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Daniel Barlet

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# Meromorphic quotients for some holomorphic G-actions.

Daniel Barlet\*

12/05/15

**ABSTRACT.** Using mainly tools from [B.13] and [B.15] we give a necessary and sufficient condition in order that a holomorphic action of a connected complex Lie group  $G$  on a reduced complex space  $X$  admits a strongly quasi-proper meromorphic quotient. We apply this characterization to obtain a result which asserts that, when  $G = K.B$  with  $B$  a closed complex subgroup of  $G$  and  $K$  a real compact subgroup of  $G$ , the existence of a strongly quasi-proper meromorphic quotient for the  $B$ -action implies, assuming moreover that there exists a  $G$ -invariant Zariski open dense subset in  $X$  which is good for the  $B$ -action, the existence of a strongly quasi-proper meromorphic quotient for the  $G$ -action on  $X$ .

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**KEY WORDS.** Strongly quasi-proper map, strongly quasi-proper meromorphic quotient, Holomorphic G-action, finite type cycles,

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## 1 Introduction

In this article we explain how the tools developed in [M.00], [B.08], [B.13] and [B.15] can be applied to produce in suitable cases a meromorphic quotient of a holomorphic action of a connected complex Lie group  $G$  on a reduced complex space  $X$ . This uses the notion of **strongly quasi-proper map** introduced in *loc. cit.* and our first goal is to give three hypotheses, called [H.1], [H.2], [H.3] on the group action which are **equivalent** to the existence of a **strongly quasi-proper meromorphic quotient**, notion defined in the section 1.2.

The proof of this equivalence is the content of proposition 2.5.1 and theorem 2.6.1. Then we discuss these hypotheses and give a simple sufficient condition [H.1str] which implies [H.1]. The existence theorem for a strongly quasi-proper meromorphic quotient under our three assumptions is applied to prove the following results :

**Theorem 1.0.1** *Assume that we have a holomorphic action of a connected complex Lie group  $G$  on a reduced complex space  $X$ . Assume that  $G = K.B$  where  $K$  is a compact (real) subgroup of  $G$  and  $B$  a connected complex closed subgroup of  $G$ . Assume that the action of  $B$  on  $X$  satisfies the condition [H.1str] on a  $G$ -invariant Zariski open dense subset  $\Omega$  in  $X$ , and the conditions [H.2] and [H.3]. Then the  $G$ -action satisfies [H.1str], [H.2] and [H.3] ; so it has a strongly quasi-proper meromorphic quotient.*

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## 2 Strongly quasi-proper meromorphic quotients.

### 2.1 Preliminaries.

For the definition of the topology on the space  $\mathcal{C}_n^f(X)$  of finite type  $n$ -cycles in  $X$  and its relationship with the (topological) space  $\mathcal{C}_n^{loc}(X)$  we refer to [B-M], [B.13] and [B.15].

For the convenience of the reader we recall shortly here the definition of a geometrically f-flat map (GF map) and of a strongly quasi-proper map (SQP map) between irreducible complex spaces and we give a short summary on some properties of the SQP maps. For more details on these notions see [B.13] and [B.15].

**Definition 2.1.1** *A holomorphic map  $f : M \rightarrow N$  between two irreducible irreducible complex spaces is called a **geometrically f-flat map (a GF-map for short)** if the following conditions are fulfilled :*

i) *The map is quasi-proper equidimensionnal and surjective.  
Let  $n := \dim M - \dim N$ .*

ii) *There exists a holomorphic map  $\varphi : N \rightarrow \mathcal{C}_n^f(M)^1$  such that for  $y$  generic in  $N$  the cycle  $\varphi(y)$  is reduced and equal to the set-theoretic fiber  $f^{-1}(y)$  of  $f$  at  $y$ .*

*A holomorphic map  $f : M \rightarrow N$  between two irreducible irreducible complex spaces will be **strongly quasi-proper (SQP map for short)** if there exists a modification<sup>2</sup>  $\tau : \tilde{N} \rightarrow N$  such that the strict transform<sup>3</sup>  $\tilde{f} : \tilde{M} \rightarrow \tilde{N}$  of  $f$  by  $\tau$  is a GF map. A meromorphic map  $M \dashrightarrow N$  will be called strongly quasi-proper when the projection on  $N$  of its graph is a SQP map.*

Note that a GF map has, by definition, a *holomorphic fiber map* and that a SQP holomorphic (or meromorphic) map has a *meromorphic fiber map* via the composition of the holomorphic fiber map of  $\tilde{f}$  with the (holomorphic) direct image map  $\tau_* : \mathcal{C}_n^f(\tilde{M}) \rightarrow \mathcal{C}_n^f(M)$ . Of course, a SQP holomorphic map is quasi-proper, but the converse is not true. The notion of strongly quasi-proper map is stable by modification of the target space, property which is not true in general for a quasi-proper map having “big fibers” (see [B.15]).

Let  $\pi : M \rightarrow N$  be a SQP map between irreducible complex spaces and define  $n := \dim M - \dim N$ . By definition of a SQP map, we can find a Zariski open dense subset  $N_0$  in  $N$  and a holomorphic map  $\varphi_0 : N_0 \rightarrow \mathcal{C}_n^f(M)$  such that

- i) For each  $y$  in  $N_0$  we have the equality of subsets  $|\varphi_0(y)| = \pi^{-1}(y)$ .
- ii) For  $y$  generic in  $N_0$  the cycle  $\varphi_0(y)$  is reduced.

Let  $\Gamma \subset N_0 \times \mathcal{C}_n^f(M)$  be the graph of  $\varphi_0$ . Then also by definition of a SQP map, the closure  $\bar{\Gamma}$  of  $\Gamma$  in  $N \times \mathcal{C}_n^f(M)$  is proper over  $N$ . Then, using the semi-proper direct image theorem 2.3.2 of [B.15], this implies that  $\tilde{N} := \bar{\Gamma}$  is an irreducible complex space (locally of finite dimension) with the structure sheaf induced by the sheaf of holomorphic functions on  $N \times \mathcal{C}_n^f(M)$ . Moreover the natural projection  $\tau : \tilde{N} \rightarrow N$  is a (proper) modification.

Let  $\tilde{M} := M \times_{N, str} \tilde{N}$  the strict transform of  $M$  by  $\tau$ , that is to say the irreducible component of  $M \times_N \tilde{N}$  containing the graph of  $\pi_0$ , the restriction of  $\pi$  to the open set  $\pi^{-1}(N_0)$ <sup>4</sup>. Then let  $\tilde{\pi} : \tilde{M} \rightarrow \tilde{N}$  the strict transform of  $\pi$  by the modification  $\tau$ , which is induced on  $\tilde{M}$  by the natural projection of  $M \times_N \tilde{N}$  onto  $\tilde{N}$ . The

<sup>1</sup>that is to say a f-analytic family of finite type  $n$ -cycles in  $M$  parametrized by  $N$ .

<sup>2</sup>a modification is, by definition, always proper.

<sup>3</sup>By definition  $\tilde{M}$  is the irreducible component of  $M \times_N \tilde{N}$  which surjects on  $\tilde{N}$  and  $\tilde{f}$  is induced by the projection.

<sup>4</sup>this graph is a Zariski open set in  $M \times_N \tilde{N}$  which is irreducible as  $N$  is irreducible.

set-theoretical fiber at  $\tilde{y} := (y, C) \in \tilde{N}$  of  $\tilde{\pi}$  is the subset  $|C| \times \tilde{y}$  in  $\tilde{M}$ . The map  $\psi : \tilde{N} \rightarrow \mathcal{C}_n^f(\tilde{M})$  given by  $(y, C) \mapsto C \times \{\tilde{y}\}$  is holomorphic and satisfies  $|\psi(\tilde{y})| = \tilde{\pi}^{-1}(\tilde{y})$  for all  $\tilde{y}$  in  $\tilde{N}$ . Moreover  $\psi(\tilde{y})$  is a reduced cycle for generic  $\tilde{y}$  in  $\tilde{N}$ . So the map  $\tilde{\pi}$  is geometrically f-flat. It is the **canonical GF-flattning** of  $\pi$ . Then we have an isomorphism induced by  $\psi$

$$\psi : \tilde{N} \longrightarrow \mathcal{C}_n^f(\tilde{\pi})$$

where  $\mathcal{C}_n^f(\tilde{\pi}) := \{C \in \mathcal{C}_n^f(\tilde{M}) / \exists \tilde{y} \in \tilde{N} \text{ s.t. } |C| \subset \tilde{\pi}^{-1}(\tilde{y})\}$  is a closed analytic subset of  $\mathcal{C}_n^f(\tilde{M})$  (see [B.15] proposition 2.1.7.); the inverse map is induced by the holomorphic map  $\hat{\pi} : \mathcal{C}_n^f(\tilde{\pi}) \rightarrow \tilde{N}$  which associates to  $\gamma \in \mathcal{C}_n^f(\tilde{\pi})$  the point in  $\tilde{N}$  whose  $\tilde{\pi}$ -fiber contains  $\gamma$  (see the proposition 2.1.7 of [B.15]).

