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# AUTOMORPHISM GROUPS OF CONFIGURATION SPACES AND DISCRIMINANT VARIETIES

VLADIMIR LIN AND MIKHAIL ZAIDENBERG

ABSTRACT. The configuration space  $\mathcal{C}^n(X)$  of an algebraic curve  $X$  is the algebraic variety consisting of all  $n$ -point subsets  $Q \subset X$ . We describe the automorphisms of  $\mathcal{C}^n(\mathbb{C})$ , deduce that the (infinite dimensional) group  $\text{Aut } \mathcal{C}^n(\mathbb{C})$  is solvable, and obtain an analog of the Mostow decomposition in this group. The Lie algebra and the Makar-Limanov invariant of  $\mathcal{C}^n(\mathbb{C})$  are also computed. We obtain similar results for the level hypersurfaces of the discriminant, including its singular zero level.

This is an extended version of our paper [39]. We strengthened the results concerning the automorphism groups of cylinders over rigid bases, replacing the rigidity assumption by the weaker assumption of tightness. We also added alternative proofs of two auxiliary results cited in [39] and due to Zinde and to the first author. This allowed us to provide the optimal dimension bounds in our theorems.

## 1. INTRODUCTION

Let  $X$  be an irreducible smooth algebraic curve over the field  $\mathbb{C}$ . The  $n$ th *configuration space*  $\mathcal{C}^n(X)$  of  $X$  is a smooth affine algebraic variety of dimension  $n$  consisting of all  $n$ -point subsets  $Q = \{q_1, \dots, q_n\} \subset X$  with distinct  $q_1, \dots, q_n$ . We would like to study its biregular automorphisms and the structure of the group  $\text{Aut } \mathcal{C}^n(X)$ .

For a hyperbolic curve  $X$  the group  $\text{Aut } \mathcal{C}^n(X)$  is finite (possibly, trivial for a generic curve). We are interested in the case where  $X$  is non-hyperbolic, i.e., one of the curves  $\mathbb{C}$ ,  $\mathbb{P}^1 = \mathbb{P}_{\mathbb{C}}^1$ ,  $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ , and an elliptic curve. In the latter two cases the groups  $\text{Aut } \mathcal{C}^n(X)$  were described in [51] and [15], respectively. Here we investigate automorphisms of the configuration space  $\mathcal{C}^n = \mathcal{C}^n(\mathbb{C})$  and of some related spaces. Our central results are Theorems 1.1 and 1.2; see also Theorems 3.13, 4.4, 4.5, and Corollary 4.2. Let us first introduce some conventions and a portion of notation.

All varieties in this paper are algebraic varieties defined over<sup>1</sup>  $\mathbb{C}$  and reduced; in general, irreducibility is not required. *Morphism* means a regular morphism of varieties.

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<sup>1</sup>However, all general results remain valid over any algebraically closed field of characteristic zero.

The same applies to the terms *automorphism* and *endomorphism*. The actions of algebraic groups are assumed to be regular. We use the standard notation  $\mathcal{O}(\mathcal{Z})$ ,  $\mathcal{O}_+(\mathcal{Z})$ , and  $\mathcal{O}^\times(\mathcal{Z})$  for the algebra of all regular functions on a variety  $\mathcal{Z}$ , the additive group of this algebra, and its group of invertible elements, respectively.

For  $z \in \mathbb{C}^n$ , let  $d_n(z)$  denote the discriminant of the monic polynomial

$$P_n(\lambda, z) = \lambda^n + z_1\lambda^{n-1} + \dots + z_n, \quad z = (z_1, \dots, z_n) \in \mathbb{C}^n = \mathbb{C}_{(z)}^n. \quad (1)$$

If  $d_n(z) \neq 0$  and  $Q \subset \mathbb{C}$  is the set of all roots of  $P_n(\cdot, z)$ , then  $Q \in \mathcal{C}^n$  and

$$D_n(Q) \stackrel{\text{def}}{=} \prod_{\{q', q''\} \subset Q} (q' - q'')^2 = d_n(z). \quad (2)$$

Denoting by  $\mathcal{P}^n$  the space of all polynomials (1) with simple roots, we have the natural identification

$$\mathcal{C}^n = \{Q \subset \mathbb{C} \mid \#Q = n\} \cong \mathcal{P}^n = \mathbb{C}_{(z)}^n \setminus \Sigma^{n-1}, \quad Q \leftrightarrow z = (z_1, \dots, z_n), \quad (3)$$

where<sup>2</sup> the *discriminant variety*  $\Sigma^{n-1}$  is defined by

$$\Sigma^{n-1} \stackrel{\text{def}}{=} \{z \in \mathbb{C}^n \mid d_n(z) = 0\}. \quad (4)$$

For  $n > 2$ , we describe the automorphisms of the configuration space  $\mathcal{C}^n$ , the discriminant variety  $\Sigma^{n-1}$ , and the *special configuration space*

$$\mathcal{SC}^{n-1} \stackrel{\text{def}}{=} \{Q \in \mathcal{C}^n \mid D_n(Q) = 1\} \cong \{z \in \mathbb{C}^n \mid d_n(z) = 1\}. \quad (5)$$

This leads to structure theorems for the automorphism groups  $\text{Aut } \mathcal{C}^n$ ,  $\text{Aut } \mathcal{SC}^{n-1}$ , and  $\text{Aut } \Sigma^{n-1}$ .

The varieties  $\mathcal{C}^n$  and  $\Sigma^{n-1}$  can be viewed as the complementary to each other parts of the symmetric power  $\text{Sym}^n \mathbb{C} = \mathbb{C}^n / \mathbf{S}(n)$ , where  $\mathbf{S}(n)$  is the symmetric group permuting the coordinates  $q_1, \dots, q_n$  in  $\mathbb{C}^n = \mathbb{C}_{(q)}^n$ . We have the natural morphisms

$$p: \mathbb{C}_{(q)}^n \rightarrow \text{Sym}^n \mathbb{C} \cong \mathbb{C}_{(z)}^n, \quad \Delta^{n-1} \rightarrow \Sigma^{n-1}, \quad \text{and} \quad \mathbb{C}_{(q)}^n \setminus \Delta^{n-1} \rightarrow \mathcal{C}^n, \quad (6)$$

where  $\Delta^{n-1} \stackrel{\text{def}}{=} \bigcup_{i \neq j} \{q = (q_1, \dots, q_n) \in \mathbb{C}_{(q)}^n \mid q_i = q_j\}$  is the big diagonal. The points  $z \in \Sigma^{n-1}$  are in one-to-one correspondence with unordered  $n$ -term multisets (or corteges)  $Q = \{q_1, \dots, q_n\}$ ,  $q_i \in \mathbb{C}$ , with at least one repetition.

The *barycenter*  $\text{bc}(Q)$  of a point  $Q \in \text{Sym}^n \mathbb{C} = \mathcal{C}^n \cup \Sigma^{n-1} \cong \mathbb{C}_{(z)}^n$  is defined as

$$\text{bc}(Q) \stackrel{\text{def}}{=} \frac{1}{n} \sum_{q \in Q} q = -z_1/n \quad (7)$$

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<sup>2</sup>The upper index will usually mean the dimension of the variety.

(if  $Q$  is a multiset, the summation takes into account multiplicities). The *balanced configuration space*  $\mathcal{C}_{\text{blc}}^{n-1} \subset \mathcal{C}^n$  is defined by

$$\mathcal{C}_{\text{blc}}^{n-1} = \{Q \in \mathcal{C}^n \mid \text{bc}(Q) = 0\}. \quad (8)$$

One defines similarly balanced hypersurfaces  $\mathcal{S}\mathcal{C}_{\text{blc}}^{n-2} \subset \mathcal{S}\mathcal{C}^{n-1}$  and  $\Sigma_{\text{blc}}^{n-2} \subset \Sigma^{n-1}$  (see also Notation 2.19).

Our main results related to automorphisms of  $\mathcal{C}^n$  are the following two theorems (for more general results see Theorems 3.13, 4.4, 4.5, and Corollary 4.2).

**Theorem 1.1.** *Assume that  $n > 2$ . A map  $F: \mathcal{C}^n \rightarrow \mathcal{C}^n$  is an automorphism if and only if it is of the form*

$$F(Q) = s \cdot \pi(Q) + A(\pi(Q)) \text{bc}(Q) \quad \text{for any } Q \in \mathcal{C}^n, \quad (9)$$

where  $\pi(Q) = Q - \text{bc}(Q)$ ,  $s \in \mathbb{C}^*$ , and  $A: \mathcal{C}_{\text{blc}}^{n-1} \rightarrow \text{Aff } \mathbb{C}$  is a regular map.

**Theorem 1.2.** *If  $n > 2$ , then the following hold.*

(a) *The group  $\text{Aut } \mathcal{C}^n$  is solvable. More precisely, it is a semi-direct product*

$$\text{Aut } \mathcal{C}^n \cong (\mathcal{O}_+(\mathcal{C}_{\text{blc}}^{n-1}) \rtimes (\mathbb{C}^*)^2) \rtimes \mathbb{Z}.$$

(b) *Any finite subgroup  $\Gamma \subset \text{Aut } \mathcal{C}^n$  is Abelian.*

(c) *Any connected algebraic subgroup  $G$  of  $\text{Aut } \mathcal{C}^n$  is either Abelian or metabelian of rank  $\leq 2$ .*<sup>3</sup>

(d) *Any two maximal tori in  $\text{Aut } \mathcal{C}^n$  are conjugated.*

Similar facts are established for  $\mathcal{S}\mathcal{C}^{n-1}$  and  $\Sigma^{n-1}$ , see loc. cit.

