad (4.4)$$

that is clearly bounded by cn for some constant c , and this applies to the three terms of the RHS of (4.3). So after time integration we can estimate, for any $a_1 > 0$,

$$\left| \int_0^t ds n^2 \beta_n \bar{\tau}(t) \int p_n f_t^n d\nu_\beta. \right| \leq \frac{C}{a_1} \left(H_n(t) + H_n(0) + \int_0^t H_n(s) ds \right) + c_{a_1} n \quad (4.5)$$

for C depending from $\beta, \bar{\tau}$ and c_{a_1} also on a_1 .

By integration by part and Young inequality, for any a_2 we have

$$\begin{aligned} \left| n \sum_{i=1}^{n-1} \nabla_n \beta(i/n) \int V'(r_{i+1}) p_i f_t^n d\nu_\beta. \right| &\leq \frac{1}{2a_2} \sum_{i=1}^{n-1} \frac{(\nabla_n \beta(i/n))^2}{\beta_i} \int V'(r_{i+1})^2 f_t^n d\nu_\beta. \\ &\quad + \frac{a_2 n^2}{2} \sum_{i=1}^{n-1} \mathcal{D}_{i,n}^p(f_t^n) \end{aligned}$$

By our assumptions on $\beta(\cdot)$ and V , we have that

$$\sum_{i=1}^{n-1} \frac{(\nabla_n \beta(i/n))^2}{\beta_i} V'(r_{i+1})^2 \leq C_\beta \sum_{i=1}^n V(r_i)$$

Again by the entropy inequality:

$$\begin{aligned} \sum_{i=1}^n \int V(r_i) f_t^n d\nu_\beta &\leq \frac{1}{\delta} \log \int e^{\delta \sum_{i=1}^n \int V(r_i)} d\nu_\beta + \frac{1}{\delta} H_n(t) \\ &= \frac{1}{\delta} \sum_{i=1}^n \log \frac{\int e^{-(\beta_i - \delta)V(r)} dr}{\int e^{-\beta_i V(r)} dr} + \frac{1}{\delta} H_n(t) \leq C_{\delta, \beta} n + \frac{1}{\delta} H_n(t) \end{aligned}$$

At this point we have obtained the inequality

$$\begin{aligned} H_n(t) - H_n(0) &\leq -n^2 \left(\gamma - \frac{a_2}{2} \right) \int_0^t \sum_{i=1}^n \mathcal{D}_{i,n}^p(f_s^n) ds + C (\delta^{-1} + a_1^{-1}) \int_0^t H_n(s) ds \\ &\quad + \frac{C}{a_1} (H_n(t) + H_n(0)) + c_{a_1, a_2, \delta} n \end{aligned} \quad (4.6)$$

consequently, choosing $a_2 = \gamma$ and $a_1 = 2C$ we have

$$H_n(t) \leq 3H_n(0) + C' \int_0^t H_n(s) ds + cn - n^2 \int_0^t \sum_{i=1}^n \mathcal{D}_{i,n}^p(f_s^n) ds$$

where C' and c are constants independent of n . Given the initial bound on $H_n(0) \leq cn$, by Gronwall inequality we have that $H_n(t) \leq c' e^{C't} n$. Inserting this in (4.6) we obtain

$$\frac{\gamma}{2} \int_0^t \sum_{i=1}^n \mathcal{D}_{i,n}^p(f_s^n) ds \leq \frac{\tilde{C}}{n} \quad (4.7)$$

The bound (4.7) gives only informations about the distribution of the velocities, but we actually need a corresponding bound of the distribution of the positions.

Define for $i = 1, \dots, n-1$

$$\mathcal{D}_{i,n}^r(f) = \beta_i^{-1} \int \frac{(\partial_{q_i} f)^2}{f} d\nu_\beta, \quad (4.8)$$

where $\partial_{q_i} = (\partial_{r_i} - \partial_{r_{i+1}})$.

Recall the definition of $I_n(t)$ given at (2.25), and that we assumed $I_n(0)$ growing less than exponential. In appendix D we prove that

$$I_n(t) = \sum_{i=1}^{n-1} \left(\mathcal{D}_{i,n}^p(f) + \mathcal{D}_{i,n}^r(f) + 2\beta_i^{-1} \int \frac{(\partial_{q_i} f)(\partial_{p_i} f)}{f} d\nu_\beta \right) \leq \frac{C}{n} \quad \forall t > 0. \quad (4.9)$$

Consequently

$$\begin{aligned} \sum_{i=1}^n \mathcal{D}_{i,n}^r(f) &\leq - \sum_{i=1}^n \left(\mathcal{D}_{i,n}^p(f) + 2\beta_i^{-1} \int \frac{(\partial_{q_i} f)(\partial_{p_i} f)}{f} d\nu_\beta \right) + \frac{C}{n} \\ &\leq \sum_{i=1}^n \left(\mathcal{D}_{i,n}^p(f) + \frac{1}{2} \mathcal{D}_{i,n}^r(f) \right) + \frac{C}{n} \end{aligned}$$

that gives

$$\sum_{i=1}^n \mathcal{D}_{i,n}^r(f) \leq 2 \sum_{i=1}^n \mathcal{D}_{i,n}^p(f) + \frac{2C}{n}$$

Since we have already the bound (4.7) we have obtained

Proposition 4.1. *There exist a constant C such that*

$$\int_0^t \sum_{i=1}^n \mathcal{D}_{i,n}^r(f_s^n) ds \leq \frac{C}{n} \quad (4.10)$$

This bound allow us to apply directly the one/two blocks method developed in [6] for the overdamped dynamics.