The direct image of  $n$ -cycles by  $\tau$  gives a holomorphic map  $\tau_* : \mathcal{C}_n^f(\tilde{M}) \rightarrow \mathcal{C}_n^f(M)$  which sends  $\tilde{N} \simeq \mathcal{C}_n^f(\tilde{\pi})$  in  $\mathcal{C}_n^f(\pi)$ . Let us show that it is an isomorphism of  $\tilde{N}$  onto its image in  $\mathcal{C}_n^f(\pi)$  :

We have an obvious holomorphic map  $\tilde{N} \rightarrow \mathcal{C}_n^f(\pi)$  given by  $(y, C) \mapsto C$ . We have also a holomorphic map  $\mathcal{C}_n^f(\pi) \rightarrow N \times \mathcal{C}_n^f(M)$  given by  $C \mapsto (\hat{\pi}, C)$  where  $\hat{\pi} : \mathcal{C}_n^f(\pi) \rightarrow N$  is the map associating to  $C \in \mathcal{C}_n^f(\pi)$  the point  $y \in N$  such that  $|C| \subset \pi^{-1}(y)$ . This proves our claim.

Remark that  $\mathcal{C}_n^f(\pi)$  is not, in general, a complex space (locally of finite dimension).

## 2.2 Action of $G$ on $\mathcal{C}_n^f(X)$ .

Let  $G$  be a Lie group. We shall say that  $G$  acts **continuously holomorphically** on the reduced complex space  $X$  when the action  $f : G \times X \rightarrow X$  is a continuous map such that for each  $g \in G$  fixed, the map  $x \mapsto f(g, x)$  is a (biholomorphic) automorphism of  $X$ . Then there is a natural action of  $G$  induced on the set  $\mathcal{C}_n^f(X)$  of finite type  $n$ -cycles given by  $(g, C) \mapsto g_*(C)$  where we denote  $g_*(C)$  the direct image of the cycle  $C$  by the automorphism of  $X$  associated to  $g \in G$ . When  $G$  is a complex Lie group and the map  $f$  is holomorphic we shall say that the action is **completely holomorphic**.

**Proposition 2.2.1** *The action of  $G$  on  $\mathcal{C}_n^f(X)$  is continuously holomorphic. This precisely means that the map  $G \times \mathcal{C}_n^f(X) \rightarrow \mathcal{C}_n^f(X)$  given by  $(g, C) \mapsto g_*(C)$  is continuous and that, for any  $f$ -analytic family of  $n$ -cycles  $(C_s)_{s \in S}$  in  $X$  parametrized by a reduced complex space  $S$ , the family  $g_*(C_s)_{(g,s) \in \{g\} \times S}$  is  $f$ -analytic for each fixed  $g \in G$ . If  $G$  is complex Lie group and the action is completely holomorphic, the action of  $G$  on  $\mathcal{C}_n^f(X)$  is completely holomorphic, so for any  $f$ -analytic family of  $n$ -cycles  $(C_s)_{s \in S}$  in  $X$  parametrized by a reduced complex space  $S$ , the family  $g_*(C_s)_{(g,s) \in G \times S}$  is  $f$ -analytic.*

**PROOF.** First we prove the continuity of the action of  $G$  on  $\mathcal{C}_n^{loc}(X)$ . To apply the theorem IV 2.5.6 de [B-M] it is enough to see that the map  $F : G \times X \rightarrow G \times X$  given by  $(g, x) \mapsto (g, g.x)$  is proper. But if  $L \subset G$  and  $K \subset X$  are compact sets,

we have  $F^{-1}(L \times K) \subset L \times (L^{-1}.K)$  which is a compact set in  $G \times X$ .

The only point left to prove the continuity statement for the topology of  $\mathcal{C}_n^f(X)$ , assuming that the continuity for the topology of  $\mathcal{C}_n^{loc}(X)$  is obtained as follows :

Let  $W$  be a relatively compact open set in  $X$  and  $\mathcal{W}$  be the open set in  $\mathcal{C}_n^f(X)$  of cycles  $C$  such any irreducible component of  $C$  meets  $W$ . Then we want to show that the set of  $(g, s) \in G \times S$  such that  $g_*(C_s)$  lies in  $\mathcal{W}$  is an open set in  $G \times S$ . As the topology of  $\mathcal{C}_n^f(X)$  has a countable basis<sup>5</sup> it is enough to show that if a sequence  $(g_\nu, s_\nu)$  converges to  $(g, s)$  with  $g_*(C_s) \in \mathcal{W}$  then for  $\nu \gg 1$  we have also  $(g_\nu)_*(C_{s_\nu}) \in \mathcal{W}$ . If this not the case, we can choose for infinitely many  $\nu$  an irreducible component  $\Gamma_\nu$  of  $(g_\nu)_*(C_{s_\nu})$  which does not meet  $W$ . Up to pass to a sub-sequence, we may assume that the sequence  $\Gamma_\nu$  converges in  $\mathcal{C}_n^{loc}(X)$  to a cycle  $\Gamma$  which does not meet  $W$  and is contained in  $g_*(C_s)$ . This is a simple consequence of the continuity of the  $G$ -action on  $\mathcal{C}_n^{loc}(X)$  and the characterization of compact subsets in  $\mathcal{C}_n^{loc}(X)$  (see [B-M] ch.IV). As any irreducible component of  $g_*(C_s)$  meets  $W$  this implies that  $\Gamma$  is the empty  $n$ -cycle. This means that for any compact  $K$  in  $X$  there exists an integer  $\nu(K)$  such that for  $\nu \geq \nu(K)$  we have  $\Gamma_\nu \cap K = \emptyset$ . Choose now a compact neighbourhood  $L$  of  $g(K)$ . For  $\nu$  large enough we shall have  $K \subset g_\nu^{-1}(L)$ . This comes from the fact that the automorphisms  $g_\nu^{-1}$  converge to  $g^{-1}$  in the compact-open topology. Then this implies that for  $\nu \geq \nu(L)$  the irreducible component  $g_\nu^{-1}(\Gamma_\nu)$  of  $C_{s_\nu}$  does not meet  $K$ . Then, when  $s_\nu \rightarrow s$  the cycles  $C_{s_\nu}$  does not converge to  $C_s$  for the topology of  $\mathcal{C}_n^f(X)$  because we have some “escape at infinity” in a well choosen sub-sequence. Contradiction.  $\blacksquare$

**Lemma 2.2.2** *Assume now that in the situation above we have a Lie group  $H$  acting continuously holomorphically on  $M$ . Assume that  $\pi$  satisfies  $\pi(h.x) = \pi(x)$  for each  $x \in M$ . Then the strict transform  $\tilde{M}$  of  $M$  by the canonical modification  $\tau : \tilde{N} \rightarrow N$  giving the canonical GF flattning of  $\pi$  has a natural continuous holomorphic action of  $H$  such that the projection  $\tilde{M} \rightarrow M$  is  $H$ -equivariant. Moreover the GF map  $\tilde{\pi} : \tilde{M} \rightarrow \tilde{N}$  satisfies  $\tilde{\pi}(h.\tilde{x}) = \tilde{\pi}(\tilde{x})$  for each  $(h, \tilde{x}) \in H \times \tilde{M}$ .*

PROOF. We have a natural action of  $H$  on  $\mathcal{C}_n^f(M)$  which is continuous and holomorphic for each fixed  $h \in H$ . As the action of  $H$  is trivial on the set of fibers of  $\pi$ , the action of  $H$  is trivial on  $\tilde{N}$ . Define the action of  $H$  on  $M \times_N \tilde{N}$  by the formula

$$h.(x, (\pi(x), C)) = (h.x, \pi(x), h_*(C)).$$

It is easy to see that this is a continuous holomorphic action, and that it leaves  $\tilde{M}$  globally invariant. Now we have for each  $h \in H$  and each  $(x, \pi(x), C) \in \tilde{M}$  :

$$\tilde{\pi}(h.(x, (\pi(x), C))) = \tilde{\pi}(h.x, (\pi(x), h_*(C))) = (\pi(x), h_*(C)) = (\pi(x), C) = \tilde{\pi}(x, (\pi(x), C))$$

because a limit of  $H$ -invariant cycles is an  $H$ -invariant cycle.  $\blacksquare$

---

<sup>5</sup>This is a corollary of the fact that this is true for  $\mathcal{C}_n^{loc}(X)$  (see [B-M] ch.IV) as the topology of  $X$  has a countable basis of open sets; see [B.15] for details.