Let us overview some results of [32, 29, 51, 35, 37, 15, 38] initiated the present paper and used in the proofs. Given a smooth irreducible non-hyperbolic algebraic curve  $X$ , consider the diagonal action of the group  $\text{Aut } X$  on the configuration space  $\mathcal{C}^n(X)$ ,

$$\text{Aut } X \ni A: \mathcal{C}^n(X) \rightarrow \mathcal{C}^n(X), \quad Q = \{q_1, \dots, q_n\} \mapsto AQ \stackrel{\text{def}}{=} \{Aq_1, \dots, Aq_n\}. \quad (10)$$

To any morphism  $T: \mathcal{C}^n(X) \rightarrow \text{Aut } X$  we assign an endomorphism  $F_T$  of  $\mathcal{C}^n(X)$  defined by

$$F_T(Q) \stackrel{\text{def}}{=} T(Q)Q \quad \text{for all } Q \in \mathcal{C}^n(X). \quad (11)$$

Such endomorphisms  $F_T$  are called *tame*. A tame endomorphism preserves each  $(\text{Aut } X)$ -orbit in  $\mathcal{C}^n(X)$ . For automorphisms, the converse is also true: an automorphism of  $\mathcal{C}^n(X)$  preserving each  $(\text{Aut } X)$ -orbit is tame, see Proposition 3.3 and the remark following this proposition. In the general case, Tame Map Theorem below implies the following: an endomorphism of  $\mathcal{C}^n(X)$  whose image is not contained in a

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<sup>3</sup>The rank of an affine algebraic group is the dimension of its maximal tori.

single  $(\text{Aut } X)$ -orbit is tame and hence preserves each  $(\text{Aut } X)$ -orbit. If the image of  $F$  is contained in a single  $(\text{Aut } X)$ -orbit, then  $F$  is called *orbit-like*.

The braid group of  $X$ ,  $B_n(X) = \pi_1(\mathcal{C}^n(X))$ , is non-Abelian for any  $n \geq 3$ . If  $X = \mathbb{C}$  then  $B_n(X) = \mathbf{A}_{n-1}$  is the Artin braid group on  $n$  strands.<sup>4</sup> An endomorphism  $F$  of  $\mathcal{C}^n(X)$  is called *non-Abelian* if the image of the induced endomorphism  $F_*: \pi_1(\mathcal{C}^n(X)) \rightarrow \pi_1(\mathcal{C}^n(X))$  is a non-Abelian group. Otherwise,  $F$  is said to be *Abelian*. Rather unexpectedly, this evident algebraic dichotomy gives rise to the following analytic one.

**Tame Map Theorem.** *Let  $X$  be a smooth irreducible non-hyperbolic algebraic curve. For  $n > 4$  any non-Abelian endomorphism of  $\mathcal{C}^n(X)$  is tame, whereas any Abelian endomorphism of  $\mathcal{C}^n(X)$  is orbit-like.*

**Remarks 1.3.** (a) A proof of Tame Map Theorem for  $X = \mathbb{C}$  is sketched in [32] and [33]; a complete proof for  $X = \mathbb{C}$  or  $\mathbb{P}^1$  in the analytic category can be found in [35], [37], and [38]. For  $X = \mathbb{C}^*$  the theorem is proved in [51]<sup>5</sup>, and for elliptic curves in [15]. The proofs apply mutatis mutandis in the algebraic setting. We use this theorem to describe automorphisms of the balanced spaces  $\mathcal{C}_{\text{blc}}^{n-1}$  and  $\Sigma_{\text{blc}}^{n-2}$  (see Theorems 4.1(a),(c) and 7.1); its analytic counterpart is involved in the proof of Theorem 10.2.

(b) A morphism  $T: \mathcal{C}^n(X) \rightarrow \text{Aut } X$  in the tame representation  $F = F_T$  is uniquely determined by a non-Abelian endomorphism  $F$ . Indeed, if  $T_1(Q)Q = T_2(Q)Q$  for all  $Q \in \mathcal{C}^n(X)$ , then the automorphism  $[T_1(Q)]^{-1}T_2(Q)$  is contained in the  $\text{Aut}(X)$ -stabilizer of  $Q$ , which is trivial for general configurations  $Q$ . Therefore,  $T_1 = T_2$ .

(c) According to Tame Map Theorem and Theorem 1.1, the map  $F$  in (9), being an automorphism, must be tame. This is indeed the case, with the morphism

$$T: \mathcal{C}^n \rightarrow \text{Aff } \mathbb{C}, \quad T(Q)\zeta = s \cdot (\zeta - \text{bc}(Q)) + A(\pi(Q)) \text{bc}(Q), \quad (12)$$

where  $\zeta \in \mathbb{C}$  and  $Q \in \mathcal{C}^n$ .

(d) Let  $X = \mathbb{C}$ . Then Tame Map Theorem holds also for  $n = 3$ , but not for  $n = 4$ . However, any *automorphism* of  $\mathcal{C}^4(\mathbb{C})$  is tame. The automorphism groups of  $\mathcal{C}^1(\mathbb{C}) \cong \mathbb{C}$  and  $\mathcal{C}^2(\mathbb{C}) \cong \mathbb{C}^* \times \mathbb{C}$  are well known, so we assume in the sequel that  $n > 2$ .

Using Tame Map Theorem, Zinde and Feler [loc. cit.] described all automorphisms of  $\mathcal{C}^n(X)$  when  $\dim_{\mathbb{C}} \text{Aut } X = 1$ , or more precisely, when  $X$  is  $\mathbb{C}^*$  and  $n > 4$ , or  $X$  is an elliptic curve and  $n > 2$ . For  $\mathbb{C}$  and  $\mathbb{P}^1$ , where the automorphism groups  $\text{Aff } \mathbb{C}$  and  $\text{PSL}(2, \mathbb{C})$  have dimension 2 and 3, respectively, the problem becomes more difficult.

<sup>4</sup>In [39] we used the notation  $B_n$  for the Artin braid group on  $n$  strands. Here we prefer the notation  $\mathbf{A}_{n-1}$  that indicates the place of this group among the Artin-Brieskorn groups of series  $A$ . A similar notation will be applied to the Artin-Brieskorn groups of other series  $B, D$  etc., while  $\mathbf{WA}_{n-1}$ ,  $\mathbf{WB}_n$ , etc. stands for the corresponding Coxeter group.

<sup>5</sup>The complex Weyl chamber of type  $B$  studied in [51] is isomorphic to  $\mathcal{C}^n(\mathbb{C}^*)$ .

The group  $\text{Aut } \mathcal{C}^n = \text{Aut } \mathcal{C}^n(\mathbb{C})$  is the subject of the present paper; the case  $X = \mathbb{P}^1$  remains open.

The content of the paper is as follows. In Sections 2 and 3 we propose an abstract scheme to study the automorphism groups of cylinders over rigid bases and, more generally, of tight cylinders. An irreducible affine variety  $\mathcal{X}$  will be called *rigid* if the images of non-constant morphisms  $\mathbb{C} \rightarrow \text{reg } \mathcal{X}$  do not cover any Zariski open dense subset in the smooth locus  $\text{reg } \mathcal{X}$ , i.e., if  $\text{reg } \mathcal{X}$  is *non- $\mathbb{C}$ -uniruled*, see Definition 2.1. Any cylinder  $\mathcal{X} \times \mathbb{C}$  over a rigid base  $\mathcal{X}$  is *tight*, meaning that its cylinder structure over  $\mathcal{X}$  is unique, see Definition 2.2 and Corollary 2.4.

We show in Section 2.6 that the bases  $\mathcal{C}_{\text{blc}}^{n-1}$ ,  $\mathcal{SC}_{\text{blc}}^{n-2}$ , and  $\Sigma_{\text{blc}}^{n-2}$  of cylinders (27) are rigid. So, the scheme of Section 2.2 applied to the latter cylinders yields that their automorphisms have a triangular form, see (18)-(19).

For a tight cylinder  $\mathcal{X} \times \mathbb{C}$  we describe in Subsection 2.4 the special automorphism group  $\text{SAut}(\mathcal{X} \times \mathbb{C})$  (see Definition 2.13), and in Section 3 the neutral component  $\text{Aut}_0(\mathcal{X} \times \mathbb{C})$  of the group  $\text{Aut}(\mathcal{X} \times \mathbb{C})$  and its algebraic subgroups. In Theorem 3.13 we establish an analog of Theorem 1.2 for such cylinders. Besides, in Sections 2.3- 2.5 we find the locally nilpotent derivations of the algebra  $\mathcal{O}(\mathcal{X} \times \mathbb{C})$  and its Makar-Limanov invariant subalgebra. In Section 3.7 we study the Lie algebra  $\text{Lie}(\text{Aut}_0(\mathcal{X} \times \mathbb{C}))$ . These results are used in the subsequent sections in the concrete setting of the varieties  $\mathcal{C}^n$ ,  $\mathcal{SC}^{n-1}$ , and  $\Sigma^{n-1}$ .

Theorems 1.1 and 1.2 are proven in Section 4, see Theorems 4.5 and 4.4, respectively. We provide analogs of our main results for the automorphism groups of the special configuration space  $\mathcal{SC}^{n-1}$ , the discriminant variety  $\Sigma^{n-1}$ , and the pair  $(\mathbb{C}^n, \Sigma^{n-1})$ . All these groups are solvable; we also find presentations of their Lie algebras. In Section 5 we show that all these groups are centerless and describe their commutator series, semisimple and torsion elements. In Section 6 we give a description of the group  $\text{Aut } \mathcal{C}^n(\mathbb{C}^*)$  due to Zinde [51]. We provide a new proof based upon Zinde's analog of Artin's Theorem on the pure braid groups. In Section 7 we apply Zinde's Theorem in order to complete the description of the automorphism group of  $\Sigma^{n-1}$ . In Section 8 we provide an alternative proof of the structure theorem for the group  $\text{Aut } \mathcal{C}_{\text{blc}}^{n-1}$ , which does not refer to Tame Map Theorem. In Section 9 we give a proof of Kaliman's Theorem on the group  $\text{Aut } \mathcal{SC}_{\text{blc}}^{n-2}$  in the exceptional case  $n = 4$ , where the original proof does not work. We reproduce an example from [33] which shows that the original Kaliman's Theorem on endomorphisms of  $\mathcal{SC}_{\text{blc}}^{n-2}$  does not hold in this generality for  $n = 4$  (see Example 9.3). Finally, in Section 10, using the analytic counterpart of Tame Map Theorem, we obtain its analog for the space  $\mathcal{C}_{\text{blc}}^{n-1}$  ( $n > 4$ ), describe the proper holomorphic self-maps of this space and the group of its biholomorphic automorphisms  $\text{Aut}_{\text{hol}} \mathcal{C}_{\text{blc}}^{n-1}$ .

