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Daniel Barlet

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Meromorphic quotients for some holomorphic G-actions.

Daniel Barlet*

12/05/15

ABSTRACT. Using mainly tools from [B.13] and [B.15] we give a necessary and sufficient condition in order that a holomorphic action of a connected complex Lie group G on a reduced complex space X admits a strongly quasi-proper meromorphic quotient. We apply this characterization to obtain a result which asserts that, when $G = K.B$ with B a closed complex subgroup of G and K a real compact subgroup of G , the existence of a strongly quasi-proper meromorphic quotient for the B -action implies, assuming moreover that there exists a G -invariant Zariski open dense subset in X which is good for the B -action, the existence of a strongly quasi-proper meromorphic quotient for the G -action on X .

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1 Introduction

In this article we explain how the tools developed in [M.00], [B.08], [B.13] and [B.15] can be applied to produce in suitable cases a meromorphic quotient of a holomorphic action of a connected complex Lie group G on a reduced complex space X . This uses the notion of **strongly quasi-proper map** introduced in *loc. cit.* and our first goal is to give three hypotheses, called [H.1], [H.2], [H.3] on the group action which are **equivalent** to the existence of a **strongly quasi-proper meromorphic quotient**, notion defined in the section 1.2.

The proof of this equivalence is the content of proposition 2.5.1 and theorem 2.6.1. Then we discuss these hypotheses and give a simple sufficient condition [H.1str] which implies [H.1]. The existence theorem for a strongly quasi-proper meromorphic quotient under our three assumptions is applied to prove the following results :

Theorem 1.0.1 *Assume that we have a holomorphic action of a connected complex Lie group G on a reduced complex space X . Assume that $G = K.B$ where K is a compact (real) subgroup of G and B a connected complex closed subgroup of G . Assume that the action of B on X satisfies the condition [H.1str] on a G -invariant Zariski open dense subset Ω in X , and the conditions [H.2] and [H.3]. Then the G -action satisfies [H.1str], [H.2] and [H.3] ; so it has a strongly quasi-proper meromorphic quotient.*

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2 Strongly quasi-proper meromorphic quotients.

2.1 Preliminaries.

For the definition of the topology on the space $\mathcal{C}_n^f(X)$ of finite type n -cycles in X and its relationship with the (topological) space $\mathcal{C}_n^{loc}(X)$ we refer to [B-M], [B.13] and [B.15].

For the convenience of the reader we recall shortly here the definition of a geometrically f-flat map (GF map) and of a strongly quasi-proper map (SQP map) between irreducible complex spaces and we give a short summary on some properties of the SQP maps. For more details on these notions see [B.13] and [B.15].

Definition 2.1.1 A holomorphic map $f : M \rightarrow N$ between two irreducible irreducible complex spaces is called a **geometrically f-flat map (a GF-map for short)** if the following conditions are fulfilled :

i) The map is quasi-proper equidimensionnal and surjective.
Let $n := \dim M - \dim N$.

ii) There exists a holomorphic map $\varphi : N \rightarrow \mathcal{C}_n^f(M)^1$ such that for y generic in N the cycle $\varphi(y)$ is reduced and equal to the set-theoretic fiber $f^{-1}(y)$ of f at y .

A holomorphic map $f : M \rightarrow N$ between two irreducible irreducible complex spaces will be **strongly quasi-proper (SQP map for short)** if there exists a modification² $\tau : \tilde{N} \rightarrow N$ such that the strict transform³ $\tilde{f} : \tilde{M} \rightarrow \tilde{N}$ of f by τ is a GF map. A meromorphic map $M \dashrightarrow N$ will be called strongly quasi-proper when the projection on N of its graph is a SQP map.

Note that a GF map has, by definition, a *holomorphic fiber map* and that a SQP holomorphic (or meromorphic) map has a *meromorphic fiber map* via the composition of the holomorphic fiber map of \tilde{f} with the (holomorphic) direct image map $\tau_* : \mathcal{C}_n^f(\tilde{M}) \rightarrow \mathcal{C}_n^f(M)$. Of course, a SQP holomorphic map is quasi-proper, but the converse is not true. The notion of strongly quasi-proper map is stable by modification of the target space, property which is not true in general for a quasi-proper map having “big fibers” (see [B.15]).

Let $\pi : M \rightarrow N$ be a SQP map between irreducible complex spaces and define $n := \dim M - \dim N$. By definition of a SQP map, we can find a Zariski open dense subset N_0 in N and a holomorphic map $\varphi_0 : N_0 \rightarrow \mathcal{C}_n^f(M)$ such that

- i) For each y in N_0 we have the equality of subsets $|\varphi_0(y)| = \pi^{-1}(y)$.
- ii) For y generic in N_0 the cycle $\varphi_0(y)$ is reduced.

Let $\Gamma \subset N_0 \times \mathcal{C}_n^f(M)$ be the graph of φ_0 . Then also by definition of a SQP map, the closure $\bar{\Gamma}$ of Γ in $N \times \mathcal{C}_n^f(M)$ is proper over N . Then, using the semi-proper direct image theorem 2.3.2 of [B.15], this implies that $\tilde{N} := \bar{\Gamma}$ is an irreducible complex space (locally of finite dimension) with the structure sheaf induced by the sheaf of holomorphic functions on $N \times \mathcal{C}_n^f(M)$. Moreover the natural projection $\tau : \tilde{N} \rightarrow N$ is a (proper) modification.

Let $\tilde{M} := M \times_{N, str} \tilde{N}$ the strict transform of M by τ , that is to say the irreducible component of $M \times_N \tilde{N}$ containing the graph of π_0 , the restriction of π to the open set $\pi^{-1}(N_0)$ ⁴. Then let $\tilde{\pi} : \tilde{M} \rightarrow \tilde{N}$ the strict transform of π by the modification τ , which is induced on \tilde{M} by the natural projection of $M \times_N \tilde{N}$ onto \tilde{N} . The

¹that is to say a f-analytic family of finite type n -cycles in M parametrized by N .

²a modification is, by definition, always proper.

³By definition \tilde{M} is the irreducible component of $M \times_N \tilde{N}$ which surjects on \tilde{N} and \tilde{f} is induced by the projection.

⁴this graph is a Zariski open set in $M \times_N \tilde{N}$ which is irreducible as N is irreducible.

set-theoretical fiber at $\tilde{y} := (y, C) \in \tilde{N}$ of $\tilde{\pi}$ is the subset $|C| \times \tilde{y}$ in \tilde{M} . The map $\psi : \tilde{N} \rightarrow \mathcal{C}_n^f(\tilde{M})$ given by $(y, C) \mapsto C \times \{\tilde{y}\}$ is holomorphic and satisfies $|\psi(\tilde{y})| = \tilde{\pi}^{-1}(\tilde{y})$ for all \tilde{y} in \tilde{N} . Moreover $\psi(\tilde{y})$ is a reduced cycle for generic \tilde{y} in \tilde{N} . So the map $\tilde{\pi}$ is geometrically f-flat. It is the **canonical GF-flattning** of π . Then we have an isomorphism induced by ψ

$$\psi : \tilde{N} \longrightarrow \mathcal{C}_n^f(\tilde{\pi})$$

where $\mathcal{C}_n^f(\tilde{\pi}) := \{C \in \mathcal{C}_n^f(\tilde{M}) / \exists \tilde{y} \in \tilde{N} \text{ s.t. } |C| \subset \tilde{\pi}^{-1}(\tilde{y})\}$ is a closed analytic subset of $\mathcal{C}_n^f(\tilde{M})$ (see [B.15] proposition 2.1.7.); the inverse map is induced by the holomorphic map $\hat{\pi} : \mathcal{C}_n^f(\tilde{\pi}) \rightarrow \tilde{N}$ which associates to $\gamma \in \mathcal{C}_n^f(\tilde{\pi})$ the point in \tilde{N} whose $\tilde{\pi}$ -fiber contains γ (see the proposition 2.1.7 of [B.15]).