5. CHARACTERIZATION OF THE LIMIT POINTS

Define the empirical measure

$$\pi_t^n(dx) := \frac{1}{n} \sum_{i=1}^n r_i(t) \delta_{i/n}(dx).$$

For any smooth function $G : [0, 1] \rightarrow \mathbb{R}$, we have:

$$\langle \pi_t^n, G \rangle - \langle \pi_0^n, G \rangle = \int_0^t \mathcal{L}_n^{\bar{r}(t)} \langle \pi_s^n, G \rangle ds. \quad (5.1)$$

By definition of the empirical measure,

$$\mathcal{L}_n^{\bar{r}(t)} \langle \pi_t^n, G \rangle = \frac{1}{n} \sum_{i=1}^n G\left(\frac{i}{n}\right) \mathcal{L}_n^{\bar{r}(t)} r_i(n^2 t) \quad (5.2)$$

Replacing in the previous sum $\mathcal{L}_n^{\bar{r}(t)} r_i(n^2 t)$, by the explicit expression

$$\mathcal{L}_n^{\bar{r}(t)} r_i = n^2(p_i - p_{i-1}), \quad i = 1, \dots, n$$

always defining $p_0 = 0$. After performing a summation by parts, we obtain

$$\mathcal{L}_n^{\bar{r}(t)} \langle \pi_t^n, G \rangle = - \sum_{i=1}^{n-1} \nabla_n G\left(\frac{i}{n}\right) p_i(t) + n p_n(t) G(1) \quad (5.3)$$

where

$$\nabla_n G\left(\frac{i}{n}\right) = n \left[G\left(\frac{i+1}{n}\right) - G\left(\frac{i}{n}\right) \right], \quad i = 1, \dots, n-1,$$

is the discrete derivative. Define also

$$\nabla_n^* G\left(\frac{i}{n}\right) = n \left[G\left(\frac{i-1}{n}\right) - G\left(\frac{i}{n}\right) \right] \quad i = 2, \dots, n.$$

Now observe that

$$\begin{aligned}
\mathcal{L}_n^{\bar{\tau}(t)} \left[\frac{1}{n^2} \sum_{i=1}^{n-1} \nabla_n G \left(\frac{i}{n} \right) p_i - \frac{1}{n} p_n(t) G(1) \right] &= -\gamma \sum_{i=1}^{n-1} \nabla_n G \left(\frac{i}{n} \right) p_i + \gamma n p_n(t) G(1) \\
&+ \sum_{i=1}^{n-1} \nabla_n G \left(\frac{i}{n} \right) (V'(r_{i+1}) - V'(r_i)) - nG(1) (\bar{\tau}(t) - V'(r_n)) \\
&= -\gamma \sum_{i=1}^{n-1} \nabla_n G \left(\frac{i}{n} \right) p_i + \gamma n p_n(t) G(1) \\
&+ \frac{1}{n} \sum_{i=2}^{n-1} \nabla_n^* \nabla_n G \left(\frac{i}{n} \right) V'(r_{i+1}) + \nabla_n G \left(\frac{n-1}{n} \right) V'(r_n) - \nabla_n G \left(\frac{1}{n} \right) V'(r_1) \\
&- nG(1) (\bar{\tau}(t) - V'(r_n))
\end{aligned} \tag{5.4}$$

It is enough to consider test functions G such that $G(1) = 0$ and $G'(0) = 0$. It follows that

$$\begin{aligned}
\mathcal{L}_n^{\bar{\tau}(t)} \langle \pi^n, G \rangle &= \frac{1}{\gamma n} \sum_{i=2}^{n-1} (-\nabla_n^* \nabla_n) G \left(\frac{i}{n} \right) V'(r_{i+1}) - \gamma^{-1} \nabla_n G \left(\frac{n-1}{n} \right) V'(r_n) \\
&+ \gamma^{-1} \nabla_n G \left(\frac{1}{n} \right) V'(r_1) + \mathcal{L}_n^{\bar{\tau}(t)} \frac{1}{\gamma n^2} \sum_{i=1}^{n-1} \nabla_n G \left(\frac{i}{n} \right) p_i
\end{aligned} \tag{5.5}$$

It is easy to show, by using the entropy inequality, that the last two terms are negligible. Since $|\nabla_n G(\frac{1}{n})| \leq C_G n^{-1}$, and

$$\int e^{\alpha|V'(r)| - \beta_1 V(r)} dr < +\infty \quad \forall \alpha > 0$$

by the entropy inequality we have for any $\alpha > 0$:

$$\begin{aligned}
\frac{C_G}{n} \int |V'(r_1)| f_s^n d\nu_\beta &\leq \frac{C_G}{n\alpha} \int e^{\alpha|V'(r_1)|} d\nu_\beta^n + \frac{C_G}{n\alpha} H_n(s) \\
&\leq \frac{C(\alpha)}{n} + \frac{C'}{\alpha}
\end{aligned}$$

that goes to 0 after taking the limit as $n \rightarrow \infty$ then $\alpha \rightarrow \infty$. About the last term of the RHS in (5.5), after time integration we have to estimate

$$\int \frac{1}{\gamma n^2} \sum_{i=1}^{n-1} \nabla_n G \left(\frac{i}{n} \right) p_i f_s^n d\nu_\beta.$$

for $s = 0, t$. This is also done by similar use of the entropy inequality.

To deal with the second term of the RHS of (5.5), we need the following lemma:

Lemma 5.1.

$$\lim_{n \rightarrow \infty} \int_0^t ds \int (V'(r_n) - \bar{\tau}(s)) f_s^n d\nu = 0 \tag{5.6}$$

Proof. Observe that

$$V'(r_n) - \bar{\tau}(s) = -L^{\bar{\tau}(s)} p_n - \gamma p_n \tag{5.7}$$

Then after time integration:

$$\begin{aligned} \int_0^t ds \int (V'(r_n) - \bar{\tau}(s)) f_s^n d\nu &= \frac{1}{n^2} \left(\int p_n f_0^n d\nu - \int p_n f_t^n d\nu \right) - \gamma \int_0^t ds \int p_n f_s^n d\nu. \\ &= \frac{1}{n^2} \left(\int p_n f_0^n d\nu - \int p_n f_t^n d\nu \right) - \gamma \int_0^t ds \int \partial_{p_n} f_s^n d\nu. \end{aligned}$$

these terms are easy to show they converge to 0 by the bounds of section 4. \square

We are finally left to deal with the first term of the RHS of (5.5). We will proceed as in [6]. For any $\varepsilon > 0$ define

$$\bar{r}_{i,\varepsilon} = \frac{1}{2n\varepsilon + 1} \sum_{|j-i| \leq n\varepsilon} r_j, \quad n\varepsilon < i < n(1 - \varepsilon). \quad (5.8)$$

We need to show that

$$\lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \int_0^t \int \frac{1}{\gamma n} \sum_{i=[n\varepsilon]+1}^{[n(1-\varepsilon)]} (-\nabla_n^* \nabla_n) G\left(\frac{i}{n}\right) (V'(r_{i+1}) - \tau(\bar{r}_{i,\varepsilon}, \beta_i)) f_s^n d\nu. ds = 0 \quad (5.9)$$

and that the boundary terms are negligible:

$$\lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \int_0^t \int \frac{1}{\gamma n} \left(\sum_{i=2}^{[n\varepsilon]} + \sum_{i=[n(1-\varepsilon)]+1}^{[n-1]} \right) (-\nabla_n^* \nabla_n) G\left(\frac{i}{n}\right) V'(r_{i+1}) f_s^n d\nu. ds = 0 \quad (5.10)$$

This last limit can be proven again by the entropy inequality.