We shall also use the following simple tool from the cycle's space.

**Proposition 2.2.3** *Let  $M$  be a reduced complex space and  $(X_s)_{s \in S}$  a  $f$ -continuous family of  $d$ -dimensional finite type cycles parametrized by a compact subset  $S$  in  $\mathcal{C}_d^f(M)$ . Let  $(C_t)_{t \in T}$  be a  $f$ -continuous family of finite type non empty  $n$ -dimensional cycles in  $M$  parametrized by a subset  $T$  in  $\mathcal{C}_n^f(M)$  which is compact in  $\mathcal{C}_n^{\text{loc}}(M)$ . We assume the following condition :*

- *There exists an open dense set  $T'$  in  $T$  such that each  $C_t, t \in T'$  is equal to the union some  $X_s$ . (@@)*

*Then  $T$  is a compact subset in  $\mathcal{C}_n^f(M)$ .*

PROOF. First remark that, as  $S$  is compact, there exists a compact set  $L \subset M$  such that any irreducible component of any  $X_s$  meets  $L$ .

Let  $(t_m)_{m \in \mathbb{N}}$  be a sequence of points in  $T'$  converging to a point  $t \in T$  and denote by  $C_m$  the cycle  $C_{t_m}$  for short and  $C_t = C_\infty$ . Now choose for each  $m$  an irreducible component  $\Gamma_m$  of some  $X_{s_m}$  contained in  $C_m$ . Up to pass to a subsequence, we may assume that  $\Gamma_m$  converges in  $\mathcal{C}_d^f(M)$  to a cycle  $\Gamma$  which is non empty (it contains a point in  $L$ ) and included in  $|C_\infty|$ . So  $C_\infty$  is not the empty cycle.

Let  $x$  be a generic point of an irreducible component  $D$  of  $C_\infty$ . Then, up to pass to a subsequence, we may choose a sequence  $(x_m)$  of points respectively in  $C_m$  which converges to  $x$ . Choose for each  $m$  an irreducible component  $\Gamma_m$  of some  $X_{s_m} \subset |C_m|$  which contains  $x_m$ . This is possible because of condition (@@). Now, again up to pass to a subsequence, we may assume that the sequence  $(\Gamma_m)_{m \in \mathbb{N}}$  converges in  $\mathcal{C}_d^f(M)$  to a cycle  $\Gamma$  containing the point  $x$  and contained in  $|C_\infty|$ . Note that  $|\Gamma|$  is contained in some  $|X_{s_\infty}|$  as we may assume, by compactness of  $S$ , that the sequence  $(s_m)$  converges to  $s_\infty \in S$ . Then we have  $|X_{s_\infty}| \subset |C_\infty|$ . As  $D$  is the only irreducible component of  $C_\infty$  containing  $x$ , it contains at least an irreducible component of  $|X_{s_\infty}|$  containing  $x$ , and so  $D$  meets  $L$ . So we have proved that  $C_\infty$  is not the the empty  $n$ -cycle and that any irreducible component of  $C_\infty$  meets the compact set  $L$ . This is enough to conclude thanks to the proposition 3.2.2 in [B.15]. ■

### 2.3 Definition of SQP meromorphic quotient.

We shall consider a complex connected Lie group  $G$  and a completely holomorphic action of  $G$  on an irreducible complex space  $X$ . It is given, by definition, by a holomorphic map  $f : G \times X \rightarrow X$  such that  $f(g.g', x) = f(g, f(g'.x))$  for all  $g, g' \in G$  and  $x \in X$ , assuming that for each  $g \in G$  the holomorphic map  $x \mapsto f(g, x)$  is an automorphism of  $X$ , and that  $f(1, x) = x$  for all  $x \in X$ .

A **strongly quasi-proper meromorphic quotient** (we shall say a **SQP-meromorphic quotient** for short) for such an action  $f : G \times X \rightarrow X$  will be the following data:

1. a  $G$ -modification<sup>6</sup>  $\tau : \tilde{X} \rightarrow X$  with center  $\Sigma$ .
2. a holomorphic  $G$ -invariant GF map  $q : \tilde{X} \rightarrow Q$  where  $Q$  is an irreducible complex space.
3. an analytic  $G$ -invariant subset  $Y \subset X$  containing  $\Sigma$ , with no interior point in  $X$ . We shall denote  $\tilde{Y} := \tau^{-1}(Y)$ ,  $\Omega := X \setminus Y$ ,  $\tilde{\Omega} := \tau^{-1}(\Omega)$  and  $Q' := q(\tilde{\Omega})$ . Note that, as  $q$  is an open surjective map,  $Q'$  is open and dense in  $Q$ .

Now we ask that these data satisfy the following properties :

- i) The restriction to  $\Omega$  of the map  $q \circ \tau^{-1}$  is a GF map onto the dense open set  $Q'$  in  $Q$  and there is an open dense set  $Q''$  in  $Q'$  such that each fiber of  $q \circ \tau^{-1}$  at a point in  $Q''$  is equal to a  $G$ -orbit in  $\Omega$ .
- ii) There exists an open dense subset  $\Omega_0 \subset \Omega$  such that for each  $\tilde{x}$  in  $\tilde{\Omega}_0 := \tau^{-1}(\Omega_0)$  the **closure**  $\overline{G.\tilde{x}}$  of  $G.\tilde{x}$  in  $\tilde{X}$  is exactly the set  $q^{-1}(q(\tilde{x}))$ .

**Proposition 2.3.1** *Let  $G$  be a complex connected Lie group which acts completely holomorphically on an irreducible complex space  $X$ . Assume that we have a SQP meromorphic quotient for this action, given by a modification  $\tau : \tilde{X} \rightarrow X$  and a  $G$ -invariant GF map  $q : \tilde{X} \rightarrow Q$ . Then let  $\psi : Q \rightarrow \mathcal{C}_n^f(X)$  be the holomorphic map obtained by the composition of the fiber map of the GF map  $q$  and the direct image map for  $n$ -cycles by the modification  $\tau$ . Define  $Q_u := \psi(Q)$ . Then we have the following properties:*

1.  $Q_u$  is a closed analytic subset in  $\mathcal{C}_n^f(X)$  which is an irreducible complex space (locally of finite dimension) with the structure sheaf induced by the sheaf of holomorphic functions on  $\mathcal{C}_n^f(X)$ .
2. Let  $\tilde{X}_u$  be the graph of the meromorphic map  $q_u : X \dashrightarrow Q_u$  given by the holomorphic map  $\psi \circ q : \tilde{X} \rightarrow Q_u$  and let  $\tau_u : \tilde{X}_u \rightarrow X$  and  $q_u : \tilde{X}_u \rightarrow Q_u$  be the projections on  $X$  and  $Q_u$  respectively of this graph. Then  $(\tau_u, q_u)$  is also a SQP meromorphic quotient for the given  $G$ -action.
3. For any SQP meromorphic quotient  $(\tau, q)$  there exists a unique holomorphic surjective map  $\eta : Q \rightarrow Q_u$  such that the meromorphic maps  $q : X \dashrightarrow Q$  and  $q_u : X \dashrightarrow Q_u$  satisfies  $\eta \circ q = q_u$ .
4. For any  $G$ -invariant holomorphic map  $h : X \rightarrow Y$  there exists a holomorphic map  $H : Q_u \rightarrow Y$  such that  $h \circ \tau_u = H \circ q_u$ .

**Definition 2.3.2** *In the situation of the previous theorem the SQP meromorphic quotient for the given  $G$ -action defined by  $(\tau_u, q_u)$  will be called **the minimal SQP meromorphic quotient** of this  $G$ -action.*

---

<sup>6</sup>This means that we have a completely holomorphic  $G$ -action on  $\tilde{X}$  and that the modification  $\tau$  is  $G$ -equivariant.

