Gradient schemes: generic tools for the numerical analysis of diffusion equations

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GRADIENT SCHEMES: GENERIC TOOLS FOR THE NUMERICAL ANALYSIS OF DIFFUSION EQUATIONS

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Abstract. The gradient scheme framework is based on a small number of properties and encompasses a large number of numerical methods. We recall these properties and develop some new generic tools associated with the gradient scheme framework. These tools enable us to prove that classical schemes are indeed gradient schemes, and allow us to perform a complete and generic study of the well-known (but rarely well-studied) mass lumping process.

1. Introduction

A great variety of schemes have been developed in the last few years for the numerical simulation of anisotropic diffusion equations on general meshes, see [18, 41, 36] and references therein. The rigorous analysis of these methods is crucial to ensure their robustness and convergence, and to avoid the pitfalls of methods seemingly well-defined but not converging to the proper model [37, Chapter III, §3.2]. The necessity to conduct this analysis for each method and each model has given rise to a number of general ideas which are re-used from one study to the other, and a set of rather informal techniques has thus emerged over the years.

It is tempting to push further this idea of "set of informal similar techniques", to try and make it a formal mathematical theory. This boils down to finding common factors in the studies for all pairs (*method*,*model*), and to extract the core properties that ensure the stability and convergence of numerical methods for a variety of models. Identifying these core properties greatly reduces the work, which then amounts to two tasks:

Task (1): establish that a given numerical method satisfies the said properties;

Task (2): prove that these properties ensure the convergence of a method for all considered models.

Thus, the number of convergence studies is reduced from ($\sharp$ *methods*)$\times$($\sharp$ *models*) – which corresponds to one per pair (*method*,*model*) – to ($\sharp$ *methods*) + ($\sharp$ *models*). ($\sharp$ *methods*) studies are needed to prove that each method satisfies the core properties, and ($\sharp$ *models*) studies are required to prove that an abstract method that satisfies the core properties is convergent for each model.

Attempts at designing rigorous theories of unified convergence analysis for families of numerical methods is not new, see e.g. [12, 16] for finite element and discontinuous Galerkin methods. Recently, the Gradient scheme framework was developed [30, 24]; Not only does this framework provide a unifying framework for a number of methods – conforming and non-conforming Finite Elements, Finite Volumes, Mimetic Finite Differences, etc. – (Task 1) but it also allows enables complete convergence analyses for a wide variety of models of 2nd order diffusion PDEs (Task 2): linear, non-linear, non-local, degenerate, etc. [24, 33, 25, 21, 20, 2, 22, 9, 34, 11] through the verification of a very small number of properties (3 for linear models, 4 or 5 for non-linear models).

The purpose of this article is to bring gradient schemes one step further towards a unification theory. Indeed, we develop a set of generic tools that make Task (1) extremely simple for a great variety of methods. In other words, using these tools we can produce short but complete proofs that several numerical methods for 2nd order diffusion problems are gradient schemes.

The paper is organised as follows. In Section 2, we present the gradient scheme framework. This framework is based on the notion of gradient discretisation, which defines discrete spaces and operators, and on five core properties, presented in Subsection 2.1: coercivity, consistency, limit-conformity,

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compactness, and piecewise constant reconstruction. A gradient scheme is then obtained by applying a gradient discretisation to a given diffusion model, which consists in a set of second order partial differential equations and boundary conditions. Depending on the considered model, a gradient discretisation must satisfy three, four or five of these core properties to give rise to a convergent gradient scheme. In the next subsections, we develop generic notions that are useful to establish that particular methods fit into the framework. More precisely, in Subsection 2.2 we introduce the concept of local and linearly exact gradients, and we show that it implies one of the core properties – the consistency of gradient discretisations. Subsection 2.3 deals with the barycentric condensation of gradient discretisations, which is a classical way to eliminate degrees of freedom. The gradient scheme framework enables us, in Subsection 2.4, to rigorously define the well-known technique of mass lumping, and to show that this technique does not affect the convergence of a given scheme, thanks to a generic result on the comparison of two gradient discretisations given in Proposition 2.22.

In Section 3, we show that all methods in the following list are gradient schemes and satisfy four of the five core properties (coercivity, consistency, limit-conformity, compactness): conforming and non conforming finite elements, RT_k mixed finite elements, multipoint flux approximation (MPFA) schemes, discrete duality finite volume (DDFV) schemes, hybrid mixed methods (HMM), nodal mimetic finite difference (nMFD) methods, vertex average gradient (VAG) methods. For all these methods, the fifth property (piecewise constant reconstruction) is either satisfied by definition, or can be satisfied by a mass-lumped version in the sense of Subsection 2.4. The mass-lumped versions are only detailed in the important cases of the conforming and non-conforming $\mathbb{P}_1$ finite elements. Some of the schemes have already been more or less formally shown to be gradient schemes in [30, 24]. Here we use the generic tools developed in Section 2 to give, for each method, a rigorous complete proof, or sketch of proof (due to space limitation).

2. Gradient discretisations: definitions and analysis tools

We present here the principle of the abstract gradient schemes framework. For simplicity we restrict ourselves to homogeneous Dirichlet boundary conditions; all other classical boundary conditions (non-homogeneous Dirichlet, Neumann, Fourier or mixed) can be dealt with seamlessly in the gradient schemes framework [22]. The principle of gradient schemes is to write the weak formulation of the PDE by replacing all continuous spaces and operators by discrete analogs. These discrete objects are described in a gradient discretisation. Once a gradient discretisation is defined, its application to a given problem then leads to a gradient scheme.

For linear models, the convergence of gradient schemes is obtained via error estimates based on the *consistency* and *limit-conformity* measures $S_{\mathcal{D}}$ and $W_{\mathcal{D}}$. For non-linear models, whose solutions may lack regularity or even be non-unique, error estimates may not always be obtained; however, convergence of approximate solutions can be obtained via compactness techniques such as those developed in the finite volume framework [26, 29, 18]. Even though they do not yield an explicit rate of convergence, these compactness techniques provide strong convergence results – such as uniform-in-time convergence [20] – under assumptions that are compatible with field applications (discontinuous data, fully non-linear models, etc.).

2.1. Definitions.

Definition 2.1 (Gradient discretisation for homogeneous Dirichlet boundary conditions)**.** *Let $p \in (1, \infty)$. A gradient discretisation for homogeneous Dirichlet boundary conditions is a triplet $\mathcal{D} = (X_{\mathcal{D},0}, \Pi_{\mathcal{D}}, \nabla_{\mathcal{D}})$, where:*

1. *$X_{\mathcal{D},0}$ is a finite dimensional space encoding the degrees of freedom (and accounting for the homogeneous Dirichlet boundary conditions),*
2. *$\Pi_{\mathcal{D}} : X_{\mathcal{D},0} \to L^p(\Omega)$ is a linear mapping reconstructing a function in $L^p(\Omega)$ from the degrees of freedom,*
3. *$\nabla_{\mathcal{D}} : X_{\mathcal{D},0} \to L^p(\Omega)^d$ is a linear mapping defining a discrete gradient from the degrees of freedom. It must be chosen such that $\|\nabla_{\mathcal{D}} \cdot\|_{L^p(\Omega)^d}$ is a norm on $X_{\mathcal{D},0}$.*

Here are the three properties a gradient discretisation needs to satisfy to enable the analysis of the corresponding gradient scheme on linear problems:

- The *coercivity* ensures uniform discrete Poincaré inequalities for the family of gradient discretisations; this is essential to obtain *a priori* estimates on the solutions to gradient schemes.
- The *consistency* states that the family of gradient discretisations "covers" the whole energy space of the model (e.g. $H_0^1(\Omega)$ for the linear equation (2.1)).
- The *limit-conformity* ensures that the family of gradient and function reconstructions asymptotically satisfies the Stokes formula.

Definition 2.2 (Coercivity)**.** *Let $\mathcal{D}$ be a gradient discretisation in the sense of Definition 2.1 and let $C_\mathcal{D}$ be the norm of the linear mapping $\Pi_\mathcal{D}$ defined by*

$$C_\mathcal{D} = \max_{v\in X_{\mathcal{D},0}\setminus\{0\}} \frac{\|\Pi_\mathcal{D} v\|_{L^p(\Omega)}}{\|\nabla_\mathcal{D} v\|_{L^p(\Omega)^d}}.$$

A sequence $(\mathcal{D}_m)_{m\in\mathbf{N}}$ of gradient discretisations in the sense of Definition 2.1 is said to be coercive if there exists $C_P \in \mathbf{R}_+$ such that $C_{\mathcal{D}_m} \le C_P$ for all $m \in \mathbf{N}$.

Definition 2.3 (Consistency)**.** *Let $\mathcal{D}$ be a gradient discretisation in the sense of Definition 2.1 and let $S_\mathcal{D} : W_0^{1,p}(\Omega) \to [0,+\infty)$ be defined by*

$$\forall \varphi \in W_0^{1,p}(\Omega)\,,\ S_\mathcal{D}(\varphi) = \min_{v\in X_{\mathcal{D},0}} \left(\|\Pi_\mathcal{D} v - \varphi\|_{L^p(\Omega)} + \|\nabla_\mathcal{D} v - \nabla\varphi\|_{L^p(\Omega)^d}\right).$$

A sequence $(\mathcal{D}_m)_{m\in\mathbf{N}}$ of gradient discretisations in the sense of Definition 2.1 is said to be consistent if for all $\varphi \in W_0^{1,p}(\Omega)$ we have $\lim_{m\to\infty} S_{\mathcal{D}_m}(\varphi) = 0$.

Definition 2.4 (Limit-conformity)**.** *Let $\mathcal{D}$ be a gradient discretisation in the sense of Definition 2.1. We set $p' = \frac{p}{p-1}$ the dual exponent of p and $W^{\mathrm{div},p'}(\Omega) = \{\boldsymbol{\varphi} \in L^{p'}(\Omega)^d\,,\ \mathrm{div}\boldsymbol{\varphi} \in L^{p'}(\Omega)\}$, and we define*

$$\forall \boldsymbol{\varphi} \in W^{\mathrm{div},p'}(\Omega)\,,\ W_\mathcal{D}(\boldsymbol{\varphi}) = \sup_{u\in X_{\mathcal{D},0}\setminus\{0\}} \frac{1}{\|\nabla_\mathcal{D} u\|_{L^p(\Omega)^d}} \left| \int_\Omega (\nabla_\mathcal{D} u(\boldsymbol{x})\cdot\boldsymbol{\varphi}(\boldsymbol{x}) + \Pi_\mathcal{D} u(\boldsymbol{x})\mathrm{div}\boldsymbol{\varphi}(\boldsymbol{x}))\,\mathrm{d}\boldsymbol{x}\right|.$$

A sequence $(\mathcal{D}_m)_{m\in\mathbf{N}}$ of gradient discretisations is said to be limit-conforming if for all $\boldsymbol{\varphi} \in W^{\mathrm{div},p'}(\Omega)$ we have $\lim_{m\to\infty} W_{\mathcal{D}_m}(\boldsymbol{\varphi}) = 0$.

To give an idea of how a gradient discretisation gives a converging gradient scheme for diffusion equations, let us consider the case of a linear elliptic equation

$$\left\{\begin{array}{ll} -\mathrm{div}(A\nabla\overline{u}) = f & \text{in } \Omega,\\ \overline{u} = 0 & \text{on } \partial\Omega,\end{array}\right. \tag{2.1}$$

where Ω is a bounded open set in $\mathbf{R}^d$, $A : \Omega \mapsto \mathcal{M}_d(\mathbf{R})$ is a measurable bounded and uniformly elliptic matrix-valued function such that $A(\boldsymbol{x})$ is symmetric for a.e. $\boldsymbol{x} \in \Omega$, and $f \in L^2(\Omega)$. The solution to problem (2.1) is understood in the weak sense:

$$\text{Find } \overline{u} \in H_0^1(\Omega) \text{ such that, for all } v \in H_0^1(\Omega),\quad \int_\Omega A(\boldsymbol{x})\nabla\overline{u}(\boldsymbol{x})\cdot\nabla v(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = \int_\Omega f(\boldsymbol{x})v(\boldsymbol{x})\mathrm{d}\boldsymbol{x}. \tag{2.2}$$

If $\mathcal{D}$ is a gradient discretisation with $p = 2$, then the corresponding gradient scheme for (2.1) consists in writing

$$\text{Find } u \in X_{\mathcal{D},0} \text{ such that, for all } v \in X_{\mathcal{D},0},\quad \int_\Omega A(\boldsymbol{x})\nabla_\mathcal{D} u(\boldsymbol{x})\cdot\nabla_\mathcal{D} v(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = \int_\Omega f(\boldsymbol{x})\Pi_\mathcal{D} v(\boldsymbol{x})\mathrm{d}\boldsymbol{x}. \tag{2.3}$$

As seen here, (2.3) consists in replacing in (2.1) the continuous space $H_0^1(\Omega)$ and the continuous gradient and function by their discrete reconstruction from $\mathcal{D}$. Reference [30] proves the following error estimate between the solution to (2.2) and its gradient scheme approximation (2.3):

$$\|\nabla\overline{u} - \nabla_\mathcal{D} u_\mathcal{D}\|_{L^2(\Omega)^d} + \|\overline{u} - \Pi_\mathcal{D} u_\mathcal{D}\|_{L^2(\Omega)} \le C_1\left[W_\mathcal{D}(A\nabla\overline{u}) + S_\mathcal{D}(\overline{u})\right], \tag{2.4}$$

where C_1 only depends on A and an upper bound of $C_\mathcal{D}$ in Definition 2.2. This shows that if a sequence $(\mathcal{D}_m)_{m\in\mathbf{N}}$ of gradient discretisations is coercive, consistent and limit-conforming, and if $(u_m)_{m\in\mathbf{N}}$ is a sequence of solutions to the corresponding gradient schemes for (2.1), then $\Pi_{\mathcal{D}_m} u_m \to \overline{u}$ in $L^2(\Omega)$ and $\nabla_{\mathcal{D}_m} u_m \to \nabla\overline{u}$ in $L^2(\Omega)^d$. The study of a scheme for (2.1) then amounts to finding a gradient discretisation $\mathcal{D}$ such that the scheme can be written under the form (2.3), and to proving that sequences of such gradient discretisations satisfy the above described properties. Establishing the consistency

and limit-conformity usually consists in obtaining estimates on $S_{\mathcal{D}}$ and $W_{\mathcal{D}}$ that give explicit rates of convergence in (2.4).

Dealing with non-linear problems might additionally require one or the other of the following properties.

- The *compactness* is used to deal with low-order non-linearities – e.g. in semi- or quasi-linear equations.
- The *piecewise constant reconstruction* corresponds to mass-lumping and is required to manage certain monotone non-linearities, or non-linearities on the time derivative as in Richards' model.

Definition 2.5 (Compactness)**.** *A sequence $(\mathcal{D}_m)_{m\in\mathbf{N}}$ of gradient discretisations in the sense of Definition 2.1 is said to be compact if, for any sequence $(u_m)_{m\in\mathbf{N}}$ such that $u_m \in X_{\mathcal{D}_m,0}$ for $m \in \mathbf{N}$ and $(\|\nabla_{\mathcal{D}_m} u_m\|_{L^p(\Omega)^d})_{m\in\mathbf{N}}$ is bounded, the sequence $(\Pi_{\mathcal{D}_m} u_m)_{m\in\mathbf{N}}$ is relatively compact in $L^p(\Omega)$.*

Definition 2.6 (Piecewise constant reconstruction)**.** *Let $\mathcal{D} = (X_{\mathcal{D},0}, \Pi_{\mathcal{D}}, \nabla_{\mathcal{D}})$ be a gradient discretisation in the sense of Definition 2.1. A linear mapping $\Pi_{\mathcal{D}}$ from $X_{\mathcal{D},0}$ to $L^p(\Omega)$ is a piecewise constant reconstruction if there exists a basis $(e_i)_{i\in I}$ of $X_{\mathcal{D},0}$ and a family of disjoint subsets $(\Omega_i)_{i\in I}$ of Ω such that for all $u = \sum_{i\in I} u_i e_i \in X_{\mathcal{D},0}$ we have $\Pi_{\mathcal{D}} u = \sum_{i\in I} u_i \chi_{\Omega_i}$, where χ_{Ω_i} is the characteristic function of Ω_i.*

As an illustration of the use of the importance of these properties for nonlinear problems, let us consider the following semi-linear modification of (2.1):

$$\left\{\begin{array}{ll} -\mathrm{div}(A\nabla \overline{u}) + \beta(\overline{u}) = f & \text{in } \Omega, \\ \overline{u} = 0 & \text{on } \partial\Omega, \end{array}\right. \tag{2.5}$$

for some function β. The gradient discretisation of this problem is pretty straightforward:

(2.6) Find $u \in X_{\mathcal{D},0}$ such that, for all $v \in X_{\mathcal{D},0}$,

$$\int_\Omega A(\boldsymbol{x})\nabla_{\mathcal{D}} u(\boldsymbol{x})\cdot\nabla_{\mathcal{D}} v(\boldsymbol{x})\mathrm{d}\boldsymbol{x} + \int_\Omega \beta(\Pi_{\mathcal{D}} u(\boldsymbol{x}))\Pi_{\mathcal{D}} v(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = \int_\Omega f(\boldsymbol{x})\Pi_{\mathcal{D}} v(\boldsymbol{x})\mathrm{d}\boldsymbol{x}.$$

The compactness property implies that an estimate on a sequence of discrete gradient $(\nabla_{\mathcal{D}_m} u)_{m\in\mathbf{N}}$ will yield relative compactness of the corresponding sequence of reconstructions $(\Pi_{\mathcal{D}_m} u)_{m\in\mathbf{N}}$, thus enabling a passing to the limit in the nonlinearity β.

Piecewise constant reconstructions are extremely useful for two reasons. First, if $\Pi_{\mathcal{D}}$ is a piecewise constant reconstruction, then $\Pi_{\mathcal{D}}(\beta(u)) = \beta(\Pi_{\mathcal{D}} u)$ (where $\beta(u) \in X_{\mathcal{D},0}$ is defined degree-of-freedom per degree-of-freedom, that is $(\beta(u))_i = \beta(u_i)$ for all $i \in I$). This property is essential when establishing *a priori* estimates on gradient schemes for non-linear elliptic and parabolic equations [20, 25, 33]. It also facilitates the computation of the solution of Problem (2.6), when using an iterative procedure for instance. Second, if we consider a time-dependent problem, the discretisation of $\partial_t u$ leads to a term of the form

$$\int_\Omega \frac{\Pi_{\mathcal{D}} u^{n+1}(\boldsymbol{x}) - \Pi_{\mathcal{D}} u^n(\boldsymbol{x})}{\delta t}\Pi_{\mathcal{D}} v(\boldsymbol{x})\mathrm{d}\boldsymbol{x}. \tag{2.7}$$

If $\Pi_{\mathcal{D}}$ is a piecewise constant reconstruction, the mass matrix multiplying the coordinates $(u_i^{n+1})_{i\in I}$ of u^{n+1} in (2.7) is diagonal, and its inversion is therefore trivial.

Remark 2.7. *(1) Equivalent statements can be written for the consistency, limit-conformity and compactness of gradient discretisations* [22]*. In particular, the consistency and limit-conformity of a coercive sequence of gradient discretisations only need to be checked for φ and $\boldsymbol{\varphi}$ in dense sets of the domains of $S_{\mathcal{D}}$ and $W_{\mathcal{D}}$ (e.g. $C_c^\infty(\Omega)$ and $C_c^\infty(\mathbf{R}^d)^d$ respectively – the second set is indeed dense in $W^{\mathrm{div},p'}(\Omega)$ when Ω is locally star-shaped, which is the case if Ω is polyhedral). Moreover, the compactness of a sequence of gradient discretisations implies its coercivity.*

(2) Gradient discretisations for time-dependent problems can be easily deduced from the gradient discretisations for steady-state problems [24, 22]*.*

(3) For piecewise constant reconstructions, the set I is usually a natural set of degrees of freedom of the method (see Section 2.4 for examples).

2.2. **Local and linearly exact gradients.** Most numerical methods for diffusion equations are based, explicitly or implicitly, on local linearly exact reconstructed gradients. The following definition gives a precise meaning to this.

Definition 2.8 (Linearly exact gradient reconstructions)**.** *Let U be a bounded set of $\mathbf{R}^d$ and let $S = (\boldsymbol{x}_i)_{i\in I}$ be a finite family of points of $\mathbf{R}^d$. A linear mapping $\mathcal{G} : \mathbf{R}^I \mapsto L^\infty(U)^d$ is a linearly exact gradient reconstruction from S if, for all linear functions $L : \mathbf{R}^d \mapsto \mathbf{R}$, if $\xi = (L(\boldsymbol{x}_i))_{i\in I}$ then $\mathcal{G}\xi = \nabla L$ on U. The norm of $\mathcal{G}$ is defined by*

$$\|\mathcal{G}\|_\infty = \operatorname{diam}(U) \max_{\xi\in\mathbf{R}^N\setminus\{0\}} \frac{||\mathcal{G}\xi||_{L^\infty(U)^d}}{\max_{i\in I}|\xi_i|}. \tag{2.8}$$

As expected, linearly exact gradient reconstructions enjoy nice approximation properties when computed from interpolants of smooth functions.