Thank to the bound (4.10), we are now in the same position as in the proof of the overdamped dynamics, as considered in [6], and by using similar argument as used there (the so called one-block/two blocks) (5.9) follows. A slight difference is due to the dependence of τ on β_i , but since this changes very slowly and smoothly in space it is easy to consider microscopic blocks of size k with constant temperature inside.

At this point the proof of theorem 2.1 follows by standard arguments. Let Q_n the probability distribution of π^n on $\mathcal{C}([0, T], \mathcal{M}_1([0, 1]))$, where $\mathcal{M}_1([0, 1])$ are the probability measures on $[0, 1]$. In appendix B we prove that the sequence Q_n is compact. Then, by the above results any limit point Q of Q_n is concentrated on absolutely continuous measures with densities $\bar{r}(y, t)$ such that for any $0 \leq t \leq T$,

$$\begin{aligned} \mathbb{E}^Q \left| \int_0^1 G(y) (\bar{r}(y, t) - \bar{r}(y, 0)) dy \right. \\ \left. - \gamma^{-1} \int_0^t ds \left[\int_0^1 G''(y) \tau(\bar{r}(y, s), \beta(y)) dy - G'(1) \bar{\tau}(s) \right] \right| = 0 \end{aligned} \quad (5.11)$$

Furthermore in appendix A we prove that Q is concentrated on densities that satisfy the regularity condition to have uniqueness of the solution of the equation.

6. APPENDIX A: PROOF OF THE REGULARITY BOUND 8.1

Proposition 6.1. *There exists a finite constant C such that for any limit point distribution Q we have the bound:*

$$\mathbb{E}^Q \left(\int_0^t ds \int_0^1 dx (\partial_x \tau(\bar{r}(s, x), \beta^{-1}(x)))^2 \right) < C \quad (6.1)$$

Proof. It is enough to prove that for any function $G \in \mathcal{C}^1([0, 1])$ such that $G(0) = 0$ the following inequality holds:

$$\mathbb{E}^Q \left(\int_0^t ds \left[\int_0^1 dx G'(x) \tau(\bar{r}(s, x), \beta^{-1}(x)) - G(1) \bar{\tau}(s) \right] \right) \leq C \left(\int_0^1 G(x)^2 dx \right)^{1/2} \quad (6.2)$$

In order to obtain such bound, compute

$$\begin{aligned} & \frac{1}{n^2} \mathcal{L}_n^{\bar{\tau}} \sum_{i=1}^n G(i/n) (p_i + \gamma q_i) = \sum_{i=1}^n G(i/n) A_n^{\bar{\tau}} p_i \\ &= \sum_{i=1}^{n-1} G(i/n) (V'(r_{i+1}) - V'(r_i)) + G(1) (\bar{\tau} - V'(r_n)) \\ &= \frac{1}{n} \sum_{i=2}^n \nabla_n^* G(i/n) V'(r_i) + G(1) \bar{\tau} - G(1/n) V'(r_1) \end{aligned}$$

and after time integration and averaging over trajectories we have

$$\begin{aligned} & \frac{1}{n^2} \int \sum_{i=1}^n G(i/n) (p_i + \gamma q_i) (f_t - f_0) d\nu_\beta \\ &= \int_0^t ds \int \frac{1}{n} \sum_{i=2}^n \nabla_n^* G(i/n) V'(r_i) f_s d\nu_\beta + G(1) \int_0^t \bar{\tau}(s) ds \\ & \quad - G(1/n) \int_0^t ds \int V'(r_1) f_s d\nu_\beta. \end{aligned} \quad (6.3)$$

It is easy to see that, since $G(0) = 0$ and differentiable, the last term of the right hand side is negligible as $n \rightarrow \infty$. By the results of section 5, it converges, through subsequences, to

$$-\mathbb{E}^Q \left(\int_0^t ds \int_0^1 dx G'(x) \tau(\bar{r}(s, x), \beta(x)) \right).$$

About the left hand side of (6.3), one can see easily that

$$\frac{1}{n^2} \int \sum_{i=1}^n G(i/n) p_i (f_t - f_0) d\nu \xrightarrow{n \rightarrow \infty} 0$$

while we can bound the other term of the LHS by

$$\begin{aligned} & \left| \frac{\gamma}{n} \int \sum_{i=1}^n G(i/n) \frac{q_i}{n} (f_t - f_0) d\nu \right| \leq \gamma \left(\frac{1}{n} \sum_{i=1}^n G(i/n)^2 \right)^{1/2} \left(\int \frac{1}{n} \sum_{i=1}^n \frac{q_i^2}{n^2} (f_t + f_0) d\nu \right)^{1/2} \\ & \leq \gamma \left(\frac{1}{n} \sum_{i=1}^n G(i/n)^2 \right)^{1/2} \left(\int \frac{1}{n} \sum_{i=1}^n r_i^2 (f_t + f_0) d\nu \right)^{1/2} \leq C \gamma \left(\frac{1}{n} \sum_{i=1}^n G(i/n)^2 \right)^{1/2}. \end{aligned}$$

Taking the limit as $n \rightarrow \infty$ we obtain (6.2). \square

7. APPENDIX B: COMPACTNESS

Proposition 7.1. *For any function $G \in \mathcal{C}^1([0, 1])$ such that $G(1) = 0$, $G'(0) = 0$ and any $\varepsilon > 0$ we have*

$$\lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} \mathbb{P}^{\mu_0} \left[\sup_{0 \leq s < t \leq T, |s-t| < \delta} | \langle \pi_n(t), G \rangle - \langle \pi_n(s), G \rangle | \geq \varepsilon \right] = 0 \quad (7.1)$$