So the proposition above says that the existence of a SQP meromorphic quotient for the given  $G$ -action implies the existence and uniqueness of a minimal meromorphic quotient for this  $G$ -action.

PROOF. To prove the point 1) we shall prove that the map  $\psi \circ q : \tilde{X} \rightarrow \mathcal{C}_n^f(X)$  is semi-proper.

Let  $C \neq \emptyset$  be in  $\mathcal{C}_n^f(X)$  and fix a relatively compact open set  $W$  in  $X$  meeting all irreducible components of  $C$ . The subset  $\mathcal{W}$  of  $\mathcal{C}_n^f(X)$  of cycles  $C'$  such that any irreducible component of  $C'$  meets  $W$  is an open set containing  $C$ . Now  $q(\tau^{-1}(\bar{W}))$  is a compact set in  $Q$ , as  $\tau$  is proper. Take any  $y \in Q$  such that  $C' := \psi(y)$  is in  $\mathcal{W}$ . The point  $y$  is the limit in  $Q$  of points  $y_\nu \in q(\tilde{\Omega}_0)$  such that the fiber of  $Q$  at  $y$  is limit in  $\mathcal{C}_n^f(\tilde{X})$  of the fibers  $q^{-1}(y_\nu) = \overline{G \cdot \tilde{x}_\nu}$  where, for  $\nu \gg 1$ , we can choose  $\tilde{x}_\nu$  in  $\tilde{\Omega}_0 \cap \tau^{-1}(W)$ . Up to pass to a sub-sequence, we may assume that  $\tilde{x}_\nu$  converges to a point  $\tilde{x}$  in  $\tau^{-1}(\bar{W})$ . Then the continuity of  $q$  implies that  $q(\tilde{x}) = y$  and  $C'$  is the limit of  $\overline{G \cdot \tilde{x}_\nu}$ . So  $|C'|$  is in the image by  $\psi$  of the compact set  $q(\tau^{-1}(\bar{W}))$  and this gives the semi-properness of  $\psi \circ q$ .

Now the direct image theorem 2.3.2 in [B.15] shows that  $Q_u$  is an irreducible complex space (locally of finite dimension) and the point 1) is proved.

To prove the second point we have to show that the map  $q_u : \tilde{X}_u \rightarrow Q_u$  is a GF map. By definition  $\tilde{X}_u$  is the closure in  $X \times Q_u$  of the graph of the map  $q|_{\Omega_0}$  where  $\Omega_0$  is an open dense set in  $X$  such that for any point  $x \in \Omega_0$  we have  $\psi(q(x)) = \overline{G \cdot x}$  (as  $\Omega_0$  is disjoint from the center of the modification  $\tau$  we identify here  $\Omega_0$  and  $\tilde{\Omega}_0 := \tau^{-1}(\Omega_0)$ ). Then by irreducibility of  $Q_u$  and  $X$  the closed analytic subset  $\tilde{X}_u \subset X \times Q_u$  is equal to the graph of the tautological family of cycles in  $X$  parametrized by  $Q_u \subset \mathcal{C}_n^f(X)$ . This proves the point 2).

Now consider a SQP meromorphic quotient of the given action given by the maps  $\tau : \tilde{X} \rightarrow X$  and  $q : \tilde{X} \rightarrow Q$ . Let  $\psi : \tilde{X} \rightarrow \mathcal{C}_n^f(X)$  the composition of the holomorphic map classifying the fibers of the GF map  $q$  with the direct image of  $n$ -cycles by the modification  $\tau$ . Then, by the construction in the proof of the point 1), we know that  $\psi(Q) = Q_u$  and then the map  $\psi$  induces a surjective holomorphic map  $\eta : Q \rightarrow Q_u$ . We may assume that  $\tilde{X}$  is in fact the graph of the meromorphic map  $q : X \dashrightarrow Q$ . Then, as  $\tilde{X}_u$  is the graph of the meromorphic map  $q_u : X \dashrightarrow Q_u$  the holomorphic map  $\text{id}_X \times \eta : X \times Q \rightarrow X \times Q_u$  sends  $\tilde{X}$  to  $\tilde{X}_u$  because this is true over a dense open set in  $X$  where the maps  $q$  and  $q_u$  are holomorphic and satisfy  $q^{-1}(q(x)) = q_u^{-1}(q_u(x)) = \overline{G \cdot x}$ . This complete the proof of 3).

Consider now a  $G$ -invariant holomorphic map  $h : X \rightarrow Y$ . Recall (see [B.15] proposition 2.1.7) that  $\mathcal{C}_n^f(h)$ , the subset of cycles  $C \in \mathcal{C}_n^f(X)$  which are contained in a fiber of  $h$  is a closed analytic subset in  $\mathcal{C}_n^f(X)$ . On an open dense subset of  $Q_u$  we know that a point corresponds to a cycle with support  $\overline{G \cdot x}$  for some  $x \in X$ . Then this means that an open dense subset in  $Q_u$  is contained in  $\mathcal{C}_n^f(h)$ . As this subset is closed we obtain  $Q_u \subset \mathcal{C}_n^f(h)$ . Now there is a holomorphic map  $\hat{h} : \mathcal{C}_n^f(h) \rightarrow Y$  associating to a cycle  $C$  the point in  $Y$  such  $|C| \subset h^{-1}(y)$  (see [B.15] proposition 2.1.7). This induces a holomorphic map  $H : Q_u \rightarrow Y$ , and it is clear that the relation  $h \circ \tau_u = H \circ q_u$  is true on a dense open set, so everywhere.  $\blacksquare$

## 2.4 Good points, good open set.

Let  $X$  be a reduced complex space and  $G$  be a connected complex Lie group. Let  $f : G \times X \rightarrow X$  be a holomorphic action of  $G$  on  $X$ .

**Definition 2.4.1** *We shall say that a point  $x \in X$  is a **good point** for the action  $f$  if the following condition is satisfied*

- *For each compact set  $K$  in  $X$  there exists an open neighbourhood  $V$  of  $x$  and a compact set  $L$  in  $G$  such that if  $y \in V$  and  $g \in G$  are such that  $g.y \in K$ , there exists  $\gamma \in L$  with  $\gamma.y = g.y$*

*We shall say that the action of  $G$  on  $X$  is **good** when each point in  $X$  is a good point. If  $\Omega$  is a  $G$ -invariant open set in  $X$ , we shall say that  $\Omega$  is a **good open set** for the action  $f$  when all points in  $\Omega$  are good points for the  $G$ -action given by  $f$  restricted to  $\Omega$ .*

REMARKS.

1. If  $x \in X$  is a good point, then for any  $g_0 \in G$   $g_0.x$  is also a good point : for  $K$  given, choose  $g_0.V$  as neighbourhood of  $g_0.x$  and the compact set  $L.g_0^{-1} \subset G$  to satisfy the needed conditions.
2. If  $\Omega$  is a good open set, then points in  $\Omega$  are not in general good points for the action on  $X$ .
3. If  $\Omega$  is a good open set and  $W \subset \Omega$  a  $G$ -invariant open set, then  $W$  is a good open set.
4. If  $M$  is a compact set of good points in  $X$  for any compact set  $K$  in  $X$  we can find a neighbourhood  $V$  of  $M$  in  $X$  and a compact set  $L$  in  $G$  such that for any point  $y \in V$  and any  $g \in G$  such that  $g.y \in K$  there exists  $\gamma \in L$  with  $\gamma.y = g.y$ . This is easily obtained by a standard compactness argument. We shall say that a compact set of good points is **uniformly good**.

**Lemma 2.4.2** *Let  $x$  be a point in  $X$ . Then  $x$  is a good point for the  $G$ -action on  $X$  if and only if the map  $F_X : G \times X \rightarrow X \times X$  given by  $(g, x) \mapsto (x, g.x)$  is semi-proper at each point of  $\{x\} \times X$ . As a consequence a  $G$ -invariant open set  $\Omega$  in  $X$  is a good open set for the  $G$ -action if and only if the map  $F_\Omega : G \times \Omega \rightarrow \Omega \times \Omega$  given by  $(g, x) \mapsto (x, g.x)$  is semi-proper.*

PROOF. Let  $x \in X$  be a good point and fix any  $z \in X$ . To prove that the map  $F_X$  is semi-proper at  $(x, z)$  choose compact neighbourhoods  $V_0$  and  $K$  of  $x$  and  $z$  in  $X$  and apply the definition of a good point to the compact set  $K$ . So we can find a neighbourhood  $V$  of  $x$ , that we may assume to be contained in  $V_0$ , and a compact set  $L$  in  $G$  such that for any  $y \in V$  such that  $g.y \in K$  we have a  $\gamma \in L$  with  $g.y = \gamma.y$ . Then we have  $F_X(G \times X) \cap (V \times K) = F_X(L \times V_0) \cap (V \times K)$  and  $L \times V_0$  is a compact set in  $G \times X$ .