Lemma 2.9 (Estimate for linearly exact gradient reconstructions)**.** *Let U be a bounded set of $\mathbf{R}^d$, let $S = (\boldsymbol{x}_i)_{i\in I} \subset \mathbf{R}^d$, and let $\mathcal{G} : \mathbf{R}^N \mapsto L^\infty(U)^d$ be a linearly exact gradient reconstruction from S in the sense of Definition 2.8. Let $\varphi \in W^{2,\infty}(\mathbf{R}^d)$ and define $v \in \mathbf{R}^N$ by $v_i = \varphi(\boldsymbol{x}_i)$ for any $i \in I$. Then*

$$|\mathcal{G}v - \nabla\varphi| \le \left(1 + \frac{1}{2}\|\mathcal{G}\|_\infty \left(\frac{\max_{i\in I}\operatorname{dist}(\boldsymbol{x}_i, U)}{\operatorname{diam}(U)} + 1\right)^2\right) \operatorname{diam}(U)||\varphi||_{W^{2,\infty}(\mathbf{R}^d)} \quad \text{a.e. on } U.$$

Proof. Take $\boldsymbol{x}_U \in U$ and let $L(\boldsymbol{x}) = \varphi(\boldsymbol{x}_U) + \nabla\varphi(\boldsymbol{x}_U)\cdot(\boldsymbol{x} - \boldsymbol{x}_U)$ be the first order Taylor expansion of φ around $\boldsymbol{x}_U$. Let $\xi = (L(\boldsymbol{x}_i))_{i\in I}$. By linear exactness of $\mathcal{G}$ we have $\mathcal{G}\xi = \nabla L = \nabla\varphi(\boldsymbol{x}_U)$ on U. Hence,

$$|\mathcal{G}\xi - \nabla\varphi| \le \operatorname{diam}(U)||\varphi||_{W^{2,\infty}(\mathbf{R}^d)} \quad \text{on } U. \tag{2.9}$$

For any $i \in I$ we have $(v-\xi)_i = \varphi(\boldsymbol{x}_i) - L(\boldsymbol{x}_i) = \varphi(\boldsymbol{x}_i) - \varphi(\boldsymbol{x}_U) - \nabla\varphi(\boldsymbol{x}_U)\cdot(\boldsymbol{x}_i - \boldsymbol{x}_U)$. Since $|\boldsymbol{x}_i - \boldsymbol{x}_U| \le \operatorname{dist}(\boldsymbol{x}_i, U) + \operatorname{diam}(U)$, we get $|(v-\xi)_i| \le \frac{1}{2}(\operatorname{dist}(\boldsymbol{x}_i, U) + \operatorname{diam}(U))^2||\varphi||_{W^{2,\infty}(\mathbf{R}^d)}$. The linearity of $\mathcal{G}$ and the definition of its norm therefore imply, for all a.e. $\boldsymbol{x} \in U$,

$$\begin{aligned}|\mathcal{G}v(\boldsymbol{x}) - \mathcal{G}\xi(\boldsymbol{x})| = |\mathcal{G}(v-\xi)(\boldsymbol{x})| &\le \frac{\|\mathcal{G}\|_\infty}{\operatorname{diam}(U)}\frac{1}{2}\left(\max_{i\in I}\operatorname{dist}(\boldsymbol{x}_i, U) + \operatorname{diam}(U)\right)^2 ||\varphi||_{W^{2,\infty}(\mathbf{R}^d)} \\ &\le \frac{1}{2}\|\mathcal{G}\|_\infty \operatorname{diam}(U)\left(\frac{\max_{i\in I}\operatorname{dist}(\boldsymbol{x}_i, U)}{\operatorname{diam}(U)} + 1\right)^2 ||\varphi||_{W^{2,\infty}(\mathbf{R}^d)}.\end{aligned}$$

Combined with (2.9), this completes the proof of the lemma. □

The consistency of gradient discretisations based on linearly exact gradient reconstructions follows. Let us first define the definition for such gradient discretisations.

Definition 2.10 (LLE Gradient discretisation)**.** *Let $\mathcal{D} = (X_{\mathcal{D},0}, \Pi_\mathcal{D}, \nabla_\mathcal{D})$ be a gradient discretisation in the sense of Definition 2.1. We say that $\mathcal{D}$ is an LLE (for "local and linearly exact") gradient discretisation if there exists a finite family of approximation points $S = (\boldsymbol{x}_i)_{i\in I} \subset \mathbf{R}^d$ and a finite partition $\mathcal{U}$ of Ω such that:*

- *$X_{\mathcal{D},0} = \mathbf{R}^{I_\Omega} \times \{0\}^{I_{\partial\Omega}}$, where the set of degrees of freedom I is partitioned into I_Ω (interior degrees of freedom) and $I_{\partial\Omega}$ (boundary degrees of freedom, such that $\boldsymbol{x}_i \notin \Omega$ for all $i \in I_{\partial\Omega}$).*
- *For all $U \in \mathcal{U}$, there exists $I_U \subset I$ and an operator $\mathcal{G}_U : \mathbf{R}^{I_U} \mapsto L^\infty(U)^d$ which is a linearly exact gradient reconstruction from $(\boldsymbol{x}_i)_{i\in I_U}$ in the sense of Definition 2.8.*
- *For all $v \in X_{\mathcal{D},0}$ and all $U \in \mathcal{U}$, $\nabla_\mathcal{D} v = \mathcal{G}_U\big((v_i)_{i\in I_U}\big)$ on U.*
- *For all $U \in \mathcal{U}$ and a.e. $\boldsymbol{x} \in U$, there exist real numbers $(\alpha_i(\boldsymbol{x}))_{i\in I_U}$ such that*

$$\sum_{i\in I_U}\alpha_i(\boldsymbol{x}) = 1 \quad \text{and} \quad \Pi_\mathcal{D} v(\boldsymbol{x}) = \sum_{i\in I_U}\alpha_i(\boldsymbol{x})v_i. \tag{2.10}$$

In that case, we define the LLE regularity of $\mathcal{D}$ by

$$\operatorname{reg}_{\mathrm{LLE}}(\mathcal{D}) = \max_{U\in\mathcal{U}}\left(\|\mathcal{G}_U\|_\infty + \max_{i\in I_U}\frac{\operatorname{dist}(\boldsymbol{x}_i, U)}{\operatorname{diam}(U)} + \sup_{\boldsymbol{x}\in U}\sum_{i\in I_U}|\alpha_i(\boldsymbol{x})|\right).$$

Remark 2.11. *Note that the existence of $i, j \in I$ with $i \neq j$ and $\boldsymbol{x}_i = \boldsymbol{x}_j$ is not excluded (see the example given in Section 3.4).*

Remark 2.12. *We do not request $\Pi_{\mathcal{D}} v$ to be linearly exact ($\alpha_i(\boldsymbol{x})$ are not necessarily linear w.r.t $\boldsymbol{x}$); this reconstruction just needs to be computable from local degrees of freedom. This enables us to consider mass-lumped gradient discretisations.*

In a number of cases, estimating $\sum_{i\in I_U} |\alpha_i(\boldsymbol{x})|$ is trivial. If $\mathcal{D}$ has a piecewise constant reconstruction, then for any $\boldsymbol{x}$ all $\alpha_i(\boldsymbol{x})$ are equal to 0 except one that is equal to 1, and thus $\sum_{i\in I_U} |\alpha_i(\boldsymbol{x})| = 1$. If $\Pi_{\mathcal{D}} v(\boldsymbol{x})$ is a convex combination of the $(v_i)_{i\in I_U}$ for all $\boldsymbol{x} \in U$ (which is the case, e.g., if $\Pi_{\mathcal{D}} v$ is linear on U and $(\boldsymbol{x}_i)_{i\in I_U}$ are extremal points of U), then $\sum_{i\in I_U} |\alpha_i(\boldsymbol{x})| = 1$.

Remark 2.13. *The term $\mathrm{diam}(U)$ in $\mathrm{reg}_{\mathrm{LLE}}(\mathcal{D})$ could be replaced with any quantity $\omega_U > 0$, the requirement to prove Proposition 2.14 below being that $\max_{U\in\mathcal{M}_m} \omega_U \to 0$ as $m \to \infty$.*

Proposition 2.14 (coercive LLE gradient discretisations are consistent)**.** *Let $(\mathcal{D}_m)_{m\in\mathbf{N}}$ be a sequence of coercive LLE gradient discretisations, associated for any $m \in \mathbf{N}$ to a partition $\mathcal{U}_m$. If $(\mathrm{reg}_{\mathrm{LLE}}(\mathcal{D}_m))_{m\in\mathbf{N}}$ is bounded and $\max_{U\in\mathcal{U}_m} \mathrm{diam}(U) \to 0$ as $m \to \infty$, then $(\mathcal{D}_m)_{m\in\mathbf{N}}$ is consistent in the sense of Definition 2.3.*

Proof. Because $(\mathcal{D}_m)_{m\in\mathbf{N}}$ is coercive, we only need to prove its consistency for smooth functions (Remark 2.7). Let $\varphi \in C_c^\infty(\Omega)$ and let $v^m = (\varphi(\boldsymbol{x}_i^m))_{i\in I^m} \in X_{\mathcal{D}_m,0}$, where $S_m = (\boldsymbol{x}_i^m)_{i\in I^m}$ is the family of approximation points of $\mathcal{D}_m$. Owing to Lemma 2.9 we have, for $U \in \mathcal{U}_m$ and a.e. $\boldsymbol{x} \in U$,

$$|\nabla_{\mathcal{D}_m} v^m(\boldsymbol{x}) - \nabla\varphi(\boldsymbol{x})| = |\mathcal{G}_U^m((v_i^m)_{i\in I_U^m})(\boldsymbol{x}) - \nabla\varphi(\boldsymbol{x})| \leq \left(1 + \frac{1}{2}\mathrm{reg}_{\mathrm{LLE}}(\mathcal{D}_m)(\mathrm{reg}_{\mathrm{LLE}}(\mathcal{D}_m)+1)^2\right)\mathrm{diam}(U)||\varphi||_{W^{2,\infty}(\mathbf{R}^d)}. \tag{2.11}$$

Let us now evaluate $|\Pi_{\mathcal{D}_m} v^m - \varphi|$. Since any $(\boldsymbol{x}_i^m)_{i\in I_U^m}$ is within distance $\mathrm{reg}_{\mathrm{LLE}}(\mathcal{D}_m)\mathrm{diam}(U)$ of U, for all $i \in I_U^m$ and all $\boldsymbol{x} \in U$ we have $|v_i^m - \varphi(\boldsymbol{x})| \leq (1 + \mathrm{reg}_{\mathrm{LLE}}(\mathcal{D}_m))\mathrm{diam}(U)||\varphi||_{W^{1,\infty}(\mathbf{R}^d)}$. By (2.10), we infer that for a.e. $\boldsymbol{x} \in U$

$$|\Pi_{\mathcal{D}_m} v^m(\boldsymbol{x}) - \varphi(\boldsymbol{x})| = \left|\sum_{i\in I_U^m} \alpha_i^m(\boldsymbol{x})(v_i^m - \varphi(\boldsymbol{x}))\right| \leq \sum_{i\in I_U^m} |\alpha_i^m(\boldsymbol{x})| \sup_{i\in I_U^m} |v_i^m - \varphi(\boldsymbol{x})| \leq \mathrm{reg}_{\mathrm{LLE}}(\mathcal{D}_m)(1 + \mathrm{reg}_{\mathrm{LLE}}(\mathcal{D}_m))\mathrm{diam}(U)||\varphi||_{W^{1,\infty}(\mathbf{R}^d)}. \tag{2.12}$$

Estimates (2.11) and (2.12) and the assumptions on $(\mathcal{D}_m)_{m\in\mathbf{N}}$ show that $\nabla_{\mathcal{D}_m} v^m \to \nabla\varphi$ in $L^\infty(\Omega)^d$ and $\Pi_{\mathcal{D}_m} v^m \to \varphi$ in $L^\infty(\Omega)$ as $m \to \infty$. □

2.3. **Barycentric elimination of degrees of freedom.** The construction of a given scheme often requires several interpolation points. However, some of these points can be eliminated afterwards to reduce the computational cost. A classical way to perform this reduction of degrees of freedom is through barycentric combinations, by replacing certain unknowns with averages of other unknowns. We describe here a way to perform this reduction in the general context of LLE gradient discretisations, while preserving the required properties (coercivity, consistency, limit-conformity and compactness).

Definition 2.15 (Barycentric condensation of an LLE gradient discretisation)**.** *Let $\mathcal{D}$ be an LLE gradient discretisation. We denote by $S = (\boldsymbol{x}_i)_{i\in I} \subset \mathbf{R}^d$ the family of approximation points of $\mathcal{D}$ and by $\mathcal{U}$ its partition. A gradient discretisation $\widetilde{\mathcal{D}}$ is a barycentric condensation of $\mathcal{D}$ if there exists $\widetilde{I} \subset I$ and, for all $i \in I\backslash\widetilde{I}$, a set $H_i \subset \widetilde{I}$ and real numbers $(\beta_j^i)_{j\in H_i}$ satisfying*

$$\sum_{j\in H_i} \beta_j^i = 1 \quad \text{and} \quad \sum_{j\in H_i} \beta_j^i \boldsymbol{x}_j = \boldsymbol{x}_i, \tag{2.13}$$

such that

- *$I_{\partial\Omega} \subset \widetilde{I}$.*
- *$X_{\widetilde{\mathcal{D}},0} = \mathbf{R}^{\widetilde{I}\cap I_\Omega} \times \{0\}^{I_{\partial\Omega}}$.*
- *For all $v \in X_{\widetilde{\mathcal{D}},0}$ we have $\Pi_{\widetilde{\mathcal{D}}} v = \Pi_{\mathcal{D}} V$ and $\nabla_{\widetilde{\mathcal{D}}} v = \nabla_{\mathcal{D}} V$, where $V \in X_{\mathcal{D},0} = \mathbf{R}^{I_\Omega} \times \{0\}^{I_{\partial\Omega}}$ is defined by*

$$\forall i \in I\,,\ V_i = \begin{cases} v_i & \text{if } i \in \widetilde{I}, \\ \sum_{j\in H_i} \beta_j^i v_j & \text{if } i \in I \setminus \widetilde{I}. \end{cases} \tag{2.14}$$

(We note that V is indeed in $X_{\mathcal{D},0}$ since $I_{\partial\Omega} \subset \widetilde{I}$ and $v_i = 0$ if $i \in I_{\partial\Omega}$.)
We define the regularity of the barycentric condensation $\widetilde{\mathcal{D}}$ by

$$\mathrm{reg}_{\mathrm{BC}}(\widetilde{\mathcal{D}}) = 1 + \max_{i\in I\setminus\widetilde{I}} \left(\sum_{j\in H_i} |\beta_j^i| + \max_{U\in\mathcal{M}\,|\,i\in I_U} \max_{j\in H_i} \frac{\mathrm{dist}(\boldsymbol{x}_j, \boldsymbol{x}_i)}{\mathrm{diam}(U)} \right).$$

It is clear that $\widetilde{\mathcal{D}}$ defined above is a gradient discretisation. Indeed, if $\nabla_{\widetilde{\mathcal{D}}} v = 0$ on Ω then $\nabla_{\mathcal{D}} V = 0$ on Ω and thus $V_i = 0$ for all $i \in S$ (since $\mathcal{D}$ is a gradient discretisation and $||\nabla_{\mathcal{D}} \cdot ||_{L^p(\Omega)^d}$ is a norm on $X_{\mathcal{D},0}$). This shows that $v_i = 0$ for all $i \in \widetilde{I}$, and thus that $||\nabla_{\widetilde{\mathcal{D}}} \cdot ||_{L^p(\Omega)^d}$ is a norm on $X_{\widetilde{\mathcal{D}},0}$.

Remark 2.16 (Localness of the barycentric elimination)**.** *Bounding the last term in $\mathrm{reg}_{\mathrm{BC}}(\widetilde{\mathcal{D}})$ consists in requiring that, if $i \in I \setminus \widetilde{I}$ is involved in the definition of $\mathcal{G}_U$, then any degree of freedom $j \in H_i$ used to eliminate the degree of freedom i lies within distance $\mathcal{O}(\mathrm{diam}(U))$ of U. This ensures that, after barycentric elimination, $\mathcal{G}_U$ is still computed using only degrees of freedom in a neighborhood of U.*

Barycentric elimination express some degrees of freedom by combinations that are linearly exact. As a consequence, the LLE property is preserved in the process, and the consistency of barycentric condensations of LLE gradient discretisations is ensured by Proposition 2.14.

Lemma 2.17 (Barycentric elimination preserves the LLE property)**.** *Let $\mathcal{D}$ be an LLE gradient discretisation in the sense of Definition 2.8, and let $\widetilde{\mathcal{D}}$ be a barycentric condensation of $\mathcal{D}$. Then $\widetilde{\mathcal{D}}$ is an LLE gradient discretisation on the same partition as $\mathcal{D}$, and* $\mathrm{reg}_{\mathrm{LLE}}(\widetilde{\mathcal{D}}) \le \mathrm{reg}_{\mathrm{BC}}(\widetilde{\mathcal{D}})\,\mathrm{reg}_{\mathrm{LLE}}(\mathcal{D}) + \mathrm{reg}_{\mathrm{BC}}(\widetilde{\mathcal{D}}) + \mathrm{reg}_{\mathrm{LLE}}(\mathcal{D})$.

Proof. Obviously, $\widetilde{I} = (\widetilde{I} \cap I_\Omega) \sqcup I_{\partial\Omega}$ forms the degrees of freedom of $\widetilde{\mathcal{D}}$ since $X_{\widetilde{\mathcal{D}},0} = \mathbf{R}^{\widetilde{I}\cap I_\Omega} \times \{0\}^{I_{\partial\Omega}}$. Let $\mathcal{U}$ be the partition corresponding to $\mathcal{D}$, and let $U \in \mathcal{U}$. Take $v \in X_{\widetilde{\mathcal{D}},0}$ and let $V \in X_{\mathcal{D},0}$ be defined by (2.14). We notice that, for any $U \in \mathcal{U}$, the values $(V_i)_{i\in I_U}$ are computed in terms of $(v_i)_{i\in\widetilde{I}_U}$ with $\widetilde{I}_U = (I_U \cap \widetilde{I}) \cup \bigcup_{i\in I_U\setminus\widetilde{I}} H_i$.

We have, for $\boldsymbol{x} \in U$,

$$\Pi_{\widetilde{\mathcal{D}}} v(\boldsymbol{x}) = \Pi_{\mathcal{D}} V(\boldsymbol{x}) = \sum_{i\in I_U} \alpha_i(\boldsymbol{x}) V_i = \sum_{i\in I_U\cap\widetilde{I}} \alpha_i(\boldsymbol{x}) v_i + \sum_{i\in I_U\setminus\widetilde{I}} \alpha_i(\boldsymbol{x}) \sum_{j\in H_i} \beta_j^i v_j = \sum_{i\in\widetilde{I}_U} \widetilde{\alpha}_i(\boldsymbol{x}) v_i$$

with

$$\widetilde{\alpha}_i(\boldsymbol{x}) = \alpha_i(\boldsymbol{x}) + \sum_{k\in I_U\setminus\widetilde{I}\,|\,i\in H_k} \beta_i^k \alpha_k(\boldsymbol{x}) \quad \text{if } i \in I_U \cap \widetilde{I},$$
$$\widetilde{\alpha}_i(\boldsymbol{x}) = \sum_{k\in I_U\setminus\widetilde{I}\,|\,i\in H_k} \beta_i^k \alpha_k(\boldsymbol{x}) \quad \text{if } i \in \widetilde{I}_U \setminus I_U.$$

Thanks to (2.13) and (2.10) we have

$$\sum_{i\in\widetilde{I}_U} \widetilde{\alpha}_i(\boldsymbol{x}) = \sum_{i\in I_U\cap\widetilde{I}} \alpha_i(\boldsymbol{x}) + \sum_{i\in\widetilde{I}_U} \sum_{k\in I_U\setminus\widetilde{I}\,|\,i\in H_k} \beta_i^k \alpha_k(\boldsymbol{x})$$
$$= \sum_{i\in I_U\cap\widetilde{I}} \alpha_i(\boldsymbol{x}) + \sum_{k\in I_U\setminus\widetilde{I}} \alpha_k(\boldsymbol{x}) \sum_{i\in H_k} \beta_i^k = \sum_{i\in I_U\cap\widetilde{I}} \alpha_i(\boldsymbol{x}) + \sum_{k\in I_U\setminus\widetilde{I}} \alpha_k(\boldsymbol{x}) = 1 \tag{2.15}$$

Hence, $\Pi_{\widetilde{\mathcal{D}}} v$ has the required form. The gradient $(\nabla_{\widetilde{\mathcal{D}}} v)_{|U} = \mathcal{G}_U((V_i)_{i\in I_U})$ only depends on $(v_i)_{i\in\widetilde{I}_U}$ and can thus be written $\widetilde{\mathcal{G}}_U((v_i)_{i\in\widetilde{I}_U})$. By (2.13) the reconstruction $v \mapsto V$ is linearly exact, that is if v interpolates the values of a linear mapping L at the points $(\boldsymbol{x}_i)_{i\in\widetilde{I}_U}$ then V interpolates the same mapping L at the points $(\boldsymbol{x}_i)_{i\in I_U}$. Hence, the linear exactness of $\mathcal{G}_U$ gives the linear exactness of $\widetilde{\mathcal{G}}_U$. This completes the proof that $\widetilde{\mathcal{D}}$ is an LLE gradient discretisation.