Proof.

$$\begin{aligned} & \langle \pi_n(t), G \rangle - \langle \pi_n(s), G \rangle = - \int_s^t du \sum_{i=1}^{n-1} \nabla_n G \left(\frac{i}{n} \right) p_i(u) \\ &= \int_s^t du \frac{1}{\gamma n} \sum_{i=2}^{n-1} (-\nabla_n^* \nabla_n) G \left(\frac{1}{n} \right) V'(r_{i+1}(u)) - \int_s^t du \frac{1}{\gamma} \nabla_n G \left(\frac{n-1}{n} \right) V'(r_n(u)) \\ & \quad + \int_s^t du \frac{1}{\gamma} \nabla_n G \left(\frac{1}{n} \right) V'(r_1(u)) + \frac{1}{\gamma n^2} \sum_{i=2}^{n-1} \nabla_n G \left(\frac{i}{n} \right) (p_i(t) - p_i(s)) \\ & \quad + \frac{1}{n} \sum_i^n \sqrt{2\gamma\beta_j^{-1}} \nabla_n G \left(\frac{i}{n} \right) (w_i(t) - w_i(s)) \\ & := I_1(s, t) + I_2(s, t) + I_3(s, t) + I_4(s, t) + I_5(s, t) \end{aligned}$$

The term I_3 is the easiest to estimate, since $G'(0) = 0$, and

$$\sup_{0 \leq s < t \leq T, |s-t| < \delta} |I_3(s, t)| \leq \frac{C}{n\gamma} \int_s^t |V'(r_1(u))| du \leq \frac{C\delta^{1/2}}{n\gamma} \left(\int_0^T |V'(r_1(u))|^2 du \right)^{1/2}$$

and

$$\begin{aligned} \mathbb{E}_{\mu_0} \left[\left(\int_0^T |V'(r_1(u))|^2 du \right)^{1/2} \right] &\leq \left[\int_0^T \mathbb{E}_{\mu_0} (|V'(r_1(u))|^2) du \right]^{1/2} \\ &\leq C \left[\int_0^T \mathbb{E}_{\mu_0} \left(\sum_{i=1}^n V(r_i(u))^2 \right) du \right]^{1/2} \leq CT^{1/2} n^{1/2} \end{aligned}$$

so that

$$\mathbb{E}_{\mu_0} \left[\sup_{0 \leq s < t \leq T, |s-t| < \delta} |I_3(s, t)| \right] \leq \frac{C\delta^{1/2} T^{1/2}}{\gamma n^{1/2}}.$$

About I_2 , this is equal to

$$-\frac{1}{\gamma} \nabla_n G \left(\frac{n-1}{n} \right) \int_s^t du (V'(r_n(u)) - \bar{\tau}(u)) - \frac{1}{\gamma} \nabla_n G \left(\frac{n-1}{n} \right) \int_s^t du \bar{\tau}(u)$$

The second term of the above expression is trivially bounded by $C\delta$ since $|t - s| \leq \delta$. For the first term, by (5.7), we have

$$\int_s^t du (V'(r_n(u)) - \bar{\tau}(u)) = \frac{p_n(s) - p_n(t)}{n^2} - \gamma \int_s^t p_n(u) du + \frac{\sqrt{2\gamma\beta_n^{-1}}}{n} (w_n(t) - w_n(s))$$

The last term of the right hand side of the above is estimated by the standard modulus of continuity of the Wiener process w_n . For the second term of the right hand side, this is bounded by

$$\begin{aligned}
& \mathbb{E}_{\mu_0} \left[\sup_{0 \leq s < t \leq T, |s-t| < \delta} \gamma \left| \int_s^t p_n(u) du \right| \right] \leq \gamma \delta^{1/2} \mathbb{E}_{\mu_0} \left[\left(\int_0^T p_n^2(u) du \right)^{1/2} \right] \\
& \leq \gamma \delta^{1/2} \left[\int_0^T \mathbb{E}_{\mu_0}(p_n^2(u)) du \right]^{1/2} = \gamma \delta^{1/2} \left[\int_0^T \mathbb{E}_{\mu_0}(p_n^2(u) - \beta_n^{-1}) du + T \beta_n^{-1} \right]^{1/2} \\
& \leq C \gamma \delta^{1/2} \left[\int_0^T \int p_n \partial_{p_n} f_u^n d\nu \cdot du + T \beta_n^{-1} \right]^{1/2} \\
& \leq C \gamma \delta^{1/2} \left[\left(\int_0^T \int p_n^2 f_u^n d\nu \cdot du \right)^{1/2} \left(\int_0^T \int \frac{(\partial_{p_n} f_u^n)^2}{f_u^n} d\nu \cdot du \right)^{1/2} + T \beta_n^{-1} \right]^{1/2} \\
& \leq C' \gamma \delta^{1/2}
\end{aligned}$$

where the last inequality is justified by the inequalities:

$$\begin{aligned}
& \int p_n^2 f_u^n d\nu \leq Cn \\
& \int_0^T \int \frac{(\partial_{p_n} f_u^n)^2}{f_u^n} d\nu \cdot du \leq \frac{C}{n}
\end{aligned}$$

To deal with the first term we have to prove that

$$\lim_{n \rightarrow \infty} \mathbb{E} \left(\sup_{0 \leq t \leq T} \frac{1}{n^2} |p_n(t)| \right) = 0 \tag{7.2}$$

Since

$$\begin{aligned}
\frac{p_n(t)}{n^2} &= \frac{1}{n^2} p_n(0) e^{-\gamma n^2 t} + \int_0^t e^{-\gamma n^2(t-u)} [\bar{\tau}(u) - V'(r_n(u))] du \\
&\quad + \sqrt{2\gamma \beta_n^{-1}} \frac{1}{n} \int_0^t e^{-\gamma n^2(t-u)} dw_n(u)
\end{aligned} \tag{7.3}$$

The stochastic integral is easy to estimate by Doob's inequality:

$$\mathbb{E} \left(\sup_{0 \leq t \leq T} \left| \sqrt{2\gamma \beta_n^{-1}} \frac{1}{n} \int_0^t e^{-\gamma n^2(t-u)} dw_n(u) \right|^2 \right) \leq \frac{CT}{n^2}$$

About the second term, by Schwarz inequality we have that

$$\begin{aligned}
& \mathbb{E} \sup_{0 \leq t \leq T} \left| \int_0^t e^{-\gamma n^2(t-u)} [\bar{\tau}(u) - V'(r_n(u))] du \right| \\
& \leq \frac{1}{n\sqrt{2\gamma}} \left(\int_0^T \mathbb{E} \left([\bar{\tau}(u) - V'(r_n(u))]^2 \right) du \right)^{1/2}
\end{aligned}$$

and by the entropy bound we have

$$\mathbb{E} \left([\bar{\tau}(u) - V'(r_n(u))]^2 \right) \leq Cn$$

so that this term goes to zero like $n^{-1/2}$. The first term in (7.3) is trivial to estimate. This concludes the estimate of I_2 .