For the sake of uniformity, we work over the field  $\mathbb{C}$ . Indeed, this is essential when dealing with configuration spaces, since in this case we employ certain topological methods. By contrast, all the results concerning rigid varieties and tight cylinders, along with their proofs, remain valid over an arbitrary algebraically closed field of characteristic zero.

This preprint is an extended version of our paper [39]. Sections 2–4 of [39] have been modified so that the results stated in [39] under the rigidity assumption are proven now under a weaker assumption of tightness. New Sections 6–9 are devoted to alternative proofs of some of the aforementioned results. This allowed to provide the optimal dimension bounds in our main theorems.

The authors thank S. Kaliman for useful discussions concerning Section 2.3, especially Example 2.9 and Remark 2.10. They are grateful also to N. Ivanov and L. Paris for providing useful information concerning different generalizations of the classical Artin’s theorem on braid groups used in Sections 6 and 8.

## 2. AUTOMORPHISMS OF CYLINDERS OVER RIGID BASES

**2.1. Triangular automorphisms.** (a) Let  $\mathfrak{C}$  be a category of sets admitting direct products  $\mathcal{X} \times \mathcal{Y}$  of its objects with the standard projections to  $\mathcal{X}$  and  $\mathcal{Y}$  being morphisms.

Suppose that the automorphism group  $\text{Aut } \mathcal{Y}$  of a certain object  $\mathcal{Y}$  is an object in  $\mathfrak{C}$  satisfying the usual axioms, namely, that the following maps are morphisms:

$$\begin{aligned} & \text{the action } (\text{Aut } \mathcal{Y}) \times \mathcal{Y} \rightarrow \mathcal{Y}, (\alpha, y) \mapsto \alpha(y), \\ & \text{the multiplication } \text{Aut } \mathcal{Y} \times \text{Aut } \mathcal{Y} \rightarrow \text{Aut } \mathcal{Y}, (\alpha, \beta) \mapsto \beta \circ \alpha \quad (13) \\ & \text{the inversion } \text{Aut } \mathcal{Y} \rightarrow \text{Aut } \mathcal{Y}, \alpha \mapsto \alpha^{-1}. \end{aligned}$$

Then for any  $S \in \text{Aut } \mathcal{X}$  and any morphism  $A: \mathcal{X} \rightarrow \text{Aut } \mathcal{Y}$  the map

$$F = F_{S,A}: \mathcal{X} \times \mathcal{Y} \rightarrow \mathcal{X} \times \mathcal{Y}, \quad F(x, y) = (Sx, A(x)y) \text{ for all } (x, y) \in \mathcal{X} \times \mathcal{Y}, \quad (14)$$

is a morphism in the category  $\mathfrak{C}$ . Moreover,  $F$  is an *automorphism* of  $\mathcal{X} \times \mathcal{Y}$ . Indeed, given  $S, S' \in \text{Aut } \mathcal{X}$  and morphisms  $A, A': \mathcal{X} \rightarrow \text{Aut } \mathcal{Y}$ , for the corresponding  $F, F'$  we have  $F'F(x, y) = (S'Sx, A'(Sx)A(x)y)$ , whereas the inverse  $F^{-1}$  of  $F$  corresponds to the couple  $(S', A')$ , where  $S' = S^{-1}$  and the morphism  $A': \mathcal{X} \rightarrow \text{Aut } \mathcal{Y}$  is defined by  $A'(x) = (A(S^{-1}x))^{-1}$  for all  $x \in \mathcal{X}$ . We call such automorphisms  $F$  of  $\mathcal{X} \times \mathcal{Y}$  *triangular* (with respect to the given product structure). All triangular automorphisms form a subgroup  $\text{Aut}_\Delta(\mathcal{X} \times \mathcal{Y}) \subset \text{Aut}(\mathcal{X} \times \mathcal{Y})$ .

(b) Suppose that an object  $\mathcal{Y} \in \mathfrak{C}$  satisfies the conditions in (a). Then  $\text{Mor}(\mathcal{X}, \text{Aut } \mathcal{Y})$  with the pointwise multiplication of morphisms can be embedded in  $\text{Aut}_\Delta(\mathcal{X} \times \mathcal{Y})$  as a normal subgroup consisting of all  $F$  of the form  $F(x, y) = (x, A(x)y)$ ,  $A \in$

$\text{Mor}(\mathcal{X}, \text{Aut } \mathcal{Y})$ . The corresponding quotient group is isomorphic to  $\text{Aut } \mathcal{X}$ , and we have the semi-direct product decomposition

$$\text{Aut}_\Delta(\mathcal{X} \times \mathcal{Y}) \cong \text{Mor}(\mathcal{X}, \text{Aut } \mathcal{Y}) \rtimes \text{Aut } \mathcal{X}. \quad (15)$$

The second factor acts by conjugation on the first via  $S.A = A \circ S^{-1}$ , where  $S \in \text{Aut } \mathcal{X}$  and  $A \in \text{Mor}(\mathcal{X}, \text{Aut } \mathcal{Y})$ .

**2.2. Automorphisms of tight cylinders.** We are interested in the case where  $\mathfrak{C}$  is the category of complex algebraic varieties and their morphisms, and  $\mathcal{Y} = \mathbb{C}$ . Thus, in the sequel we deal with *cylinders*  $\mathcal{X} \times \mathbb{C}$ . Since  $\text{Aut } \mathcal{Y} = \text{Aff } \mathbb{C} \in \mathfrak{C}$ , the conditions (13) are fulfilled. In fact, we deal mainly with affine varieties; moreover, in all our results these varieties are assumed to be irreducible.

Let us introduce the following notions.

**Definition 2.1.** An irreducible variety  $\mathcal{X}$  is called  $\mathbb{C}$ -*uniruled* if for some variety  $\mathcal{V}$  there is a dominant morphism  $\mathcal{V} \times \mathbb{C} \rightarrow \mathcal{X}$  non-constant on a general ruling  $\{v\} \times \mathbb{C}$ ,  $v \in \mathcal{V}$  ([27, Definition 5.2 and Proposition 5.1]). We say that  $\mathcal{X}$  is *rigid* if its smooth locus  $\text{reg } \mathcal{X}$  is non- $\mathbb{C}$ -uniruled. For such  $\mathcal{X}$ , the variety  $\mathcal{X} \times \mathbb{C}$  is said to be the *cylinder over a rigid base*.

**Definition 2.2.** For an irreducible  $\mathcal{X}$ , we call the cylinder  $\mathcal{X} \times \mathbb{C}$  *tight* if its cylinder structure over  $\mathcal{X}$  is unique, that is, if for any automorphism  $F \in \text{Aut}(\mathcal{X} \times \mathbb{C})$  there is a (unique) automorphism  $S \in \text{Aut } \mathcal{X}$  that fits in the commutative diagram

$$\begin{array}{ccc} \mathcal{X} \times \mathbb{C} & \xrightarrow{F} & \mathcal{X} \times \mathbb{C} \\ \text{pr}_1 \downarrow & & \downarrow \text{pr}_1 \\ \mathcal{X} & \xrightarrow{S} & \mathcal{X} \end{array} \quad (16)$$

Thus,  $\mathcal{X} \times \mathbb{C}$  is tight if and only if every  $F \in \text{Aut}(\mathcal{X} \times \mathbb{C})$  is triangular, so that

$$\text{Aut}(\mathcal{X} \times \mathbb{C}) = \text{Aut}_\Delta(\mathcal{X} \times \mathbb{C}). \quad (17)$$

For a cylinder  $\mathcal{X} \times \mathbb{C}$  formula (14) takes the form

$$F(x, y) = (Sx, A(x)y) = (Sx, ay + b) \text{ for any } (x, y) \in \mathcal{X} \times \mathbb{C} \quad (18)$$

with  $a \in \mathcal{O}^\times(\mathcal{X})$  and  $b \in \mathcal{O}_+(\mathcal{X})$ . If  $\mathcal{X} \times \mathbb{C}$  is tight, then, by (17) and (18), we have

$$\text{Aut}(\mathcal{X} \times \mathbb{C}) \cong \text{Mor}(\mathcal{X}, \text{Aff } \mathbb{C}) \rtimes \text{Aut } \mathcal{X} \text{ and } \text{Mor}(\mathcal{X}, \text{Aff } \mathbb{C}) \cong \mathcal{O}_+(\mathcal{X}) \rtimes \mathcal{O}^\times(\mathcal{X}), \quad (19)$$

where  $\mathcal{O}^\times(\mathcal{X})$  acts on  $\mathcal{O}_+(\mathcal{X})$  by multiplication  $b \mapsto ab$  for  $a \in \mathcal{O}^\times(\mathcal{X})$  and  $b \in \mathcal{O}_+(\mathcal{X})$ . The group  $\text{Aut}(\mathcal{X} \times \mathbb{C})$  of a tight cylinder is solvable as long as  $\text{Aut } \mathcal{X}$  is.

Thus the tightness is an important property. In Theorem 2.18 we give several useful criteria of tightness.



**Definition 2.3.** One says that a variety  $\mathcal{X}$  possesses the *strong cancellation property* if for any  $m > 0$ , any variety  $\mathcal{Y}$ , and any isomorphism  $F: \mathcal{X} \times \mathbb{C}^m \xrightarrow{\cong} \mathcal{Y} \times \mathbb{C}^m$  there is an isomorphism  $S: \mathcal{X} \xrightarrow{\cong} \mathcal{Y}$  that fits in the commutative diagram

$$\begin{array}{ccc} \mathcal{X} \times \mathbb{C}^m & \xrightarrow{F} & \mathcal{Y} \times \mathbb{C}^m \\ \text{pr}_1 \downarrow & & \downarrow \text{pr}_1 \\ \mathcal{X} & \xrightarrow{S} & \mathcal{Y} \end{array} \quad (20)$$

**Dryło's Theorem I** ([11, (I)]). *The strong cancellation holds for any rigid affine variety.*

For the reader's convenience, we provide a short argument for the next corollary.