Conversely, assume that the map  $F_X$  is semi-proper at each point of  $\{x\} \times X$ . Take a compact set  $K$  in  $X$  and apply the semi-properness to each point  $(x, z)$  where  $z$  is in  $K$ . For each  $z \in K$  we obtain open neighbourhoods  $V_z$  and  $W_z$  of  $x$  and  $z$  in  $X$  and a compact set  $L_z \times M_z$  in  $G \times X$  such that

$$F_X(G \times X) \cap (V_z \times W_z) = F_X(L_z \times M_z) \cap (V_z \times W_z).$$

Extract a finite sub-cover  $W_1, \dots, W_N$  of  $K$  by the open sets  $W_z$  and define the compact set  $L := \cup_{i \in [1, N]} L_{z_i}$  and the neighbourhood  $V := \cap_{i \in [1, N]} V_{z_i}$  of  $x$ . Then if  $y$  is in  $V$  and  $g.y$  is in  $K$  there exists  $i \in [1, N]$  such that  $g.y \in W_i$ . As  $y$  is in  $V_i$  we can find a  $\gamma \in L_{z_i} \subset L$  with  $F_X(g, y) = F_X(\gamma, y)$  and this implies that  $x$  is a good point.

The second assertion is an easy consequence of the first one. ■

**Proposition 2.4.3** *Let  $G$  be a connected complex Lie group. Let  $f : G \times X \rightarrow X$  be an holomorphic action of  $G$  on a reduced complex space  $X$ . Then we have the following properties :*

- i) If  $x$  is a good point for  $f$  the orbit  $G.x$  is a closed analytic subset of  $X$ .*
- ii) If  $x$  is a good point for  $f$  and if  $(G.x) \cap K = \emptyset$  where  $K \subset X$  is a compact set, there exists a neighbourhood  $V$  of  $x$  in  $X$  such that  $(\overline{G.x'}) \cap K = \emptyset$  for any  $x'$  in  $V$  ( $x'$  is not assume here to be a good point).*
- iii) If  $x$  is a good point for  $f$  there exists a neighbourhood  $V$  of  $x$  such that any good point  $x' \in V$  has an orbit which is a closed analytic subset of the same dimension than  $G.x$ .*
- iv) Let  $\Omega$  be a good connected open set in  $X$  which is normal and let  $n$  be the dimension of  $G.x$  for  $x \in \Omega$ . Then there exists a holomorphic map<sup>7</sup>  $\varphi : \Omega \rightarrow \mathcal{C}_n^f(\Omega)$  given generically by  $\varphi(x) := G.x$  as a reduced  $n$ -cycle in  $\Omega$ .*
- v) When we have a good open set  $\Omega$  in  $X$  which is normal, there exists a quasi-proper equidimensional holomorphic quotient of  $\Omega$  for the action restricted to  $\Omega$ .*

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<sup>7</sup>This means that we have an  $f$ -analytic family of  $n$ -cycles in  $\Omega$  parametrized by  $\Omega$ .

PROOF. We already proved that  $x$  is a good point if the map  $G \rightarrow X$  given by  $g \mapsto g.x$  is semi-proper in lemma 1.3.2. Now Kuhlmann's theorem [K.64], [K.66] gives that  $f_x(G) = G.x$  is a closed analytic subset of  $X$ . This proved i)

Assume ii) is not true ; then we have a compact set  $K$  such that  $(G.x) \cap K = \emptyset$  and a sequence  $(x_\nu)_{\nu \in \mathbb{N}}$  converging to  $x$  and such that  $(\overline{G.x_\nu}) \cap K$  is not empty for each  $\nu$ . Fix a compact neighbourhood  $\tilde{K}$  of  $K$  such that  $(G.x) \cap \tilde{K} = \emptyset$ . This is possible thanks to i). Pick a point  $y_\nu = \lim_{\alpha \rightarrow \infty} g_{\nu,\alpha}.x_\nu$  in  $(\overline{G.x_\nu}) \cap K$  for each  $\nu$ . Up to pass to a subsequence we may assume that sequence  $(y_\nu)$  converges to  $y \in K$  when  $\nu \rightarrow +\infty$ . So, for  $\alpha \geq \alpha(\nu)$ , we can assume that  $g_{\nu,\alpha}.x_\nu$  is in  $\tilde{K}$ . But, as  $x$  is a good point, for the given  $\tilde{K}$  there exists a neighbourhood  $V$  of  $x$  and a compact set  $L \subset G$  as in the definition. We may assume that  $x_\nu$  is in  $V$  for  $\nu \geq \nu_0$  and so we may find, for  $\nu \geq \nu_0, \alpha \geq \alpha(\nu)$ , elements  $\gamma_{\nu,\alpha} \in L$  such that  $\lim_{\alpha \rightarrow \infty} \gamma_{\nu,\alpha}.x_\nu = y_\nu \quad \forall \nu \geq \nu_0$ .

Up to pass to a subsequence for each given  $\nu \geq \nu_0$ , we may assume that the sequence  $(\gamma_{\nu,\alpha})$  converges to some  $\gamma_\nu \in L$ . And again, that the sequence  $\gamma_\nu$  converges to some  $\gamma \in L$ . So the continuity of  $f$  gives  $y_\nu = \gamma_\nu.x_\nu \rightarrow \gamma.x = y \in \tilde{K}$  giving a contradiction because we assume  $(G.x) \cap \tilde{K} = \emptyset$ . This proves ii).

Let  $E := (U, B, j)$  be a  $n$ -scale on  $\Omega$  adapted to the  $n$ -cycle  $G.x$ . Then the compact set  $K := j^{-1}(\bar{U} \times \partial B)$  does not meet  $G.x$ , by definition of an adapted scale. Using ii), there exists a neighbourhood  $V$  of  $x$  such that for any  $x' \in V$  we have  $(\overline{G.x'}) \cap K = \emptyset$ . As for a good point  $x' \in V$  we know that  $G.x'$  is a closed analytic subset, the  $n$ -scale is then adapted to  $G.x'$ . This implies that the dimension of  $G.x'$  is at most equal to  $n$ . But the semi-continuity of the dimension of the stabilizers implies that the dimension of  $G.x' \simeq G/St(x')$  is at least equal to  $n = \dim(G/St(x))$ . This proves iii).

Remark that for any  $x' \in V$  such that  $\overline{G.x'}$  is a closed analytic subset in  $\Omega$ , the previous proof shows also that  $\overline{G.x'}$  is of pure dimension  $n$ .

To prove iv) fix a good connected open set  $\Omega$  and define

$$Z := \{(g, x, y) \in G \times \Omega \times \Omega / y = g.x.\}$$

This is a closed analytic subset in  $G \times \Omega \times \Omega$ . Let us show that the projection  $p : Z \rightarrow \Omega \times \Omega$  is semi-proper. Pick a point  $(x, y) \in \Omega \times \Omega$  and choose compact neighbourhoods  $V$  and  $K$  respectively of  $x$  and of  $y$  in  $\Omega$ . Using the fact that any open set  $\Omega' \subset \subset \Omega$  is uniformly good, for the compact  $K$  in  $\Omega$  we find a compact set  $L$  in  $G$  such that for  $x' \in \Omega'$  and  $g \in G$  with  $g.x' \in K$  there exists  $\gamma \in L$  with  $\gamma.x' = g.x'$ . Choosing  $\Omega'$  containing  $V$  this implies that we have

$$p(Z) \cap (V \times K) = p((L \times V \times K) \cap Z).$$

So the projection  $p$  is semi-proper. Its image  $p(Z)$  is then a closed analytic subset

of  $\Omega \times \Omega$  by Kuhlmann's theorem [K.64], [K.66]. But now the projection

$$\pi : p(Z) \rightarrow \Omega$$

is  $n$ -equidimensional, thanks to iii), and has irreducible generic fibers on a normal basis  $\Omega$ . So its fibers (with generic multiplicity equal to 1) define an analytic family of  $n$ -cycles of  $X$  parametrized by  $\Omega$ . It is clearly  $f$ -analytic because each fiber is irreducible<sup>8</sup> and we have an holomorphic section because each  $x$  lies in  $G.x$ .