Let us now establish the upper bound on $\mathrm{reg}_{\mathrm{LLE}}(\widetilde{\mathcal{D}})$. For all $i \in I_U \setminus \widetilde{I}$ we have $|V_i| \le \sum_{j\in H_i} |\beta_j^i|\,|v_j| \le \mathrm{reg}_{\mathrm{BC}}(\widetilde{\mathcal{D}}) \max_{j\in\widetilde{I}_U} |v_j|$. This also holds for $i \in I_U \cap \widetilde{I}$ since $\mathrm{reg}_{\mathrm{BC}}(\widetilde{\mathcal{D}}) \ge 1$. Hence, a.e. on U,

$$\left| \widetilde{\mathcal{G}}_U \left((v_i)_{i\in\widetilde{I}_U} \right) \right| = |\mathcal{G}_U\left((V_i)_{i\in I_U}\right)| \le \frac{\|\mathcal{G}_U\|_\infty\, \mathrm{reg}_{\mathrm{BC}}(\widetilde{\mathcal{D}})}{\mathrm{diam}(U)} \max_{i\in\widetilde{I}_U} |v_i|$$

and thus

$$\|\widetilde{\mathcal{G}}_U\|_\infty \le \|\mathcal{G}_U\|_\infty \mathrm{reg}_{\mathrm{BC}}(\widetilde{\mathcal{D}}). \tag{2.16}$$

Reproducing the reasoning in the first two inequalities in (2.15), with absolute values and inequalities, we see that

$$\sum_{i\in\widetilde{I}_U} |\widetilde{\alpha}_i(\boldsymbol{x})| \le \sum_{i\in I_U\cap\widetilde{I}} |\alpha_i(\boldsymbol{x})| + \sum_{k\in I_U\setminus\widetilde{I}} |\alpha_k(\boldsymbol{x})| \sum_{i\in H_k} |\beta_i^k| \le \mathrm{reg}_{\mathrm{BC}}(\widetilde{\mathcal{D}}) \sum_{i\in I_U} |\alpha_i(\boldsymbol{x})|. \tag{2.17}$$

Finally, for $j\in\widetilde{I}_U$ we estimate $\frac{\mathrm{dist}(\boldsymbol{x}_j,U)}{\mathrm{diam}(U)}$ by studying two cases. If $j\in I_U$ then $\mathrm{dist}(\boldsymbol{x}_j,U)\le \mathrm{reg}_{\mathrm{LLE}}(\mathcal{D})\mathrm{diam}(U)$. If $j\notin I_U$ then there exists $i\in I_U\setminus\widetilde{I}$ such that $j\in H_i$, and thus $\mathrm{dist}(\boldsymbol{x}_j,\boldsymbol{x}_i)\le \mathrm{reg}_{\mathrm{BC}}(\widetilde{\mathcal{D}})\mathrm{diam}(U)$; this gives $\mathrm{dist}(\boldsymbol{x}_j,U)\le (\mathrm{reg}_{\mathrm{BC}}(\widetilde{\mathcal{D}})+\mathrm{reg}_{\mathrm{LLE}}(\mathcal{D}))\mathrm{diam}(U)$. Combined with (2.16) and (2.17), these estimates on $\mathrm{dist}(\boldsymbol{x}_j,U)$ prove the bound on $\mathrm{reg}_{\mathrm{LLE}}(\widetilde{\mathcal{D}})$ stated in the lemma. □

Barycentric condensation of LLE gradient discretisations satisfy the same properties (coercivity, consistency, compactness, limit-conformity) as the original gradient discretisation. The coercivity, limit-conformity and compactness properties result from the fact that $X_{\widetilde{\mathcal{D}},0}$ is (roughly) a subspace of $X_{\mathcal{D},0}$, and the consistency is a consequence of Lemma 2.17 and Proposition 2.14.

Theorem 2.18 (Properties of barycentric condensation of gradient discretisations)**.** *Let $(\mathcal{D}_m)_{m\in\mathbf{N}}$ be a sequence of LLE gradient discretisations that is coercive, consistent, limit-conforming and compact in the sense of the definitions in Section 2.1. Let $\mathcal{U}_m$ be the partition of $\mathcal{D}_m$. We assume that $\max_{U\in\mathcal{U}_m}\mathrm{diam}(U)\to 0$ as $m\to\infty$, and that $(\mathrm{reg}_{\mathrm{LLE}}(\mathcal{D}_m))_{m\in\mathbf{N}}$ is bounded. For any $m\in\mathbf{N}$ we take a barycentric condensation $\widetilde{\mathcal{D}}_m$ of $\mathcal{D}_m$ such that $(\mathrm{reg}_{\mathrm{BC}}(\widetilde{\mathcal{D}}_m))_{m\in\mathbf{N}}$ is bounded.*

Then $(\widetilde{\mathcal{D}}_m)_{m\in\mathbf{N}}$ is coercive, consistent, limit-conforming and compact.

Proof. For any $v\in X_{\widetilde{\mathcal{D}}_m,0}$, with V defined by (2.14) we have

$$||\Pi_{\widetilde{\mathcal{D}}_m} v||_{L^p(\Omega)} = ||\Pi_{\mathcal{D}_m} V||_{L^p(\Omega)} \le C_{\mathcal{D}_m}||\nabla_{\mathcal{D}_m} V||_{L^p(\Omega)^d} = C_{\mathcal{D}_m}||\nabla_{\widetilde{\mathcal{D}}_m} v||_{L^p(\Omega)^d},$$

which shows that $C_{\widetilde{\mathcal{D}}_m}\le C_{\mathcal{D}_m}$ and thus that $(\widetilde{\mathcal{D}}_m)_{m\in\mathbf{N}}$ is coercive. To prove the compactness, we take $(\nabla_{\widetilde{\mathcal{D}}_m} v_m)_{m\in\mathbf{N}} = (\nabla_{\mathcal{D}_m} V_m)_{m\in\mathbf{N}}$ bounded in $L^p(\Omega)^d$, and we use the compactness of $(\mathcal{D}_m)_{m\in\mathbf{N}}$ to see that $(\Pi_{\mathcal{D}_m} V_m)_{m\in\mathbf{N}} = (\Pi_{\widetilde{\mathcal{D}}_m} v_m)_{m\in\mathbf{N}}$ is relatively compact in $L^p(\Omega)$. The limit conformity follows by writing

$$\begin{aligned}\frac{1}{||\nabla_{\widetilde{\mathcal{D}}} v||_{L^p(\Omega)^d}} &\left|\int_\Omega \left(\nabla_{\widetilde{\mathcal{D}}} v(\boldsymbol{x})\cdot\boldsymbol{\varphi}(\boldsymbol{x}) + \Pi_{\widetilde{\mathcal{D}}} v(\boldsymbol{x})\mathrm{div}\boldsymbol{\varphi}(\boldsymbol{x})\right)\mathrm{d}\boldsymbol{x}\right| \\ &= \frac{1}{||\nabla_{\mathcal{D}} V||_{L^p(\Omega)^d}}\left|\int_\Omega \left(\nabla_{\mathcal{D}} V(\boldsymbol{x})\cdot\boldsymbol{\varphi}(\boldsymbol{x}) + \Pi_{\mathcal{D}} V(\boldsymbol{x})\mathrm{div}\boldsymbol{\varphi}(\boldsymbol{x})\right)\mathrm{d}\boldsymbol{x}\right|,\end{aligned}$$

which shows that $W_{\widetilde{\mathcal{D}}}(\boldsymbol{\varphi})\le W_{\mathcal{D}}(\boldsymbol{\varphi})$. Finally, by Lemma 2.17 each $\widetilde{\mathcal{D}}_m$ is an LLE gradient discretisation and the boundedness of $(\mathrm{reg}_{\mathrm{LLE}}(\mathcal{D}_m))_{m\in\mathbf{N}}$ and $(\mathrm{reg}_{\mathrm{BC}}(\widetilde{\mathcal{D}}_m))_{m\in\mathbf{N}}$ show that $(\mathrm{reg}_{\mathrm{LLE}}(\widetilde{\mathcal{D}}_m))_{m\in\mathbf{N}}$ is bounded. Proposition 2.14 then gives the consistency of $(\widetilde{\mathcal{D}}_m)_{m\in\mathbf{N}}$. □

2.4. **Mass lumping.** "Mass-lumping" is the generic name of the process applied to modify schemes that do not have a built-in piecewise constant reconstruction, say for instance the $\mathbb{P}_1$ finite element scheme (see Section 3.1). This is often done on a case-by-case basis, with *ad hoc* studies. The gradient scheme framework provides an efficient generic setting for performing this mass-lumping. The idea is to modify the reconstruction operator so that it becomes a piecewise constant reconstruction; under an assumption that is easy to verify in practice, this "mass-lumped" gradient discretisation can be compared with the original gradient discretisation, which ensures that all properties required for the convergence of the mass-lumped scheme are satisfied.

Definition 2.19 (Mass-lumped gradient discretisation)**.** *Let $\mathcal{D}=(X_{\mathcal{D},0},\Pi_{\mathcal{D}},\nabla_{\mathcal{D}})$ be a gradient discretisation in the sense of Definition 2.1. A mass-lumped version of $\mathcal{D}$ is a gradient discretisation $\mathcal{D}^{\mathrm{ML}}=(X_{\mathcal{D},0},\Pi_{\mathcal{D}}^{\mathrm{ML}},\nabla_{\mathcal{D}})$, where $\Pi_{\mathcal{D}}^{\mathrm{ML}}$ is obtained the following way. We select a basis $(e_i)_{i\in I}$ of $X_{\mathcal{D},0}$, we take disjoints subsets $(\Omega_i)_{i\in I}$ of Ω (some of them can be empty), and we set*

$$\forall v=\sum_{i\in I} v_i e_i\in X_{\mathcal{D},0}\,,\ \Pi_{\mathcal{D}}^{\mathrm{ML}} v=\sum_{i\in I} v_i\chi_{\Omega_i}$$

where χ_{Ω_i} is the characteristic function of Ω_i.

Remark 2.20 (Mass lumping with respect to a non canonical basis)**.** *The basis $(e_i)_{i\in I}$ of $X_{\mathcal{D},0}$ is usually chosen in a canonical way, each component of this basis corresponding to a natural degree of freedom of $\mathcal{D}$ (see examples in the $\mathbb{P}_1$ and non-conforming $\mathbb{P}_1$ case in Section 3.1). Mass-lumping could be done with respect to a non-standard basis, but this might lead to additional numerical cost if the computation of $\nabla_{\mathcal{D}}$ in this non-standard basis is complex; the scheme implementation might require to perform changes of basis, possibly with full transition matrices, to compute $\Pi_{\mathcal{D}}^{\rm ML}$ and $\nabla_{\mathcal{D}}$.*

Theorem 2.21 (Properties of mass-lumped gradient discretisations)**.** *Let $(\mathcal{D}_m)_{m\in\mathbf{N}}$ be a sequence of gradient discretisations in the sense of Definition 2.1, that is coercive, consistent, limit-conforming and compact in the sense of the definitions in Section 2.1. For any $m\in\mathbf{N}$ we take $\mathcal{D}_m^{\rm ML}$ a mass-lumped version of $\mathcal{D}_m$. If there exists $\omega_m\to 0$ as $m\to\infty$ such that*

$$\forall m\in\mathbf{N}\,,\ \forall v\in X_{\mathcal{D}_m,0}\,,\ ||\Pi_{\mathcal{D}_m}^{\rm ML}v-\Pi_{\mathcal{D}_m}v||_{L^p(\Omega)}\le\omega_m||\nabla_{\mathcal{D}_m}v||_{L^p(\Omega)^d}, \tag{2.18}$$

then $(\mathcal{D}_m^{\rm ML})_{m\in\mathbf{N}}$ is coercive, consistent, limit-conforming, and compact. The reconstruction $\Pi_{\mathcal{D}_m}^{\rm ML}$ is also piecewise constant.

This theorem is a direct consequence of Proposition 2.22, which gives a general setting for proving the properties of a gradient discretisation by comparing it with another gradient discretisation. This proposition generalises e.g. [9].

Proposition 2.22 (Comparison of gradient discretisations)**.** *Let $(\mathcal{D}_m)_{m\in\mathbf{N}}=(X_{\mathcal{D}_m,0},\Pi_{\mathcal{D}_m},\nabla_{\mathcal{D}_m})_{m\in\mathbf{N}}$ be a sequence of gradient discretisations in the sense of Definition 2.1. Let $(\mathcal{D}'_m)_{m\in\mathbf{N}}=(X_{\mathcal{D}_m,0},\Pi_{\mathcal{D}'_m},\nabla_{\mathcal{D}'_m})_{m\in\mathbf{N}}$ be a sequence of gradient discretisations having the same spaces of degrees of freedom, but possibly different gradient and reconstruction operators. We assume that there exists C_2 not depending on m, and a sequence of numbers $(\omega_m)_{m\in\mathbf{N}}$ such that $\omega_m\to 0$ and, for all $v\in X_{\mathcal{D}_m,0}$,*

$$||\Pi_{\mathcal{D}_m}v-\Pi_{\mathcal{D}'_m}v||_{L^p(\Omega)}+||\nabla_{\mathcal{D}_m}v-\nabla_{\mathcal{D}'_m}v||_{L^p(\Omega)^d}\le\omega_m||\nabla_{\mathcal{D}_m}v||_{L^p(\Omega)^d}\,, \tag{2.19}$$

$$||\nabla_{\mathcal{D}_m}v||_{L^p(\Omega)^d}\le C_2||\nabla_{\mathcal{D}'_m}v||_{L^p(\Omega)^d}. \tag{2.20}$$

Assume that $(\mathcal{D}_m)_{m\in\mathbf{N}}$ is coercive in the sense of Definition 2.2 (resp. consistent in the sense of Definition 2.3, limit-conforming in the sense of Definition 2.4, or compact in the sense of Definition 2.5). Then $(\mathcal{D}'_m)_{m\in\mathbf{N}}$ is also coercive (resp. consistent, limit-conforming, or compact).

Proof. Assume that $(\mathcal{D}_m)_{m\in\mathbf{N}}$ is coercive. By (2.19) we have $||\nabla_{\mathcal{D}'_m}v||_{L^p(\Omega)^d}\le(1+\sup_m\omega_m)||\nabla_{\mathcal{D}_m}v||_{L^p(\Omega)^d}$. Hence, using (2.20),

$$\begin{aligned}||\Pi_{\mathcal{D}'_m}v||_{L^p(\Omega)}&\le(1+\sup_m\omega_m)||\Pi_{\mathcal{D}_m}v||_{L^p(\Omega)}\\&\le(1+\sup_m\omega_m)C_P||\nabla_{\mathcal{D}_m}v||_{L^p(\Omega)^d}\le(1+\sup_m\omega_m)C_PC_2||\nabla_{\mathcal{D}'_m}v||_{L^p(\Omega)^d},\end{aligned}$$

which gives the coercivity of $(\mathcal{D}'_m)_{m\in\mathbf{N}}$.

Assume now that $(\mathcal{D}_m)_{m\in\mathbf{N}}$ is consistent. Thanks to the triangular inequality and (2.19),

$$\begin{aligned}&||\Pi_{\mathcal{D}'_m}v-\varphi||_{L^p(\Omega)}+||\nabla_{\mathcal{D}'_m}v-\nabla\varphi||_{L^p(\Omega)^d}\\&\quad\le 2\omega_m||\nabla_{\mathcal{D}_m}v||_{L^p(\Omega)^d}+||\Pi_{\mathcal{D}_m}v-\varphi||_{L^p(\Omega)}+||\nabla_{\mathcal{D}_m}v-\nabla\varphi||_{L^p(\Omega)^d}\\&\quad\le 2\omega_m||\nabla\varphi||_{L^p(\Omega)^d}+||\Pi_{\mathcal{D}_m}v-\varphi||_{L^p(\Omega)}+(1+2\omega_m)||\nabla_{\mathcal{D}_m}v-\nabla\varphi||_{L^p(\Omega)^d}.\end{aligned}$$

Hence $S_{\mathcal{D}'_m}(\varphi)\le 2\omega_m||\nabla\varphi||_{L^p(\Omega)^d}+(1+2\omega_m)S_{\mathcal{D}_m}(\varphi)\to 0$ as $m\to\infty$ and the the consistency of $(\mathcal{D}'_m)_{m\in\mathbf{N}}$ is proved.

Let now $(\mathcal{D}_m)_{m\in\mathbf{N}}$ be limit-conforming. By the triangular inequality and (2.19),

$$\begin{aligned}&\left|\int_\Omega\left(\nabla_{\mathcal{D}'_m}v(\boldsymbol{x})\cdot\boldsymbol{\varphi}(\boldsymbol{x})+\Pi_{\mathcal{D}'_m}v(\boldsymbol{x})\mathrm{div}\boldsymbol{\varphi}(\boldsymbol{x})\right)\mathrm{d}\boldsymbol{x}\right|\\&\quad\le||\boldsymbol{\varphi}||_{L^{p'}(\Omega)^d}\omega_m||\nabla_{\mathcal{D}_m}v||_{L^p(\Omega)^d}+||\mathrm{div}\boldsymbol{\varphi}||_{L^{p'}(\Omega)}\omega_m||\nabla_{\mathcal{D}_m}v||_{L^p(\Omega)^d}\\&\qquad+\left|\int_\Omega(\nabla_{\mathcal{D}_m}v(\boldsymbol{x})\cdot\boldsymbol{\varphi}(\boldsymbol{x})+\Pi_{\mathcal{D}_m}v(\boldsymbol{x})\mathrm{div}\boldsymbol{\varphi}(\boldsymbol{x}))\,\mathrm{d}\boldsymbol{x}\right|.\end{aligned}$$

Since $||\nabla_{\mathcal{D}'_m} v||_{L^p(\Omega)^d} \geq C_2^{-1} ||\nabla_{\mathcal{D}_m} v||_{L^p(\Omega)^d}$, we infer that

$$W_{\mathcal{D}'_m}(\boldsymbol{\varphi}) \leq C_2 \omega_m (||\boldsymbol{\varphi}||_{L^{p'}(\Omega)^d} + ||\mathrm{div}\boldsymbol{\varphi}||_{p'(\Omega)^d}) + C_2 W_{\mathcal{D}_m}(\boldsymbol{\varphi}) \to 0 \text{ as } m \to \infty$$

and the limit conformity of $(\mathcal{D}'_m)_{m\in\mathbf{N}}$ is established.

Let us finally assume that $(\mathcal{D}_m)_{m\in\mathbf{N}}$ is compact. By (2.20), if $(\nabla_{\mathcal{D}'_m} v_m)_{m\in\mathbf{N}}$ is bounded in $L^p(\Omega)^d$ then $(\nabla_{\mathcal{D}_m} v_m)_{m\in\mathbf{N}}$ is also bounded in $L^p(\Omega)^d$. The compactness of $(\mathcal{D}_m)_{m\in\mathbf{N}}$ then ensures that $(\Pi_{\mathcal{D}_m} v_m)_{m\in\mathbf{N}}$ is relatively compact in $L^p(\Omega)$. Since $||\Pi_{\mathcal{D}_m} v_m - \Pi_{\mathcal{D}'_m} v_m||_{L^p(\Omega)} \to 0$ as $m \to \infty$ by (2.19), we deduce that $(\Pi_{\mathcal{D}_m} v_m)_{m\in\mathbf{N}}$ is relatively compact in $L^p(\Omega)$. □

2.5. **Polyhedral meshes and Discrete Functional Analysis.** Although Gradient discretisations are not limited to mesh-based methods, a large number of schemes for (2.1) are built on meshes. We introduce here a general definition and some notations related to these meshes. Whatever the dimension, we use the word "polyhedral" to denote sets with piecewise flat boundaries (in 2D, we should actually write "polygonal").