The estimate of I_4 is similar to the proof of (7.2), but require a little extra work. We need to prove that

$$\lim_{n \rightarrow \infty} \mathbb{E} \sup_{0 \leq t \leq T} \left| \frac{1}{n^2} \sum_{i=2}^{n-1} \nabla_n G \left(\frac{i}{n} \right) p_i(t) \right| = 0. \quad (7.4)$$

By the evolution equations we have

$$\begin{aligned} \frac{1}{n^2} \sum_{i=2}^{n-1} \nabla_n G \left(\frac{i}{n} \right) p_i(t) &= \frac{1}{n^2} \sum_{i=2}^{n-1} \nabla_n G \left(\frac{i}{n} \right) p_i(0) e^{-\gamma n^2 t} \\ &\quad + \int_0^t ds e^{-\gamma n^2 (t-s)} \frac{1}{n} \sum_{i=3}^{n-1} \nabla_n^* \nabla_n G \left(\frac{i}{n} \right) V'(r_i(s)) \\ &\quad + \int_0^t ds e^{-\gamma n^2 (t-s)} \left(\nabla_n G(1) V'(r_n(s)) - \nabla_n G \left(\frac{2}{n} \right) V'(r_2(s)) \right) \end{aligned}$$

and all these terms can be estimated as in the proof of (7.2), so that (7.4) follows.

Also I_5 can be easily estimated by Doob inequality and using the independence of $w_i(t)$. \square

8. APPENDIX C: UNIQUENESS OF WEAK SOLUTIONS

Proposition 8.1. *The weak solution of (2.18) is unique in the class of function such that*

$$\int_0^t ds \int_0^1 (\partial_x \tau(r(x, s), \beta(x)))^2 dx < +\infty \quad (8.1)$$

Proof. Let $g(x) \geq 0$ a smooth function with compact support contained in $[-1/4, 1/4]$ such that $\int_{\mathbb{R}} g(y) dy = 1$. Then for $\lambda > 0$ large enough, define the function

$$G_\lambda(y, x) = 1 - \int_{-\infty}^y \lambda g(\lambda(z - x)) dz$$

Then for $1/(4\lambda) < x < 1 - 1/(4\lambda)$, we have $G_\lambda(0, x) = 0$ and $\partial_y G_\lambda(1, x) = 0$, and it can be used as test function in (2.18). So if $r(x, t)$ is a solution in the given class, we have

$$\int_0^1 G_\lambda(y, x) (r(y, t) - r_0(y)) dx = \gamma^{-1} \int_0^t ds \left[\int_0^1 \lambda g(\lambda(y - x)) \partial_y \tau(r(y, s), \beta(y)) dy \right].$$

Letting $\lambda \rightarrow +\infty$ we obtain:

$$\int_0^x (r(y, t) - r_0(y)) dx = \gamma^{-1} \int_0^t ds \partial_y \tau(r(x, s), \beta(x)), \quad \forall x \in (0, 1).$$

Let $r_1(x, t), r_2(x, t)$ two solutions in the class considered, and define

$$R_j(x, t) = \int_0^x r_j(y, t) dy, \quad j = 1, 2.$$

By the approximation argument done at the beginning of the proof, we have that

$$\partial_t R_j(x, t) = \gamma^{-1} \partial_x \tau(r_j(x, s), \beta(x))$$

for every $x \in (0, 1)$ and $t > 0$.

Since $\tau(r_j(1, t), \beta(1)) = \bar{\tau}(t)$, and since $\tau(r, \beta)$ is a strictly increasing function of r ,

$$\begin{aligned} & \frac{d}{dt} \int_0^1 (R_1(x, t) - R_2(x, t))^2 dx \\ &= 2\gamma^{-1} \int_0^1 (R_1(x, t) - R_2(x, t)) \partial_x (\tau(r_1(x, t), \beta(x)) - \tau(r_2(x, t), \beta(x))) dx \\ &= -2\gamma^{-1} \int_0^1 (r_1(x, t) - r_2(x, t)) (\tau(r_1(x, t), \beta(x)) - \tau(r_2(x, t), \beta(x))) dx \leq 0 \end{aligned}$$

□

9. APPENDIX D: PROOF OF THE HYPOCOERCIVE BOUND

In order to shorten formulas, we introduce here some vectorial notation:

$$\begin{aligned} u \odot v &= \sum_{i=1}^n \beta_i^{-1} u_i v_i, & u \tilde{\odot} v &= \sum_{i=1}^{n-1} \beta_i^{-1} u_i v_i, & |u|_{\odot}^2 &= u \odot u, & |u|_{\tilde{\odot}}^2 &= u \tilde{\odot} u. \\ \partial_p &= (\partial_{p_1}, \dots, \partial_{p_n}) & \partial_p^* &= (\partial_{p_1}^*, \dots, \partial_{p_n}^*), & \partial_{p_i}^* &= \beta_i p_i - \partial_{p_i} \\ \partial_q &= (\partial_{q_1}, \dots, \partial_{q_n}), & \partial_{q_i} &= \partial_{r_i} - \partial_{r_{i+1}}, & \partial_{q_n} &= \partial_{r_n} \end{aligned} \quad (9.1)$$

Using this notation we have:

$$\begin{aligned} I_n(f) &= \int \frac{|\partial_p f + \partial_q f|_{\tilde{\odot}}^2}{f} d\nu_{\beta}. \\ &= \sum_{i=1}^{n-1} \left(\mathcal{D}_{i,n}^p(f) + \mathcal{D}_{i,n}^r(f) \right) + 2 \int \frac{\partial_q f \tilde{\odot} \partial_p f}{f} d\nu_{\beta}. \geq 0 \end{aligned} \quad (9.2)$$

Furthermore

$$\mathcal{S}_n = -\partial_p^* \odot \partial_p - \beta_n^{-1} \partial_{p_n}^* \partial_{p_n}, \quad \mathcal{A}_n^{\tau} = p \cdot \partial_q - \partial_q \mathcal{V} \cdot \partial_p + \tau \partial_{p_n} \quad (9.3)$$

where $\mathcal{V} = \sum_i V(r_i)$ and the \cdot denotes the usual scalar product in \mathbb{R}^n .