**Corollary 2.4.** *If  $\mathcal{X}$  is rigid, then  $\mathcal{X} \times \mathbb{C}$  is tight, i.e.,  $\text{Aut}(\mathcal{X} \times \mathbb{C}) = \text{Aut}_\Delta(\mathcal{X} \times \mathbb{C})$ .*

*Proof.* Let us show that any  $F \in \text{Aut}(\mathcal{X} \times \mathbb{C})$  sends the rulings  $\{x\} \times \mathbb{C}$  into rulings. Then the same holds for  $F^{-1}$ , and so  $S \stackrel{\text{def}}{=} \text{pr}_1 \circ F|_{\mathcal{X} \times \{0\}} \in \text{Aut} \mathcal{X}$  fits in diagram (16).

Assuming the contrary, we consider the family  $\{F(\{x\} \times \mathbb{C})\}_{x \in \text{reg} \mathcal{X}}$ . Projecting it to  $\text{reg} \mathcal{X}$  we get a contradiction with the rigidity assumption.  $\square$

**Definition 2.5.** We say that the *strong 1-cancellation* holds for  $\mathcal{X}$  if one has diagram (20) with  $m = 1$ .

Clearly, the strong 1-cancellation property implies the tightness. The converse is also true, see Proposition 2.8.

**Remark 2.6.** The tightness of the cylinder  $\mathcal{X} \times \mathbb{C}$  does not imply the rigidity of  $\mathcal{X}$ , in general. Indeed, there are examples of non-rigid smooth, affine surfaces  $\mathcal{X}$  with a tight cylinder  $\mathcal{X} \times \mathbb{C}$ , cf., e.g., [5, Example 3] and Theorem 2.18.

**2.3.  $\mathbb{C}_+$ -actions and LND's on tight cylinders.** In this subsection we propose several criteria of tightness. Let us recall the necessary notions.

**Definition 2.7.** A derivation  $\partial$  of a ring  $A$  is *locally nilpotent* if  $\partial^n a = 0$  for any  $a \in A$  and for some  $n \in \mathbb{N}$  depending on  $a$ . For any  $f \in \ker \partial$  the derivation  $f\partial \in \text{Der} A$  is again locally nilpotent; it is called a *replica* of  $\partial$  (see [3]). We let  $\text{LND}(A)$  denote the set of all nonzero locally nilpotent derivations of  $A$ .

Let  $\mathcal{X}$  be an affine variety. For any  $\partial \in \text{LND}(\mathcal{O}(\mathcal{X}))$  the correspondence  $\mathbb{C}_+ \ni t \mapsto \exp(t\partial) \in \text{Aut} \mathcal{X}$  defines a unipotent one-parameter group of automorphisms of  $\mathcal{X}$ , or, in other words, an effective  $\mathbb{C}_+$ -action on  $\mathcal{X}$ . In fact, any  $\mathbb{C}_+$ -action on  $\mathcal{X}$  arises in this way (see e.g., [19]).

**Proposition 2.8.** (a) *If the cylinder  $\mathcal{X} \times \mathbb{C}$  is tight, then  $\mathcal{X}$  does not admit any nontrivial  $\mathbb{C}_+$ -action, i.e.,  $\text{LND}(\mathcal{O}(\mathcal{X})) = \emptyset$ .*

(b) *The cylinder  $\mathcal{X} \times \mathbb{C}$  is tight if and only if the strong 1-cancellation holds for  $\mathcal{X}$ .*

*Proof.* (a) Assume to the contrary that  $\mathcal{X} \times \mathbb{C}$  is tight, while there is a nontrivial  $\mathbb{C}_+$ -action  $t \mapsto \exp(t\partial)$  on  $\mathcal{X}$ , where  $\partial \in \text{LND}(\mathcal{O}(\mathcal{X}))$ . The induced  $\mathbb{C}_+$ -action on the cylinder  $\mathcal{X} \times \mathbb{C}$  has the same infinitesimal generator  $\partial$ , regarded this time as a locally nilpotent derivation of the algebra  $\mathcal{O}(\mathcal{X})[u]$  with  $\partial u = 0$ . The derivation  $\partial_1 = u\partial$  is again locally nilpotent. The associated vector field on  $\mathcal{X} \times \mathbb{C}$  vanishes on the section  $\mathcal{X} \times \{0\}$ . Hence the induced  $\mathbb{C}_+$ -action  $t \mapsto \exp(t\partial_1) \in \text{Aut}(\mathcal{X} \times \mathbb{C})$  fixes each point  $Q_0 \in \mathcal{X} \times \{0\}$ , and moves a general point  $Q_1 \in \mathcal{X} \times \mathbb{C}$  in the horizontal direction. Taking the points  $Q_0$  and  $Q_1$  on the same ruling  $\{q\} \times \mathbb{C}$ , where  $q \in \mathcal{X}$  is general, we see that their images under the automorphism  $\alpha = \exp(\partial_1)$  do not belong any longer to the same ruling. It follows that  $\alpha \in \text{Aut}(\mathcal{X} \times \mathbb{C})$  is not triangular. Therefore, the cylinder  $\mathcal{X} \times \mathbb{C}$  cannot be tight, a contradiction.

(b) Clearly, the strong 1-cancellation property implies tightness. To prove the converse, suppose that the strong 1-cancellation fails for  $\mathcal{X}$ , i.e., that  $\mathcal{X} \times \mathbb{C} \simeq \mathcal{Y} \times \mathbb{C}$ , where the rulings of these cylinders are different. Let us show that in this case  $\mathcal{X} \times \mathbb{C}$  cannot be tight.

Indeed, assume to the contrary that  $\mathcal{X} \times \mathbb{C}$  is tight. Let  $\tau'$  be the free  $\mathbb{C}_+$ -action on  $\mathcal{X} \times \mathbb{C} \simeq \mathcal{Y} \times \mathbb{C}$  via the shifts along the rulings of the cylinder  $\mathcal{Y} \times \mathbb{C}$ . Since  $\mathcal{X} \times \mathbb{C}$  is tight, this action is triangular, and so, it induces a  $\mathbb{C}_+$ -action on  $\mathcal{X}$ . The latter action is non-trivial, since otherwise the both families of rulings would be the same. By (a), this implies that  $\mathcal{X} \times \mathbb{C}$  is not tight, a contradiction.  $\square$

**Example 2.9.** Let  $\mathcal{X}$  be the surface given in  $\mathbb{C}^3$  by equation  $x^n y = p(x, z)$ , where  $n \in \mathbb{N}$ ,  $p \in \mathbb{C}[x, y]$ , and  $p(0, z) \neq 0$ . Then  $\mathcal{X}$  admits a nontrivial  $\mathbb{C}_+$ -action  $t \mapsto \exp(t\partial)$ , where  $\partial = (\partial p / \partial z)\partial / \partial y + x^n \partial / \partial z$ . Hence by Proposition 2.8(a) the cylinder  $\mathcal{X} \times \mathbb{C}$  is not tight.

This applies, in particular, to the Danielewski surfaces  $\mathcal{X}_n = \{x^n y - z^2 + 1 = 0\}$  in  $\mathbb{C}^3$ ,  $n \in \mathbb{N}$ . By Danielewski's Theorem [10], the cylinders  $\mathcal{X}_n \times \mathbb{C}$  are all isomorphic, whereas, according to Fieseler [16],  $\mathcal{X}_n$  is not even homeomorphic to  $\mathcal{X}_m$  for  $n \neq m$ . Thus these surfaces provide counterexamples to cancellation<sup>6</sup>. In addition, the cylinders over the Danielewski surfaces are not tight.

To make the latter example more explicit, take two points  $Q_0 = (q, 0)$  and  $Q_1 = (q, 1)$  on the same ruling  $\{q\} \times \mathbb{C}$  of the cylinder  $\mathcal{X}_n \times \mathbb{C}$ , where  $q = (0, 0, 1) \in \mathcal{X}_n$ . Letting  $\partial = 2z\partial / \partial y + x^n \partial / \partial z \in \text{LND}(\mathbb{C}[x, y, z])$ , we consider the locally nilpotent derivation  $\partial_1 = u\partial$  of the algebra  $\mathbb{C}[x, y, z, u]$ . Since  $\partial(x^n y - z^2 + 1) = 0$ , the derivation  $\partial_1$  descends to a locally nilpotent derivation of the quotient  $\mathbb{C}[x, y, z, u] / (x^n y - z^2 + 1) \simeq \mathcal{O}(\mathcal{X}_n \times \mathbb{C})$ . The triangular automorphism

$$\alpha = \exp(\partial_1) \in \text{Aut } \mathbb{C}^4, \quad (x, y, z, u) \mapsto (x, y + 2zu + x^n u^2, z + x^n u, u),$$

<sup>6</sup>See e.g., [28] for further examples of non-cancellation.

preserves the hypersurface  $\{x^n y - z^2 + 1 = 0\} \simeq \mathcal{X}_n \times \mathbb{C}$  in  $\mathbb{C}^4$ . This action fixes  $Q_0$  and sends  $Q_1$  to  $\alpha(Q_1) = (0, 2, 1, 1)$ , so that the points  $\alpha(Q_0)$  and  $\alpha(Q_1)$  do not belong any more to the same ruling of the cylinder. Therefore,  $\alpha \in \text{Aut}(\mathcal{X}_n \times \mathbb{C}) \setminus \text{Aut}_\Delta(\mathcal{X}_n \times \mathbb{C})$ .

**Remark 2.10.** By Theorem 3.1 in [4], the surface  $\mathcal{X}_1$  is *flexible*, i.e., the tangent vectors to the orbits of the  $\mathbb{C}_+$ -actions on  $\mathcal{X}_1$  generate the tangent space at any point of  $\mathcal{X}_1$ . It follows that the cylinder  $\mathcal{X}_1 \times \mathbb{C}$  is also flexible. By Theorem 0.1 in [3], the flexibility implies the  $k$ -transitivity of the automorphism group  $\text{Aut}(\mathcal{X}_1 \times \mathbb{C})$  for any  $k \geq 1$ . In particular, for any  $n \geq 1$ , there are automorphisms of the cylinder  $\mathcal{X}_n \times \mathbb{C} \cong \mathcal{X}_1 \times \mathbb{C}$  that do not preserve the cylinder structure, and so send it to another such structure over the same base  $\mathcal{X}_n$ . This shows again that none of the cylinders  $\mathcal{X}_n \times \mathbb{C}$  is tight.