To prove v) let us prove that the holomorphic map  $\varphi : \Omega \rightarrow \mathcal{C}_n^f(\Omega)$  classifying the fibers of  $p(Z)$  is semi-proper. Fix a point  $C \in \mathcal{C}_n^f(\Omega)$  and a point  $x_1, \dots, x_k$  in each irreducible component of  $|C|$ . Let  $W$  a relatively compact open neighbourhood of  $\{x_1, \dots, x_k\}$  in  $\Omega$  and let  $\mathcal{W}$  be the open set in  $\mathcal{C}_n^f(\Omega)$  of cycles such that each irreducible component meets  $W$ . Let  $C'$  be in  $\mathcal{W} \cap \varphi(\Omega)$  we know that if  $C' = \varphi(z)$  that  $|C'| = G.z$ . So  $G.z$  has to meet  $W$  and we can choose  $y$  in the compact set  $\bar{W}$  such that  $C' = \varphi(y)$  and this gives the semi-properness of  $\varphi$ . Now the semi-proper direct image theorem 2.3.2 of [B.15] implies that the image  $Q$  of  $\varphi$  is a finite dimensional complex space.

Then the holomorphic map  $q : \Omega \rightarrow Q$  which is a quasi-proper equidimensional holomorphic quotient for the action  $f$  on  $\Omega$  as **each** fiber of  $q$  is set-theoretically a  $G$ -orbit. ■

## 2.5 The conditions [H.1], [H.2] and [H.3].

Now we shall consider the following conditions on the action  $f$ .

- **There exists a  $G$ -invariant dense open set  $\Omega_1$  in  $X$  which admits a GF- holomorphic quotient.** [H.1]

Recall that this means that there exists a  $G$ -invariant geometrically  $f$ -flat holomorphic map  $q : \Omega_1 \rightarrow Q_1$  onto a reduced complex space  $Q_1$  such that each fiber of  $q$  over a point in  $Q_1$  is set-theoretically an orbit in  $\Omega_1$ .

The following stronger condition will be useful in the sequel:

- **There exists a  $G$ -invariant dense open set  $\Omega$  in  $X$  which is good for the action  $f$  (on  $\Omega$ ).** [H.1str]

An immediate consequence of the proposition 2.4.3 is that the condition [H.1str] below is sufficient to satisfy [H.1] by taking  $\Omega_1$  as the set of normal points in  $\Omega$  (which is open dense and  $G$ -invariant in  $\Omega$ ).

Now assume [H.1] and define  $\mathcal{R} := \{(x, y) \in \Omega_1 \times \Omega_1 / y \in G.x\}$ . It is a closed analytic set in  $\Omega_1 \times \Omega_1$ . We shall denote  $\overline{\mathcal{R}}$  the closure of  $\mathcal{R}$  in  $X \times X$ . Our second assumption will be :

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<sup>8</sup>But some multiplicities may occur.

- **The subset  $\overline{\mathcal{R}}$  is analytic in  $X \times X$  and there exists an open dense subset  $\Omega_0 \subset \Omega_1$  such that for each  $x \in \Omega_0$  we have**

$$\overline{G.x} = \overline{\mathcal{R}} \cap (\{x\} \times X). \quad [\text{H.2}]$$

Remark that the first projection  $p_1 : \overline{\mathcal{R}} \cap (\Omega_0 \times X) \rightarrow \Omega_0$  is quasi-proper because we have a holomorphic section of this map (with irreducible fibers, thanks to [H.2]) which is given by  $x \mapsto (x, x)$ .

Assuming that  $\Omega_0$  contains only normal points<sup>9</sup> in  $X$ , the equidimensionality and quasi-properness on  $\Omega_0$  of the projection of  $\overline{\mathcal{R}}$  implies that there exists a holomorphic map

$$\bar{\varphi}_0 : \Omega_0 \rightarrow \mathcal{C}_n^f(X)$$

where the supports are given by  $x \mapsto \overline{G.x}$  and where the multiplicity is generically equal to 1.

We shall denote  $\Gamma \subset \Omega_0 \times \mathcal{C}_n^f(X)$  the graph of the map  $\bar{\varphi}_0$  and  $\bar{\Gamma}$  its closure in  $X \times \mathcal{C}_n^f(X)$ . We shall denote  $\theta : \bar{\Gamma} \rightarrow X$  the map induced by the first projection. Our last hypothesis is :

- **The map  $\theta : \bar{\Gamma} \rightarrow X$  is proper .** [H.3]

This hypothesis [H.3], assuming [H.1] and [H.2] is in fact equivalent to ask that the projection  $\overline{\mathcal{R}} \rightarrow X$  is strongly quasi-proper: this is an immediate consequence of the proposition 3.2.2 of [B.15].

The following proposition shows that these conditions [H.1], [H.2] and [H.3] are necessary for the existence of a SQP-meromorphic quotient for a completely holomorphic action of  $G$  on  $X$ .

**Proposition 2.5.1** *Assuming that the completely holomorphic action  $f : G \times X \rightarrow X$  of the connected complex Lie group  $G$  on the irreducible complex space  $X$  has a SQP-meromorphic quotient, then the conditions [H.1], [H.2] and [H.3] are satisfied.*

PROOF. The conditions to be a SQP-meromorphic quotient gives an open set  $\Omega_1$  which is dense,  $G$ -stable and which admits a GF holomorphic quotient for the action on  $\Omega_1$ . So [H.1] is clear.

Let  $S$  be the graph of the equivalence relation given by  $q$  on  $\tilde{X}$ . Then the proper direct image  $(\tau \times \tau)(S)$  is  $\overline{\mathcal{R}}$  and so the condition [H.2] is satisfied as soon as we can find an open and dense subset  $\Omega_0$  in  $\Omega_1$  such that for each  $x \in \Omega_0$  we have the equality

$$\overline{G.x} = \overline{\mathcal{R}} \cap (\{x\} \times X).$$

But this property is given by the condition ii) in the definition of a SQP-meromorphic quotient.

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<sup>9</sup>This not restrictive, as we may always assume that  $X \setminus \Omega_0$  contains the non normal points in  $X$ . We shall always assume that  $\Omega_0$  is normal in the sequel, without any more comment.

The composition of  $q$  with the classifying map  $Q \rightarrow \mathcal{C}_n^f(\tilde{X})$  for the fibers of  $q$  gives a holomorphic map  $\tilde{\psi} : \tilde{X} \rightarrow \mathcal{C}_n^f(\tilde{X})$ . Composed with the direct image map, which is holomorphic (see [B.M] ch.IV ; the “quasi-proper” part of this result is easy, as  $\tau$  is proper)  $\tau_* : \mathcal{C}_n^f(\tilde{X}) \rightarrow \mathcal{C}_n^f(X)$ , we obtain the fact that  $\overline{\mathcal{R}}$  is strongly quasi-proper on  $X$  which is the condition [H.3]. ■

## 2.6 Existence of SQP-meromorphic quotient

Now we shall prove that conditions [H.1], [H.2] and [H.3] on a completely holomorphic action of a connected complex Lie group  $G$  on an irreducible complex space  $X$  are sufficient for the existence of a SQP meromorphic quotient.

**Theorem 2.6.1** *Under the hypothesis [H.1], [H.2] and [H.3] there exists a proper  $G$ -equivariant modification  $\tau : \tilde{X} \rightarrow X$  with center contained in  $X \setminus \Omega_0^{10}$  and a geometrically  $f$ -flat holomorphic map*

$$q : \tilde{X} \rightarrow Q$$

*on a reduced complex space, which give a strongly quasi-proper meromorphic quotient for the given  $G$ -action.*

Of course the complex space  $\tilde{X}$  is the topological space  $\bar{\Gamma}$  with a structure of a reduced complex space such that the projection on  $X$  is a proper modification. Then the space  $Q$  is the image of  $\tilde{X}$  in  $\mathcal{C}_n^f(X)$ . So we need some semi-proper direct image theorem for such a map to prove this result. Such a result is the content of the theorem 2.3.2 of [B.15]

**PROOF.** The first remark is that the hypothesis [H.3] says that the projection  $p : \bar{\Gamma} \rightarrow X$  is a proper topological modification of  $X$ . But to apply directly the part ii) of the theorem 2.3.6 of [B.13] to the projection  $p_1 : \overline{\mathcal{R}} \rightarrow X$  we need quasi-properness of this map. This is given by the proposition 3.2.2 of [B.15] as we have the condition [H.3].