Definition 2.23 (Polyhedral mesh)**.** *Let Ω be a bounded connected polyhedral open subset of $\mathbf{R}^d$ ($d \geq 1$). A polyedral mesh of Ω is given by $\mathcal{T} = (\mathcal{M}, \mathcal{E}, \mathcal{P}, \mathcal{V})$, where:*

1. *$\mathcal{M}$ is a finite family of non empty connected polyhedral open disjoint subsets of Ω (the cells) such that $\overline{\Omega} = \cup_{K\in\mathcal{M}} \overline{K}$. For any $K \in \mathcal{M}$, $|K| > 0$ is the measure of K and h_K denotes the diameter of K.*
2. *$\mathcal{E}$ is a finite family of disjoint subsets of $\overline{\Omega}$ (the edges of the mesh in 2D, the faces in 3D), such that any $\sigma \in \mathcal{E}$ is a non empty open subset of a hyperplane of $\mathbf{R}^d$ and $\sigma \subset \overline{\Omega}$. We assume that for all $K \in \mathcal{M}$ there exists a subset $\mathcal{E}_K$ of $\mathcal{E}$ such that $\partial K = \cup_{\sigma\in\mathcal{E}_K} \overline{\sigma}$. We then denote by $\mathcal{M}_\sigma = \{K \in \mathcal{M} : \sigma \in \mathcal{E}_K\}$. We then assume that, for all $\sigma \in \mathcal{E}$, $\mathcal{M}_\sigma$ has exactly one element and $\sigma \subset \partial\Omega$, or $\mathcal{M}_\sigma$ has two elements and $\sigma \subset \Omega$. We let $\mathcal{E}_{\rm int}$ be the set of all interior faces, i.e. $\sigma \in \mathcal{E}$ such that $\sigma \subset \Omega$, and $\mathcal{E}_{\rm ext}$ the set of boundary faces, i.e. $\sigma \in \mathcal{E}$ such that $\sigma \subset \partial\Omega$. For $\sigma \in \mathcal{E}$, the $(d-1)$-dimensional measure of σ is $|\sigma|$, the centre of gravity of σ is $\overline{\boldsymbol{x}}_\sigma$*
3. *$\mathcal{P} = (\boldsymbol{x}_K)_{K\in\mathcal{M}}$ is a family of points of Ω indexed by $\mathcal{M}$ and such that, for all $K \in \mathcal{M}$, $\boldsymbol{x}_K \in K$ ($\boldsymbol{x}_K$ is sometimes called the "centre" of K). We then assume that all cells $K \in \mathcal{M}$ are strictly $\boldsymbol{x}_K$-star-shaped, meaning that if $\boldsymbol{x}$ is in the interior of K then the line segment $[\boldsymbol{x}_K, \boldsymbol{x}]$ is included in the interior of K.*
4. *$\mathcal{V}$ is the set of vertices of the mesh. The vertices that belong to $\overline{K}$, for $K \in \mathcal{M}$, are gathered in $\mathcal{V}_K$; the set of vertices of $\sigma \in \mathcal{E}$ is denoted by $\mathcal{V}_\sigma$.*

For all $K \in \mathcal{M}$ and for any $\sigma \in \mathcal{E}_K$, we denote by $\mathbf{n}_{K,\sigma}$ the (constant) unit vector normal to σ outward to K. We also let $\mathrm{d}_{K,\sigma}$ be the signed orthogonal distance between $\boldsymbol{x}_K$ and σ (see Figure 1), that is:

$$\mathrm{d}_{K,\sigma} = (\boldsymbol{x} - \boldsymbol{x}_K) \cdot \mathbf{n}_{K,\sigma}\,, \quad \forall \boldsymbol{x} \in \sigma \tag{2.21}$$

(note that $(\boldsymbol{x} - \boldsymbol{x}_K) \cdot \mathbf{n}_{K,\sigma}$ is constant for $\boldsymbol{x} \in \sigma$). The fact that K is strictly star-shaped with respect to $\boldsymbol{x}_K$ is equivalent to $\mathrm{d}_{K,\sigma} > 0$ for all $\sigma \in \mathcal{E}_K$. For all $K \in \mathcal{M}$ and $\sigma \in \mathcal{E}_K$, we denote by $D_{K,\sigma}$ the cone with apex $\boldsymbol{x}_K$ and base σ, that is $D_{K,\sigma} = \{t\boldsymbol{x}_K + (1-t)\boldsymbol{y} : t \in (0,1),\, \boldsymbol{y} \in \sigma\}$. The diamond associated to a face $\sigma \in \mathcal{E}$ is $D_\sigma = \bigcup_{K\in\mathcal{M}_\sigma} D_{K,\sigma}$.

The size of the discretisation is $h_\mathcal{M} = \sup\{h_K : K \in \mathcal{M}\}$ and the regularity factor $\theta_\mathcal{T}$ is

$$\theta_\mathcal{T} = \max\left\{\frac{h_K}{d_{K,\sigma}} : K \in \mathcal{M}\,,\ \sigma \in \mathcal{E}_K\right\} + \max\left\{\frac{d_{K,\sigma}}{d_{L,\sigma}} : \sigma \in \mathcal{E}_{\rm int}\,,\ \mathcal{M}_\sigma = \{K, L\}\right\}. \tag{2.22}$$

Remark 2.24 (Generalised hexahedra)**.** *This definition covers a wide variety of meshes, including those with non-convex cells and cells sharing more than one face; in particular, "generalised hexahedra" with non planar faces can be handled: such cells have 12 faces if each non planar face is split in two triangles, but only 6 neighbouring cells.*

In the examples of gradient discretisations given in Section 3, the following notion is used.

Definition 2.25 (Regularity of a sequence of polyhedral meshes)**.** *A sequence of polyhedral meshes $(\mathcal{T}_m)_{m\in\mathbf{N}}$ in the sense of Definition 2.23 is regular if $(\theta_{\mathcal{T}_m})_{m\in\mathbf{N}}$ is bounded and $h_{\mathcal{M}_m} \to 0$ as $m \to +\infty$.*

Remark 2.26. *Since $\min_{\sigma\in\mathcal{E}_K} d_{K,\sigma}$ is the radius of the largest ball centred at $\boldsymbol{x}_K$ and contained in K, an upper bound on $\theta_\mathcal{T}$ imposes that the interior and exterior diameters of each cell are comparable.*

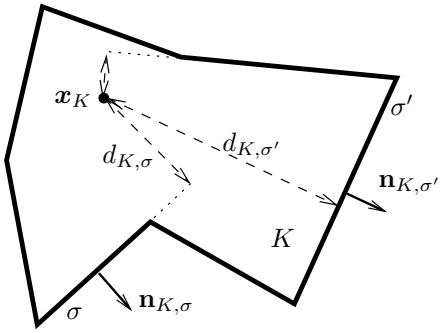

FIGURE 1. A cell K of a polyhedral mesh

Discrete functional analysis is the translation to the discrete setting of results of functional analysis. The lemmas presented below are extremely helpful in establishing the coercivity, compactness and limit-conformity of sequences of gradient discretisation. Their proof can be found in [27, 22].

We define the vector spaces $X_{\mathcal{M}}$ of degrees of freedom in the mesh, and $X_{\mathcal{T},0}$ of degrees of freedom in the mesh and on the edges (with homogeneous Dirichlet boundary conditions), by:

$$X_{\mathcal{M}} = \{v = (v_K)_{K\in\mathcal{M}} \,:\, v_K \in \mathbf{R}\}, \tag{2.23}$$

$$X_{\mathcal{T},0} = \{v = ((v_K)_{K\in\mathcal{M}}, (v_\sigma)_{\sigma\in\mathcal{E}} \,:\, v_K \in \mathbf{R}\,,\ v_\sigma \in \mathbf{R}\,,\ v_\sigma = 0 \text{ if } \sigma \in \mathcal{E}_{\rm ext}\}. \tag{2.24}$$

They are endowed with the following discrete $W_0^{1,p}$ norms, for $p \in (1,\infty)$:

$$\forall v \in X_{\mathcal{M}} \,: ||v||^p_{X_{\mathcal{M}},p} = \sum_{\sigma\in\mathcal{E}} |\sigma| d_\sigma \left|\frac{v_K - v_L}{d_\sigma}\right|^p, \tag{2.25}$$

$$\forall v \in X_{\mathcal{T},0} \,: ||v||^p_{X_{\mathcal{T},0},p} = \sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_K} |\sigma| d_{K,\sigma} \left|\frac{v_\sigma - v_K}{d_{K,\sigma}}\right|^p. \tag{2.26}$$

Here, if $\sigma \in \mathcal{E}_{\rm int}$ then K and L are the cells on each side of σ and $d_\sigma = d_{K,\sigma} + d_{L,\sigma}$; if $\sigma \in \mathcal{E}_{\rm ext} \cap \mathcal{E}_K$ then $v_L = 0$ and $d_\sigma = d_{K,\sigma}$. There is a natural restriction mapping $v \in X_{\mathcal{T},0} \mapsto \widetilde{v} \in X_{\mathcal{M}}$ obtained by simply considering the cell-centred degrees of freedom: $\widetilde{v} = (v_K)_{K\in\mathcal{M}}$. For all $v \in X_{\mathcal{T},0}$ we have $|v_K - v_L|^p \le 2^{p-1}|v_K - v_\sigma|^p + 2^{p-1}|v_L - v_\sigma|^p$ and thus

$$||\widetilde{v}||_{X_{\mathcal{M}},p} \le 2^{p-1}(1+\theta_{\mathcal{T}})||v||_{X_{\mathcal{T},0},p}. \tag{2.27}$$

We define the linear mapping $\Pi_{\mathcal{T}} : X_{\mathcal{M}} \mapsto L^\infty(\Omega)$ by

$$\forall v \in X_{\mathcal{M}}\,,\ \forall K \in \mathcal{M}\,,\ \Pi_{\mathcal{T}} v = v_K \text{ on } K. \tag{2.28}$$

$\Pi_{\mathcal{T}}$ is also considered as a linear mapping on $X_{\mathcal{T},0}$ by setting $\Pi_{\mathcal{T}} v = \Pi_{\mathcal{T}}\widetilde{v}$ if $v \in X_{\mathcal{T},0}$. The discrete gradient $\nabla_{\mathcal{T}} : X_{\mathcal{T},0} \mapsto L^p(\Omega)^d$ is defined by

$$\forall K \in \mathcal{M}\,,\ \nabla_{\mathcal{T}} v = \frac{1}{|K|}\sum_{\sigma\in\mathcal{E}_K} |\sigma| v_\sigma \mathbf{n}_{K,\sigma} \quad \text{on } K. \tag{2.29}$$

Lemma 2.27 (Discrete Poincaré inequality)**.** *Let $\mathcal{T}$ be a polyhedral discretisation of Ω in the sense of Definition 2.23, and let $\theta \ge \theta_{\mathcal{T}}$. There exists C_3 only depending on θ and p such that for all $v \in X_{\mathcal{M}}$ we have $||\Pi_{\mathcal{T}} v||_{L^p(\Omega)} \le C_3 ||v||_{X_{\mathcal{M}},p}$.*

Lemma 2.28 (Discrete Rellich theorem)**.** *Let $(\mathcal{T}_m)_{m\in\mathbf{N}}$ be a sequence of polyhedral discretisations of Ω in the sense of Definition 2.23, such that $(\theta_{\mathcal{T}_m})_{m\in\mathbf{N}}$ is bounded. If $v_m \in X_{\mathcal{M}_m}$ is such that $(||v_m||_{X_{\mathcal{T}_m,0},p})_{m\in\mathbf{N}}$ is bounded, then $(\Pi_{\mathcal{T}_m} v_m)_{m\in\mathbf{N}}$ is relatively compact in $L^p(\Omega)$.*

Lemma 2.29 (Discrete approximate Stokes formula)**.** *Let $\mathcal{T}$ be a polyhedral discretisation of Ω in the sense of Definition 2.23. If $\boldsymbol{\varphi} \in C_c^\infty(\mathbf{R}^d)^d$ and $v \in X_{\mathcal{T},0}$, then*

$$\left|\int_\Omega [\nabla_{\mathcal{T}} v(\boldsymbol{x})\cdot\boldsymbol{\varphi}(\boldsymbol{x}) + \Pi_{\mathcal{T}} v(\boldsymbol{x}) \mathrm{div}\boldsymbol{\varphi}(\boldsymbol{x})]\,\mathrm{d}\boldsymbol{x}\right| \le (d|\Omega|)^{\frac{p-1}{p}} ||\boldsymbol{\varphi}||_{W^{1,\infty}(\mathbf{R}^d)^d} ||v||_{X_{\mathcal{T},0},p} h_{\mathcal{M}}. \tag{2.30}$$

Moreover, if $(v_\sigma)_{\sigma\in\mathcal{E}_K}$ are the exact values at $(\overline{\boldsymbol{x}}_\sigma)_{\sigma\in\mathcal{E}_K}$ of a linear mapping L, then $\nabla_{\mathcal{T}} v = \nabla L$ on K.

3. Review of gradient discretisations

We now study a number of known methods among finite element, finite volume methods, mimetic methods and related discretisation schemes which are all based on polyhedral meshes. Each of the following sections is devoted to one method (called "α method" in the following theorem) which is shown to be a gradient discretisation referred to as $\mathcal{D}$; for each method we define the notion of regular sequence of gradient discretisations, based on the method itself and on the regularity of the polyhedral mesh in the sense of Definition 2.25.

The following theorem is for each method a rather direct consequence of the results on the notions of LLE gradient discretisations (Section 2.2), barycentric condensation (Section 2.3), mass lumping (Section 2.4) and Discrete Functional Analysis and approximate Stokes formula (Section 2.5).

Theorem 3.1 (Properties of the "α method")**.** *Let $(\mathcal{D}_m)_{m\in\mathbf{N}}$ be a regular sequence of gradient discretisations defined by the "α method". Then $(\mathcal{D}_m)_{m\in\mathbf{N}}$ is coercive, consistent, limit-conforming and compact in the sense of the definitions in Section 2.1.*

Hence this theorem only focuses on the four core properties, among the five ones, which need a proof. The piecewise constant reconstruction property is imposed by the construction of $\mathcal{D}$ only in some of the following methods.

3.1. $\mathbb{P}_k$ finite element methods.

3.1.1. *Conforming methods: $\mathbb{P}_k$ finite elements.* Let $\mathcal{T}$ be a simplicial discretisation of Ω, that is a polyhedral discretisation in the sense of Definition 2.23 such that for any $K\in\mathcal{M}$ we have $\mathrm{Card}(\mathcal{E}_K) = d+1$. In 2D, $\mathcal{T}$ is therefore a triangulation of Ω without hanging nodes. Let $k\in\mathbf{N}\setminus\{0\}$. We then follow Definition 2.10 for the construction of $\mathcal{D} = (X_{\mathcal{D},0}, \nabla_{\mathcal{D}}, \Pi_{\mathcal{D}})$, by describing the partition of Ω, the local linearly exact gradients and the functions α_i in the elements of the partition.

(1) The set of degrees of freedom and the approximation points are $I = S = \mathcal{V}^{(k)}$, where $\mathcal{V}^{(k)} = \bigcup_{K\in\mathcal{M}} \mathcal{V}_K^{(k)}$ and $\mathcal{V}_K^{(k)}$ is the set of the points $\boldsymbol{x}$ of the form

$$\boldsymbol{x} = \sum_{\mathsf{v}\in\mathcal{V}_K} \frac{i_{\mathsf{v}}}{k}\mathsf{v}, \tag{3.1}$$

where $i_{\mathsf{v}} = 0,\dots,k$ and $\sum_{\mathsf{v}\in\mathcal{V}_K} i_{\mathsf{v}} = k$. Then $I_\Omega = \mathcal{V}_{\rm int}^{(k)}$ (the subset of the interior vertices) and $I_{\partial\Omega} = \mathcal{V}_{\rm ext}^{(k)}$ (boundary vertices), and the partition of Ω is given by $\mathcal{U} = \mathcal{M}$. For $U = K\in\mathcal{U}$, we let $I_U = \mathcal{V}_K^{(k)}$.

(2) For any $\mathsf{v}\in\mathcal{V}_K^{(k)}$, we let α_{v} be the polynomial function of $\boldsymbol{x} = (x_1,\dots,x_d)$ with degree k, such that $\alpha_{\mathsf{v}}(\mathsf{v}) = 1$ and $\alpha_{\mathsf{v}}(\mathsf{v}') = 0$ for all $\mathsf{v}'\in\mathcal{V}_K^{(k)}\setminus\{\mathsf{v}\}$. Then the linearly exact gradient reconstruction in K is

$$\forall \boldsymbol{x}\in K,\ \mathcal{G}_K v(\boldsymbol{x}) = \sum_{\mathsf{v}\in\mathcal{V}_K} v_{\mathsf{v}} \nabla\alpha_{\mathsf{v}}(\boldsymbol{x}).$$

(3) The reconstruction (2.10) is defined using the basis functions $\alpha_{\mathsf{v}}(\boldsymbol{x})$. This leads to

$$\forall v\in X_{\mathcal{D},0},\ \forall \boldsymbol{x}\in\Omega,\ \Pi_{\mathcal{D}} v(\boldsymbol{x}) = \sum_{\mathsf{v}\in\mathcal{V}^{(k)}} v_{\mathsf{v}}\alpha_{\mathsf{v}}(\boldsymbol{x}).$$

Defining the regularity of a sequence of $\mathbb{P}_k$ discretisations $(\mathcal{D}_m)_{m\in\mathbf{N}}$ merely as the regularity of the underlying polyhedral meshes (Definition 2.25) is sufficient to get that $\mathrm{reg}_{\rm LLE}(\mathcal{D}_m)$ remains bounded. This allows to prove the consistency by Proposition 2.14. The remaining of the proof of Theorem 3.1 follows from the fact that the method is conforming. Since $\Pi_{\mathcal{D}} v\in W_0^{1,p}(\Omega)$ and $\nabla\Pi_{\mathcal{D}} v = \nabla_{\mathcal{D}} v$, we have $W_{\mathcal{D}}\equiv 0$ and the limit conformity is trivial; the coercivity and the compactness are a consequences of the Poincaré inequality and the Rellich theorem in $W_0^{1,p}(\Omega)$.

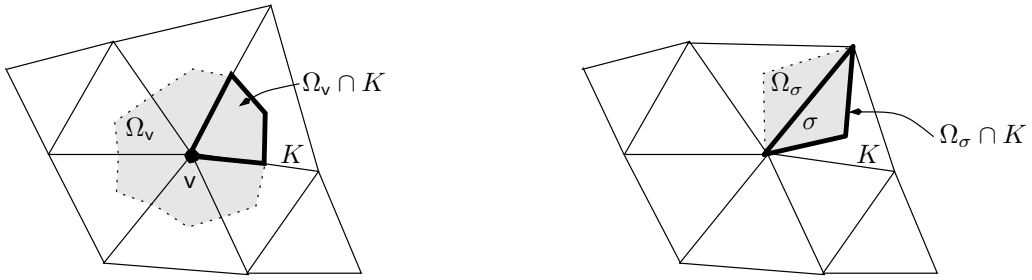

FIGURE 2. Partitions for mass-lumping of the $\mathbb{P}_1$ (left) and non-conforming $\mathbb{P}_1$ (right) finite element methods.

3.1.2. *Mass-lumped $\mathbb{P}_1$ finite elements.* We construct a mass-lumped version of the $\mathbb{P}_1$ gradient discretisation as per Definition 2.19, with the natural degrees of freedom $\mathcal{V}^{(1)} = \mathcal{V}$. Subdomains $(\Omega_{\mathsf{v}})_{\mathsf{v}\in\mathcal{V}}$ with points of $\mathcal{V}$ as centres can be constructed in various ways. One way is to define Ω_{v} such that, for all $K \in \mathcal{M}$ with $\mathsf{v} \in \mathcal{V}_K$, $\Omega_{\mathsf{v}} \cap K$ is the set of all $\boldsymbol{y} \in K$ such that, if $\boldsymbol{y}$ is written as the convex combination $\boldsymbol{y} = \sum_{\mathsf{v}\in\mathcal{V}_K} \lambda_{\mathsf{v}}\mathsf{v}$, then $\lambda_{\mathsf{v}} \geq \lambda_{\mathsf{v}'}$ for any other $\mathsf{v}' \in \mathcal{V}_K$. The left part of Figure 2 illustrates the construction of the partitions $(\Omega_{\mathsf{v}})_{\mathsf{v}\in\mathcal{V}}$ in the case $d = 2$.

A Taylor expansion in each $\Omega_{\mathsf{v}} \cap K$ then shows that Estimate (2.18) holds with $\omega_m = h_{\mathcal{M}_m}$, and thus by Theorem 2.21 we see that the mass-lumped $\mathbb{P}_1$ gradient discretisation satisfies Theorem 3.1.

3.2. Non-conforming $\mathbb{P}_1$ finite elements.

3.2.1. *Standard non-conforming $\mathbb{P}_1$ reconstruction.* Non-conforming $\mathbb{P}_1$ finite elements consist in approximating the solution to (2.2) by functions that are piecewise linear on triangles and continuous at the edge midpoints – but not necessarily continuous on the whole edge. These approximating functions therefore do not lie in $H^1_0(\Omega)$, and do not satisfy the exact Stokes formula; hence the name "non-conforming".

Let $\mathcal{T}$ be a simplicial discretisation of Ω, that is a polyhedral discretisation in the sense of Definition 2.23 such that for any $K \in \mathcal{M}$ we have $\mathrm{Card}(\mathcal{E}_K) = d+1$. We refer to Definition 2.10 for the construction of $\mathcal{D}$.