We call the *rulings* of a cylinder  $\mathcal{X} \times \mathbb{C}$  the fibers of the first projection  $\text{pr}_1: \mathcal{X} \times \mathbb{C} \rightarrow \mathcal{X}$ . These are the orbits of the ‘vertical’ free  $\mathbb{C}_+$ -action  $\tau$  on  $\mathcal{X} \times \mathbb{C}$  via translations along the second factor. In addition to the criterion of tightness in Proposition 2.8(b), we have the following one. Consider the locally nilpotent derivation  $\partial/\partial y$  on the algebra  $\mathcal{O}(\mathcal{X} \times \mathbb{C}) \cong \mathcal{O}(\mathcal{X})[y]$  with the phase flow  $(x, y) \mapsto (x, y + tx)$ . Its replica  $b(x)\partial/\partial y$ , where  $b \in \mathcal{O}(\mathcal{X})$ , has the phase flow  $(x, y) \mapsto (x, y + tb(x))$ .

**Proposition 2.11.** *The cylinder  $\mathcal{X} \times \mathbb{C}$  is tight if and only if any  $\mathbb{C}_+$ -action on this cylinder preserves each ruling, i.e., is of the form  $(x, y) \mapsto (x, y + tb(x))$ , where  $b \in \mathcal{O}(\mathcal{X})$ .*

*Proof.* Suppose first that the cylinder  $\mathcal{X} \times \mathbb{C}$  is not tight, and so, admits a second, different cylinder structure with a different family of rulings. Then the induced  $\mathbb{C}_+$ -action, say,  $\tau'$  on  $\mathcal{X} \times \mathbb{C}$  has different orbits, and so, does not preserve the rulings of the original cylinder.

Conversely, suppose that the cylinder  $\mathcal{X} \times \mathbb{C}$  is tight. Consider a  $\mathbb{C}_+$ -action  $\varphi$  on this cylinder. Since  $\varphi$  is triangular, it induces a  $\mathbb{C}_+$ -action, say,  $\psi$  on  $\mathcal{X}$ . Assuming to the contrary that  $\varphi$  is not vertical, i.e., does not preserve the rulings  $\{x\} \times \mathbb{C}$ ,  $x \in \mathcal{X}$ , the action  $\psi$  is nontrivial. By Proposition 2.8(a), the cylinder  $\mathcal{X} \times \mathbb{C}$  cannot be tight, a contradiction.  $\square$

Let us show further that an arbitrary locally nilpotent derivation on a tight cylinder is a replica of the derivation  $\partial/\partial y$ .

**Proposition 2.12.** *If  $\mathcal{X} \times \mathbb{C}$  is tight then any  $\partial \in \text{LND}(\mathcal{O}(\mathcal{X} \times \mathbb{C}))$  is a replica of the derivation  $\partial/\partial y$ , i.e.,*

$$\partial = f\partial/\partial y, \quad \text{where } f \in \mathcal{O}^\tau(\mathcal{X} \times \mathbb{C}) = \text{pr}_1^*(\mathcal{O}(\mathcal{X})) = \ker \partial/\partial y.$$

Consequently, any  $\mathbb{C}_+$ -action on  $\mathcal{X} \times \mathbb{C}$  is of the form

$$(x, y) \mapsto (x, y + tb(x)), \quad \text{where } t \in \mathbb{C} \text{ and } b \in \mathcal{O}(\mathcal{X}).$$

*Proof.* Indeed, both  $\partial$  and  $\partial/\partial y$  can be viewed as regular vector fields on  $\mathcal{X} \times \mathbb{C}$ , where the latter field is non-vanishing. By Proposition 2.11, these vector fields are proportional. That is, there exists a function  $f \in \mathcal{O}^\tau(\mathcal{X} \times \mathbb{C})$  such that  $\partial = f\partial/\partial y$ , which proves the first assertion. Now the second follows.  $\square$

#### 2.4. The group $\text{SAut}(\mathcal{X} \times \mathbb{C})$ .

**Definition 2.13.** Let  $\mathcal{Z}$  be an irreducible algebraic variety. A subgroup  $G \subset \text{Aut } \mathcal{Z}$  is called *algebraic* if it admits a structure of an algebraic group such that the natural map  $G \times \mathcal{Z} \rightarrow \mathcal{Z}$  is a morphism. The *special automorphism group*  $\text{SAut } \mathcal{Z}$  is the subgroup of  $\text{Aut } \mathcal{Z}$  generated by all the algebraic subgroups of  $\text{Aut } \mathcal{Z}$  isomorphic to  $\mathbb{C}_+$  (see e.g. [3]). Clearly,  $\text{SAut } \mathcal{Z}$  is a normal subgroup of  $\text{Aut } \mathcal{Z}$ .

Assume that  $\mathcal{X}$  is tight. Due to (19) we have the decomposition

$$\text{Aut}(\mathcal{X} \times \mathbb{C}) \cong (\mathcal{O}_+(\mathcal{X}) \rtimes \mathcal{O}^\times(\mathcal{X})) \rtimes \text{Aut } \mathcal{X}. \quad (21)$$

In the next corollary we show that the group  $\text{SAut}(\mathcal{X} \times \mathbb{C})$  corresponds to the factor  $\mathcal{O}_+(\mathcal{X})$  in (21).

**Proposition 2.14.** *If the cylinder  $\mathcal{X} \times \mathbb{C}$  is tight, then  $G = \text{SAut}(\mathcal{X} \times \mathbb{C})$  is an Abelian group with Lie algebra<sup>7</sup>*

$$L = \text{Lie } G = \mathcal{O}_+^\tau(\mathcal{X} \times \mathbb{C})\partial/\partial y = \mathcal{O}_+(\mathcal{X})\partial/\partial y.$$

Furthermore, the exponential map  $\exp: L \rightarrow G$  yields an isomorphism of groups

$$\mathcal{O}_+(\mathcal{X}) \xrightarrow{\cong} \text{SAut}(\mathcal{X} \times \mathbb{C}).$$

*Proof.* This follows easily from Proposition 2.12.  $\square$

From Proposition 2.11 we deduce such a corollary.

**Corollary 2.15.** *The cylinder  $\mathcal{X} \times \mathbb{C}$  is tight if and only if the orbits of the group  $\text{SAut}(\mathcal{X} \times \mathbb{C})$  are the rulings of this cylinder.*

**2.5. The Makar-Limanov invariant of a cylinder.** The subalgebra of  $\tau$ -invariants  $\mathcal{O}^\tau(\mathcal{X} \times \mathbb{C}) \subset \mathcal{O}(\mathcal{X} \times \mathbb{C})$  admits yet another interpretation.

**Definition 2.16.** Let  $\mathcal{Z}$  be an affine algebraic variety over  $\mathbb{C}$ . The ring of invariants  $\mathcal{O}(\mathcal{Z})^{\text{SAut } \mathcal{Z}}$  is called the *Makar-Limanov invariant* of  $\mathcal{Z}$  and is denoted by  $\text{ML}(\mathcal{Z})$ . This ring is invariant under the induced action of the group  $\text{Aut } \mathcal{Z}$  on  $\mathcal{O}(\mathcal{Z})$ .

Consider a locally nilpotent derivation  $\partial \in \text{LND}(\mathcal{O}(\mathcal{Z}))$  and the corresponding unipotent one-parameter algebraic group  $H = \exp(\mathbb{C}\partial) \subset \text{Aut } \mathcal{Z}$ . We have  $\mathcal{O}(\mathcal{Z})^H = \ker \partial$  and so

$$\text{ML}(\mathcal{Z}) = \bigcap_{\partial \in \text{LND}(\mathcal{O}(\mathcal{Z}))} \ker \partial. \quad (22)$$

<sup>7</sup>See §3.7.

Since the  $\text{SAut}(\mathcal{X} \times \mathbb{C})$ -invariant functions are exactly the functions constant on each ruling of the cylinder, the next corollary is immediate from Corollary 2.15.

**Corollary 2.17.** *The cylinder  $\mathcal{X} \times \mathbb{C}$  over an affine variety  $\mathcal{X}$  is tight if and only if*

$$\text{ML}(\mathcal{X} \times \mathbb{C}) = \mathcal{O}^\tau(\mathcal{X} \times \mathbb{C}) = \mathcal{O}(\mathcal{X}). \quad (23)$$

It is worthwhile mentioning the following related result.

**Dryło's Theorem II** ([12]). Let  $\mathcal{X}$  and  $\mathcal{Y}$  be irreducible affine varieties over an algebraically closed field  $k$ . If  $\mathcal{X}$  is rigid, then  $\text{ML}(\mathcal{X} \times \mathcal{Y}) = \mathcal{O}(\mathcal{X}) \otimes_k \text{ML}(\mathcal{Y})$ .

There is a stronger statement in the case where  $\mathcal{Y}$  is the affine line.

**Bandman, Makar-Limanov, and Crachiola's Theorem** ([5, Lemma 2], [8, Theorem 3.1]). *Let  $\mathcal{X}$  be an affine variety over an arbitrary field  $k$ . If  $\text{ML}(\mathcal{X}) = \mathcal{O}(\mathcal{X})$ , then the equality  $\text{ML}(\mathcal{X} \times \mathbb{A}_k^1) = \mathcal{O}(\mathcal{X})$  holds.*

Summarizing several previous results, we can deduce the following tightness criteria<sup>8</sup>.