We will prove now that there exists constants $\lambda > 0$ and $C > 0$ independent of n such that

$$\frac{d}{dt} I_n(f) \leq -\lambda n^2 I_n(f) + Cn. \quad (9.4)$$

We will use the following commutation relations:

$$[\partial_{p_i}, \beta_j^{-1} \partial_{p_j}^*] = \delta_{i,j}, \quad [\partial_{p_i}, \mathcal{A}_n^{\tau}] = \partial_{q_i}, \quad [\partial_{q_i}, \mathcal{A}_n^{\tau}] = -(\partial_q^2 \mathcal{V} \partial_p)_i \quad (9.5)$$

where $\partial_q^2 \mathcal{V}$ is the hessian matrix of \mathcal{V} .

Denote $g_t = \sqrt{f_t^n}$ and observe that

$$I_n(g_t^2) = 4 \int \left(|\partial_p g_t|_{\tilde{\odot}}^2 + |\partial_q g_t|_{\tilde{\odot}}^2 + 2 \partial_q g_t \tilde{\odot} \partial_p g_t \right) d\nu_{\beta}. \quad (9.6)$$

Recall that

$$n^2 \mathcal{A}_n^{\tau,*} = -n^2 \mathcal{A}_n^{\tau} + B_n^{\tau}$$

where

$$B_n^{\tau} = -n \sum_{i=1}^{n-1} \nabla_n \beta(i/n) p_i V'(r_{i+1}) + n^2 \beta(1) p_n \tau$$

Consequently g_t solves the equation:

$$\partial_t g = -n^2 \mathcal{A}_n^{\bar{r}(t)} g_t + n^2 \gamma \mathcal{S}_n g_t + n^2 \gamma \frac{|\partial_p g_t|_{\odot}^2}{g_t} + \frac{1}{2} B_n^{\bar{r}(t)} g_t$$

We then compute the time derivative of $I_n(g_t^2)$ by considering the three terms separately. The first one gives:

$$\begin{aligned} \frac{d}{dt} \int |\partial_p g_t|_{\odot}^2 d\nu_{\beta} &= -2n^2 \int \partial_p g_t \tilde{\odot} \partial_p (\mathcal{A}^{\bar{r}(t)} g_t) d\nu_{\beta} \\ &\quad - 2n^2 \gamma \int \partial_p g_t \tilde{\odot} \partial_p (\partial_p^* \odot \partial_p g_t) d\nu_{\beta} \\ &\quad + 2n^2 \gamma \int \partial_p g_t \tilde{\odot} \partial_p \left(\frac{|\partial_p g_t|_{\odot}^2}{g_t} \right) d\nu_{\beta} \\ &\quad + \int \partial_p g_t \tilde{\odot} \partial_p (B_n^{\bar{r}(t)} g_t) d\nu_{\beta}. \end{aligned} \tag{9.7}$$

Then the first term of the RHS of (9.7) is equal to

$$\begin{aligned} &-2n^2 \int \partial_p g_t \tilde{\odot} \partial_q g_t d\nu_{\beta} - 2n^2 \int \partial_p g_t \tilde{\odot} \mathcal{A}^{\bar{r}(t)} \partial_p g_t d\nu_{\beta} \\ &= -2n^2 \int \partial_p g_t \tilde{\odot} \partial_q g_t d\nu_{\beta} - \int \partial_p g_t \tilde{\odot} B_n^{\bar{r}(t)} \partial_p g_t d\nu_{\beta}. \end{aligned}$$

Then the RHS of (9.7) is equal to

$$\begin{aligned} &-2n^2 \int \partial_p g_t \tilde{\odot} \partial_q g_t d\nu_{\beta} - 2n^2 \gamma \int \partial_p g_t \tilde{\odot} \partial_p (\partial_p^* \odot \partial_p g_t) d\nu_{\beta} \\ &+ 2n^2 \gamma \int \partial_p g_t \tilde{\odot} \partial_p \left(\frac{|\partial_p g_t|_{\odot}^2}{g_t} \right) d\nu_{\beta} + \int g_t \partial_p g_t \tilde{\odot} \partial_p B_n^{\bar{r}(t)} d\nu_{\beta}. \end{aligned}$$

The last term of the above equation is equal to

$$\int g_t \partial_p g_t \tilde{\odot} \partial_p B_n d\nu_{\beta} = -n \int g_t \sum_{i=1}^{n-1} \beta_i^{-1} \nabla_n \beta \left(\frac{i}{n} \right) V'(r_{i+1}) \partial_{p_i} g_t d\nu_{\beta}. \tag{9.8}$$

For any $\alpha_1 > 0$, by the similar argument as in section 4, is bounded by

$$\begin{aligned} &\frac{1}{2\alpha_1} \int g_t^2 \sum_{i=1}^{n-1} \frac{(\nabla_n \beta \left(\frac{i}{n} \right))^2}{\beta_i} V'(r_{i+1})^2 d\nu_{\beta} + \frac{\alpha_1 n^2}{2} \int |\partial_p g_t|_{\odot}^2 d\nu_{\beta} \\ &\leq \frac{Cn}{\alpha_1} + \frac{\alpha_1 n^2}{2} \int |\partial_p g_t|_{\odot}^2 d\nu_{\beta}. \end{aligned}$$

Computing second term of the RHS of (9.7) we have:

$$\begin{aligned} \int \partial_p g_t \tilde{\odot} \partial_p (\partial_p^* \odot \partial_p g_t) d\nu_{\beta} &= \int \sum_{j=1}^{n-1} \beta_j^{-1} |\partial_p \partial_{p_j} g|_{\odot}^2 d\nu_{\beta} + \int |\partial_p g|_{\odot}^2 d\nu_{\beta} \\ &= \int \sum_{i=1}^n \sum_{j=1}^{n-1} \beta_j^{-1} \beta_i^{-1} (\partial_{p_i} \partial_{p_j} g)^2 d\nu_{\beta} + \int |\partial_p g|_{\odot}^2 d\nu_{\beta}. \end{aligned}$$