The direct image of n -cycles by τ gives a holomorphic map $\tau_* : \mathcal{C}_n^f(\tilde{M}) \rightarrow \mathcal{C}_n^f(M)$ which sends $\tilde{N} \simeq \mathcal{C}_n^f(\tilde{\pi})$ in $\mathcal{C}_n^f(\pi)$. Let us show that it is an isomorphism of \tilde{N} onto its image in $\mathcal{C}_n^f(\pi)$:

We have an obvious holomorphic map $\tilde{N} \rightarrow \mathcal{C}_n^f(\pi)$ given by $(y, C) \mapsto C$. We have also a holomorphic map $\mathcal{C}_n^f(\pi) \rightarrow N \times \mathcal{C}_n^f(M)$ given by $C \mapsto (\hat{\pi}, C)$ where $\hat{\pi} : \mathcal{C}_n^f(\pi) \rightarrow N$ is the map associating to $C \in \mathcal{C}_n^f(\pi)$ the point $y \in N$ such that $|C| \subset \pi^{-1}(y)$. This proves our claim.

Remark that $\mathcal{C}_n^f(\pi)$ is not, in general, a complex space (locally of finite dimension).

2.2 Action of G on $\mathcal{C}_n^f(X)$.

Let G be a Lie group. We shall say that G acts **continuously holomorphically** on the reduced complex space X when the action $f : G \times X \rightarrow X$ is a continuous map such that for each $g \in G$ fixed, the map $x \mapsto f(g, x)$ is a (biholomorphic) automorphism of X . Then there is a natural action of G induced on the set $\mathcal{C}_n^f(X)$ of finite type n -cycles given by $(g, C) \mapsto g_*(C)$ where we denote $g_*(C)$ the direct image of the cycle C by the automorphism of X associated to $g \in G$. When G is a complex Lie group and the map f is holomorphic we shall say that the action is **completely holomorphic**.

Proposition 2.2.1 *The action of G on $\mathcal{C}_n^f(X)$ is continuously holomorphic. This precisely means that the map $G \times \mathcal{C}_n^f(X) \rightarrow \mathcal{C}_n^f(X)$ given by $(g, C) \mapsto g_*(C)$ is continuous and that, for any f -analytic family of n -cycles $(C_s)_{s \in S}$ in X parametrized by a reduced complex space S , the family $g_*(C_s)_{(g,s) \in \{g\} \times S}$ is f -analytic for each fixed $g \in G$. If G is complex Lie group and the action is completely holomorphic, the action of G on $\mathcal{C}_n^f(X)$ is completely holomorphic, so for any f -analytic family of n -cycles $(C_s)_{s \in S}$ in X parametrized by a reduced complex space S , the family $g_*(C_s)_{(g,s) \in G \times S}$ is f -analytic.*

PROOF. First we prove the continuity of the action of G on $\mathcal{C}_n^{loc}(X)$. To apply the theorem IV 2.5.6 de [B-M] it is enough to see that the map $F : G \times X \rightarrow G \times X$ given by $(g, x) \mapsto (g, g.x)$ is proper. But if $L \subset G$ and $K \subset X$ are compact sets,

we have $F^{-1}(L \times K) \subset L \times (L^{-1}.K)$ which is a compact set in $G \times X$.

The only point left to prove the continuity statement for the topology of $\mathcal{C}_n^f(X)$, assuming that the continuity for the topology of $\mathcal{C}_n^{loc}(X)$ is obtained as follows :

Let W be a relatively compact open set in X and \mathcal{W} be the open set in $\mathcal{C}_n^f(X)$ of cycles C such any irreducible component of C meets W . Then we want to show that the set of $(g, s) \in G \times S$ such that $g_*(C_s)$ lies in \mathcal{W} is an open set in $G \times S$. As the topology of $\mathcal{C}_n^f(X)$ has a countable basis⁵ it is enough to show that if a sequence (g_ν, s_ν) converges to (g, s) with $g_*(C_s) \in \mathcal{W}$ then for $\nu \gg 1$ we have also $(g_\nu)_*(C_{s_\nu}) \in \mathcal{W}$. If this not the case, we can choose for infinitely many ν an irreducible component Γ_ν of $(g_\nu)_*(C_{s_\nu})$ which does not meet W . Up to pass to a sub-sequence, we may assume that the sequence Γ_ν converges in $\mathcal{C}_n^{loc}(X)$ to a cycle Γ which does not meet W and is contained in $g_*(C_s)$. This is a simple consequence of the continuity of the G -action on $\mathcal{C}_n^{loc}(X)$ and the characterization of compact subsets in $\mathcal{C}_n^{loc}(X)$ (see [B-M] ch.IV). As any irreducible component of $g_*(C_s)$ meets W this implies that Γ is the empty n -cycle. This means that for any compact K in X there exists an integer $\nu(K)$ such that for $\nu \geq \nu(K)$ we have $\Gamma_\nu \cap K = \emptyset$. Choose now a compact neighbourhood L of $g(K)$. For ν large enough we shall have $K \subset g_\nu^{-1}(L)$. This comes from the fact that the automorphisms g_ν^{-1} converge to g^{-1} in the compact-open topology. Then this implies that for $\nu \geq \nu(L)$ the irreducible component $g_\nu^{-1}(\Gamma_\nu)$ of C_{s_ν} does not meet K . Then, when $s_\nu \rightarrow s$ the cycles C_{s_ν} does not converge to C_s for the topology of $\mathcal{C}_n^f(X)$ because we have some “escape at infinity” in a well choosen sub-sequence. Contradiction. ■

Lemma 2.2.2 *Assume now that in the situation above we have a Lie group H acting continuously holomorphically on M . Assume that π satisfies $\pi(h.x) = \pi(x)$ for each $x \in M$. Then the strict transform \tilde{M} of M by the canonical modification $\tau : \tilde{N} \rightarrow N$ giving the canonical GF flattning of π has a natural continuous holomorphic action of H such that the projection $\tilde{M} \rightarrow M$ is H -equivariant. Moreover the GF map $\tilde{\pi} : \tilde{M} \rightarrow \tilde{N}$ satisfies $\tilde{\pi}(h.\tilde{x}) = \tilde{\pi}(\tilde{x})$ for each $(h, \tilde{x}) \in H \times \tilde{M}$.*

PROOF. We have a natural action of H on $\mathcal{C}_n^f(M)$ which is continuous and holomorphic for each fixed $h \in H$. As the action of H is trivial on the set of fibers of π , the action of H is trivial on \tilde{N} . Define the action of H on $M \times_N \tilde{N}$ by the formula

$$h.(x, (\pi(x), C)) = (h.x, \pi(x), h_*(C)).$$

It is easy to see that this is a continuous holomorphic action, and that it leaves \tilde{M} globally invariant. Now we have for each $h \in H$ and each $(x, \pi(x), C) \in \tilde{M}$:

$$\tilde{\pi}(h.(x, (\pi(x), C))) = \tilde{\pi}(h.x, (\pi(x), h_*(C))) = (\pi(x), h_*(C)) = (\pi(x), C) = \tilde{\pi}(x, (\pi(x), C))$$

because a limit of H -invariant cycles is an H -invariant cycle. ■

⁵This is a corollary of the fact that this is true for $\mathcal{C}_n^{loc}(X)$ (see [B-M] ch.IV) as the topology of X has a countable basis of open sets; see [B.15] for details.