(1) The set of degrees of freedom is $I = \mathcal{E}$ and the approximation points are $S = (\overline{\boldsymbol{x}}_\sigma)_{\sigma\in\mathcal{E}}$. Then $I_\Omega = \mathcal{E}_{\rm int}$ and $I_{\partial\Omega} = \mathcal{E}_{\rm ext}$ (boundary vertices), and the partition of Ω is given by $\mathcal{U} = \mathcal{M}$.
(2) For all $K \in \mathcal{M}$, $I_K = \mathcal{E}_K$ is a family of $d+1$ elements and the linearly exact gradient reconstruction in K is defined by the constant value
$$\forall \boldsymbol{x} \in K\,,\ \mathcal{G}_K v(\boldsymbol{x}) = \sum_{\sigma\in\mathcal{E}_K} v_\sigma \nabla\alpha_\sigma(\boldsymbol{x}),$$
where $\alpha_\sigma(\boldsymbol{x})$ is the affine non-conforming finite element basis function associated with σ.
(3) The reconstruction operator $\Pi_\mathcal{D}$ is then obtained by using these basis functions $\alpha_\sigma(\boldsymbol{x})$ in (2.10), which leads to
$$\forall v \in X_{\mathcal{D},0},\ \forall \boldsymbol{x} \in \Omega,\ \Pi_\mathcal{D} v(\boldsymbol{x}) = \sum_{\sigma\in\mathcal{E}} v_\sigma \alpha_\sigma(\boldsymbol{x}).$$

Let us prove that Theorem 3.1 holds for a regular sequence for gradient discretisations defined by the non-conforming $\mathbb{P}_1$ method, defining this regularity merely as the regularity of the underlying polyhedral meshes (Definition 2.25).

Proof of Theorem 3.1 for non-conforming $\mathbb{P}_1$ gradient discretisations. We drop the index m from time to time for sake of legibility, and all constants below do not depend on m or the considered cells/edges. For $v \in X_{\mathcal{D},0}$ we define $\widetilde{v} \in X_{\mathcal{T},0}$ (see notations in Section 2.5) by: for all $\sigma \in \mathcal{E}$, $\widetilde{v}_\sigma = v_\sigma = \Pi_\mathcal{D} v(\overline{\boldsymbol{x}}_\sigma)$, and for all $K \in \mathcal{M}$, $\widetilde{v}_K = \frac{1}{d+1}\sum_{\sigma\in\mathcal{E}_K} v_\sigma = \Pi_\mathcal{D} v(\overline{\boldsymbol{x}}_K)$, where $\overline{\boldsymbol{x}}_K$ is the centre of gravity of K. Let us first prove the following properties (in which C_4 does not depend on v or $\mathcal{T}$):

$$\forall v \in X_{\mathcal{D},0},\ ||\widetilde{v}||_{X_{\mathcal{T},0},p} \leq C_4 \|\nabla_\mathcal{D} v\|_{L^p(\Omega)}, \tag{3.2}$$

$$\forall v \in X_{\mathcal{D},0},\ ||\Pi_\mathcal{T}\widetilde{v} - \Pi_\mathcal{D} v||_{L^p(\Omega)} \leq h_\mathcal{M} \|\nabla_\mathcal{D} v\|_{L^p(\Omega)}, \tag{3.3}$$

$$\forall v \in X_{\mathcal{D},0},\ \nabla_\mathcal{D} v = \nabla_\mathcal{T}\widetilde{v}. \tag{3.4}$$

Let $K \in \mathcal{M}$. Since $\Pi_{\mathcal{D}} v$ is linear in K we have $|\widetilde{v}_\sigma - \widetilde{v}_K| \leq |\overline{\boldsymbol{x}}_\sigma - \overline{\boldsymbol{x}}_K| \, |(\nabla_{\mathcal{D}} v)_K| \leq C_5 d_{K,\sigma} |(\nabla_{\mathcal{D}} v)_K|$ for all $\sigma \in \mathcal{E}_K$, where C_5 does not depend on v and $(\nabla_{\mathcal{D}} v)_K$ is the constant value of $\nabla_{\mathcal{D}} v$ on K. Since $\sum_{\sigma \in \mathcal{E}_K} |\sigma| d_{K,\sigma} = d|K|$, the definition (2.26) of the norm $||\cdot||_{X_{\mathcal{T},0},p}$ gives (3.2) with $C_4 = d^{1/p} C_5$. The proof of (3.3) follows by observing that $|\Pi_{\mathcal{D}} v(\boldsymbol{x}) - \Pi_{\mathcal{T}} \widetilde{v}(\boldsymbol{x})| = |\Pi_{\mathcal{D}} v(\boldsymbol{x}) - \widetilde{v}_K| = |\Pi_{\mathcal{D}} v(\boldsymbol{x}) - \Pi_{\mathcal{D}} v(\overline{\boldsymbol{x}}_K)| \leq h_K |(\nabla_{\mathcal{D}} v)_K|$ for any $x \in K$ and any $K \in \mathcal{M}$. As for (3.4), it follows from the last conclusion in Lemma 2.29, the fact that $\Pi_{\mathcal{D}} v$ is linear in each cell K, and the fact that $\widetilde{v}_\sigma = \Pi_{\mathcal{D}} v(\overline{\boldsymbol{x}}_\sigma)$ for any $\sigma \in \mathcal{E}_K$. We can now prove the theorem.

Coercivity: applying Estimate (3.3), Lemma 2.27, and Estimates (2.27) and (3.2) we have, for all $v \in X_{\mathcal{D},0}$, $||\Pi_{\mathcal{D}} v||_{L^p(\Omega)} \leq ||\Pi_{\mathcal{T}} \widetilde{v}||_{L^p(\Omega)} + \mathrm{diam}(\Omega) ||\nabla_{\mathcal{D}} v||_{L^p(\Omega)^d} \leq (C_4 C_3 2^{p-1}(1+\theta_{\mathcal{T}}) + \mathrm{diam}(\Omega)) ||\nabla_{\mathcal{D}} v||_{L^p(\Omega)^d}$, and the coercivity follows from the upper bound on $\theta_{\mathcal{T}}$.

Consistency: The estimate on $\theta_{\mathcal{T}}$ ensure that $||\alpha_\sigma||_{L^\infty(\Omega)} \leq C_6$ and $||\nabla \alpha_\sigma||_{L^\infty(K)} \leq C_6 \mathrm{diam}(K)^{-1}$. Hence $\mathrm{reg}_{\mathrm{LLE}}(\mathcal{D})$ remains bounded and the consistency is a consequence of Proposition 2.14.

Limit-conformity: we use Lemma 2.29 with $\widetilde{v}$ instead of v, Formula (3.4), and Estimates (3.2) and (3.3) to write

$$\left| \int_\Omega \nabla_{\mathcal{D}} v(\boldsymbol{x}) \cdot \boldsymbol{\varphi}(\boldsymbol{x}) + \Pi_{\mathcal{D}} v(\boldsymbol{x}) \mathrm{div}(\boldsymbol{\varphi})(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} \right| \leq C_7 h_{\mathcal{M}} ||\boldsymbol{\varphi}||_{W^{1,\infty}(\mathbf{R}^d)^d} ||\nabla_{\mathcal{D}} v||_{L^p(\Omega)^d}$$

for any $\boldsymbol{\varphi} \in C^\infty(\mathbf{R}^d)^d$ and any $v \in X_{\mathcal{D},0}$. Hence $W_{\mathcal{D}}(\boldsymbol{\varphi}) \leq C_7 h_{\mathcal{M}} ||\boldsymbol{\varphi}||_{W^{1,\infty}(\mathbf{R}^d)^d} \to 0$ as $h_{\mathcal{M}} \to 0$.

Compactness: if $(\nabla_{\mathcal{D}_m} v_m)_{m \in \mathbf{N}}$ is bounded in $L^p(\Omega)^d$ then (3.2) and (2.27) show that $||\widetilde{v}_m||_{X_{\mathcal{M}_m},p}$ is bounded and therefore, by Lemma 2.28, that $(\Pi_{\mathcal{T}_m} \widetilde{v}_m)_{m \in \mathbf{N}}$ is relatively compact in $L^p(\Omega)$. Combined with (3.3), this shows the relative compactness in $L^p(\Omega)$ of $(\Pi_{\mathcal{D}_m} v_m)_{m \in \mathbf{N}}$, as required. □

3.2.2. *Mass-lumped non-conforming $\mathbb{P}_1$ reconstruction.* We apply to the preceding gradient discretisation the technique of mass lumping as in Definition 2.19. Recalling that the set of degrees of freedom is $I = \mathcal{E}$ here, the subdomains $(\Omega_\sigma)_{\sigma \in \mathcal{E}}$ are the diamonds $(D_\sigma)_{\sigma \in \mathcal{E}}$ around the edges. The right part of Figure 2 illustrates the construction of this partition.

Since $\Pi_{\mathcal{D}} v$ is linear in each cell and $\Pi_{\mathcal{D}^{\mathrm{ML}}} v = \Pi_{\mathcal{D}} v(\overline{\boldsymbol{x}}_\sigma)$ on D_σ, an order 1 Taylor's expansion immediately provides Estimate (2.18), and Theorem 3.1 for the mass-lumped non-conforming $\mathbb{P}_1$ gradient discretisation is a consequence of Theorem 2.21.

3.3. **Mixed Finite Element RT_k schemes.** Let $\mathcal{T}$ be a simplicial discretisation of Ω as for the non-conforming $\mathbb{P}_1$ scheme. We fix $k \in \mathbf{N}$ and introduce the following spaces

$$\boldsymbol{V}_h = \{\boldsymbol{w} \in (L^2(\Omega))^d \,:\, \boldsymbol{w}|_K \in \mathrm{RT}_k(K),\ \forall K \in \mathcal{M}\}, \qquad \boldsymbol{V}_h^{\mathrm{div}} = \boldsymbol{V}_h \cap H_{\mathrm{div}}(\Omega),$$

$$W_h = \{p \in L^2(\Omega) \,:\, p|_K \in \mathcal{P}_k(K),\ \forall K \in \mathcal{M}\}, \qquad M_h^0 = \left\{ \mu \,:\, \bigcup_{\sigma \in \mathcal{E}} \overline{\sigma} \to \mathbf{R}, \mu|_\sigma \in \mathcal{P}_k(\sigma), \mu|_{\partial\Omega} = 0 \right\},$$

where

- $\mathcal{P}_k(K)$ is the space of polynomials of d variables of degree less than or equal to k.
- $\mathcal{P}_k(\sigma)$ is the space of polynomials of $d-1$ variables of degree less than or equal to k.
- $\mathrm{RT}_k(K) = P_k(K)^d + \boldsymbol{x} \overline{P}_k(K)$ is the Raviart-Thomas space of order k defined on K. Here, $\overline{P}_k(K) \subset P_k(K)$ is the set of homogeneous polynomials of degree k.

Here we construct a gradient discretisation inspired by the dual mixed finite element formulation of Problem (2.1), as given in [5], assuming that for a.e. $\boldsymbol{x} \in \Omega$, $A(\boldsymbol{x})$ is a self-adjoint positive definite linear operator with bounded eigenvalues, constant in each $K \in \mathcal{M}$:

(3.5)

$$(\boldsymbol{v}, q, \lambda) \in \boldsymbol{V}_h \times W_h \times M_h^0,$$

$$\int_K \boldsymbol{w}(\boldsymbol{x}) \cdot A(\boldsymbol{x})^{-1} \boldsymbol{v}(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} - \int_K q(\boldsymbol{x}) \mathrm{div} \boldsymbol{w}(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} + \sum_{\sigma \in \mathcal{E}_K} \int_\sigma \lambda(\boldsymbol{x}) \boldsymbol{w}|_K(\boldsymbol{x}) \cdot \mathbf{n}_{K,\sigma} \mathrm{d}s(\boldsymbol{x}) = 0,\ \forall \boldsymbol{w} \in \boldsymbol{V}_h,$$

$$\int_K \psi(\boldsymbol{x}) \mathrm{div} \boldsymbol{v}(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} = \int_K \psi(\boldsymbol{x}) f(\boldsymbol{x}) \mathrm{d}\boldsymbol{x},\ \forall \psi \in W_h, \forall K \in \mathcal{M},$$

$$\int_\sigma \mu(\boldsymbol{x}) \boldsymbol{v}|_K(\boldsymbol{x}) \cdot \mathbf{n}_{K,\sigma} \mathrm{d}s(\boldsymbol{x}) + \int_\sigma \mu(\boldsymbol{x}) \boldsymbol{v}|_L(\boldsymbol{x}) \cdot \mathbf{n}_{L,\sigma} \mathrm{d}s(\boldsymbol{x}) = 0,\ \forall \sigma \in \mathcal{E}_{\mathrm{int}} \text{ with } \mathcal{M}_\sigma = \{K, L\}, \forall \mu \in M_h^0.$$

We again refer to Definition 2.10 for the construction of $\mathcal{D} = (X_{\mathcal{D},0}, \nabla_{\mathcal{D}}, \Pi_{\mathcal{D}})$. We consider $(\psi_i)_{i\in I_W}$ the standard basis of W_h, and $(\xi_j)_{j\in I_M}$ the standard basis of M_h^0. These two standard bases are respectively associated to points $\boldsymbol{x}_i$ located in the cells and points $\boldsymbol{s}_j$ located on the faces of the cells.

(1) The set of degrees of freedom is $I = I_W \cup I_M$, and the approximation points S are the corresponding $(\boldsymbol{x}_i)_{i\in I_W}$ and $(\boldsymbol{s}_j)_{j\in I_M}$. Then $I_\Omega = I_W \cup (I_M)_{\rm int}$ and $I_{\partial\Omega} = (I_M)_{\rm ext}$, where $(I_M)_{\rm int}$ and $(I_M)_{\rm ext}$ respectively correspond to points $\boldsymbol{s}_j$ in Ω and on $\partial\Omega$. The partition of Ω is given by $\mathcal{U} = \mathcal{M}$. We denote by $(I_W)_K$ the set of all $i \in I_W$ such that $\boldsymbol{x}_i \in K$, and by $(I_M)_\sigma$ the set of all $j \in I_M$ such that $\boldsymbol{s}_j \in \sigma$. Then, for all $K \in \mathcal{M} = \mathcal{U}$, $I_K = (I_W)_K \cup \bigcup_{\sigma\in\mathcal{E}_K}(I_M)_\sigma$.

(2) For all $K \in \mathcal{M}$, the linearly exact gradient reconstruction in K is defined by: $\mathcal{G}_K v$ is the function such that $A\mathcal{G}_K v \in \boldsymbol{V}_h$ and

$$\forall \boldsymbol{w} \in \boldsymbol{V}_h\,,\ \int_K \boldsymbol{w}(\boldsymbol{x})\cdot\mathcal{G}_K v(\boldsymbol{x})\mathrm{d}\boldsymbol{x} + \int_K \left(\sum_{i\in(I_W)_K} v_i\psi_i(\boldsymbol{x})\right)\mathrm{div}\boldsymbol{w}(\boldsymbol{x})\mathrm{d}\boldsymbol{x} - \sum_{\sigma\in\mathcal{E}_K}\int_\sigma\left(\sum_{i\in(I_M)_\sigma} v_i\xi_i(\boldsymbol{x})\right)\boldsymbol{w}|_K(\boldsymbol{x})\cdot\mathbf{n}_{K,\sigma}\mathrm{d}\gamma(\boldsymbol{x}) = 0.$$

(3) The reconstruction (2.10) is applied with $\alpha_i(\boldsymbol{x}) = \psi_i(x)$ for all $i \in (I_W)_K$ and $\alpha_i(\boldsymbol{x}) = 0$ for all $i \in \bigcup_{\sigma\in\mathcal{E}_K}(I_M)_\sigma$. This leads to

$$\forall v \in X_{\mathcal{D},0},\ \forall \boldsymbol{x}\in\Omega,\ \Pi_{\mathcal{D}}v(\boldsymbol{x}) = \sum_{i\in I_W} v_i\psi_i(\boldsymbol{x}).$$

The proof of the equivalence between the corresponding gradient scheme (2.3) and the Arnold-Brezzi mixed hybrid formulation (3.5) is found in [28], along with the proof of Theorem 3.1 for a regular sequence of discretisations in the sense of Definition 2.25. Note that, in the case $k = 0$, the property of piecewise constant reconstruction holds.

3.4. **Multi-Point Flux Approximations.** We consider in this section two particular cases of the Multi-Point Flux Approximation schemes [1]. They are based on particular polyhedral meshes of Ω in the sense of Definition 2.23: Cartesian for the first MPFA scheme, and simplicial for the second one. In each of these cases, for $K \in \mathcal{M}$ we select $\boldsymbol{x}_K$ the centre of gravity of K and we define a partition $(V_{K,\mathsf{v}})_{\mathsf{v}\in\mathcal{V}_K}$ of K the following way (see Figure 3):

- *Cartesian meshes*: $V_{K,\mathsf{v}}$ is the parallelepipedic polyhedron whose faces are parallel to the faces of K and that has $\boldsymbol{x}_K$ and v as vertices. We define, for $\sigma \in \mathcal{E}$ and $\mathsf{v} \in \mathcal{V}_\sigma$, $\boldsymbol{x}_{(\sigma,\mathsf{v})} = \overline{\boldsymbol{x}}_\sigma$ (note that these points are identical for all $\mathsf{v} \in \mathcal{V}_\sigma$, see Remark 2.11).
- *Simplicial meshes*: we denote by $(\beta_{\mathsf{v}}^K(\boldsymbol{x}))_{\mathsf{v}\in\mathcal{V}_K}$ the barycentric coordinates of $\boldsymbol{x}$ in K (that is $\boldsymbol{x} - \boldsymbol{x}_K = \sum_{\mathsf{v}\in\mathcal{V}_K}\beta_{\mathsf{v}}^K(\boldsymbol{x})(\mathsf{v}' - \boldsymbol{x}_K)$, $\beta_{\mathsf{v}}^K(\boldsymbol{x}) \geq 0$ and $\sum_{\mathsf{v}'\in\mathcal{V}_K}\beta_{\mathsf{v}'}^K(\boldsymbol{x}) = 1$) and we define $V_{K,\mathsf{v}}$ as the set of $\boldsymbol{x} \in K$ whose barycentric coordinates $(\beta_{\mathsf{v}'}^K(\boldsymbol{x}))_{\mathsf{v}'\in\mathcal{V}_K}$ satisfy $\beta_{\mathsf{v}}^K(\boldsymbol{x}) > \beta_{\mathsf{v}'}^K(\boldsymbol{x})$ for all $\mathsf{v}' \in \mathcal{V}_K \setminus \{\mathsf{v}\}$. For $\sigma \in \mathcal{E}$ and $\mathsf{v} \in \mathcal{V}_\sigma$, $\boldsymbol{x}_{(\sigma,\mathsf{v})}$ is the point of σ whose barycentric coordinates in σ are $\beta_{\mathsf{v}'}^\sigma(\boldsymbol{x}_{(\sigma,\mathsf{v})}) = 1/(d+1)$ for all $\mathsf{v}' \in \mathcal{V}_\sigma \setminus \{\mathsf{v}\}$, and $\beta_{\mathsf{v}}^\sigma(\boldsymbol{x}_{(\sigma,\mathsf{v})}) = 2/(d+1)$.

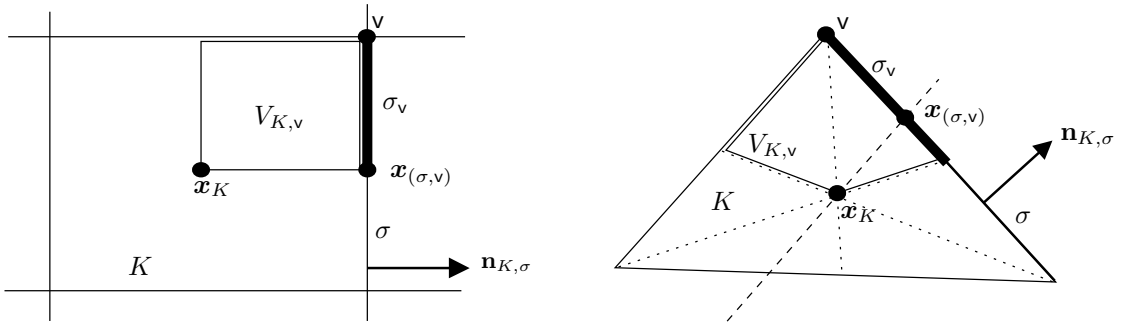

FIGURE 3. Notations for MPFA schemes defined on cartesian (left) and simplicial (right) meshes.