**Theorem 2.18.** *For an affine variety  $\mathcal{X}$ , the following conditions are equivalent:*

- (i) *the cylinder  $\mathcal{X} \times \mathbb{C}$  is tight, i.e.,  $\text{Aut}(\mathcal{X} \times \mathbb{C}) = \text{Aut}_\Delta(\mathcal{X} \times \mathbb{C})$ ;*
- (ii) *the strong 1-cancellation holds for  $\mathcal{X}$ ;*
- (iii) *any  $\mathbb{G}_a$ -action on  $\mathcal{X} \times \mathbb{C}$  is of the form  $(x, y) \mapsto (x, y + tb(x))$ , where  $t \in \mathbb{C}$  and  $b \in \mathcal{O}(\mathcal{X})$ ;*
- (iv) *any  $\partial \in \text{LND}(\mathcal{O}(\mathcal{X} \times \mathbb{C}))$  is of the form  $\partial = b(x)\partial/\partial y$ , where  $b \in \mathcal{O}(\mathcal{X})$ ;*
- (v) *the orbits of the group  $\text{SAut}(\mathcal{X} \times \mathbb{C})$  are the rulings of this cylinder;*
- (vi)  $\text{ML}(\mathcal{X} \times \mathbb{C}) = \mathcal{O}(\mathcal{X})$ ;
- (vii)  $\text{ML}(\mathcal{X}) = \mathcal{O}(\mathcal{X})$ , i.e.,  $\text{LND}(\mathcal{O}(\mathcal{X})) = \emptyset$ .

*Proof.* Equivalences (i) $\Leftrightarrow$ (ii), (i) $\Leftrightarrow$ (iii), (i) $\Leftrightarrow$ (v), and (i) $\Leftrightarrow$ (vi) are established in Propositions 2.8 and 2.11 and Corollaries 2.15 and 2.17, respectively. Equivalence (iii) $\Leftrightarrow$ (iv) is easy, and implication (i) $\Rightarrow$ (iv) follows from Proposition 2.12. Thus the conditions (i)-(vi) are all equivalent. Implication (vii) $\Rightarrow$ (vi) follows from the Bandman-Makar-Limanov-Crachiola Theorem. To show (vi) $\Rightarrow$ (vii) suppose that  $\text{ML}(\mathcal{X}) \neq \mathcal{O}(\mathcal{X})$ . Then  $\text{LND}(\mathcal{O}(\mathcal{X})) \neq \emptyset$ , and so, there is a nontrivial  $\mathbb{C}_+$ -action on  $\mathcal{X}$ . It follows by Proposition 2.8(a) that the cylinder  $\mathcal{X} \times \mathbb{C}$  admits a non-triangular automorphism. Then it possesses a second cylinder structure (over the same base  $\mathcal{X}$ ). Let  $\partial$  be the locally nilpotent derivation of  $\mathcal{O}(\mathcal{X} \times \mathbb{C})$  generating the one-parameter group of shifts along the rulings of the new cylinder. The kernel  $\ker \partial$  is different from  $\mathcal{O}^\tau(\mathcal{X} \times \mathbb{C})$ . Hence the intersection  $\text{ML}(\mathcal{X} \times \mathbb{C})$  of the kernels of the locally nilpotent derivations on  $\mathcal{O}(\mathcal{X} \times \mathbb{C})$

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<sup>8</sup>As we already mentioned, these criteria remain valid over any algebraically closed field  $k$  of zero characteristic.

is strictly smaller than  $\mathcal{O}^\tau(\mathcal{X} \times \mathbb{C})$ , that is,  $\text{ML}(\mathcal{X} \times \mathbb{C}) \neq \mathcal{O}^\tau(\mathcal{X} \times \mathbb{C})$  (see (22)). This provides (vi) $\Rightarrow$ (vii).  $\square$

**2.6. Configuration spaces and discriminant levels as cylinders over rigid bases.** In Proposition 2.21 we show that the bases  $\mathcal{C}_{\text{blc}}^{n-1}$ ,  $\mathcal{SC}_{\text{blc}}^{n-2}$ , and  $\Sigma_{\text{blc}}^{n-2}$  of the cylinders (27) possess a property, which implies the rigidity. Therefore, all automorphisms of these cylinders are triangular (Corollary 2.22). The latter applies as well to the hypersurfaces  $D_n(Q) = c \neq 0$ . Indeed, since  $D_n$  is homogeneous, any such hypersurface is isomorphic to  $\mathcal{SC}^{n-1}$ .

**Notation 2.19.** For any  $X$  and any  $n \in \mathbb{N}$ , let  $\mathcal{C}_{\text{ord}}^n(X)$  denote the *ordered configuration space* of  $X$ , i.e.,  $\mathcal{C}_{\text{ord}}^n(X) = \{(q_1, \dots, q_n) \in X^n \mid q_i \neq q_j \text{ for all } i \neq j\}$ . The symmetric group  $\mathbf{S}(n)$  acts freely on  $\mathcal{C}_{\text{ord}}^n(X)$  permuting the coordinates  $q_1, \dots, q_n$ . By definition,  $\mathcal{C}_{\text{ord}}^n(X)/\mathbf{S}(n) = \mathcal{C}^n(X)$ . We let

$$\mathcal{C}_{\text{ord}}^n \stackrel{\text{def}}{=} \mathcal{C}_{\text{ord}}^n(\mathbb{C}) \quad \text{and} \quad \mathcal{C}_{\text{ord,blc}}^{n-1} \stackrel{\text{def}}{=} \{q = \{q_1, \dots, q_n\} \in \mathcal{C}_{\text{ord}}^n \mid q_1 + \dots + q_n = 0\}.$$

Clearly  $\mathcal{C}_{\text{ord}}^n/\mathbf{S}(n) = \mathcal{C}^n$  (see (3)) and  $\mathcal{C}_{\text{ord,blc}}^{n-1}/\mathbf{S}(n) = \mathcal{C}_{\text{blc}}^{n-1}$ . The variety  $\mathcal{C}^n$  and the discriminant variety

$$\Sigma^{n-1} = \{z \in \mathbb{C}^n \mid d_n(z) = 0\}$$

(see (4)) can be viewed as complementary to each other parts of the symmetric power  $\text{Sym}^n \mathbb{C} = \mathbb{C}_{(q)}^n/\mathbf{S}(n)$ . We have the quotient morphisms

$$p: \mathbb{C}_{(q)}^n \rightarrow \text{Sym}^n \mathbb{C} \cong \mathbb{C}_{(z)}^n, \quad \Delta^{n-1} \rightarrow \Sigma^{n-1}, \quad \text{and} \quad \mathbb{C}_{(q)}^n \setminus \Delta^{n-1} \rightarrow \mathcal{C}^n, \quad (24)$$

where  $\Delta^{n-1} \stackrel{\text{def}}{=} \bigcup_{i \neq j} \{q = (q_1, \dots, q_n) \in \mathbb{C}^n \mid q_i = q_j\}$  is the big diagonal. The points  $z \in \Sigma^{n-1}$  are in one-to-one correspondence with unordered  $n$ -term multisets (or corteges)  $Q = \{q_1, \dots, q_n\}$ ,  $q_i \in \mathbb{C}$ , with at least one repetition.

Let  $\mathcal{Z}$  be one of the varieties  $\mathcal{C}^n$ ,  $\mathcal{SC}^{n-1}$ , or  $\Sigma^{n-1}$ . The corresponding *balanced* variety  $\mathcal{Z}_{\text{blc}} \subset \mathcal{Z}$  is defined by

$$\mathcal{Z}_{\text{blc}} = \{Q \in \mathcal{Z} \mid \text{bc}(Q) = 0\}, \quad \dim_{\mathbb{C}} \mathcal{Z}_{\text{blc}} = \dim_{\mathbb{C}} \mathcal{Z} - 1 \quad (25)$$

(cf. (8)). The free regular  $\mathbb{C}_+$ -action  $\tau$  on  $\mathbb{C}^n$  defined by  $\tau_\zeta Q = Q + \zeta = \{q_1 + \zeta, \dots, q_n + \zeta\}$  for  $\zeta \in \mathbb{C}$  and  $Q \in \mathbb{C}^n$  preserves  $\mathcal{Z}$ . The orbit map of  $\tau$  gives the morphism

$$\pi: \mathcal{Z} \rightarrow \mathcal{Z}_{\text{blc}}, \quad Q \mapsto Q^\circ \stackrel{\text{def}}{=} Q - \text{bc}(Q), \quad \text{and} \quad \pi': \mathcal{Z} \rightarrow \mathbb{C}, \quad Q \mapsto \text{bc}(Q) \quad (26)$$

with all fibers reduced and isomorphic to  $\mathbb{C}$ . The corresponding *cylindrical* direct decomposition  $\mathcal{Z} = \mathcal{Z}_{\text{blc}} \times \mathbb{C}$  leads to decompositions of our varieties

$$\mathcal{C}^n = \mathcal{C}_{\text{blc}}^{n-1} \times \mathbb{C}, \quad \mathcal{SC}^{n-1} = \mathcal{SC}_{\text{blc}}^{n-2} \times \mathbb{C}, \quad \text{and} \quad \Sigma^{n-1} = \Sigma_{\text{blc}}^{n-2} \times \mathbb{C}, \quad (27)$$

which play an important part in what follows.

Note that the regular part  $\text{reg } \Sigma^{n-1}$  of the discriminant variety  $\Sigma^{n-1}$  consists of all the unordered  $n$ -multisets  $Q = \{q_1, \dots, q_{n-2}, u, u\} \subset \mathbb{C}$  with  $q_i \neq q_j$  for  $i \neq j$  and  $q_i \neq u$  for all  $i$ . Since the hyperplane  $q_1 + \dots + q_n = 0$  is transversal to each of the hyperplanes  $q_i = q_j$ , the regular part  $\text{reg } \Sigma_{\text{blc}}^{n-2}$  of  $\Sigma_{\text{blc}}^{n-2}$  consists of all the multisets  $Q = \{q_1, \dots, q_{n-2}, u, u\}$  as above that satisfy the additional condition  $\sum_{i=1}^{n-2} q_i + 2u = 0$ . In the proofs of Proposition 2.21 and Theorem 7.1 we need the following lemma.