We shall also use the following simple tool from the cycle's space.

Proposition 2.2.3 *Let M be a reduced complex space and $(X_s)_{s \in S}$ a f -continuous family of d -dimensional finite type cycles parametrized by a compact subset S in $\mathcal{C}_d^f(M)$. Let $(C_t)_{t \in T}$ be a f -continuous family of finite type non empty n -dimensional cycles in M parametrized by a subset T in $\mathcal{C}_n^f(M)$ which is compact in $\mathcal{C}_n^{\text{loc}}(M)$. We assume the following condition :*

- *There exists an open dense set T' in T such that each $C_t, t \in T'$ is equal to the union some X_s . (@@)*

Then T is a compact subset in $\mathcal{C}_n^f(M)$.

PROOF. First remark that, as S is compact, there exists a compact set $L \subset M$ such that any irreducible component of any X_s meets L .

Let $(t_m)_{m \in \mathbb{N}}$ be a sequence of points in T' converging to a point $t \in T$ and denote by C_m the cycle C_{t_m} for short and $C_t = C_\infty$. Now choose for each m an irreducible component Γ_m of some X_{s_m} contained in C_m . Up to pass to a subsequence, we may assume that Γ_m converges in $\mathcal{C}_d^f(M)$ to a cycle Γ which is non empty (it contains a point in L) and included in $|C_\infty|$. So C_∞ is not the empty cycle.

Let x be a generic point of an irreducible component D of C_∞ . Then, up to pass to a subsequence, we may choose a sequence (x_m) of points respectively in C_m which converges to x . Choose for each m an irreducible component Γ_m of some $X_{s_m} \subset |C_m|$ which contains x_m . This is possible because of condition (@@). Now, again up to pass to a subsequence, we may assume that the sequence $(\Gamma_m)_{m \in \mathbb{N}}$ converges in $\mathcal{C}_d^f(M)$ to a cycle Γ containing the point x and contained in $|C_\infty|$. Note that $|\Gamma|$ is contained in some $|X_{s_\infty}|$ as we may assume, by compactness of S , that the sequence (s_m) converges to $s_\infty \in S$. Then we have $|X_{s_\infty}| \subset |C_\infty|$. As D is the only irreducible component of C_∞ containing x , it contains at least an irreducible component of $|X_{s_\infty}|$ containing x , and so D meets L . So we have proved that C_∞ is not the the empty n -cycle and that any irreducible component of C_∞ meets the compact set L . This is enough to conclude thanks to the proposition 3.2.2 in [B.15]. ■

2.3 Definition of SQP meromorphic quotient.

We shall consider a complex connected Lie group G and a completely holomorphic action of G on an irreducible complex space X . It is given, by definition, by a holomorphic map $f : G \times X \rightarrow X$ such that $f(g.g', x) = f(g, f(g'.x))$ for all $g, g' \in G$ and $x \in X$, assuming that for each $g \in G$ the holomorphic map $x \mapsto f(g, x)$ is an automorphism of X , and that $f(1, x) = x$ for all $x \in X$.

A **strongly quasi-proper meromorphic quotient** (we shall say a **SQP-meromorphic quotient** for short) for such an action $f : G \times X \rightarrow X$ will be the following data:

1. a G -modification⁶ $\tau : \tilde{X} \rightarrow X$ with center Σ .
2. a holomorphic G -invariant GF map $q : \tilde{X} \rightarrow Q$ where Q is an irreducible complex space.
3. an analytic G -invariant subset $Y \subset X$ containing Σ , with no interior point in X . We shall denote $\tilde{Y} := \tau^{-1}(Y)$, $\Omega := X \setminus Y$, $\tilde{\Omega} := \tau^{-1}(\Omega)$ and $Q' := q(\tilde{\Omega})$. Note that, as q is an open surjective map, Q' is open and dense in Q .

Now we ask that these data satisfy the following properties :

- i) The restriction to Ω of the map $q \circ \tau^{-1}$ is a GF map onto the dense open set Q' in Q and there is an open dense set Q'' in Q' such that each fiber of $q \circ \tau^{-1}$ at a point in Q'' is equal to a G -orbit in Ω .
- ii) There exists an open dense subset $\Omega_0 \subset \Omega$ such that for each \tilde{x} in $\tilde{\Omega}_0 := \tau^{-1}(\Omega_0)$ the **closure** $\overline{G.\tilde{x}}$ of $G.\tilde{x}$ in \tilde{X} is exactly the set $q^{-1}(q(\tilde{x}))$.

Proposition 2.3.1 *Let G be a complex connected Lie group which acts completely holomorphically on an irreducible complex space X . Assume that we have a SQP meromorphic quotient for this action, given by a modification $\tau : \tilde{X} \rightarrow X$ and a G -invariant GF map $q : \tilde{X} \rightarrow Q$. Then let $\psi : Q \rightarrow \mathcal{C}_n^f(X)$ be the holomorphic map obtained by the composition of the fiber map of the GF map q and the direct image map for n -cycles by the modification τ . Define $Q_u := \psi(Q)$. Then we have the following properties:*

1. Q_u is a closed analytic subset in $\mathcal{C}_n^f(X)$ which is an irreducible complex space (locally of finite dimension) with the structure sheaf induced by the sheaf of holomorphic functions on $\mathcal{C}_n^f(X)$.
2. Let \tilde{X}_u be the graph of the meromorphic map $q_u : X \dashrightarrow Q_u$ given by the holomorphic map $\psi \circ q : \tilde{X} \rightarrow Q_u$ and let $\tau_u : \tilde{X}_u \rightarrow X$ and $q_u : \tilde{X}_u \rightarrow Q_u$ be the projections on X and Q_u respectively of this graph. Then (τ_u, q_u) is also a SQP meromorphic quotient for the given G -action.
3. For any SQP meromorphic quotient (τ, q) there exists a unique holomorphic surjective map $\eta : Q \rightarrow Q_u$ such that the meromorphic maps $q : X \dashrightarrow Q$ and $q_u : X \dashrightarrow Q_u$ satisfies $\eta \circ q = q_u$.
4. For any G -invariant holomorphic map $h : X \rightarrow Y$ there exists a holomorphic map $H : Q_u \rightarrow Y$ such that $h \circ \tau_u = H \circ q_u$.

Definition 2.3.2 *In the situation of the previous theorem the SQP meromorphic quotient for the given G -action defined by (τ_u, q_u) will be called **the minimal SQP meromorphic quotient** of this G -action.*

⁶This means that we have a completely holomorphic G -action on \tilde{X} and that the modification τ is G -equivariant.

So the proposition above says that the existence of a SQP meromorphic quotient for the given G -action implies the existence and uniqueness of a minimal meromorphic quotient for this G -action.

PROOF. To prove the point 1) we shall prove that the map $\psi \circ q : \tilde{X} \rightarrow \mathcal{C}_n^f(X)$ is semi-proper.