We then follow the notations in Definition 2.10 to construct the MPFA-O gradient discretisations in both cases:

(1) The set of degrees of freedom is $I = \mathcal{M} \cup \{(\sigma, \mathsf{v}) : \sigma \in \mathcal{E},\ \mathsf{v} \in \mathcal{V}_\sigma\}$, the family of approximation points is $S = ((\boldsymbol{x}_K)_{K\in\mathcal{M}}, (\boldsymbol{x}_{(\sigma,\mathsf{v})})_{\sigma\in\mathcal{E},\ \mathsf{v}\in\mathcal{V}_\sigma})$ and the partition is $\mathcal{U} = (V_{K,\mathsf{v}})_{K\in\mathcal{M},\mathsf{v}\in\mathcal{V}_K}$. For any $U = V_{K,\mathsf{v}}$, we set $\mathcal{E}_K^{\mathsf{v}} = \{\sigma \in \mathcal{E}_K \text{ such that } \mathsf{v} \in \mathcal{V}_\sigma\}$ and $I_U = \{K\} \cup \{(\sigma, \mathsf{v}) : \sigma \in \mathcal{E}_K^{\mathsf{v}}\}$.

(2) Setting $\sigma_{\mathsf{v}} = \overline{V_{K,\mathsf{v}}} \cap \sigma$, the gradient reconstruction on $U = V_{K,\mathsf{v}}$ is

$$\forall \boldsymbol{x} \in V_{K,\mathsf{v}},\ \mathcal{G}_{V_{K,\mathsf{v}}} v(\boldsymbol{x}) = \frac{1}{|V_{K,\mathsf{v}}|} \sum_{\sigma\in\mathcal{E}_K^{\mathsf{v}}} |\sigma_{\mathsf{v}}|(v_{(\sigma,\mathsf{v})} - v_K)\mathbf{n}_{K,\sigma}.$$

(3) The functions α_i are defined by $\alpha_i = 1$ for $i = K$ and $\alpha_i = 0$ for $i = (\sigma, \mathsf{v})$, which means that

$$\forall v \in X_{\mathcal{D},0},\ \forall K \in \mathcal{M},\ \forall \boldsymbol{x} \in K,\ \Pi_{\mathcal{D}} v(\boldsymbol{x}) = v_K.$$

In this case, the gradient scheme (2.3) is equivalent to a finite volume scheme. Indeed, by selecting a test function with only non-zero value $v_K = 1$ in the corresponding gradient scheme (2.3), we obtain the flux balance

$$\sum_{\sigma\in\mathcal{E}_K} \sum_{\mathsf{v}\in\mathcal{V}_\sigma} F_{K,\sigma,\mathsf{v}}(u) = \int_K f(\boldsymbol{x})\mathrm{d}\boldsymbol{x}, \tag{3.6}$$

where $F_{K,\sigma,\mathsf{v}}(u) = \int_{\sigma_{\mathsf{v}}} \mathcal{G}_{V_{K,\mathsf{v}}} u(\boldsymbol{x}) \cdot \mathbf{n}_{K,\sigma} \mathrm{d}\gamma(\boldsymbol{x})$. Selecting a test function with only non-zero value $v_{(\sigma,\mathsf{v})} = 1$ in (2.3) leads to the conservativity of the fluxes:

$$F_{K,\sigma,\mathsf{v}}(u) + F_{L,\sigma,\mathsf{v}}(u) = 0 \text{ for all } \sigma \in \mathcal{E}_{\rm int} \text{ with } \mathcal{M}_\sigma = \{K, L\}, \text{ and all } \mathsf{v} \in \mathcal{V}_\sigma. \tag{3.7}$$

In the gradient scheme (2.3) we can locally express the degree of freedom $u_{(\sigma,\mathsf{v})}$ in terms of $(u_K)_{K|\mathsf{v}\in\mathcal{V}_K}$. For a given $\mathsf{v} \in \mathcal{V}$ this is done by solving the local linear system issued from (3.7) written for all σ such that $\mathsf{v} \in \mathcal{V}_\sigma$. After these local eliminations of $u_{(\sigma,\mathsf{v})}$, the resulting linear system only involves the cell unknowns. This property is a foundation of the MPFA O-scheme [1].

It is then proved in [35, 34] that the above definitions for the families S of approximation points gives the LLE property in both cases (cartesian and simplicial). References [35, 34] also prove Theorem 3.1, with the regularity of the MPFA-O gradient discretisations defined as the regularity of $\mathcal{T}$ (see Definition 2.25).

3.5. **Discrete Duality Finite Volumes.** The principle of Discrete Duality Finite Volume (DDFV) schemes [38, 17, 8, 39, 3, 13] is to design discrete divergence and gradient operators that are linked in duality through a discrete Stokes formula. Since discrete operators and an asymptotic Stokes formula are at the core of gradient schemes (see Definitions 2.1 and 2.4), it is not a surprise that they should contain DDFV methods. This was already briefly noticed in [30]; we give here a precise construction and proof of this result.

3.5.1. *2D case.* We consider a polygonal open subset Ω of $\mathbf{R}^2$, and we take $\mathcal{T}$ a polygonal discretisation of Ω in the sense of Definition 2.23. We refer to Figure 4 for some notations; the dual cell $\mathcal{K}_{\mathsf{v}}$ around a vertex v is obtained by joining all cell centres around v, or the cell centres and boundary edge midpoints around v if $\mathsf{v} \in \partial\Omega$. We define the following elements, using the notations in Definition 2.10:

(1) The set of degrees of freedom is $I = \mathcal{M} \cup \mathcal{V}$ and the approximation points are $S = \mathcal{P} \cup \mathcal{V}$. The partition $\mathcal{U}$ is the set of all diamonds $(D_\sigma)_{\sigma\in\mathcal{E}}$.

(2) For $U = D_\sigma$, the linearly exact gradient is constructed by imposing finite difference formulas in the two linearly independent directions $\overrightarrow{\boldsymbol{x}_K\boldsymbol{x}_L}$ and $\overrightarrow{\mathsf{v}\mathsf{v}'}$. We therefore set $I_{D_\sigma} = \{K, L, \mathsf{v}, \mathsf{v}'\}$ and, for $u \in X_{\mathcal{D},0}$ and $\boldsymbol{x} \in D_\sigma$, we define $\mathcal{G}_{D_\sigma} u(\boldsymbol{x})$ as the unique vector in $\mathbf{R}^2$ such that

$$u_K - u_L = \mathcal{G}_{D_\sigma} u(\boldsymbol{x}) \cdot (\boldsymbol{x}_K - \boldsymbol{x}_L) \text{ and } u_{\mathsf{v}} - u_{\mathsf{v}'} = \mathcal{G}_{D_\sigma} u(\boldsymbol{x}) \cdot (\mathsf{v} - \mathsf{v}'). \tag{3.8}$$

(3) In the definition (2.10) of the reconstruction operator we let $\alpha_K(\boldsymbol{x}) = \frac{1}{2}\chi_K(\boldsymbol{x})$, $\alpha_L(\boldsymbol{x}) = \frac{1}{2}\chi_L(\boldsymbol{x})$, $\alpha_{\mathsf{v}}(x) = \frac{1}{2}\chi_{\mathcal{K}_{\mathsf{v}}}(\boldsymbol{x})$, and $\alpha_{\mathsf{v}'}(x) = \frac{1}{2}\chi_{\mathcal{K}_{\mathsf{v}'}}(\boldsymbol{x})$, where for any $V \subset \Omega$ we denote by χ_V the characteristic function of V. We therefore have

$$\forall v \in X_{\mathcal{D},0},\ \forall \boldsymbol{x} \in \Omega,\ \Pi_{\mathcal{D}} v(\boldsymbol{x}) = \frac{1}{2} \sum_{K\in\mathcal{M}} v_K \chi_K(\boldsymbol{x}) + \frac{1}{2} \sum_{\mathsf{v}\in\mathcal{V}} v_{\mathsf{v}} \chi_{\mathcal{K}_{\mathsf{v}}}(\boldsymbol{x}).$$

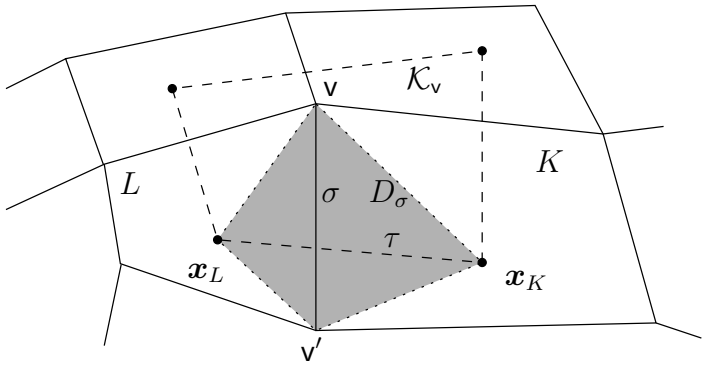

FIGURE 4. DDFV primal meshes (continuous lines: K, L), dual meshes (dashed lines: $\mathcal{K}_\mathsf{v}$) and diamonds (filled: D_σ).

With this gradient discretisation, the gradient scheme (2.3) is the 2D DDFV scheme for the linear diffusion problem (2.1), and Theorem 3.1 holds with the following definition. We skip the proof of this theorem since it is identical to the proof in the 3D case, given below.

Definition 3.2 (Regularity of 2D DDFV gradient discretisations)**.** *A sequence $(\mathcal{D}_m)_{m\in\mathbf{N}}$ of 2D DDFV gradient discretisations is regular if $h_{\mathcal{M}_m} \to 0$ as $m \to \infty$ and if $(\mathrm{reg}_{\mathrm{ddfv}}(\mathcal{D}_m))_{m\in\mathbf{N}}$ remains bounded, where*

$$\mathrm{reg}_{\mathrm{ddfv}}(\mathcal{D}) = \theta_\mathcal{T} + \theta_{\mathcal{T}'} + \max_{\sigma\in\mathcal{E}_{\mathrm{int}}} \frac{|\boldsymbol{x}_K - \boldsymbol{x}_L|}{|\mathsf{v}-\mathsf{v}'|} + \max_{\sigma\in\mathcal{E}_{\mathrm{int}}} \frac{|\mathsf{v}-\mathsf{v}'|}{|\boldsymbol{x}_K - \boldsymbol{x}_L|} + \max_{\sigma\in\mathcal{E}} \frac{1}{\widehat{\sigma\tau}}$$

with $\widehat{\sigma\tau} \in (0,\pi/2)$ the angle between σ and τ, the regularity factor θ defined by (2.22), and $\mathcal{T}'$ is the dual polygonal discretisation of Ω defined by the cells $(\mathcal{K}_\mathsf{v})_{\mathsf{v}\in\mathcal{V}}$.

3.5.2. *3D DDFV and gradient schemes on general octohedral meshes.* Two 3D DDFV versions have been developed: the CeVe-DDFV, which uses cell and vertex unknowns [40, 15, 4], and the CeVeFE-DDFV, which uses cell, vertex, faces and edges unknowns [13, 14]. The coercivity properties of the two methods differ: the CeVe-DDFV does not seem to be unconditionally coercive on generic meshes, whereas the CeVeFE-DDFV scheme is unconditionally coercive [18]. We show here that this latter method is a gradient scheme. To do so, we adopt a different stance than in the 2D case. Instead of starting from the three "main" meshes of the CeVeFE-DDFV method, as we did with the primal and dual mesh in the 2D case, we introduce a gradient discretisation on a general octahedral mesh (possibly including degenerate cells), and we show that when the octahedral cells of this mesh are the "diamond cells" of a CeVeFE-DDFV method, then the gradient scheme corresponding to this gradient discretisation is the CeVeFE-DDFV scheme. This construction therefore yields a new scheme, which can be used on octahedral meshes that are more general than the CeVeFE-DDFV diamond meshes; in particular, these octahedral can be constructed on meshes which are compatible with ground heterogeneities. Incidentally, our presentations also gives a complete description of the CeVeFE-DDFV scheme using only one mesh instead of four.

We use Definition 2.10 to construct an "octahedral" gradient discretisation $\mathcal{D} = (X_{\mathcal{D},0}, \nabla_\mathcal{D}, \Pi_\mathcal{D})$ (see Figure 5 for some notations):

(1) $\mathcal{T} = (\mathcal{M}, \mathcal{E}, \mathcal{P}, \mathcal{V})$ a polyhedral mesh of Ω in the sense of Definition 2.23, such that the elements of $\mathcal{M}$ are octahedra (open polyhedron with eight triangular faces and six vertices, not necessarily convex; five vertices may be coplanar), and the element of $\mathcal{E}$ are the triangular faces of the elements of $\mathcal{M}$. Each $\mathcal{E}_K$ has 8 elements, each $\mathcal{V}_K$ has 6 elements, and each $\mathcal{V}_\sigma$ has 3 elements.
(2) The set of degrees of freedom and the approximation points are $I = S = \mathcal{V}$. The partition is $\mathcal{U} = \mathcal{M}$. For $U = K \in \mathcal{M}$, we set $I_K = \mathcal{V}_K$ and, if $u \in X_{\mathcal{D},0}$, the cell gradient is defined by

$$\forall \boldsymbol{x} \in K,\ \mathcal{G}_K u(\boldsymbol{x}) = \frac{1}{|K|} \sum_{\sigma\in\mathcal{E}_K} |\sigma| u_\sigma \mathbf{n}_{K,\sigma}\,, \quad \text{where } u_\sigma = \frac{1}{3}\sum_{\mathsf{v}\in\mathcal{V}_\sigma} u_\mathsf{v} \text{ for all } \sigma \in \mathcal{E}_K. \tag{3.9}$$

(3) For $K \in \mathcal{M}$ and $\mathsf{v} \in \mathcal{V}_K$, we denote by $V_{K,s}$ the octahedron formed by $\boldsymbol{x}_K$, v, and the four other vertices of K that share a face of K with v. We then define the functions $(\alpha_i)_{i\in I_K}$ by

$$\forall \boldsymbol{x} \in K\,,\ \forall \mathsf{v} \in \mathcal{V}_K\,,\ \alpha_{\mathsf{v}}(\boldsymbol{x}) = \frac{1}{3}\chi_{V_{K,\mathsf{v}}}(\boldsymbol{x}), \tag{3.10}$$

where $\chi_{V_{K,\mathsf{v}}}$ denotes the characteristic function of $V_{K,\mathsf{v}}$. This leads to

$$\forall u \in X_{\mathcal{D},0}\,,\ \forall K \in \mathcal{M}\,, \forall \boldsymbol{x} \in K\,,\ \Pi_{\mathcal{D}} u(\boldsymbol{x}) = \frac{1}{3}\sum_{\mathsf{v}\in\mathcal{V}_K} u_{\mathsf{v}}\chi_{V_{K,\mathsf{v}}}(\boldsymbol{x}).$$

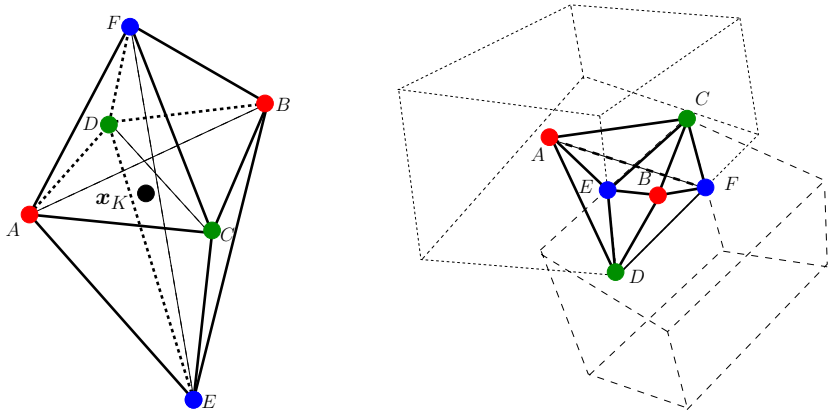

FIGURE 5. Left: octahedral cell K for the CeVeFE-DDFV scheme. Right: construction of a degenerate octahedron from a non-conforming hexahedral mesh in an heterogeneous medium.

Remark 3.3 (Octohedra and heterogeneous media)**.** *A discretisation of Ω as in Item (1) is easy to obtain in the case where the domain Ω is the disjoint union of star-shaped hexahedra. It suffices to consider the octahedra obtained from the centres of neighbourhing hexahedral cells, and the four vertices of their interface. This also works for non-conforming hexahedral meshes, for which interfaces between cells may be different from the physical faces of the cells.*

In the case of an heterogeneous media, in which the material properties (e.g. the permeability A in (2.1)*) are constant inside each hexahedral cell but may be discontinuous from one cell to the other, it is usually preferable to construct octahedral cells that are compatible with these heterogeneities (i.e. such that the material properties are constant inside each octahedron). This prevents from introducing a non-physical average of A in the gradient scheme* (2.3)*, which would lead to a loss of accuracy of the approximate solutions. Such an octahedral discretisation can be constructed fairly easily as illustrated in Figure 5 (right). Each of these octahedra is built from the centre of an hexahedral cell, the four vertices of the interface between this cell and a neighbouring hexahedral cell, and a point selected on this interface. This interface need not be planar.*

Remark 3.4. *The choice* (3.10) *of $\Pi_{\mathcal{D}}$ is driven by our desire to construct a gradient discretisation whose gradient scheme is exactly the CeVeFE-DDFV method, for particular octahedral meshes; this choice ensures that the discrete duality formula holds true. We could as well take other choices for $\Pi_{\mathcal{D}}$, e.g. the piecewise constant reconstruction such that $\Pi_{\mathcal{D}} u = \frac{1}{6}\sum_{\mathsf{v}\in\mathcal{V}_K} u_{\mathsf{v}}$ in K, and the corresponding gradient discretisation would still satisfy all required properties.*

The following lemma is useful to prove that this gradient discretisation gives back the CeVeFE-DDFV method, and to establish Theorem 3.1.

Lemma 3.5. *Let $\mathcal{D}$ be the octahedral gradient discretisation defined as above. For any $u \in X_{\mathcal{D},0}$ and any $K \in \mathcal{M}$, the constant discrete gradient $\mathcal{G}_K u$ is characterised by*

$$\text{For all opposite vertices } (\mathsf{v},\mathsf{v}') \text{ of } K,\ \mathcal{G}_K u \cdot (\mathsf{v}-\mathsf{v}') = u_{\mathsf{v}} - u_{\mathsf{v}'}. \tag{3.11}$$

Remark 3.6. *The opposite vertices in the octehedra in Figure 5 are (A,B), (C,D) and (E,F).*

Proof. We first note that since the three directions defined by the three pairs of opposite vertices in K are linearly independent, (3.11) indeed characterises one and only one vector $\mathcal{G}_K u \in \mathbf{R}^3$. We therefore just have to show that the gradient defined by (3.9) satisfies (3.11). We have

$$\mathcal{G}_K u = \frac{1}{|K|}\frac{1}{3}\sum_{\mathsf{v}\in\mathcal{V}_K} u_{\mathsf{v}} \sum_{\sigma\in\mathcal{E}_K|\mathsf{v}\in\mathcal{V}_\sigma} |\sigma|\mathbf{n}_{K,\sigma}. \tag{3.12}$$

Let us consider for example the case where $\mathsf{v} = A$ in Figure 5. For a triangular face σ we can write $|\sigma|\mathbf{n}_{K,\sigma}$ as the exterior product of two of the edges of σ (with proper orientation). This gives

$$\begin{aligned}\sum_{\sigma\in\mathcal{E}_K|\mathsf{v}\in\mathcal{V}_\sigma} |\sigma|\mathbf{n}_{K,\sigma} &= \frac{1}{2}(\overrightarrow{AC}\times\overrightarrow{AF} + \overrightarrow{AF}\times\overrightarrow{AD} + \overrightarrow{AD}\times\overrightarrow{AE} + \overrightarrow{AE}\times\overrightarrow{AC})\\ &= \frac{1}{2}(\overrightarrow{DC}\times\overrightarrow{AF} + \overrightarrow{CD}\times\overrightarrow{AE}) = -\frac{1}{2}\overrightarrow{CD}\times\overrightarrow{EF}.\end{aligned}$$

Applying this to all vertices of K, and since $|K| = \frac{1}{6}\Delta_K$ with $\Delta_K = \det(\overrightarrow{AB},\overrightarrow{CD},\overrightarrow{EF})$, we deduce from (3.12) that

$$\mathcal{G}_K u = \frac{1}{\Delta_K}\left((u_B-u_A)\overrightarrow{CD}\times\overrightarrow{EF} + (u_D-u_C)\overrightarrow{EF}\times\overrightarrow{AB} + (u_F-u_E)\overrightarrow{AB}\times\overrightarrow{CD}\right).$$

Property (3.11) is then straightforward. Considering for example the case $(\mathsf{v},\mathsf{v}') = (B,A)$, the formula follows from $(\overrightarrow{EF}\times\overrightarrow{AB})\cdot\overrightarrow{AB} = (\overrightarrow{AB}\times\overrightarrow{CD})\cdot\overrightarrow{AB} = 0$ and $(\overrightarrow{CD}\times\overrightarrow{EF})\cdot\overrightarrow{AB} = \det(\overrightarrow{CD},\overrightarrow{EF},\overrightarrow{AB}) = \Delta_K$. □

It is now easy to detail the relationship between the octahedral gradient discretisation $\mathcal{D}$ and the CeVeFE-DDFV scheme.