Then we obtain a proper (holomorphic) modification with center in  $\Sigma \subset X \setminus \Omega_0$ ,  $\tau : \tilde{X} \rightarrow X$ , and a  $f$ -analytic family of cycles in  $X$  parametrized by  $\tilde{X}$  extending the family  $(\overline{G.x})_{x \in \Omega_0}$ , corresponding to a “holomorphic” map extending  $\tilde{\varphi}_0$  :

$$\tilde{\varphi} : \tilde{X} \rightarrow \mathcal{C}_n^f(X).$$

Now let us prove that this map  $\tilde{\varphi}$  is quasi-proper<sup>11</sup>. This will allow us to apply the theorem 2.3.2 of *loc. cit.* and to define the reduced complex space  $Q$  as the image  $\tilde{\varphi}(\tilde{X})$ . Then it will be easy to check that the map  $\tilde{\varphi} : \tilde{X} \rightarrow Q$  is a strongly quasi-proper meromorphic quotient for the  $G$ -action we consider.

If  $C_0$  is in  $\mathcal{C}_n^f(X)$  and is not the empty cycle, choose a relatively compact open

<sup>10</sup>The dense open subset  $\Omega_0 \subset \Omega_1$  is defined in the condition [H.2]

<sup>11</sup>This makes sense as the fibers are closed analytic subsets of  $\tilde{X}$ .

set  $W$  in  $X$  such that any irreducible component of  $|C_0|$  meets  $W$ . Then let  $\mathcal{W}$  be the open set in  $\mathcal{C}_n^f(X)$  defined by the condition that any irreducible component of  $C$  meets  $W$  for  $C \in \mathcal{W}$ . Then we shall prove that there exists a compact set  $K$  in  $\tilde{X}$  such that any irreducible component of the fiber of  $\tilde{\varphi}$  at a point in  $\mathcal{W} \cap \tilde{\varphi}(\tilde{X})$  meets  $K$ . Let  $K := \tau^{-1}(\bar{W})$ . If  $(y, C)$  is in  $\tilde{X}^{12}$  with  $C \in \mathcal{W}$ , each irreducible component of  $C$  meets  $W$ . But the fiber of  $\tilde{\varphi}$  at  $C$  is equal to  $|C|$ , and the quasi-properness is proved. ■

### 3 Application.

#### 3.1 The sub-analytic lemma.

We shall use the following lemma (see [G-M-O]) in our application.

**Lemma 3.1.1** *Let  $M$  be a reduced complex space and  $Y \subset M$  a closed analytic subset with no interior point in  $M$ . Let  $R$  be a closed (complex) analytic subset in  $M \setminus Y$  such that  $\bar{R}$  is a sub-analytic set in  $M$ . Then  $\bar{R}$  is a (complex) analytic subset in  $M$ .*

This important lemma is a consequence of Bishop's theorem (see [Bi.64]) and of a classical result on sub-analytic subsets (see [G-M-O]).

#### 3.2 Proof of the theorem 1.0.1.

Now we shall assume that  $G$  is a connected complex Lie group such that  $G = K.B$  where  $B$  is a closed complex connected subgroup of  $G$  and  $K$  a compact real subgroup of  $G$ .

The first condition [H.1str] for the  $G$ -action is given by the following lemma :

**Lemma 3.2.1** *In the situation of the theorem 1.0.1, assume that we have a  $G$ -invariant open set  $\Omega$  which is a good open set for the  $B$ -action, then  $\Omega$  is a good open set for the  $G$ -action.*

PROOF. Consider a point  $x \in \Omega$  and a compact set  $M$  in  $\Omega$ . Then there exists a neighbourhood  $V$  of  $x$  in  $\Omega$  and a compact set  $L$  in  $B$  such that  $b.y \in M$  for some  $y \in V$  and some  $b \in B$  implies that we can find  $\beta \in L$  with  $b.y = \beta.y$ . Now assume that  $M$  is  $K$ -invariant (here we use the  $G$ -invariance of  $\Omega$ ) and that  $g.y$  is in  $M$  for some  $g \in G$  and some  $y \in V$ . Write  $g = k.b$  for some  $k \in K$  and  $b \in B$ . Then  $b.y$  is again in  $M$  so we can find  $\beta \in L$  with  $\beta.y = b.y$  and then  $g.y = k.\beta.y$  with  $k.\beta \in K.L$  which is a compact set in  $G$ . So  $x$  is a good point for the  $G$ -action on  $\Omega$ . ■

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<sup>12</sup>Recall that, as a topological space,  $\tilde{X} = \bar{\Gamma}$ .

The corollary of the next lemma will give the first part of [H.2] for the  $G$ -action assuming that we have a  $G$ -invariant dense good Zariski open set  $\Omega$  for the  $B$ -action with the condition [H.2] for the  $B$ -action. In order to use the sub-analytic lemma 3.1.1 in this situation we shall need the following lemma.

**Lemma 3.2.2** *Let  $\Omega$  be an open  $G$ -invariant good set for the  $B$ -action, and then also good for the  $G$ -action thanks to the previous lemma.*

*Let  $\chi : K \times X \times X \rightarrow X \times X$  the map given by  $(k, x, y) \mapsto (k.x, y)$  and let  $p : K \times X \times X \rightarrow X \times X$  be the natural projection. Then we have*

$$p(\chi^{-1}(\overline{\mathcal{R}_B})) = \overline{\mathcal{R}_G}$$

where we define

$$\mathcal{R}_B := \{(x, y) \in \Omega \times \Omega / B.x = B.y\} \quad \text{and} \quad \mathcal{R}_G := \{(x, y) \in \Omega \times \Omega / G.x = G.y\},$$

and where the closures are taken in  $X \times X$ .

PROOF. Remark first that

$$p(\chi^{-1}(\mathcal{R}_B)) = \{(x, y) \in \Omega \times \Omega / \exists k \in K \ B.k.x = B.y\}.$$

So  $(x, y) \in p(\chi^{-1}(\mathcal{R}_B))$  implies  $y \in B.k.x \subset G.x$  and also  $k.x \in B.y$ ; we conclude that  $x$  is in  $K.B.y = G.y$ . This gives the inclusion  $p(\chi^{-1}(\mathcal{R}_B)) \subset \mathcal{R}_G$ . The opposite inclusion is easy because  $G.x = G.y$  implies that  $x \in K.B.y$  so there exists  $k \in K$  such that  $k.x \in B.y$ .

Now the map  $\chi$  and  $p$  are proper, so we obtain the inclusion  $p(\chi^{-1}(\overline{\mathcal{R}_B})) \subset \overline{\mathcal{R}_G}$ .

Let  $(x, y) := \lim_{\nu \rightarrow \infty} (x_\nu, y_\nu)$  where  $(x_\nu, y_\nu)$  is in  $\mathcal{R}_G$  for each  $\nu \in \mathbb{N}$ . Then for each  $\nu$  there exists  $k_\nu \in K, b_\nu \in B$  such that  $x_\nu = k_\nu.b_\nu.y_\nu$ ; then  $(k_\nu^{-1}, x_\nu, y_\nu) \in \mathcal{R}_B$ . Up to pass to a subsequence we may assume that the sequence  $(k_\nu)$  converges to some  $k \in K$ . As  $(k_\nu^{-1}, x_\nu, y_\nu)$  is in  $\chi^{-1}(\mathcal{R}_B)$  for each  $\nu$ , the point  $(k^{-1}, x, y)$  is in

$$\overline{\chi^{-1}(\mathcal{R}_B)} = \chi^{-1}(\overline{\mathcal{R}_B})$$

and so  $(x, y)$  is in  $p(\chi^{-1}(\overline{\mathcal{R}_B}))$  proving the opposite inclusion. ■

**Corollaire 3.2.3** *In the situation of the previous lemma, assume that  $X \setminus \Omega$  is a (complex) analytic subset; then if the subset  $\overline{\mathcal{R}_B}$  is (complex) analytic in  $X \times X$ , the subset  $\overline{\mathcal{R}_G}$  is also a (complex) analytic subset of  $X \times X$ .*

PROOF. Note first that the maps  $\chi$  and  $p$  are real analytic, so assuming that  $\overline{\mathcal{R}_B}$  is analytic implies that  $p(\chi^{-1}(\overline{\mathcal{R}_B}))$  is sub-analytic. Then, as we know that  $\mathcal{R}_G$  is an irreducible locally closed complex analytic subset, the conclusion follows from the lemma 3.1.1, as our assumption that  $\Omega$  is a Zariski (dense) open set in  $X$  implies that  $\Omega \times \Omega$  is Zariski open (and dense) in  $X \times X$ . ■

A first step to prove the quasi-properness of  $\overline{\mathcal{R}_G}$  is our next result.