Let $C \neq \emptyset$ be in $\mathcal{C}_n^f(X)$ and fix a relatively compact open set W in X meeting all irreducible components of C . The subset \mathcal{W} of $\mathcal{C}_n^f(X)$ of cycles C' such that any irreducible component of C' meets W is an open set containing C . Now $q(\tau^{-1}(\bar{W}))$ is a compact set in Q , as τ is proper. Take any $y \in Q$ such that $C' := \psi(y)$ is in \mathcal{W} . The point y is the limit in Q of points $y_\nu \in q(\tilde{\Omega}_0)$ such that the fiber of Q at y is limit in $\mathcal{C}_n^f(\tilde{X})$ of the fibers $q^{-1}(y_\nu) = \overline{G.\tilde{x}_\nu}$ where, for $\nu \gg 1$, we can choose \tilde{x}_ν in $\tilde{\Omega}_0 \cap \tau^{-1}(W)$. Up to pass to a sub-sequence, we may assume that \tilde{x}_ν converges to a point \tilde{x} in $\tau^{-1}(\bar{W})$. Then the continuity of q implies that $q(\tilde{x}) = y$ and C' is the limit of $\overline{G.\tilde{x}_\nu}$. So $|C'|$ is in the image by ψ of the compact set $q(\tau^{-1}(\bar{W}))$ and this gives the semi-properness of $\psi \circ q$.

Now the direct image theorem 2.3.2 in [B.15] shows that Q_u is an irreducible complex space (locally of finite dimension) and the point 1) is proved.

To prove the second point we have to show that the map $q_u : \tilde{X}_u \rightarrow Q_u$ is a GF map. By definition \tilde{X}_u is the closure in $X \times Q_u$ of the graph of the map $q|_{\Omega_0}$ where Ω_0 is an open dense set in X such that for any point $x \in \Omega_0$ we have $\psi(q(x)) = \overline{G.x}$ (as Ω_0 is disjoint from the center of the modification τ we identify here Ω_0 and $\tilde{\Omega}_0 := \tau^{-1}(\Omega_0)$). Then by irreducibility of Q_u and X the closed analytic subset $\tilde{X}_u \subset X \times Q_u$ is equal to the graph of the tautological family of cycles in X parametrized by $Q_u \subset \mathcal{C}_n^f(X)$. This proves the point 2).

Now consider a SQP meromorphic quotient of the given action given by the maps $\tau : \tilde{X} \rightarrow X$ and $q : \tilde{X} \rightarrow Q$. Let $\psi : \tilde{X} \rightarrow \mathcal{C}_n^f(X)$ the composition of the holomorphic map classifying the fibers of the GF map q with the direct image of n -cycles by the modification τ . Then, by the construction in the proof of the point 1), we know that $\psi(Q) = Q_u$ and then the map ψ induces a surjective holomorphic map $\eta : Q \rightarrow Q_u$. We may assume that \tilde{X} is in fact the graph of the meromorphic map $q : X \dashrightarrow Q$. Then, as \tilde{X}_u is the graph of the meromorphic map $q_u : X \dashrightarrow Q_u$ the holomorphic map $\text{id}_X \times \eta : X \times Q \rightarrow X \times Q_u$ sends \tilde{X} to \tilde{X}_u because this is true over a dense open set in X where the maps q and q_u are holomorphic and satisfy $q^{-1}(q(x)) = q_u^{-1}(q_u(x)) = \overline{G.x}$. This complete the proof of 3).

Consider now a G -invariant holomorphic map $h : X \rightarrow Y$. Recall (see [B.15] proposition 2.1.7) that $\mathcal{C}_n^f(h)$, the subset of cycles $C \in \mathcal{C}_n^f(X)$ which are contained in a fiber of h is a closed analytic subset in $\mathcal{C}_n^f(X)$. On an open dense subset of Q_u we know that a point corresponds to a cycle with support $\overline{G.x}$ for some $x \in X$. Then this means that an open dense subset in Q_u is contained in $\mathcal{C}_n^f(h)$. As this subset is closed we obtain $Q_u \subset \mathcal{C}_n^f(h)$. Now there is a holomorphic map $\hat{h} : \mathcal{C}_n^f(h) \rightarrow Y$ associating to a cycle C the point in Y such $|C| \subset h^{-1}(y)$ (see [B.15] proposition 2.1.7). This induces a holomorphic map $H : Q_u \rightarrow Y$, and it is clear that the relation $h \circ \tau_u = H \circ q_u$ is true on a dense open set, so everywhere. \blacksquare

2.4 Good points, good open set.

Let X be a reduced complex space and G be a connected complex Lie group. Let $f : G \times X \rightarrow X$ be a holomorphic action of G on X .

Definition 2.4.1 *We shall say that a point $x \in X$ is a **good point** for the action f if the following condition is satisfied*

- *For each compact set K in X there exists an open neighbourhood V of x and a compact set L in G such that if $y \in V$ and $g \in G$ are such that $g.y \in K$, there exists $\gamma \in L$ with $\gamma.y = g.y$*

*We shall say that the action of G on X is **good** when each point in X is a good point. If Ω is a G -invariant open set in X , we shall say that Ω is a **good open set** for the action f when all points in Ω are good points for the G -action given by f restricted to Ω .*

REMARKS.

1. If $x \in X$ is a good point, then for any $g_0 \in G$ $g_0.x$ is also a good point : for K given, choose $g_0.V$ as neighbourhood of $g_0.x$ and the compact set $L.g_0^{-1} \subset G$ to satisfy the needed conditions.
2. If Ω is a good open set, then points in Ω are not in general good points for the action on X .
3. If Ω is a good open set and $W \subset \Omega$ a G -invariant open set, then W is a good open set.
4. If M is a compact set of good points in X for any compact set K in X we can find a neighbourhood V of M in X and a compact set L in G such that for any point $y \in V$ and any $g \in G$ such that $g.y \in K$ there exists $\gamma \in L$ with $\gamma.y = g.y$. This is easily obtained by a standard compactness argument. We shall say that a compact set of good points is **uniformly good**.

Lemma 2.4.2 *Let x be a point in X . Then x is a good point for the G -action on X if and only if the map $F_X : G \times X \rightarrow X \times X$ given by $(g, x) \mapsto (x, g.x)$ is semi-proper at each point of $\{x\} \times X$. As a consequence a G -invariant open set Ω in X is a good open set for the G -action if and only if the map $F_\Omega : G \times \Omega \rightarrow \Omega \times \Omega$ given by $(g, x) \mapsto (x, g.x)$ is semi-proper.*

PROOF. Let $x \in X$ be a good point and fix any $z \in X$. To prove that the map F_X is semi-proper at (x, z) choose compact neighbourhoods V_0 and K of x and z in X and apply the definition of a good point to the compact set K . So we can find a neighbourhood V of x , that we may assume to be contained in V_0 , and a compact set L in G such that for any $y \in V$ such that $g.y \in K$ we have a $\gamma \in L$ with $g.y = \gamma.y$. Then we have $F_X(G \times X) \cap (V \times K) = F_X(L \times V_0) \cap (V \times K)$ and $L \times V_0$ is a compact set in $G \times X$.

Conversely, assume that the map F_X is semi-proper at each point of $\{x\} \times X$. Take a compact set K in X and apply the semi-properness to each point (x, z) where z is in K . For each $z \in K$ we obtain open neighbourhoods V_z and W_z of x and z in X and a compact set $L_z \times M_z$ in $G \times X$ such that

$$F_X(G \times X) \cap (V_z \times W_z) = F_X(L_z \times M_z) \cap (V_z \times W_z).$$

Extract a finite sub-cover W_1, \dots, W_N of K by the open sets W_z and define the compact set $L := \cup_{i \in [1, N]} L_{z_i}$ and the neighbourhood $V := \cap_{i \in [1, N]} V_{z_i}$ of x . Then if y is in V and $g.y$ is in K there exists $i \in [1, N]$ such that $g.y \in W_i$. As y is in V_i we can find a $\gamma \in L_{z_i} \subset L$ with $F_X(g, y) = F_X(\gamma, y)$ and this implies that x is a good point.