Lemma 3.7 (CeVeFE-DDFV is a gradient scheme)**.** *For any polyhedral discretisation $\widetilde{\mathcal{T}}$ of Ω, there exists an octahedral discretisation $\mathcal{T}$ of Ω such that, if $\mathcal{D}$ is the octahedral gradient discretisation defined as above from $\mathcal{T}$, then the gradient scheme* (2.3) *for $\mathcal{D}$ is the CeVeFE-DDFV method on $\widetilde{\mathcal{T}}$.*

Proof. The CeVeFE-DDFV method on $\widetilde{\mathcal{T}}$ has cell, edge and face unknowns, and its discrete gradient is piecewise constant on so-called "diamond cells". A diamond cell is an octahedra as in Figure 5 (left), but with E chosen on the segment $[A,B]$ – hence, the octahedra actually degenerates into an hexahedra. The segment $[A,B]$ corresponds to an edge of the primal mesh of the CeVeFE-DDFV method, F is a point on a face of this mesh that contains $[A,B]$, and D and C are points inside the cells on each side of this face. Let us take $\mathcal{T}$ the polyhedral discretisation of Ω made of the diamond cells (degenerate octahedra). It is proved in [13, Lemma 3.1] that the CeVeFE-DDFV discrete gradient satisfies (3.11); hence, this gradient is $\nabla_{\mathcal{D}}$. It is then just a matter of applying the discrete duality formula [13, Theorem 4.1] on the formulation [13, Eq. (5.4)] of the scheme to see that the CeVeFE-DDFV scheme for (2.1) is indeed the gradient scheme (2.3) for $\mathcal{D}$. □

Remark 3.8 (CeVeFE-DDFV and heterogeneities)**.** *Except on the boundary of the domain, the diamond cells of the CeVeFE-DDFV method are always spread on two neighbouring cells. If the primal mesh is aligned with heterogeneities of the medium, then these heterogeneities are actually averaged in the formulation of the CeVeFE-DDFV scheme (see* [13, Eq. (5.2)]*). The octahedral gradient discretisation defined here can manage octahedral cells that are aligned with the heterogeneities of the medium, even starting from a primal hexahedral mesh also aligned with the ground properties (see Remark 3.3). This seems to ensure a better accuracy in case of strong heterogeneities, but at the expense of more degrees of freedom around the heterogeneities (two octahedra need to be constructed, where only one diamond is constructed in the CeVeFE-DDFV method). It is also not clear if, on generic octahedral meshes, the octahedral gradient discretisation satisfies the CeVeFE-DDFV discrete duality formula.*

Let us now turn to the analysis of the properties of the octahedral gradient discretisation. This analysis is done by introducing three polyhedral discretisations $(\mathcal{T}^i)_{i=1,2,3}$, each of these associated with pairs of opposite vertices in each cell $K\in\mathcal{M}$. Consider two opposite vertices in an octahedron, say A and B in Figure 5; none of the other octahedra that contain A also contain B. This allows to partition the set of vertices $\mathcal{V}$ into three subsets $(\mathcal{V}^i)_{i=1,2,3}$, such that if v,v' are different vertices in $\mathcal{V}^i$ then either v,v' do not belong to the same octahedra in $\mathcal{M}$, or they are opposite vertices in the octahedron to which they

both belong. The centres of $\mathcal{T}^i$ are $\mathcal{V}^i$, and the cells are $(P_\mathsf{v})_{\mathsf{v}\in\mathcal{V}^i}$ defined the following way: if, say, $\mathsf{v}=A$ in Figure 5 then the intersection between P_v and K is the poyhedron built from A, E, C, $\boldsymbol{x}_K$, F, and D. The faces of P_v contained in K are therefore the four triangles $\boldsymbol{x}_KDF$, $\boldsymbol{x}_KFC$, $\boldsymbol{x}_KCE$, and $\boldsymbol{x}_KED$. In particular, the interface between two neighbourhing cells in $\mathcal{T}^i$ is not planar, but made of four triangles.

Definition 3.9 (Regularity of octahedral gradient discretisations)**.** *A sequence of octahedral gradient discretisations $(\mathcal{D}_m)_{m\in\mathbf{N}}$ as above is regular if the sequences of polyhedral discretisations $(\mathcal{T}_m)_{m\in\mathbf{N}}$, $(\mathcal{T}_m^1)_{m\in\mathbf{N}}$, $(\mathcal{T}_m^2)_{m\in\mathbf{N}}$ and $(\mathcal{T}_m^3)_{m\in\mathbf{N}}$ are regular in the sense of Definition 2.25.*

Under this definition of regularity, Theorem 3.1 holds for octahedral gradient discretisations.

Proof of Theorem 3.1 for octahedral gradient discretisations. As usual, we sometimes drop the indices m for sake of legibility, and we write C for a generic positive constant that does not depend on m, the considered cells/faces/vertices or the vector in $X_{\mathcal{D},0}$. We start with two preliminary considerations. From the bound on $\mathrm{reg}_{\mathrm{oct}}(\mathcal{T})$ we deduce that for any i, any two vertices $\mathsf{v},\mathsf{v}'\in\mathcal{V}_i$, and any τ face between P_v and $P_{\mathsf{v}'}$, we have $|\mathsf{v}-\mathsf{v}'|\le Cd_\tau$. Hence, since v,v' are opposite in an octahedral cell of $\mathcal{T}$, (3.11) and (2.25) show that

$$||u_{\mathcal{M}^i}||_{X_{\mathcal{M}^i},p}\le C||\nabla_{\mathcal{D}}u||_{L^p(\Omega)^d}, \tag{3.13}$$

where any vector $u\in X_{\mathcal{D},0}$ is partitioned into the three vectors $u_{\mathcal{M}^i}=(u_\mathsf{v})_{\mathsf{v}\in\mathcal{V}^i}\in X_{\mathcal{M}^i}$, each one corresponding to the vertices associated with $\mathcal{T}^i$. Here, we used the fact that, for opposite vertices v, v' in some octahedra $K\in\mathcal{M}$, $\sum_{\tau\text{ face of }P_\mathsf{v}\text{ and }P_{\mathsf{v}'}}|\tau|d_\tau=3|K|$.

The second preliminary we need is the relation (in which $\Pi_{\mathcal{T}^i}$ is defined by (2.28))

$$\Pi_{\mathcal{D}}u=\frac{1}{3}\sum_{i=1,2,3}\Pi_{\mathcal{T}^i}u_{\mathcal{M}^i}. \tag{3.14}$$

This equation is obtained by considering each subcell of $K\in\mathcal{M}$ obtained by joining a face of K and the centre $\boldsymbol{x}_K$. Consider for example the subcell created by $S=(\boldsymbol{x}_K,A,D,F)$ in Figure 5. The only sets $V_{K,\mathsf{v}}$ that intersect S correspond to $\mathsf{v}=A,D,F$ and therefore $\Pi_{\mathcal{D}}u=\frac{1}{3}(u_A+u_D+u_F)$ on S. But no pair of A,D,F is made of opposite vertices in K, so all these vertices belong to separate $(\mathcal{V}^i)_{i=1,2,3}$ – say $A\in\mathcal{V}^1$, $D\in\mathcal{V}^2$ and $F\in\mathcal{V}^3$. On S we thus have $\Pi_{\mathcal{T}^1}u_{\mathcal{M}^1}=u_A$, $\Pi_{\mathcal{T}^2}u_{\mathcal{M}^2}=u_D$ and $\Pi_{\mathcal{T}^3}u_{\mathcal{M}^3}=u_F$. This shows that (3.14) holds on S, and thus everywhere since the subcell S and the cell $K\in\mathcal{M}$ were arbitrary.

Coercivity: by Lemma 2.27 and (3.13) we have $||\Pi_{\mathcal{T}^i}u_{\mathcal{T}^i}||_{L^p(\Omega)}\le C||\nabla_{\mathcal{D}}u||_{L^p(\Omega)^d}$, and (3.14) thus shows hat $||\Pi_{\mathcal{D}}u||_{L^p(\Omega)}\le C||\nabla_{\mathcal{D}}u||_{L^p(\Omega)^d}$.

Compactness: if $(||\nabla_{\mathcal{D}_m}u^m||_{L^p(\Omega)^d})_{m\in\mathbf{N}}$ is bounded then (3.13) and Lemma 2.28 show that, for any $i=1,2,3$, $(\Pi_{\mathcal{T}_m^i}u^m\mathcal{T}_m^i)_{m\in\mathbf{N}}$ is relatively compact in $L^p(\Omega)$. The relative compactness of $(\Pi_{\mathcal{D}_m}u^m)_{m\in\mathbf{N}}$ in $L^p(\Omega)$ follows from Equation (3.14).

Consistency: the characterisation (3.11) of the local gradients show that $\mathcal{D}$ has is an LLE gradient discretization, and the regularity factor $\mathrm{reg}_{\mathrm{LLE}}(\mathcal{D})$ is bounded thanks to the bound on $\mathrm{reg}_{\mathrm{oct}}(\mathcal{T})$. Indeed, this bounds forces the three vectors $(\mathsf{v}-\mathsf{v}')_{(\mathsf{v},\mathsf{v}')\text{ opposite in }K}$ to be of similar length and "well non-coplanar" – that is to say with a determinant of order the cube of their similar length – which gives an estimate on $||\mathcal{G}_K||_\infty$.

Limit-conformity: comparing the definition (3.9) of the discrete gradient with the definition (2.29), it seems natural to invoke Lemma 2.29 in order to prove the limit-conformity. We first need to establish some estimates. For $u\in X_{\mathcal{D},0}$ we define $\widetilde{u}=((\widetilde{u}_K)_{K\in\mathcal{M}},(\widetilde{u}_\sigma)_{\sigma\in\mathcal{E}})\in X_{\mathcal{T},0}$ (see (2.24)) by $\widetilde{u}_K=\frac{1}{6}\sum_{\mathsf{v}\in\mathcal{V}_K}u_\mathsf{v}$ and $\widetilde{u}_\sigma=u_\sigma=\frac{1}{3}\sum_{\mathsf{v}\in\mathcal{V}_\sigma}u_\mathsf{v}$. Note that if $\sigma\in\mathcal{E}_{\mathrm{ext}}$ then all vertices of σ are on $\partial\Omega$ and therefore, as expected, $\widetilde{u}_\sigma=0$. Then

$$\nabla_{\mathcal{D}}u=\nabla_{\mathcal{T}}\widetilde{u}. \tag{3.15}$$

We now estimate $||\widetilde{u}||_{X_{\mathcal{T},0},p}$ in terms of $\nabla_{\mathcal{D}}u$. For any $K\in\mathcal{M}$ and any $\sigma\in\mathcal{E}_K$,

$$\widetilde{u}_\sigma-\widetilde{u}_K=\frac{1}{3}\sum_{\mathsf{v}\in\mathcal{V}_\sigma}u_\mathsf{v}-\frac{1}{6}\sum_{\mathsf{v}'\in\mathcal{V}_K}u_{\mathsf{v}'}=\frac{1}{6}\sum_{\mathsf{v}\in\mathcal{V}_\sigma}u_\mathsf{v}-\frac{1}{6}\sum_{\mathsf{v}'\in\mathcal{V}_K\setminus\mathcal{V}_\sigma}u_{\mathsf{v}'}.$$

The vertices of $\mathcal{V}_\sigma$ can be grouped with those of $\mathcal{V}_K\backslash\mathcal{V}_\sigma$ in pairs of opposite vertices in K. For two such opposite vertices v,v' we use (3.11) and $|\mathsf{v}-\mathsf{v}'|\le h_K\le\theta_{\mathcal{T}}d_{K,\sigma}$ and we arrive at

$$|\widetilde{u}_\sigma-\widetilde{u}_K|=\Big|\frac{1}{6}\sum_{(\mathsf{v},\mathsf{v}')\text{ opposite in }K}(u_{\mathsf{v}}-u_{\mathsf{v}'})\Big|\le\frac{\theta_{\mathcal{T}}}{6}|\mathcal{G}_Ku|d_{K,\sigma}. \tag{3.16}$$

Hence,

$$||\widetilde{u}||_{X_{\mathcal{T},0},p}\le C||\nabla_{\mathcal{D}}u||_{L^p(\Omega)^d}. \tag{3.17}$$

Let us compare $\Pi_{\mathcal{D}}u$ and $\Pi_{\mathcal{T}}\widetilde{u}$, using the subcells considered to prove (3.14). We saw that in the subcell $S=(\boldsymbol{x}_K,A,D,F)$ we have $\Pi_{\mathcal{D}}u=\frac{1}{3}(u_A+u_D+u_F)=\widetilde{u}_\sigma$, if σ is the triangle (A,D,F). Hence, by using (3.16) we have $|\Pi_{\mathcal{D}}u-\Pi_{\mathcal{T}}\widetilde{u}|=|\widetilde{u}_\sigma-\widetilde{u}_K|\le Ch_{\mathcal{M}}|\mathcal{G}_Ku|$ in S. Since this is valid for any subcell S of any mesh K, this leads to

$$||\Pi_{\mathcal{D}}u-\Pi_{\mathcal{T}}\widetilde{u}||_{L^p(\Omega)}\le Ch_{\mathcal{M}}||\nabla_{\mathcal{D}}u||_{L^p(\Omega)^d}. \tag{3.18}$$

The limit-conformity can now be proved. By Remark 2.7 we only need to prove that, as $h_{\mathcal{M}}\to0$, we have $W_{\mathcal{D}}(\boldsymbol{\varphi})\to0$ for all $\boldsymbol{\varphi}\in C_c^\infty(\mathbf{R}^d)^d$. Let $u\in X_{\mathcal{D},0}$. By applying (2.30) to $v=\widetilde{u}$ and using (3.15), (3.17) and (3.18) we find

$$\left|\int_\Omega[\nabla_{\mathcal{D}}u(\boldsymbol{x})\cdot\boldsymbol{\varphi}(\boldsymbol{x})+\Pi_{\mathcal{D}}u(\boldsymbol{x})\mathrm{div}\boldsymbol{\varphi}(\boldsymbol{x})]\,\mathrm{d}\boldsymbol{x}\right|\le C||\boldsymbol{\varphi}||_{W^{1,\infty}(\mathbf{R}^d)^d}||\nabla_{\mathcal{D}}u||_{L^p(\Omega)^d}h_{\mathcal{M}}.$$

Hence, $W_{\mathcal{D}}(\boldsymbol{\varphi})\le C||\boldsymbol{\varphi}||_{W^{1,\infty}(\mathbf{R}^d)^d}h_{\mathcal{M}}\to0$ as $h_{\mathcal{M}}\to0$ and the proof is complete. □

3.6. **Hybrid Mixed Mimetic schemes.**

3.6.1. *Fully hybrid scheme.* Since the 50's, several schemes have been developed with the objective to satisfy some form of calculus formula at the discrete level. These schemes are called Mimetic Finite Difference (MFD) or Compatible Discrete Operator (CDO) schemes. Contrary to DDFV methods that design a discrete operators and duality products to satisfy fully discrete calculus formula, MFD/CDO methods design discrete operators that satisfy a Stokes formula that involves both continuous and discrete functions. Depending on the choice of the location of the main degrees of freedom (faces or vertices), two different MFD/CDO families exist. We refer to [41] for a review on MFD methods, and to [7, 6] (and reference therein) for CDO methods.

A first MFD method can be designed by using the fluxes through the mesh faces as initial unknowns. This requires to recast (2.1) in a mixed form, i.e. to write $\overline{q}=A\nabla\overline{u}$ and $\mathrm{div}(\overline{q})=f$, and to discretise this set of two equations. The resulting scheme takes a form that is apparently far from the gradient scheme (2.3). It was however proved in [23] that this form of MFD can be actually embedded in a slightly larger family that also contains Hybrid Finite Volume (HFV) methods [27] and Mixed Finite Volume (MFV) methods [19]. This family has been called Hybrid Mimetic Mixed (HMM) schemes; each scheme in this family can be written in three different ways, depending on the considered approach (MFD, HFV or MFV). The HFV formulation of an HMM scheme is very close to the weak formulation (2.2) of the elliptic PDE; it actually consists in writing this weak formulation with a discrete gradient and a stabilisation term (bilinear form on (u,v)). It was proved in [24] that the discrete gradient can be modified to include the stabilisation terms, and thus that all HMM methods – which means all (face-based) MFD methods also – are actually gradient schemes.

The discrete elements that define an HMM gradient discretisation are the following. We again refer to Definition 2.10 for the construction of $\mathcal{D}$.

(1) Let $\mathcal{T}$ be a polyhedral mesh of Ω as in Definition 2.23. The degrees of freedom are $I=\mathcal{M}\cup\mathcal{E}$, the approximation points are $S=((\boldsymbol{x}_K)_{K\in\mathcal{M}},(\overline{\boldsymbol{x}}_\sigma)_{\sigma\in\mathcal{E}})$ and the partition is $\mathcal{U}=\{D_{K,\sigma}:K\in\mathcal{M},\sigma\in\mathcal{E}_K\}$.

(2) If $U=D_{K,\sigma}$ we set $I_U=\{K\}\cup\mathcal{E}_K$ and we start the construction of the discrete gradient by defining

$$\forall v\in X_{\mathcal{D},0}\,,\ \forall K\in\mathcal{M}\,,\ \nabla_Kv=\frac{1}{|K|}\sum_{\sigma\in\mathcal{E}_K}|\sigma|(v_\sigma-v_K)\mathbf{n}_{K,\sigma}=\frac{1}{|K|}\sum_{\sigma\in\mathcal{E}_K}|\sigma|v_\sigma\mathbf{n}_{K,\sigma}$$

(the second equality is due to $\sum_{\sigma\in\mathcal{E}_K}|\sigma|\mathbf{n}_{K,\sigma}=0$). This gradient is linearly exact, but it is not coercive: it can vanish everywhere even for $v\neq 0$. We therefore add a stabilisation that is constant in each half-diamond, and that vanishes on interpolation of linear functions.

$$\forall v\in X_{\mathcal{D},0}\,,\ \forall D_{K,\sigma}\in\mathcal{U}\,,\ \forall \boldsymbol{x}\in D_{K,\sigma}\,,\ \mathcal{G}_{D_{K,\sigma}}v(\boldsymbol{x})=\nabla_K v+\frac{\sqrt{d}}{d_{K,\sigma}}[\mathcal{L}_K R_K(V_K(v))]_\sigma \mathbf{n}_{K,\sigma}, \tag{3.19}$$

where

- $V_K(v)=(v_\sigma-v_K)_{\sigma\in\mathcal{E}_K}$,
- $R_K:\mathbf{R}^{\mathcal{E}_K}\mapsto\mathbf{R}^{\mathcal{E}_K}$ is the linear mapping defined by $R_K(\xi)=(R_{K,\sigma}(\xi))_{\sigma\in\mathcal{E}_K}$ with
$$R_{K,\sigma}(\xi)=\xi_\sigma-\left(\frac{1}{|K|}\sum_{\sigma'\in\mathcal{E}_K}|\sigma'|\xi_{\sigma'}\mathbf{n}_{K,\sigma'}\right)\cdot(\overline{\boldsymbol{x}}_\sigma-\boldsymbol{x}_K),$$
- $\mathcal{L}_K$ is an isomorphism of the vector space $\mathrm{Im}(R_K)\subset\mathbf{R}^{\mathcal{E}_K}$.

(3) For $U=D_{K,\sigma}$ we let $\alpha_K\equiv 1$ and $\alpha_{\sigma'}\equiv 0$, which means that
$$\forall v\in X_{\mathcal{D},0},\ \forall K\in\mathcal{M},\ \forall\boldsymbol{x}\in K,\ \Pi_{\mathcal{D}}v(\boldsymbol{x})=v_K.$$

Remark 3.10. *The face degree of freedom v_σ corresponds to the hybridisation of the face-based MFD methods.*

The freedom of choice of the isomorphisms $(\mathcal{L}_K)_{K\in\mathcal{M}}$ ensures that all MFD, HFV and MFV schemes are covered by the framework (there are several such schemes, due to different possible stabilisation parameters). More precisely, [24] proves that for any HMM scheme $\mathcal{S}$ on $\mathcal{T}$ there exists a family of isomorphism $(\mathcal{L}_K)_{K\in\mathcal{M}}$ such that, if $\mathcal{D}$ is defined as above, then the gradient scheme (2.3) is $\mathcal{S}$. It is also proved in [24] that Theorem 3.1 holds with the following definition of regularity, which is both used for bounding $\mathrm{reg}_{\mathrm{LLE}}(\mathcal{D})$ and for allowing the use of the Discrete Functional Analysis results.