**Lemma 3.2.4** *Let assume that the  $B$ -action  $f$  on  $X$  satisfies [H.1] and [H.2]. Let  $\Omega_0 \subset \Omega_1$  be an open set on which the fiber at any  $x \in \Omega_0$  of  $\overline{\mathcal{R}_B}$  is equal to  $\overline{B.x}$  (with some multiplicity). Then the fiber at any  $x \in \Omega_0$  of  $\overline{\mathcal{R}_G}$  is equal to  $\overline{G.x}$  (with some multiplicity).*

PROOF. As we know that the map  $x \mapsto \overline{B.x}$ , with generic multiplicity 1, extends to is a  $f$ -analytic family of cycles of  $X$  parametrized by  $\Omega_0$ , for each sequence  $(x_\nu)_{\nu \in \mathbb{N}}$  of points in  $\Omega_0$  converging to a point  $x \in \Omega_0$  we have (with suitable multiplicity)  $\overline{B.x} = \lim_{\nu \rightarrow \infty} \overline{B.x_\nu}$  in the topology of  $\mathcal{C}_d^f(X)$ . We shall show that this implies, also with suitable multiplicity, the equality  $\overline{G.x} = \lim_{\nu \rightarrow \infty} \overline{G.x_\nu}$  in the topology of  $\mathcal{C}_n^f(X)$ . As we have  $G = K.B$  with  $K$  compact, for any  $y \in X$  we have  $\overline{G.y} = \overline{K.B.y}$ . So the inclusion of  $\lim_{\nu \rightarrow \infty} \overline{G.x_\nu}$  in the fiber at  $x$  of  $\overline{\mathcal{R}_G}$  is clear. The point is to prove the opposite inclusion. Let  $y$  be a point in the fiber at  $x \in \Omega_0$  of  $\overline{\mathcal{R}_G}$ . It is a limit of a sequence  $y_\nu \in \overline{G.x_\nu}$  where  $x_\nu \in \Omega_0$  converges to  $x$ . Write  $y_\nu = k_\nu.b_\nu.x_\nu$  with  $k_\nu \in K$  and  $b_\nu \in B$ . Up to pass to a subsequence, we may assume that the sequence  $(k_\nu)$  converges to a point  $k \in K$ . So we have  $k^{-1}.y$  which is the limit of the sequence  $b_\nu.x_\nu$ . We obtain that  $k^{-1}.y$  is in the limit of  $\overline{B.x_\nu}$  which has support equal to  $\overline{B.x}$ . Then  $y$  is in  $\overline{K.B.x} = \overline{G.x}$ , concluding the proof.  $\blacksquare$

PROOF OF THE THEOREM 1.0.1. The fact that the  $G$ -action satisfies [H.1str] is consequence of lemma 3.2.1. The analyticity of  $\overline{\mathcal{R}_G}$  in  $X \times X$  is proved at corollary 3.2.3. The lemma 3.2.4 gives a dense open set  $\Omega_0$  where the fiber of the projection  $p_1$  of  $\overline{\mathcal{R}_G}$  at each point  $x \in \Omega_0$  is equal to  $\overline{G.x}$  as a set. This implies the quasi-properness of  $p_1$  over  $\Omega_0$ , because  $x$  is in  $\overline{G.x}$  and  $G$  is connected; assuming (which is not restrictive) that  $\Omega_0$  is normal, we obtain a holomorphic map

$$\Phi : \Omega_0 \longrightarrow \mathcal{C}_n^f(X)$$

where the support of  $\Phi(x)$  is equal to  $\overline{G.x}$  for each  $x \in \Omega_0$  and with generic multiplicity equal to 1. This complete the proof of [H.2] for the  $G$ -action.

Thanks to proposition 3.2.2 of [B.15], to prove [H.3] it is enough to show that the closure of the graph  $\Gamma_G$  of  $\Phi$  in  $X \times \mathcal{C}_n^f(X)$  is proper on  $X$ .

The projection  $p_B : \overline{\mathcal{R}_B} \rightarrow X$  is strongly quasi-proper so, for  $V$  compact in  $X$ , the limits of cycles (with convenient multiplicity)  $\overline{B.x}$  for  $x \in V \cap \Omega_0$  stay in a compact subset  $S_0$  in  $\mathcal{C}_d^f(X)$ . Then let  $S := K.S_0$ ; this is again a compact subset in  $\mathcal{C}_d^f(X)$ . But the subset  $T$  of limits of the generic fibers of the projection  $p_G : \overline{\mathcal{R}_G} \rightarrow X$  for  $x \in V$  is a compact set of  $\mathcal{C}_n^{loc}(X)$  thanks to [B.13] theorem 2.3.6 i). For  $x \in V \cap \Omega_0$  we have  $p_G^{-1}(x) = \overline{G.x}$ , and let  $T'$  be the open dense set in  $T$  described by the cycles  $\overline{\varphi_0(x)}$ ,  $x \in V \cap \Omega_0$ . As we know that for  $x$  in  $\Omega_0$  we have  $|\overline{\varphi_0(x)}| = \overline{G.x}$  we know that each of them is an union of  $d$ -cycles in  $S$ , thanks to the lemma 3.2.5 below. The proposition 2.2.3 implies that the set of limits of  $\overline{G.x_\nu}$  for  $x_\nu \in V \cap \Omega_0$  is a compact set in  $\mathcal{C}_n^f(X)$ , proving [H.3].  $\blacksquare$

**Lemma 3.2.5** *Let  $K$  be a compact Lie group with a holomorphic action on a reduced complex space  $M$ . Let  $X$  be an irreducible complex space and let  $\varphi : X \rightarrow \mathcal{C}_d^f(M)$  and  $\psi : X \rightarrow \mathcal{C}_n^f(M)$  two holomorphic maps. Assume that for each  $x$  in a dense open subset  $\Omega_0$  in  $X$  we have*

$$|\psi(x)| = \cup_{k \in K} k \cdot |\varphi(x)| = K \cdot |\varphi(x)| \quad (\text{@@@})$$

where  $K$  acts holomorphically on  $\mathcal{C}_d^f(M)$  via the action of  $K$  on  $M$ . Then the relation (@@@) holds for each  $x \in X$ .

PROOF. Let  $x \in X$  and choose a sequence  $(x_\nu)_{\nu \geq 0}$  in  $\Omega_0$  converging to  $x$  and let  $y$  a point in  $|\psi(x)|$ . Then we can choose a sequence  $(y_\nu)_{\nu \geq 0}$  of points in  $|\psi(x_\nu)|$  converging to  $y$ . As  $x_\nu$  is in  $\Omega_0$  we can write  $y_\nu = k_\nu \cdot z_\nu$  with  $z_\nu \in |\varphi(x_\nu)|$  and  $k_\nu \in K$ . Up to pass to a subsequence we can assume that the sequence  $(k_\nu)$  converges to some  $k \in K$ . So the sequence  $(z_\nu)$  converges to  $k^{-1} \cdot y$  which is in  $|\varphi(x)|$ . This gives that  $y$  is in  $k \cdot |\varphi(x)|$  and we have proved the inclusion  $|\psi(x)| \subset K \cdot |\varphi(x)|$ . Conversely, if  $z$  is in  $|\varphi(x)|$  and  $k$  is in  $K$ , write again  $z = \lim z_\nu$  with  $z_\nu \in |\varphi(x_\nu)|$  where the sequence  $(x_\nu)$  of points in  $\Omega_0$  converges to  $x$ . We have  $k \cdot z_\nu \in |\psi(x_\nu)|$ , and then  $k \cdot z$  is in  $|\psi(x)|$  concluding the proof. ■

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