The second assertion is an easy consequence of the first one. ■

Proposition 2.4.3 *Let G be a connected complex Lie group. Let $f : G \times X \rightarrow X$ be an holomorphic action of G on a reduced complex space X . Then we have the following properties :*

- i) If x is a good point for f the orbit $G.x$ is a closed analytic subset of X .*
- ii) If x is a good point for f and if $(G.x) \cap K = \emptyset$ where $K \subset X$ is a compact set, there exists a neighbourhood V of x in X such that $(\overline{G.x'}) \cap K = \emptyset$ for any x' in V (x' is not assume here to be a good point).*
- iii) If x is a good point for f there exists a neighbourhood V of x such that any good point $x' \in V$ has an orbit which is a closed analytic subset of the same dimension than $G.x$.*
- iv) Let Ω be a good connected open set in X which is normal and let n be the dimension of $G.x$ for $x \in \Omega$. Then there exists a holomorphic map⁷ $\varphi : \Omega \rightarrow \mathcal{C}_n^f(\Omega)$ given generically by $\varphi(x) := G.x$ as a reduced n -cycle in Ω .*
- v) When we have a good open set Ω in X which is normal, there exists a quasi-proper equidimensional holomorphic quotient of Ω for the action restricted to Ω .*

⁷This means that we have an f -analytic family of n -cycles in Ω parametrized by Ω .

PROOF. We already proved that x is a good point if the map $G \rightarrow X$ given by $g \mapsto g.x$ is semi-proper in lemma 1.3.2. Now Kuhlmann's theorem [K.64], [K.66] gives that $f_x(G) = G.x$ is a closed analytic subset of X . This proved i)

Assume ii) is not true ; then we have a compact set K such that $(G.x) \cap K = \emptyset$ and a sequence $(x_\nu)_{\nu \in \mathbb{N}}$ converging to x and such that $(\overline{G.x_\nu}) \cap K$ is not empty for each ν . Fix a compact neighbourhood \tilde{K} of K such that $(G.x) \cap \tilde{K} = \emptyset$. This is possible thanks to i). Pick a point $y_\nu = \lim_{\alpha \rightarrow \infty} g_{\nu,\alpha}.x_\nu$ in $(\overline{G.x_\nu}) \cap K$ for each ν . Up to pass to a subsequence we may assume that sequence (y_ν) converges to $y \in K$ when $\nu \rightarrow +\infty$. So, for $\alpha \geq \alpha(\nu)$, we can assume that $g_{\nu,\alpha}.x_\nu$ is in \tilde{K} . But, as x is a good point, for the given \tilde{K} there exists a neighbourhood V of x and a compact set $L \subset G$ as in the definition. We may assume that x_ν is in V for $\nu \geq \nu_0$ and so we may find, for $\nu \geq \nu_0, \alpha \geq \alpha(\nu)$, elements $\gamma_{\nu,\alpha} \in L$ such that $\lim_{\alpha \rightarrow \infty} \gamma_{\nu,\alpha}.x_\nu = y_\nu \quad \forall \nu \geq \nu_0$.

Up to pass to a subsequence for each given $\nu \geq \nu_0$, we may assume that the sequence $(\gamma_{\nu,\alpha})$ converges to some $\gamma_\nu \in L$. And again, that the sequence γ_ν converges to some $\gamma \in L$. So the continuity of f gives $y_\nu = \gamma_\nu.x_\nu \rightarrow \gamma.x = y \in \tilde{K}$ giving a contradiction because we assume $(G.x) \cap \tilde{K} = \emptyset$. This proves ii).

Let $E := (U, B, j)$ be a n -scale on Ω adapted to the n -cycle $G.x$. Then the compact set $K := j^{-1}(\bar{U} \times \partial B)$ does not meet $G.x$, by definition of an adapted scale. Using ii), there exists a neighbourhood V of x such that for any $x' \in V$ we have $(\overline{G.x'}) \cap K = \emptyset$. As for a good point $x' \in V$ we know that $G.x'$ is a closed analytic subset, the n -scale is then adapted to $G.x'$. This implies that the dimension of $G.x'$ is at most equal to n . But the semi-continuity of the dimension of the stabilizers implies that the dimension of $G.x' \simeq G/St(x')$ is at least equal to $n = \dim(G/St(x))$. This proves iii).

Remark that for any $x' \in V$ such that $\overline{G.x'}$ is a closed analytic subset in Ω , the previous proof shows also that $\overline{G.x'}$ is of pure dimension n .

To prove iv) fix a good connected open set Ω and define

$$Z := \{(g, x, y) \in G \times \Omega \times \Omega / y = g.x.\}$$

This is a closed analytic subset in $G \times \Omega \times \Omega$. Let us show that the projection $p : Z \rightarrow \Omega \times \Omega$ is semi-proper. Pick a point $(x, y) \in \Omega \times \Omega$ and choose compact neighbourhoods V and K respectively of x and of y in Ω . Using the fact that any open set $\Omega' \subset \subset \Omega$ is uniformly good, for the compact K in Ω we find a compact set L in G such that for $x' \in \Omega'$ and $g \in G$ with $g.x' \in K$ there exists $\gamma \in L$ with $\gamma.x' = g.x'$. Choosing Ω' containing V this implies that we have

$$p(Z) \cap (V \times K) = p((L \times V \times K) \cap Z).$$

So the projection p is semi-proper. Its image $p(Z)$ is then a closed analytic subset

of $\Omega \times \Omega$ by Kuhlmann's theorem [K.64], [K.66]. But now the projection

$$\pi : p(Z) \rightarrow \Omega$$

is n -equidimensional, thanks to iii), and has irreducible generic fibers on a normal basis Ω . So its fibers (with generic multiplicity equal to 1) define an analytic family of n -cycles of X parametrized by Ω . It is clearly f -analytic because each fiber is irreducible⁸ and we have an holomorphic section because each x lies in $G.x$.

To prove v) let us prove that the holomorphic map $\varphi : \Omega \rightarrow \mathcal{C}_n^f(\Omega)$ classifying the fibers of $p(Z)$ is semi-proper. Fix a point $C \in \mathcal{C}_n^f(\Omega)$ and a point x_1, \dots, x_k in each irreducible component of $|C|$. Let W a relatively compact open neighbourhood of $\{x_1, \dots, x_k\}$ in Ω and let \mathcal{W} be the open set in $\mathcal{C}_n^f(\Omega)$ of cycles such that each irreducible component meets W . Let C' be in $\mathcal{W} \cap \varphi(\Omega)$ we know that if $C' = \varphi(z)$ that $|C'| = G.z$. So $G.z$ has to meet W and we can choose y in the compact set \bar{W} such that $C' = \varphi(y)$ and this gives the semi-properness of φ . Now the semi-proper direct image theorem 2.3.2 of [B.15] implies that the image Q of φ is a finite dimensional complex space.

Then the holomorphic map $q : \Omega \rightarrow Q$ which is a quasi-proper equidimensional holomorphic quotient for the action f on Ω as **each** fiber of q is set-theoretically a G -orbit. ■

2.5 The conditions [H.1], [H.2] and [H.3].

Now we shall consider the following conditions on the action f .

- **There exists a G -invariant dense open set Ω_1 in X which admits a GF- holomorphic quotient.** [H.1]

Recall that this means that there exists a G -invariant geometrically f -flat holomorphic map $q : \Omega_1 \rightarrow Q_1$ onto a reduced complex space Q_1 such that each fiber of q over a point in Q_1 is set-theoretically an orbit in Ω_1 .