About the third term on the RHS:

$$\partial_p g_t \tilde{\odot} \partial_p \left(\frac{|\partial_p g_t|_{\odot}^2}{g_t} \right) = \frac{2 \sum_{j=1}^{n-1} \sum_{i=1}^n \beta_j^{-1} \beta_i^{-1} \partial_{p_j} g_t \partial_{p_i} g_t \partial_{p_i} \partial_{p_j} g_t}{g_t} - \frac{|\partial_p g_t|_{\odot}^2 |\partial_p g_t|_{\odot}^2}{g_t^2}$$

Summing all together we have obtained

$$\begin{aligned} \frac{d}{dt} \int |\partial_p g_t|_{\odot}^2 d\nu_{\beta} &= -2n^2 \int \partial_p g_t \tilde{\odot} \partial_q g_t d\nu_{\beta} - n^2 \left(2\gamma - \frac{\alpha_1}{2} \right) \int |\partial_p g_t|_{\odot}^2 d\nu_{\beta} \\ &\quad - 2n^2 \gamma \int \sum_{j=1}^{n-1} \sum_{i=1}^n \beta_j^{-1} \beta_i^{-1} (\partial_{p_i} \partial_{p_j} g_t - g_t^{-1} \partial_{p_i} g_t \partial_{p_j} g_t)^2 d\nu_{\beta} + \frac{Cn}{\alpha_1}. \end{aligned} \quad (9.9)$$

Now we deal with the derivative of the second term:

$$\begin{aligned} \frac{d}{dt} \int |\partial_q g_t|_{\odot}^2 d\nu_{\beta} &= -2n^2 \int \partial_q g_t \tilde{\odot} \partial_q (\mathcal{A}^{\bar{r}(t)} g_t) d\nu_{\beta} - 2n^2 \gamma \int \partial_q g_t \tilde{\odot} \partial_q (\partial_p^* \odot \partial_p g_t) d\nu_{\beta} \\ &\quad + 2n^2 \gamma \int \partial_q g_t \tilde{\odot} \partial_q \left(\frac{|\partial_p g_t|_{\odot}^2}{g_t} \right) d\nu_{\beta} + \int \partial_q g_t \tilde{\odot} \partial_q (B_n g_t) d\nu_{\beta} \\ &= -2n^2 \int \partial_q g_t \tilde{\odot} \partial_q (\mathcal{A}^{\bar{r}(t)} g_t) d\nu_{\beta} - 2n^2 \gamma \int \sum_{j=1}^{n-1} \sum_{i=1}^n \beta_i^{-1} \beta_j^{-1} (\partial_{p_i} \partial_{q_j} g - g_t^{-1} \partial_{p_i} g \partial_{q_j} g)^2 d\nu_{\beta} \\ &\quad + \int \partial_q g_t \tilde{\odot} \partial_q (B_n g_t) d\nu_{\beta}. \end{aligned} \quad (9.10)$$

The first and the last term give:

$$\begin{aligned} &-2n^2 \int \partial_q g_t \tilde{\odot} \partial_q (\mathcal{A}^{\bar{r}(t)} g_t) d\nu_{\beta} + \int \partial_q g_t \tilde{\odot} \partial_q (B_n g_t) d\nu_{\beta} \\ &= 2n^2 \int \partial_q g_t \tilde{\odot} (\partial_q^2 \mathcal{V} \partial_p) g_t d\nu_{\beta} + \int g_t \partial_q g_t \tilde{\odot} \partial_q B_n d\nu_{\beta}. \end{aligned}$$

The last term of the RHS of the above is equal to

$$\begin{aligned} &\int g_t \partial_q g_t \tilde{\odot} \partial_q B_n d\nu_{\beta} \\ &= n \sum_{i=1}^{n-1} \int \beta_i^{-1} g_t (\partial_{q_i} g_t) \left[\nabla \beta \left(\frac{i}{n} \right) V''(r_{i+1}) p_i - \nabla \beta \left(\frac{i-1}{n} \right) V''(r_i) p_{i-1} \right] d\nu_{\beta}. \end{aligned}$$

Since V'' and $\nabla_n \beta$ are bounded and $\beta > \beta_0$, this last quantity is bounded for any $\alpha_2 > 0$ by

$$n^2 \alpha_2 \int |\partial_q g_t|_{\odot}^2 d\nu_{\beta} + C \alpha_2^{-1} \int \sum_{i=1}^{n-1} p_i^2 g_t^2 d\nu_{\beta} \leq n^2 \alpha_2 \int |\partial_q g_t|_{\odot}^2 d\nu_{\beta} + C' \alpha_2^{-1} n$$

Again since V'' is bounded we can bound for any $\alpha_3 > 0$

$$2n^2 \int \partial_q g_t \tilde{\odot} (\partial_q^2 \mathcal{V} \partial_p) g_t d\nu_{\beta} \leq \alpha_3 n^2 \int |\partial_q g_t|_{\odot}^2 d\nu_{\beta} + \frac{V_{\infty}^2 n^2}{\alpha_3} \int |\partial_p g_t|_{\odot}^2 d\nu_{\beta}.$$

All together the time derivative of the second term is bounded by

$$\begin{aligned} \frac{d}{dt} \int |\partial_q g_t|_{\odot}^2 d\nu_\beta &\leq (\alpha_2 + \alpha_3)n^2 \int |\partial_q g_t|_{\odot}^2 d\nu_\beta + \frac{Cn^2}{\alpha_3} \int |\partial_p g_t|_{\odot}^2 d\nu_\beta \\ &- 2n^2\gamma \int \sum_{j=1}^{n-1} \sum_{i=1}^n \beta_i^{-1} \beta_j^{-1} (\partial_{p_i} \partial_{q_j} g - g_t^{-1} \partial_{p_i} g \partial_{q_j} g)^2 d\nu_\beta + C' \alpha_2^{-1} n \end{aligned} \quad (9.11)$$