**Lemma 2.20.** *For  $n > 2$  the regular part  $\text{reg } \Sigma_{\text{blc}}^{n-2}$  of  $\Sigma_{\text{blc}}^{n-2}$  is isomorphic to the configuration space  $\mathcal{C}^{n-2}(\mathbb{C}^*)$ . Consequently,  $\text{Aut}(\text{reg } \Sigma_{\text{blc}}^{n-2}) \cong \text{Aut } \mathcal{C}^{n-2}(\mathbb{C}^*)$ .*

*Proof.* An isomorphism  $\text{reg } \Sigma_{\text{blc}}^{n-2} \cong \mathcal{C}^{n-2}(\mathbb{C}^*)$  does exist since both these varieties are cross-sections of the standard  $\mathbb{C}_+$ -action  $\tau$  on the cylinder  $\text{reg } \Sigma^{n-2} = (\text{reg } \Sigma_{\text{blc}}^{n-2}) \times \mathbb{C}$  (see (27)). To construct such an isomorphism explicitly, for any

$$Q = \{q_1, \dots, q_{n-2}, u, u\} \in \text{reg } \Sigma_{\text{blc}}^{n-2}, \quad \text{where } u = -\frac{1}{2} \sum_{i=1}^{n-2} q_i,$$

we let  $\tilde{Q} = \{q_1 - u, \dots, q_{n-2} - u\}$ . Then  $\tilde{Q} \in \mathcal{C}^{n-2}(\mathbb{C}^*)$ , and we have an epimorphism

$$\varphi: \text{reg } \Sigma_{\text{blc}}^{n-2} \rightarrow \mathcal{C}^{n-2}(\mathbb{C}^*), \quad \varphi(Q) = \tilde{Q}. \quad (28)$$

To show that  $\varphi$  is an isomorphism, for any  $Q' = \{q'_1, \dots, q'_{n-2}\} \in \mathcal{C}^{n-2}(\mathbb{C}^*)$  take

$$v = -\frac{1}{n} \sum_{i=1}^{n-2} q'_i \quad \text{and} \quad Q'' = \{q'_1 + v, \dots, q'_{n-2} + v, v, v\};$$

note that  $v = u$  for  $Q' = \tilde{Q}$  as above. Then  $Q'' \in \text{reg } \Sigma_{\text{blc}}^{n-2}$  and the morphism

$$\psi: \mathcal{C}^{n-2}(\mathbb{C}^*) \rightarrow \text{reg } \Sigma_{\text{blc}}^{n-2}, \quad \psi(Q') = Q'', \quad (29)$$

is the inverse of  $\varphi$ . □

**Proposition 2.21.** *For  $n > 2$ , let  $\mathcal{X}$  be one of the varieties  $\mathcal{C}_{\text{blc}}^{n-1}$ ,  $\mathcal{S}\mathcal{C}_{\text{blc}}^{n-2}$ , or  $\Sigma_{\text{blc}}^{n-2}$ . Then any morphism  $\mathbb{C} \rightarrow \text{reg } \mathcal{X}$  is constant. In particular, these varieties are rigid, and the cylinders  $\mathcal{C}_{\text{blc}}^{n-1} \times \mathbb{C}$ ,  $\mathcal{S}\mathcal{C}_{\text{blc}}^{n-2} \times \mathbb{C}$ , and  $\Sigma_{\text{blc}}^{n-2} \times \mathbb{C}$  in (27) are tight.*

*Proof.* Let us show first that any morphism  $f: \mathbb{C} \rightarrow \mathcal{C}_{\text{blc}}^{n-1}$  is constant. Consider the unramified  $\mathbf{S}(n)$ -covering  $p: \mathcal{C}_{\text{blc,ord}}^{n-1} \rightarrow \mathcal{C}_{\text{blc}}^{n-1}$ . By the monodromy theorem  $f$  can be lifted to a morphism  $g = (g_1, \dots, g_n): \mathbb{C} \rightarrow \mathcal{C}_{\text{blc,ord}}^{n-1}$ . For any  $i \neq j$  the regular function  $g_i - g_j$  on  $\mathbb{C}$  does not vanish, hence is constant. In particular,  $g_i = g_1 + c_i$ , where  $c_i \in \mathbb{C}$ ,  $i = 1, \dots, n$ , and so,  $0 = \sum_{i=1}^n g_i = ng_1 + c$ , where  $c = \sum_{i=1}^n c_i$ . Thus,  $g_1 = \text{const}$ , so  $g_i = \text{const}$  for all  $i = 1, \dots, n$ . Hence  $f = \text{const}$ , and the variety  $\mathcal{C}_{\text{blc}}^{n-1}$  is rigid.

Since  $\mathcal{S}\mathcal{C}_{\text{blc}}^{n-2} \subset \mathcal{C}_{\text{blc}}^{n-1}$ , any morphism  $\mathbb{C} \rightarrow \mathcal{S}\mathcal{C}_{\text{blc}}^{n-2}$  is constant and  $\mathcal{S}\mathcal{C}_{\text{blc}}^{n-2}$  is rigid.

It remains to show that any morphism  $\mathbb{C} \rightarrow \text{reg } \Sigma_{\text{blc}}^{n-2}$  is constant. For  $n = 3$  we have  $\text{reg } \Sigma_{\text{blc}}^{n-2} \cong \mathbb{C}^*$ , hence the claim follows.

For  $n \geq 4$ , by Lemma 2.20, it suffices to show that any morphism  $f: \mathbb{C} \rightarrow \mathcal{C}^{n-2}(\mathbb{C}^*)$  is constant. By monodromy theorem  $f$  admits a lift  $g: \mathbb{C} \rightarrow \mathcal{C}_{\text{ord}}^{n-2}(\mathbb{C}^*) \subset (\mathbb{C}^*)^{n-2}$  to the unramified  $\mathbf{S}(n-2)$ -covering  $\mathcal{C}_{\text{ord}}^{n-2}(\mathbb{C}^*) \rightarrow \mathcal{C}^{n-2}(\mathbb{C}^*)$ . This implies that both  $g$  and  $f$  are constant, since any morphism  $\mathbb{C} \rightarrow \mathbb{C}^*$  is.  $\square$

Using Corollary 2.4 and Proposition 2.21, we can deduce such a corollary.

**Corollary 2.22.** *For  $n > 2$ , all automorphisms of the cylinders*

$$\mathcal{C}^n \cong \mathcal{C}_{\text{blc}}^{n-1} \times \mathbb{C}, \quad \mathcal{S}\mathcal{C}^{n-1} \cong \mathcal{S}\mathcal{C}_{\text{blc}}^{n-2} \times \mathbb{C}, \quad \text{and} \quad \Sigma^{n-1} \cong \Sigma_{\text{blc}}^{n-2} \times \mathbb{C}$$

*are triangular, and (17)-(19) hold for the corresponding automorphism groups.*

### 3. THE AUTOMORPHISM GROUPS OF TIGHT CYLINDERS

**3.1. The structure of the orbits.** Let  $\tau$  stands as before for the standard  $\mathbb{C}_+$ -action on  $\mathcal{X} \times \mathbb{C}$  by shifts along the second factor, and let  $U = \exp(\mathbb{C}\partial/\partial y)$  be the corresponding one-parameter unipotent subgroup of  $\text{SAut}(\mathcal{X} \times \mathbb{C})$ . Consider also the subgroup  $B \stackrel{\text{def}}{=} U \cdot \text{Aut } \mathcal{X} \cong U \rtimes \text{Aut } \mathcal{X}$  of  $\text{Aut}(\mathcal{X} \times \mathbb{C})$ , and let  $B_0 \cong U \rtimes \text{Aut}_0 \mathcal{X}$  be its neutral component.

More generally, given a character  $\chi$  of  $\text{Aut } \mathcal{X}$  (of  $\text{Aut}_0 \mathcal{X}$ , respectively) we let

$$B(\chi) = \{F \in \text{Aut}(\mathcal{X} \times \mathbb{C}) \mid F : (x, y) \mapsto (Sx, \chi(S)y + b), S \in \text{Aut } \mathcal{X}, b \in \mathbb{C}\},$$

and let  $B_0(\chi)$  be the neutral component of  $B(\chi)$ . Thus,  $B = B(1)$  and  $B_0 = B_0(1)$ . Clearly,  $B(\chi)$  ( $B_0(\chi)$ , respectively) is algebraic as soon as  $\text{Aut } \mathcal{X}$  ( $\text{Aut}_0 \mathcal{X}$ , respectively) is.

From Proposition 2.12 we deduce the following result.

**Corollary 3.1.** *If the cylinder  $\mathcal{X} \times \mathbb{C}$  over an affine variety  $\mathcal{X}$  is tight, then the orbits of the automorphism group  $\text{Aut}(\mathcal{X} \times \mathbb{C})$  (of  $\text{Aut}_0(\mathcal{X} \times \mathbb{C})$ , respectively) coincide with the orbits of the group  $B(\chi)$  ( $B_0(\chi)$ , respectively), whatever is the character  $\chi$  of  $\text{Aut } \mathcal{X}$  (of  $\text{Aut}_0 \mathcal{X}$ , respectively).*

*Proof.* We give a proof for the group  $\text{Aut}(\mathcal{X} \times \mathbb{C})$ ; that for  $\text{Aut}_0(\mathcal{X} \times \mathbb{C})$  is similar. Recall that any automorphism  $F$  of the tight cylinder  $\mathcal{X} \times \mathbb{C}$  is triangular, and so, can be written as

$$F(x, y) = (Sx, A(x)y) \text{ for any } (x, y) \in \mathcal{X} \times \mathbb{C}, \quad (30)$$

where  $S \in \text{Aut } \mathcal{X}$  and  $A \in \text{Mor}(\mathcal{X}, \text{Aff } \mathbb{C})$ . It follows that the  $B(\chi)$ -orbit  $B(\chi)Q$  of a point  $Q = (x, y)$  in  $\mathcal{X} \times \mathbb{C}$  is  $B(\chi)Q = [(\text{Aut } \mathcal{X})x] \times \mathbb{C}$ . By virtue of Proposition 2.12 the  $\text{SAut}(\mathcal{X} \times \mathbb{C})$ -orbits in  $\mathcal{X} \times \mathbb{C}$  coincide with the  $\tau$ -orbits, i.e., with the rulings of the cylinder  $\mathcal{X} \times \mathbb{C}$ . Now the assertion follows from decomposition (21) and the isomorphism  $\mathcal{O}_+(\mathcal{X}) \cong \text{SAut}(\mathcal{X} \times \mathbb{C})$  of Proposition 2.14.  $\square$



**Remark 3.2.** Let  $\mathcal{X}$  be an affine variety with  $\text{LND}(\mathcal{X}) = \emptyset$ . If  $\text{Aut}_0 \mathcal{X}$  is an algebraic group, then this is an algebraic torus, see [25, Lemma 3]. In this case  $B_0(\chi)$  is a metabelian linear algebraic group isomorphic to a semi-direct product  $\mathbb{C}_+ \rtimes (\mathbb{C}^*)^r$ , where  $r \geq 0$  and  $(\mathbb{C}^*)^r$  acts on  $\mathbb{C}_+$  via multiplication by the character  $\chi$  of the torus  $(\mathbb{C}^*)^r$ .