The following stronger condition will be useful in the sequel:

- **There exists a G -invariant dense open set Ω in X which is good for the action f (on Ω).** [H.1str]

An immediate consequence of the proposition 2.4.3 is that the condition [H.1str] below is sufficient to satisfy [H.1] by taking Ω_1 as the set of normal points in Ω (which is open dense and G -invariant in Ω).

Now assume [H.1] and define $\mathcal{R} := \{(x, y) \in \Omega_1 \times \Omega_1 / y \in G.x\}$. It is a closed analytic set in $\Omega_1 \times \Omega_1$. We shall denote $\overline{\mathcal{R}}$ the closure of \mathcal{R} in $X \times X$. Our second assumption will be :

⁸But some multiplicities may occur.

- **The subset $\overline{\mathcal{R}}$ is analytic in $X \times X$ and there exists an open dense subset $\Omega_0 \subset \Omega_1$ such that for each $x \in \Omega_0$ we have**

$$\overline{G.x} = \overline{\mathcal{R}} \cap (\{x\} \times X). \quad [\text{H.2}]$$

Remark that the first projection $p_1 : \overline{\mathcal{R}} \cap (\Omega_0 \times X) \rightarrow \Omega_0$ is quasi-proper because we have a holomorphic section of this map (with irreducible fibers, thanks to [H.2]) which is given by $x \mapsto (x, x)$.

Assuming that Ω_0 contains only normal points⁹ in X , the equidimensionality and quasi-properness on Ω_0 of the projection of $\overline{\mathcal{R}}$ implies that there exists a holomorphic map

$$\bar{\varphi}_0 : \Omega_0 \rightarrow \mathcal{C}_n^f(X)$$

where the supports are given by $x \mapsto \overline{G.x}$ and where the multiplicity is generically equal to 1.

We shall denote $\Gamma \subset \Omega_0 \times \mathcal{C}_n^f(X)$ the graph of the map $\bar{\varphi}_0$ and $\bar{\Gamma}$ its closure in $X \times \mathcal{C}_n^f(X)$. We shall denote $\theta : \bar{\Gamma} \rightarrow X$ the map induced by the first projection. Our last hypothesis is :

- **The map $\theta : \bar{\Gamma} \rightarrow X$ is proper .** [H.3]

This hypothesis [H.3], assuming [H.1] and [H.2] is in fact equivalent to ask that the projection $\overline{\mathcal{R}} \rightarrow X$ is strongly quasi-proper: this is an immediate consequence of the proposition 3.2.2 of [B.15].

The following proposition shows that these conditions [H.1], [H.2] and [H.3] are necessary for the existence of a SQP-meromorphic quotient for a completely holomorphic action of G on X .

Proposition 2.5.1 *Assuming that the completely holomorphic action $f : G \times X \rightarrow X$ of the connected complex Lie group G on the irreducible complex space X has a SQP-meromorphic quotient, then the conditions [H.1], [H.2] and [H.3] are satisfied.*

PROOF. The conditions to be a SQP-meromorphic quotient gives an open set Ω_1 which is dense, G -stable and which admits a GF holomorphic quotient for the action on Ω_1 . So [H.1] is clear.

Let S be the graph of the equivalence relation given by q on \tilde{X} . Then the proper direct image $(\tau \times \tau)(S)$ is $\overline{\mathcal{R}}$ and so the condition [H.2] is satisfied as soon as we can find an open and dense subset Ω_0 in Ω_1 such that for each $x \in \Omega_0$ we have the equality

$$\overline{G.x} = \overline{\mathcal{R}} \cap (\{x\} \times X).$$

But this property is given by the condition ii) in the definition of a SQP-meromorphic quotient.

⁹This not restrictive, as we may always assume that $X \setminus \Omega_0$ contains the non normal points in X . We shall always assume that Ω_0 is normal in the sequel, without any more comment.

The composition of q with the classifying map $Q \rightarrow \mathcal{C}_n^f(\tilde{X})$ for the fibers of q gives a holomorphic map $\tilde{\psi} : \tilde{X} \rightarrow \mathcal{C}_n^f(\tilde{X})$. Composed with the direct image map, which is holomorphic (see [B.M] ch.IV ; the “quasi-proper” part of this result is easy, as τ is proper) $\tau_* : \mathcal{C}_n^f(\tilde{X}) \rightarrow \mathcal{C}_n^f(X)$, we obtain the fact that $\overline{\mathcal{R}}$ is strongly quasi-proper on X which is the condition [H.3]. \blacksquare

2.6 Existence of SQP-meromorphic quotient

Now we shall prove that conditions [H.1], [H.2] and [H.3] on a completely holomorphic action of a connected complex Lie group G on an irreducible complex space X are sufficient for the existence of a SQP meromorphic quotient.

Theorem 2.6.1 *Under the hypothesis [H.1], [H.2] and [H.3] there exists a proper G -equivariant modification $\tau : \tilde{X} \rightarrow X$ with center contained in $X \setminus \Omega_0^{10}$ and a geometrically f -flat holomorphic map*

$$q : \tilde{X} \rightarrow Q$$

on a reduced complex space, which give a strongly quasi-proper meromorphic quotient for the given G -action.

Of course the complex space \tilde{X} is the topological space $\bar{\Gamma}$ with a structure of a reduced complex space such that the projection on X is a proper modification. Then the space Q is the image of \tilde{X} in $\mathcal{C}_n^f(X)$. So we need some semi-proper direct image theorem for such a map to prove this result. Such a result is the content of the theorem 2.3.2 of [B.15]

PROOF. The first remark is that the hypothesis [H.3] says that the projection $p : \bar{\Gamma} \rightarrow X$ is a proper topological modification of X . But to apply directly the part ii) of the theorem 2.3.6 of [B.13] to the projection $p_1 : \overline{\mathcal{R}} \rightarrow X$ we need quasi-properness of this map. This is given by the proposition 3.2.2 of [B.15] as we have the condition [H.3].

Then we obtain a proper (holomorphic) modification with center in $\Sigma \subset X \setminus \Omega_0$, $\tau : \tilde{X} \rightarrow X$, and a f -analytic family of cycles in X parametrized by \tilde{X} extending the family $(\overline{G.x})_{x \in \Omega_0}$, corresponding to a “holomorphic” map extending $\tilde{\varphi}_0$:

$$\tilde{\varphi} : \tilde{X} \rightarrow \mathcal{C}_n^f(X).$$

Now let us prove that this map $\tilde{\varphi}$ is quasi-proper¹¹. This will allow us to apply the theorem 2.3.2 of *loc. cit.* and to define the reduced complex space Q as the image $\tilde{\varphi}(\tilde{X})$. Then it will be easy to check that the map $\tilde{\varphi} : \tilde{X} \rightarrow Q$ is a strongly quasi-proper meromorphic quotient for the G -action we consider.

If C_0 is in $\mathcal{C}_n^f(X)$ and is not the empty cycle, choose a relatively compact open

¹⁰The dense open subset $\Omega_0 \subset \Omega_1$ is defined in the condition [H.2]

¹¹This makes sense as the fibers are closed analytic subsets of \tilde{X} .

set W in X such that any irreducible component of $|C_0|$ meets W . Then let \mathcal{W} be the open set in $\mathcal{C}_n^f(X)$ defined by the condition that any irreducible component of C meets W for $C \in \mathcal{W}$. Then we shall prove that there exists a compact set K in \tilde{X} such that any irreducible component of the fiber of $\tilde{\varphi}$ at a point in $\mathcal{W} \cap \tilde{\varphi}(\tilde{X})$ meets K . Let $K := \tau^{-1}(\bar{W})$. If (y, C) is in \tilde{X}^{12} with $C \in \mathcal{W}$, each irreducible component of C meets W . But the fiber of $\tilde{\varphi}$ at C is equal to $|C|$, and the quasi-properness is proved. ■

3 Application.

3.1 The sub-analytic lemma.

We shall use the following lemma (see [G-M-O]) in our application.