About the derivative of the third term

$$\begin{aligned} \frac{d}{dt} 2 \int \partial_q g_t \tilde{\odot} \partial_p g_t d\nu_\beta &= -2n^2 \int \left[\partial_q (A^{\tau(t)} g_t) \tilde{\odot} \partial_p g_t + \partial_q g_t \tilde{\odot} \partial_p (A^{\tau(t)} g_t) \right] d\nu_\beta \\ &+ \int \left[\partial_q (B_n g_t) \tilde{\odot} \partial_p g_t + \partial_q g_t \tilde{\odot} \partial_p (B_n g_t) \right] d\nu_\beta \\ &- 2n^2\gamma \int \left[\partial_q g_t \tilde{\odot} \partial_p (\partial_p^* \odot \partial_p g_t) + \partial_q (\partial_p^* \odot \partial_p g_t) \tilde{\odot} \partial_p g_t \right] d\nu_\beta \\ &+ 2n^2\gamma \int \left[\partial_q g_t \tilde{\odot} \partial_p \left(\frac{|\partial_p g_t|_{\odot}^2}{g_t} \right) + \partial_q \left(\frac{|\partial_p g_t|_{\odot}^2}{g_t} \right) \tilde{\odot} \partial_p g_t \right] d\nu_\beta \\ &= 2n^2 \int (\partial_q^2 \mathcal{V} \partial_p) g_t \tilde{\odot} \partial_p g_t d\nu_\beta - 2n^2 \int |\partial_q g_t|_{\odot}^2 d\nu_\beta \\ &+ \frac{1}{2} \int g_t \left[\partial_q B_n \tilde{\odot} \partial_p g_t + \partial_q g_t \tilde{\odot} \partial_p B_n \right] d\nu_\beta \\ &- 4n^2\gamma \int \sum_{j=1}^{n-1} \sum_{i=1}^n \beta_i^{-1} \beta_j^{-1} [(\partial_{p_i} \partial_{q_j} g) (\partial_{p_i} \partial_{p_j} g)] d\nu_\beta \\ &+ 2n^2\gamma \int \sum_{j=1}^{n-1} \sum_{i=1}^n 2\beta_i^{-1} \beta_j^{-1} g_t^{-1} [(\partial_{p_i} \partial_{p_j} g) (\partial_{p_i} g_t) (\partial_{q_j} g_t) + (\partial_{q_j} \partial_{p_i} g) (\partial_{p_j} g_t) (\partial_{p_i} g_t)] d\nu_\beta \\ &- 4n^2\gamma \int \sum_{j=1}^{n-1} \sum_{i=1}^n \beta_i^{-1} \beta_j^{-1} g_t^{-2} (\partial_{p_i} g_t)^2 (\partial_{q_j} g_t) (\partial_{p_j} g_t) d\nu_\beta. \end{aligned} \quad (9.12)$$

The last three terms of the RHS of the above can be written as

$$-4n^2\gamma \int \sum_{j=1}^{n-1} \sum_{i=1}^n \beta_i^{-1} \beta_j^{-1} [(\partial_{p_i} \partial_{q_j} g_t - g_t^{-1} \partial_{p_i} g_t \partial_{q_j} g_t) (\partial_{p_i} \partial_{p_j} g_t - g_t^{-1} \partial_{p_i} g_t \partial_{p_j} g_t)] d\nu_\beta.$$

so they combine with the corresponding terms coming from the time derivative of the first two terms of I_n giving an exact square.

The second term by the same arguments used before, can be bounded by

$$n^2\alpha_4 \int |\partial_q g_t|_{\odot}^2 d\nu_\beta + n^2\alpha_5 \int |\partial_p g_t|_{\odot}^2 d\nu_\beta + Cn(\alpha_4^{-1} + \alpha_5^{-1})$$

About the first term, since V'' is bounded, it is bounded by $V''_\infty n^2 \int |\partial_p g_t|_{\odot}^2 d\nu_\beta$.

Putting all these bounds together we obtain that

$$\frac{d}{dt} I_n(f_t) \leq -n^2\kappa_p \int |\partial_p g_t|_{\odot}^2 d\nu_\beta - n^2\kappa_q \int |\partial_q g_t|_{\odot}^2 d\nu_\beta - 2n^2 \int \partial_p g_t \tilde{\odot} \partial_q g_t d\nu_\beta + Cn$$

with

$$\begin{aligned}\kappa_p &= 2\gamma - \frac{\alpha_1}{2} - \frac{C}{\alpha_3} - \alpha_5 - V_\infty'' \\ \kappa_q &= 2 - \alpha_2 - \alpha_3 - \alpha_4\end{aligned}$$

By choosing $\alpha_2 + \alpha_3 + \alpha_4 \leq 1$ we have obtained that for some constants $C_1, C_2 > 0$ independent of n

$$\frac{d}{dt} I_n(f_t) \leq -n^2 I_n(f_t) + C_1 n + C_2 \int |\partial_p g_t|_{\otimes}^2 d\nu_\beta.$$

By recalling that

$$\int_0^t ds \int |\partial_p g_s|_{\otimes}^2 d\nu_\beta \leq \frac{C'}{n}$$

after time integration we have for some constant C_3 :

$$I_n(f_t) \leq e^{-n^2 t} I_n(f_0) + \frac{C_3}{n} (1 - e^{-n^2 t})$$

that implies

$$I_n(f_t) \leq \frac{C_4}{n} \tag{9.13}$$

for any reasonable initial conditions such that $I_n(f_0)$ is finite and not growing too fast with n .

Remark 9.1. *An important example for understanding the meaning of a density with small I_n functional, consider the inhomogeneous Gibbs density:*

$$f = \exp \left(\sum_{i=1}^n \beta_i \tau_i r_i + \sum_{i=1}^{n-1} \frac{1}{n} \nabla_n(\beta_i \tau_i) p_i \right) / \mathcal{N} \tag{9.14}$$

where \mathcal{N} is a normalization constant. In the case of constant temperature these densities play an important role in the relative entropy method (cf [13, 8]), as to a nonconstant profile of tension corresponds a profile of small damped velocities averages. Computing I_n on f we have

$$I_n(f) = \sum_{i=1}^{n-1} \left[\beta_i \tau_i - \beta_{i+1} \tau_{i+1} + \frac{1}{n} \nabla_n(\beta_i \tau_i) \right] = 0$$

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