Definition 3.11 (Regularity of HMM gradient discretisations)**.** *For $\mathcal{D}$ an HMM gradient discretisation as above, we define $\zeta_{\mathcal{D}}$ as the smallest number such that, for all $K\in\mathcal{M}$ and all $\xi\in\mathbf{R}^{\mathcal{E}_K}$,*

$$\zeta_{\mathcal{D}}^{-1}\sum_{\sigma\in\mathcal{E}_K}|D_{K,\sigma}|\left|\frac{R_{K,\sigma}(\xi)}{d_{K,\sigma}}\right|^p\le\sum_{\sigma\in\mathcal{E}_K}|D_{K,\sigma}|\left|\frac{[\mathcal{L}_K R_K(\xi)]_\sigma}{d_{K,\sigma}}\right|^p\le\zeta_{\mathcal{D}}\sum_{\sigma\in\mathcal{E}_K}|D_{K,\sigma}|\left|\frac{R_{K,\sigma}(\xi)}{d_{K,\sigma}}\right|^p.$$

A sequence $(\mathcal{D}_m)_{m\in\mathbf{N}}$ of HMM gradient discretisations is regular if the sequence $(\mathcal{T}_m)_{m\in\mathbf{N}}$ is regular in the sense of Definition 2.25 and if $(\zeta_{\mathcal{D}_m})_{m\in\mathbf{N}}$ is bounded.

3.6.2. *The SUSHI scheme.* The SUSHI scheme [27] is obtained from the HFV by eliminating some of the face unknowns. With the notions of Section 2, this simply means that the SUSHI method is the gradient scheme of a barycentric condensation of the HMM gradient discretisation. For simplicity, we only consider here the case when all face unknowns are eliminated (the "SUCCES" version in [27]), although more accurate methods could be used in the case of coarse meshes in heterogeneous domains.

(1) Let $\mathcal{T}$ be a polyhedral discretisation of Ω in the sense of Definition 2.23. We first define the gradient discretisation $\overline{\mathcal{D}}=(X_{\overline{\mathcal{D}},0},\nabla_{\overline{\mathcal{D}}},\Pi_{\overline{\mathcal{D}}})$ by the HMM method, as introduced in the above section, for which $I=\mathcal{M}\cup\mathcal{E}$.
(2) We introduce $\widetilde{I}=\mathcal{M}\cup\mathcal{E}_{\mathrm{ext}}$, and for all $\sigma\in\mathcal{E}_{\mathrm{int}}$ we select $H_\sigma\subset\widetilde{I}$ and introduce barycentric coefficients β_i^σ such that
$$\sum_{i\in H_\sigma}\beta_i^\sigma=1\quad\text{and}\quad\overline{\boldsymbol{x}}_\sigma=\sum_{i\in H_\sigma}\beta_i^\sigma\boldsymbol{x}_i,$$
which corresponds to (2.13).
(3) $\mathcal{D}$ is the corresponding barycentric condensation in the sense of Definition 2.15 of $\overline{\mathcal{D}}$.
(4) We have $\Pi_{\mathcal{D}}=\Pi_{\overline{\mathcal{D}}}$ since this reconstruction is only built from the values at the centre of the cells.

Definition 3.12 (Regularity of SUSHI gradient discretisations)**.** *A sequence $(\mathcal{D}_m)_{m\in\mathbf{N}}$ of SUSHI gradient discretisations is regular if the sequence $(\mathcal{T}_m)_{m\in\mathbf{N}}$ is regular in the sense of Definition 2.25 and if $(\mathrm{reg}_{\mathrm{BC}}(\mathcal{D}_m)+\zeta_{\overline{\mathcal{D}}_m})_{m\in\mathbf{N}}$ is bounded, where $\mathrm{reg}_{\mathrm{BC}}$ is defined in Definition 2.15 and $\zeta_{\mathcal{D}}$ is defined in Definition 3.11 for the HMM gradient discretisations $(\overline{\mathcal{D}}_m)_{m\in\mathbf{N}}$.*

Theorem 3.1 for this barycentric condensation of the HMM method is a straightforward consequence of Theorem 3.1 for the HMM method and of Theorem 2.18.

3.7. **Nodal Mimetic Finite Difference methods.** The nodal MFD method (nMFD) is described in [10]. We present here a gradient discretisation such that the nMFD method for (2.1) is the gradient scheme (2.3). We use again Definition 2.10 to construct this gradient discretisation $\mathcal{D} = (X_{\mathcal{D},0}, \nabla_{\mathcal{D}}, \Pi_{\mathcal{D}})$.

(1) Let $\mathcal{T}$ be a polyhedral discretisation of Ω in the sense of Definition 2.23. The degrees of freedom and approximation points are defined by $I = S = \mathcal{V}$. For each $K \in \mathcal{M}$ we choose non-negative weights $(\omega_K^{\mathsf{v}})_{\mathsf{v}\in\mathcal{V}_K}$ such that the quadrature

$$\int_K w(\boldsymbol{x})\mathrm{d}\boldsymbol{x} \approx \sum_{\mathsf{v}\in\mathcal{V}_K} \omega_K^{\mathsf{v}} w(\mathsf{v}) \tag{3.20}$$

is exact for constant functions w, which means that $\sum_{\mathsf{v}\in\mathcal{V}_K} \omega_K^{\mathsf{v}} = |K|$. For each face $\sigma \in \mathcal{E}_K \cap \mathcal{E}_{\rm int}$, we also choose non-negative weights $(\omega_\sigma^{\mathsf{v}})_{\mathsf{v}\in\mathcal{V}_\sigma}$ such that the quadrature

$$\int_\sigma w(\boldsymbol{x})\mathrm{d}s(\boldsymbol{x}) \approx \sum_{\mathsf{v}\in\mathcal{V}_\sigma} \omega_\sigma^{\mathsf{v}} w(\mathsf{v}) \tag{3.21}$$

is exact for *linear* functions w. This is equivalent to $\sum_{\mathsf{v}\in\mathcal{V}_\sigma} \omega_\sigma^{\mathsf{v}} = |\sigma|$ and $\sum_{\mathsf{v}\in\mathcal{V}_\sigma} \omega_\sigma^{\mathsf{v}}\mathsf{v} = |\sigma|\overline{\boldsymbol{x}}_\sigma$. We will assume the following property on these weights. This property is not required in the construction of the nMFD, but it is used to identify the nMFD with a gradient scheme. We note that this assumption is not very restrictive, since it holds for any natural choice of weights for (3.21).

$$\forall K \in \mathcal{M}\,,\ \forall \mathsf{v} \in \mathcal{V}_K\,,\ \exists \sigma \in \mathcal{E}_{K,\mathsf{v}} \text{ such that } \omega_\sigma^{\mathsf{v}} \neq 0, \tag{3.22}$$

where $\mathcal{E}_{K,\mathsf{v}}$ is the set of faces of K that have v as one of their vertices. For each cell K, we select a partition $(P_{K,\mathsf{v}})_{\mathsf{v}\in\mathcal{V}_K}$ such that

$$\forall \mathsf{v} \in \mathcal{V}_K\,,\ |P_{K,\mathsf{v}}| = \sum_{\sigma\in\mathcal{E}_{K,\mathsf{v}}} \omega_\sigma^{\mathsf{v}} \frac{|D_{K,\sigma}|}{|\sigma|} = \frac{1}{d}\sum_{\sigma\in\mathcal{E}_{K,\mathsf{v}}} \omega_\sigma^{\mathsf{v}} d_{K,\sigma}. \tag{3.23}$$

The partition of Ω is then $\mathcal{U} = (P_{K,\mathsf{v}})_{K\in\mathcal{M},\,\mathsf{v}\in\mathcal{V}_K}$, and for $U = P_{K,\mathsf{v}}$ we let $I_U = \mathcal{V}_K$.

(2) In a similar way as for the HMM method, the reconstructed gradient is the sum of a constant gradient in each cell and of stabilisation terms in each $P_{K,\mathsf{v}}$. We set

$$\forall v \in X_{\mathcal{D},0}\,,\ \forall K \in \mathcal{M}\,,\ \nabla_K v = \frac{1}{|K|}\sum_{\sigma\in\mathcal{E}_K}\left(\sum_{\mathsf{v}\in\mathcal{V}_\sigma} \omega_\sigma^{\mathsf{v}} v_{\mathsf{v}}\right)\mathbf{n}_{K,\sigma}. \tag{3.24}$$

and

$$\forall P_{K,\mathsf{v}} \in \mathcal{U}\,,\ \forall x \in P_{K,\mathsf{v}}\,,\ \mathcal{G}_{P_{K,\mathsf{v}}} v(\boldsymbol{x}) = \nabla_K v + \frac{1}{h_K}[\mathcal{L}_K R_K(V_K(v))]_{\mathsf{v}}\mathbf{N}_{K,\mathsf{v}} \tag{3.25}$$

where

- $\mathbf{N}_{K,\mathsf{v}} = \frac{h_K}{d|P_{K,\mathsf{v}}|}\sum_{\sigma\in\mathcal{E}_{K,\mathsf{v}}} \omega_\sigma^{\mathsf{v}}\mathbf{n}_{K,\sigma}$,
- $V_K(v) = (v_{\mathsf{v}} - v_K)_{\mathsf{v}\in\mathcal{V}_K}$ with $v_K = \frac{1}{|K|}\sum_{\mathsf{v}\in\mathcal{V}_K} \omega_K^{\mathsf{v}} v_{\mathsf{v}}$,
- $R_K : \mathbf{R}^{\mathcal{V}_K} \mapsto \mathbf{R}^{\mathcal{V}_K}$ is the linear mapping described by $R_K(\xi) = (R_{K,\mathsf{v}}(\xi))_{\mathsf{v}\in\mathcal{V}_K}$ with

$$R_{K,\mathsf{v}}(\xi) = \xi_{\mathsf{v}} - \nabla_K\xi\cdot(\mathsf{v} - \widetilde{\boldsymbol{x}}_K), \tag{3.26}$$

where $\nabla_K\xi$ is defined as in (3.24), and $\widetilde{\boldsymbol{x}}_K$ is the "centre" of K with respect to the weights in (3.20):

$$\widetilde{\boldsymbol{x}}_K = \frac{1}{|K|}\sum_{\mathsf{v}\in\mathcal{V}_K} \omega_K^{\mathsf{v}}\mathsf{v}. \tag{3.27}$$

- $\mathcal{L}_K$ is an isomorphism of the space $\mathrm{Im}(R_K) \subset \mathbf{R}^{\mathcal{V}_K}$.

(3) For $U = P_{K,\mathsf{v}}$ we choose in the reconstruction (2.10) the weights

$$\forall \boldsymbol{x} \in U\,,\ \forall \mathsf{v}' \in \mathcal{V}_K\,,\ \alpha_{\mathsf{v}'}(\boldsymbol{x}) := \frac{1}{|K|}\omega_K^{\mathsf{v}'}. \tag{3.28}$$

This leads to

$$\forall v \in X_{\mathcal{D},0},\ \forall K \in \mathcal{M},\ \forall \boldsymbol{x} \in K,\ \Pi_{\mathcal{D}} v(\boldsymbol{x}) = v_K = \frac{1}{|K|}\sum_{\mathsf{v}\in\mathcal{V}_K} \omega_K^{\mathsf{v}} v_{\mathsf{v}}.$$

Under Assumption (3.22) it is proved in [22] that the gradient scheme (2.3) obtained with the gradient discretisation above is equivalent to the nMFD method of [10] for (2.1).

Remark 3.13. *The second equality in* (3.23) *comes from* $|D_{K,\sigma}| = \frac{|\sigma| d_{K,\sigma}}{d}$*, and this choice of* $|P_{K,\mathsf{v}}|$ *is compatible with the requirement that* $\sum_{\mathsf{v}\in\mathcal{V}_K} |P_{K,\mathsf{v}}| = |K|$.

The actual construction and geometric properties of $P_{K,\mathsf{v}}$ *are irrelevant. For the analysis of the method as well as its implementation, only the measure of this set is useful. Other choices of* $P_{K,\mathsf{v}}$ *are possible. For example, we could take all* $(P_{K,\mathsf{v}})_{\mathsf{v}\in\mathcal{V}_K}$ *of the same measure* $|K|/\sharp\mathcal{V}_K$*. Theorem 3.1 would still be valid. However, a stronger assumption than* (3.22) *would be required to ensure the coercivity of the corresponding gradient discretisations; we would need* $\sum_{\sigma\in\mathcal{E}_{K,\mathsf{v}}} \omega_\sigma^{\mathsf{v}} \geq c h_K^{d-1}$ *with* $c > 0$ *not depending on* K *or* v.

Definition 3.14 (Regularity of the nMFD gradient discretisation)**.** *If* $\mathcal{D}$ *is an nMFD gradient discretisation as above, we define* $\zeta_{\mathcal{D}}$ *as the smallest number such that, for all* $K \in \mathcal{M}$ *and all* $\xi \in \mathbf{R}^{\mathcal{V}_K}$*,*

$$\zeta_{\mathcal{D}}^{-1} \sum_{\mathsf{v}\in\mathcal{V}_K} |P_{K,\mathsf{v}}| \left|\frac{R_{K,\mathsf{v}}(\xi)}{h_K}\right|^p \leq \sum_{\mathsf{v}\in\mathcal{V}_K} |P_{K,\mathsf{v}}| \left|\frac{[\mathcal{L}_K R_K(\xi)]_{\mathsf{v}}}{h_K}\right|^p \leq \zeta_{\mathcal{D}} \sum_{\mathsf{v}\in\mathcal{V}_K} |P_{K,\mathsf{v}}| \left|\frac{R_{K,\mathsf{v}}(\xi)}{h_K}\right|^p. \tag{3.29}$$

A sequence $(\mathcal{D}_m)_{m\in\mathbf{N}}$ *of nMFD gradient discretisations is regular if the sequence* $(\mathcal{T}_m)_{m\in\mathbf{N}}$ *is regular in the sense of Definition 2.25 and if* $(\zeta_{\mathcal{D}_m})_{m\in\mathbf{N}}$ *is bounded.*

Remark 3.15. *Contrary to the HMM gradient discretisation, the nMFD gradient discretisation is not piecewise constant for the natural choice of unknowns, or for any obvious choice of unknowns. It should therefore be modified, e.g. by mass-lumping as in Section 2.4, to be applicable in practice to certain non-linear models.*

The proof of Theorem 3.1 for the nMFD method follows the same steps as the proof of the same theorem for the HMM method (Section 3.6), as detailed in [30] in the case where $\mathcal{L}_K = \beta_K \mathrm{Id}$ and in [23] for general isomorphisms. It relies on the tools given in Section 2.1, applicable thanks to the above notion of regularity (linearly local exactness for consistency, discrete functional analysis for coercivity and compactness, Stokes formula for limit-conformity), and in comparisons between $||\widetilde{v}||_{X_{\mathcal{T},0},p}$ and $||\nabla_{\mathcal{D}} v||_{L^p(\Omega)^d}$, where for $v \in X_{\mathcal{D},0}$ we define $\widetilde{v} \in X_{\mathcal{T},0}$ by $\widetilde{v}_\sigma = \frac{1}{|\sigma|}\sum_{\mathsf{v}\in\mathcal{V}_\sigma} \omega_\sigma^{\mathsf{v}} v_{\mathsf{v}}$ and $\widetilde{v}_K = v_K = (\Pi_{\mathcal{D}} v)_{|K}$.

3.8. **Vertex Approximate Gradient (VAG) methods.** Successive versions of the VAG scheme have been described in several papers [30, 32]. VAG methods stem from the idea that it is often computationally efficient to have all unknowns located at the vertices of the mesh, especially with tetrahedral meshes (which have much less vertices than cells). It is however known that schemes with degrees of freedom at the vertices may lead to unacceptable results for the transport of a species in an heterogeneous domain, in particular for coarse meshes (one layer of mesh for one homogeneous layer, for example). The VAG scheme is an answer to this conundrum. After all possible local eliminations, the VAG schemes only has vertex unknowns, and it has been shown to cure the numerical issues for coarse meshes and heterogeneous media [32, 31, 33]; this is due to a specific mass-lumping that spreads the reconstructed function between the centre of the control volumes and the vertices. Let us remark that the original version of the VAG scheme in [30] uses the same nodal formalism as Section 3.7, but has been shown in the FVCA6 3D Benchmark [36] to be less precise than the version presented here.

The VAG scheme is defined as a barycentric condensation and mass-lumping of the $\mathbb{P}_1$ gradient discretisation on a sub-tetrahedral mesh.

(1) Let $\mathcal{T} = (\mathcal{M}, \mathcal{E}, \mathcal{P}, \mathcal{V})$ be a polyhedral discretisation of Ω in the sense of Definition 2.23, except the hypothesis that the faces $\sigma \in \mathcal{E}$ are planar, which is not necessary here. We define a tetrahedral mesh by the following procedure. For any $K \in \mathcal{M}$, $\sigma \in \mathcal{E}_K$, and $\mathsf{v}, \mathsf{v}' \in \mathcal{V}_\sigma$ such that $[\mathsf{v}, \mathsf{v}']$ is an

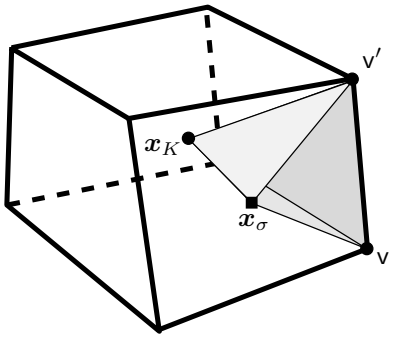

FIGURE 6. Definition of tetrahedra from a mesh.

edge of σ, we define the tetrahedron $T_{K,\sigma,\mathsf{v},\mathsf{v}'}$ by its four vertices $\boldsymbol{x}_K, \boldsymbol{x}_\sigma, \mathsf{v}, \mathsf{v}'$ (see Figure 3.8), where the point $\boldsymbol{x}_\sigma$ corresponding to the face σ is

$$\boldsymbol{x}_\sigma = \frac{1}{\mathrm{Card}(\mathcal{V}_\sigma)} \sum_{\mathsf{v}\in\mathcal{V}_\sigma} \mathsf{v}, \tag{3.30}$$

We denote by $\mathcal{T}^T$ the simplicial polyhedral discretisation corresponding to these $T_{K,\sigma,\mathsf{v},\mathsf{v}'}$.

(2) We let $\overline{\mathcal{D}} = (X_{\overline{\mathcal{D}},0}, \nabla_{\overline{\mathcal{D}}}, \Pi_{\overline{\mathcal{D}}})$ be the $\mathbb{P}_1$ gradient discretisation defined from $\mathcal{T}^T$ as in Section 3.1.1 for $k = 1$. The degrees of freedom and approximation points of $\overline{\mathcal{D}}$ are therefore $I = \mathcal{M} \cup \mathcal{V} \cup \mathcal{E}$ and $S = ((\boldsymbol{x}_K)_{K\in\mathcal{M}}, (\mathsf{v})_{\mathsf{v}\in\mathcal{V}}, (\boldsymbol{x}_\sigma)_{\sigma\in\mathcal{E}})$. We define $\widetilde{\mathcal{D}}$ as the barycentric condensation of $\overline{\mathcal{D}}$ (see Definition 2.15) such that $\widetilde{I} = \mathcal{M} \cup \mathcal{V}$ and the degrees of freedom $\mathcal{E}$ are eliminated by setting $H_\sigma = \mathcal{V}_\sigma$ and the coefficients $\beta^\sigma_\mathsf{v} = 1/\mathrm{Card}(\mathcal{V}_\sigma)$ for all $\mathsf{v} \in \mathcal{V}_\sigma$, which are precisely the coefficients in (3.30).

(3) The VAG scheme is the gradient discretisation $\mathcal{D}$ obtained from the gradient discretisation $\widetilde{\mathcal{D}}$ by performing a mass-lumping in the sense of Definition 2.19. We split each tetrahedron $T_{K,\sigma,\mathsf{v},\mathsf{v}'}$ into three parts $T^K_{K,\sigma,\mathsf{v},\mathsf{v}'}$, $T^\mathsf{v}_{K,\sigma,\mathsf{v},\mathsf{v}'}$, and $T^{\mathsf{v}'}_{K,\sigma,\mathsf{v},\mathsf{v}'}$ (whose geometry is irrelevant), and we let Ω_K be the union of all $(T^K_{K,\sigma,\mathsf{v},\mathsf{v}'})_{\sigma,\mathsf{v},\mathsf{v}'}$ and Ω_v be the union of all $(T^\mathsf{v}_{K,\sigma,\mathsf{v},\mathsf{v}'} \cup T^\mathsf{v}_{K,\sigma,\mathsf{v}',\mathsf{v}})_{K,\sigma,\mathsf{v}'}$. This leads to

$$\forall v \in X_{\mathcal{D},0} \,:\, \Pi_\mathcal{D} v = \sum_{K\in\mathcal{M}} v_K \chi_{\Omega_K} + \sum_{\mathsf{v}\in\mathcal{V}} v_\mathsf{v} \chi_{\Omega_\mathsf{v}}.$$

The regularity of a sequence of VAG gradient discretisations is defined as the regularity of the underlying tetrahedral meshes $\mathcal{T}^T$ in the sense of Definition 2.25. We can check that $\mathrm{reg}_{\mathrm{BC}}(\mathcal{D})$ remains bounded by a non-decreasing function of $\theta_{\mathcal{T}^T}$, and the proof of Theorem 3.1 for VAG gradient discretisation is thus a direct consequence of the results in Section 2, especially Theorem 2.18 (properties of the barycentric condensation) and Theorem 2.21 (properties of mass-lumped gradient discretisations).

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