Lemma 3.1.1 *Let M be a reduced complex space and $Y \subset M$ a closed analytic subset with no interior point in M . Let R be a closed (complex) analytic subset in $M \setminus Y$ such that \bar{R} is a sub-analytic set in M . Then \bar{R} is a (complex) analytic subset in M .*

This important lemma is a consequence of Bishop's theorem (see [Bi.64]) and of a classical result on sub-analytic subsets (see [G-M-O]).

3.2 Proof of the theorem 1.0.1.

Now we shall assume that G is a connected complex Lie group such that $G = K.B$ where B is a closed complex connected subgroup of G and K a compact real subgroup of G .

The first condition [H.1str] for the G -action is given by the following lemma :

Lemma 3.2.1 *In the situation of the theorem 1.0.1, assume that we have a G -invariant open set Ω which is a good open set for the B -action, then Ω is a good open set for the G -action.*

PROOF. Consider a point $x \in \Omega$ and a compact set M in Ω . Then there exists a neighbourhood V of x in Ω and a compact set L in B such that $b.y \in M$ for some $y \in V$ and some $b \in B$ implies that we can find $\beta \in L$ with $b.y = \beta.y$. Now assume that M is K -invariant (here we use the G -invariance of Ω) and that $g.y$ is in M for some $g \in G$ and some $y \in V$. Write $g = k.b$ for some $k \in K$ and $b \in B$. Then $b.y$ is again in M so we can find $\beta \in L$ with $\beta.y = b.y$ and then $g.y = k.\beta.y$ with $k.\beta \in K.L$ which is a compact set in G . So x is a good point for the G -action on Ω . ■

¹²Recall that, as a topological space, $\tilde{X} = \bar{\Gamma}$.

The corollary of the next lemma will give the first part of [H.2] for the G -action assuming that we have a G -invariant dense good Zariski open set Ω for the B -action with the condition [H.2] for the B -action. In order to use the sub-analytic lemma 3.1.1 in this situation we shall need the following lemma.

Lemma 3.2.2 *Let Ω be an open G -invariant good set for the B -action, and then also good for the G -action thanks to the previous lemma.*

Let $\chi : K \times X \times X \rightarrow X \times X$ the map given by $(k, x, y) \mapsto (k.x, y)$ and let $p : K \times X \times X \rightarrow X \times X$ be the natural projection. Then we have

$$p(\chi^{-1}(\overline{\mathcal{R}_B})) = \overline{\mathcal{R}_G}$$

where we define

$$\mathcal{R}_B := \{(x, y) \in \Omega \times \Omega / B.x = B.y\} \quad \text{and} \quad \mathcal{R}_G := \{(x, y) \in \Omega \times \Omega / G.x = G.y\},$$

and where the closures are taken in $X \times X$.

PROOF. Remark first that

$$p(\chi^{-1}(\mathcal{R}_B)) = \{(x, y) \in \Omega \times \Omega / \exists k \in K \ B.k.x = B.y\}.$$

So $(x, y) \in p(\chi^{-1}(\mathcal{R}_B))$ implies $y \in B.k.x \subset G.x$ and also $k.x \in B.y$; we conclude that x is in $K.B.y = G.y$. This gives the inclusion $p(\chi^{-1}(\mathcal{R}_B)) \subset \mathcal{R}_G$. The opposite inclusion is easy because $G.x = G.y$ implies that $x \in K.B.y$ so there exists $k \in K$ such that $k.x \in B.y$.

Now the map χ and p are proper, so we obtain the inclusion $p(\chi^{-1}(\overline{\mathcal{R}_B})) \subset \overline{\mathcal{R}_G}$.

Let $(x, y) := \lim_{\nu \rightarrow \infty} (x_\nu, y_\nu)$ where (x_ν, y_ν) is in \mathcal{R}_G for each $\nu \in \mathbb{N}$. Then for each ν there exists $k_\nu \in K, b_\nu \in B$ such that $x_\nu = k_\nu.b_\nu.y_\nu$; then $(k_\nu^{-1}, x_\nu, y_\nu) \in \mathcal{R}_B$. Up to pass to a subsequence we may assume that the sequence (k_ν) converges to some $k \in K$. As $(k_\nu^{-1}, x_\nu, y_\nu)$ is in $\chi^{-1}(\mathcal{R}_B)$ for each ν , the point (k^{-1}, x, y) is in

$$\overline{\chi^{-1}(\mathcal{R}_B)} = \chi^{-1}(\overline{\mathcal{R}_B})$$

and so (x, y) is in $p(\chi^{-1}(\overline{\mathcal{R}_B}))$ proving the opposite inclusion. ■

Corollaire 3.2.3 *In the situation of the previous lemma, assume that $X \setminus \Omega$ is a (complex) analytic subset; then if the subset $\overline{\mathcal{R}_B}$ is (complex) analytic in $X \times X$, the subset $\overline{\mathcal{R}_G}$ is also a (complex) analytic subset of $X \times X$.*

PROOF. Note first that the maps χ and p are real analytic, so assuming that $\overline{\mathcal{R}_B}$ is analytic implies that $p(\chi^{-1}(\overline{\mathcal{R}_B}))$ is sub-analytic. Then, as we know that \mathcal{R}_G is an irreducible locally closed complex analytic subset, the conclusion follows from the lemma 3.1.1, as our assumption that Ω is a Zariski (dense) open set in X implies that $\Omega \times \Omega$ is Zariski open (and dense) in $X \times X$. ■

A first step to prove the quasi-properness of $\overline{\mathcal{R}_G}$ is our next result.

Lemma 3.2.4 *Let assume that the B -action f on X satisfies [H.1] and [H.2]. Let $\Omega_0 \subset \Omega_1$ be an open set on which the fiber at any $x \in \Omega_0$ of $\overline{\mathcal{R}_B}$ is equal to $\overline{B.x}$ (with some multiplicity). Then the fiber at any $x \in \Omega_0$ of $\overline{\mathcal{R}_G}$ is equal to $\overline{G.x}$ (with some multiplicity).*

PROOF. As we know that the map $x \mapsto \overline{B.x}$, with generic multiplicity 1, extends to is a f -analytic family of cycles of X parametrized by Ω_0 , for each sequence $(x_\nu)_{\nu \in \mathbb{N}}$ of points in Ω_0 converging to a point $x \in \Omega_0$ we have (with suitable multiplicity) $\overline{B.x} = \lim_{\nu \rightarrow \infty} \overline{B.x_\nu}$ in the topology of $\mathcal{C}_d^f(X)$. We shall show that this implies, also with suitable multiplicity, the equality $\overline{G.x} = \lim_{\nu \rightarrow \infty} \overline{G.x_\nu}$ in the topology of $\mathcal{C}_n^f(X)$. As we have $G = K.B$ with K compact, for any $y \in X$ we have $\overline{G.y} = \overline{K.B.y}$. So the inclusion of $\lim_{\nu \rightarrow \infty} \overline{G.x_\nu}$ in the fiber at x of $\overline{\mathcal{R}_G}$ is clear. The point is to prove the opposite inclusion. Let y be a point in the fiber at $x \in \Omega_0$ of $\overline{\mathcal{R}_G}$. It is a limit of a sequence $y_\nu \in \overline{G.x_\nu}$ where $x_\nu \in \Omega_0$ converges to x . Write $y_\nu = k_\nu.b_\nu.x_\nu$ with $k_\nu \in K$ and $b_\nu \in B$. Up to pass to a subsequence, we may assume that the sequence (k_ν) converges to a point $k \in K$. So we have $k^{-1}.y$ which is the limit of the sequence $b_\nu.x_\nu$. We obtain that $k^{-1}.y$ is in the limit of $\overline{B.x_\nu}$ which has support equal to $\overline{B.x}$. Then y is in $\overline{K.B.x} = \overline{G.x}$, concluding the proof. \blacksquare

PROOF OF THE THEOREM 1.0.1. The fact that the G -action satisfies [H.1str] is consequence of lemma 3.2.1. The analyticity of $\overline{\mathcal{R}_G}$ in $X \times X$ is proved at corollary 3.2.3. The lemma 3.2.4 gives a dense open set Ω_0 where the fiber of the projection p_1 of $\overline{\mathcal{R}_G}$ at each point $x \in \Omega_0$ is equal to $\overline{G.x}$ as a set. This implies the quasi-properness of p_1 over Ω_0 , because x is in $\overline{G.x}$ and G is connected; assuming (which is not restrictive) that Ω_0 is normal, we obtain a holomorphic map

$$\Phi : \Omega_0 \longrightarrow \mathcal{C}_n^f(X)$$

where the support of $\Phi(x)$ is equal to $\overline{G.x}$ for each $x \in \Omega_0$ and with generic multiplicity equal to 1. This complete the proof of [H.2] for the G -action.

Thanks to proposition 3.2.2 of [B.15], to prove [H.3] it is enough to show that the closure of the graph Γ_G of Φ in $X \times \mathcal{C}_n^f(X)$ is proper on X .

The projection $p_B : \overline{\mathcal{R}_B} \rightarrow X$ is strongly quasi-proper so, for V compact in X , the limits of cycles (with convenient multiplicity) $\overline{B.x}$ for $x \in V \cap \Omega_0$ stay in a compact subset S_0 in $\mathcal{C}_d^f(X)$. Then let $S := K.S_0$; this is again a compact subset in $\mathcal{C}_d^f(X)$. But the subset T of limits of the generic fibers of the projection $p_G : \overline{\mathcal{R}_G} \rightarrow X$ for $x \in V$ is a compact set of $\mathcal{C}_n^{loc}(X)$ thanks to [B.13] theorem 2.3.6 i). For $x \in V \cap \Omega_0$ we have $p_G^{-1}(x) = \overline{G.x}$, and let T' be the open dense set in T described by the cycles $\overline{\varphi_0(x)}$, $x \in V \cap \Omega_0$. As we know that for x in Ω_0 we have $|\overline{\varphi_0(x)}| = \overline{G.x}$ we know that each of them is an union of d -cycles in S , thanks to the lemma 3.2.5 below. The proposition 2.2.3 implies that the set of limits of $\overline{G.x_\nu}$ for $x_\nu \in V \cap \Omega_0$ is a compact set in $\mathcal{C}_n^f(X)$, proving [H.3]. \blacksquare

Lemma 3.2.5 *Let K be a compact Lie group with a holomorphic action on a reduced complex space M . Let X be an irreducible complex space and let $\varphi : X \rightarrow \mathcal{C}_d^f(M)$ and $\psi : X \rightarrow \mathcal{C}_n^f(M)$ two holomorphic maps. Assume that for each x in a dense open subset Ω_0 in X we have*

$$|\psi(x)| = \cup_{k \in K} k \cdot |\varphi(x)| = K \cdot |\varphi(x)| \quad (\text{@@@})$$

where K acts holomorphically on $\mathcal{C}_d^f(M)$ via the action of K on M . Then the relation (@@@) holds for each $x \in X$.

PROOF. Let $x \in X$ and choose a sequence $(x_\nu)_{\nu \geq 0}$ in Ω_0 converging to x and let y a point in $|\psi(x)|$. Then we can choose a sequence $(y_\nu)_{\nu \geq 0}$ of points in $|\psi(x_\nu)|$ converging to y . As x_ν is in Ω_0 we can write $y_\nu = k_\nu \cdot z_\nu$ with $z_\nu \in |\varphi(x_\nu)|$ and $k_\nu \in K$. Up to pass to a subsequence we can assume that the sequence (k_ν) converges to some $k \in K$. So the sequence (z_ν) converges to $k^{-1} \cdot y$ which is in $|\varphi(x)|$. This gives that y is in $k \cdot |\varphi(x)|$ and we have proved the inclusion $|\psi(x)| \subset K \cdot |\varphi(x)|$. Conversely, if z is in $|\varphi(x)|$ and k is in K , write again $z = \lim z_\nu$ with $z_\nu \in |\varphi(x_\nu)|$ where the sequence (x_ν) of points in Ω_0 converges to x . We have $k \cdot z_\nu \in |\psi(x_\nu)|$, and then $k \cdot z$ is in $|\psi(x)|$ concluding the proof. ■

4 Bibliography

- [B.75] Barlet, D. *Espace analytique réduit des cycles analytiques complexes compacts d'un espace analytique complexe de dimension finie*, Fonctions de plusieurs variables complexes, II (Séminaire François Norguet, 1974-1975), pp.1-158. Lecture Notes in Math., Vol. 482, Springer, Berlin, 1975
- [B.78] Barlet, D. *Majoration du volume des fibres génériques et forme géométrique du théorème d'aplatissement*, Seminaire Lelong-Skoda 78-79, Lect. Notes in Math. 822, p.1-17, Springer-Verlag, Berlin.
- [B.08] Barlet, D. *Reparamétrisation universelle de familles f -analytiques de cycles et f -aplatissement géométrique* Comment. Math. Helv. 83 (2008), p. 869-888.
- [B.13] Barlet, D. *Quasi-proper meromorphic equivalence relations*, Math. Z. (2013), vol. 273, p. 461-484.
- [B.15] Barlet, D. *Strongly quasi-proper maps and f -flattning theorem*, arXiv.1504.01579v1 [math. CV].
- [Bi.64] Bishop, E. Condition for the analyticity of certain sets, Michigan Math. J. 11 (1964), p.289-304.
- [B-M] Barlet, D. et Magnússon, J. *Cycles analytiques complexes I : Théorèmes de préparation des cycles*, Cours Spécialisés 22, Société mathématique de France, (2014).
- [K.64] Kuhlmann, N. *Über holomorphe Abbildungen komplexer Räume* Archiv der Math. 15 (1964), p.81-90.
- [K.66] Kuhlmann, N. *Bemerkungen über holomorphe Abbildungen komplexer Räume* Wiss. Abh. Arbeitsgemeinschaft Nordrhein-Westfalen 33, Festschr. Gedächtnisfeier K. Weierstrass (1966), p.495-522.
- [G-M-O] B. Gilligan, C. Miebach and K. Oeljeklaus *Homogeneous Kähler and Hamiltonian manifolds*, Math. Ann. (2011), p.889-901.
- [M.00] Mathieu, D. *Universal reparametrization of a family of cycles : a new approach to meromorphic equivalence relations*, Ann. Inst. Fourier (Grenoble) t. 50, fasc.4 (2000) p.1155-1189.
- [R.57] Remmert, R. *Holomorphe und meromorphe Abbildungen komplexer Räume*, Math. Ann. 133 (1957), p.328-370.