In the spirit of Tame Map Theorem, the following holds.

**Proposition 3.3.** *Given a tight cylinder  $\mathcal{X} \times \mathbb{C}$  over an affine variety  $\mathcal{X}$  and a character  $\chi$  of  $\text{Aut} \mathcal{X}$ , any automorphism  $F$  of  $\mathcal{X} \times \mathbb{C}$  admits a unique factorization*

$$F: \mathcal{X} \times \mathbb{C} \xrightarrow{T \times \{\text{id}\}} B(\chi) \times (\mathcal{X} \times \mathbb{C}) \xrightarrow{\alpha} \mathcal{X} \times \mathbb{C}, \quad (31)$$

where  $\alpha$  stands for the  $B(\chi)$ -action on  $\mathcal{X} \times \mathbb{C}$ , and  $T: \mathcal{X} \times \mathbb{C} \rightarrow B(\chi)$  is a morphism with a constant  $(\text{Aut} \mathcal{X})$ -component.

*Proof.* By Corollary 3.1 for any point  $Q = (x, y) \in \mathcal{X} \times \mathbb{C}$  there exists an element  $T(Q) \in B(\chi)$ ,  $T(Q): (x', y') \mapsto (S(Q)x', \chi(S(Q))y' + f(Q))$  for some  $f(Q) \in \mathbb{C}$ , such that

$$F(Q) = T(Q)Q = (S(Q)x, \chi(S(Q))y + f(Q)) \in \mathcal{X} \times \mathbb{C}. \quad (32)$$

On the other hand, according to (30),

$$F(Q) = (Sx, a(x)y + b(x)) \in \mathcal{X} \times \mathbb{C}, \quad (33)$$

where  $S \in \text{Aut} \mathcal{X}$ ,  $a \in \mathcal{O}^\times(\mathcal{X})$ , and  $b \in \mathcal{O}_+(\mathcal{X})$  are uniquely determined by  $F$ . Comparing (32) and (33) yields  $S(Q)x = Sx$  and  $f(Q) = (a(x) - \chi(S))y + b(x)$  for any  $Q \in \mathcal{X} \times \mathbb{C}$ . Vice versa, the latter equalities define unique  $f \in \mathcal{O}(\mathcal{X} \times \mathbb{C})$  and  $S \in \text{Aut} \mathcal{X}$  such that  $T: \mathcal{X} \times \mathbb{C} \rightarrow B(\chi)$ ,  $Q \mapsto (S, z \mapsto \chi(S)z + f(Q))$ , fits in (31) i.e.,  $F = \alpha \circ (T \times \text{id})$ , as required.  $\square$

Formula (12) corresponds to the particular case where  $\mathcal{X} \times \mathbb{C} = \mathcal{C}_{\text{blc}}^{n-1} \times \mathbb{C} \cong \mathcal{C}^n$ . In this case  $\text{Aut} \mathcal{X} = \text{Aut} \mathcal{C}_{\text{blc}}^{n-1} = \mathbb{C}^*$  (see Theorem 4.1(a)), and the character  $\chi: \mathbb{C}^* \rightarrow \mathbb{C}^*$  is the identity.

**3.2.  $\text{Aut}(\mathcal{X} \times \mathbb{C})$  as ind-group.** Recall the following notions (see [30], [49]).

**Definition 3.4.** An *ind-group* is a group  $G$  equipped with an increasing filtration  $G = \bigcup_{i \in \mathbb{N}} G_i$ , where the components  $G_i$  are algebraic varieties (and not necessarily algebraic groups) such that the natural inclusion  $G_i \hookrightarrow G_{i+1}$ , the multiplication map  $G_i \times G_j \rightarrow G_{m(i,j)}$ ,  $(g_i, g_j) \mapsto g_i g_j$ , and the inversion  $G_i \rightarrow G_{k(i)}$ ,  $g_i \mapsto g_i^{-1}$ , are morphisms for any  $i, j \in \mathbb{N}$  with a suitable choice of  $m(i, j)$ ,  $k(i) \in \mathbb{N}$ .

**Examples 3.5.** (a) (*Ind-structure on  $\mathcal{O}_+(\mathcal{X})$* ). Given an affine variety  $\mathcal{X}$  we fix a closed embedding  $\mathcal{X} \hookrightarrow \mathbb{C}^N$ . For  $f \in \mathcal{O}_+(\mathcal{X})$  we define its degree  $\deg f$  as the minimal degree of a polynomial extension of  $f$  to  $\mathbb{C}^N$ . Letting

$$G_i = \{f \in \mathcal{O}_+(\mathcal{X}) \mid \deg f \leq i\} \tag{34}$$

we obtain a filtration of the group  $\mathcal{O}_+(\mathcal{X})$  by an increasing sequence of connected Abelian algebraic subgroups  $G_i$  ( $i \in \mathbb{N}$ ), hence an ind-structure on  $\mathcal{O}_+(\mathcal{X})$ .

(b) (*Ind-structure on  $\text{Aut } \mathcal{X}$* ). Given a closed embedding  $\mathcal{X} \hookrightarrow \mathbb{C}^N$ , any automorphism  $F \in \text{Aut } \mathcal{X}$  can be written as  $F = (f_1, \dots, f_N)$ , where  $f_j \in \mathcal{O}_+(\mathcal{X})$ . Letting

$$\deg F = \max_{j=1, \dots, N} \{\deg f_j\} \quad \text{and} \quad G_i = \{F \in \text{Aut } \mathcal{X} \mid \deg F \leq i\}$$

we obtain an ind-group structure  $\text{Aut } \mathcal{X} = \bigcup_{i \in \mathbb{N}} G_i$  compatible with the action of  $\text{Aut } \mathcal{X}$  on  $\mathcal{X}$ . The latter means that the maps  $G_i \times \mathcal{X} \rightarrow \mathcal{X}$ ,  $(F, x) \mapsto F(x)$ , are morphisms of algebraic varieties. It is well known that any two such ind-structures on  $\text{Aut } \mathcal{X}$  are equivalent.

(c) (*Ind-structure on  $\text{Aut}(\mathcal{X} \times \mathbb{C})$* ). For a tight cylinder  $\mathcal{X} \times \mathbb{C}$  an ind-structure on the group  $\text{Aut}(\mathcal{X} \times \mathbb{C})$  can be defined via the ind-structures on the factors  $\mathcal{O}_+(\mathcal{X})$ ,  $\mathcal{O}^\times(\mathcal{X})$ , and  $\text{Aut } \mathcal{X}$  in decomposition (21).

**3.3.  $\mathcal{O}^\times(\mathcal{X})$  as ind-group.** Any extension of an algebraic group by a countable group is an ind-group. In particular, the group  $\mathcal{O}^\times(\mathcal{X})$  is an ind-group due to the following well-known fact (see [48, Lemme 1]; see also [42, Ch. 3, Lemma 1.2.1] or [17, Lemma 1.1]).

**Lemma 3.6** (*Samuel's Lemma*). *For any irreducible algebraic variety  $\mathcal{X}$  defined over an algebraically closed field  $k$  we have*

$$\mathcal{O}^\times(\mathcal{X}) \cong k^* \times \mathbb{Z}^m \quad \text{for some } m \geq 0.$$

*If  $k = \mathbb{C}$ , then  $m \leq \text{rank } H^1(\mathcal{X}, \mathbb{Z})$ .*

We provide an argument for  $k = \mathbb{C}$ , which follows the sheaf-theoretic proofs of the topological Bruschi's [7] and Eilenberg [13], [14] theorems.

*Proof.* The sheaves  $\mathcal{Z}_\mathcal{X}$ ,  $\mathcal{C}_\mathcal{X}$ , and  $\mathcal{C}_\mathcal{X}^*$  of germs of continuous functions with values in  $\mathbb{Z}$ ,  $\mathbb{C}$ , and  $\mathbb{C}^*$ , respectively, form the exact sequence  $0 \rightarrow \mathcal{Z}_\mathcal{X} \xrightarrow{\cdot 2\pi i} \mathcal{C}_\mathcal{X} \xrightarrow{\text{exp}} \mathcal{C}_\mathcal{X}^* \rightarrow 1$ . As  $\mathcal{X}$  is connected and paracompact, and the sheaf  $\mathcal{C}_\mathcal{X}$  is fine,  $H^1(\mathcal{X}, \mathcal{C}_\mathcal{X}) = 0$  and the corresponding exact cohomology sequence takes the form

$$0 \rightarrow \mathbb{Z} \xrightarrow{\cdot 2\pi i} C(\mathcal{X}) \xrightarrow{\text{exp}} C^*(\mathcal{X}) \xrightarrow{\rho} H^1(\mathcal{X}, \mathbb{Z}) \rightarrow 0.$$

Restricting the homomorphism  $\rho$  to  $\mathcal{O}^\times(\mathcal{X}) \subset \mathcal{C}^*(\mathcal{X})$  and taking into account that the conditions  $\varphi \in \mathcal{O}(\mathcal{X})$  and  $e^\varphi \in \mathcal{O}^\times(\mathcal{X})$  imply  $\varphi = \text{const}$ , we obtain the exact sequence

$$\mathbb{C} \xrightarrow{\text{exp}} \mathcal{O}^\times(\mathcal{X}) \xrightarrow{\rho} H^1(\mathcal{X}, \mathbb{Z}).$$

Since  $H^1(\mathcal{X}, \mathbb{Z})$  is a free Abelian group of finite rank, the image of  $\rho$  is isomorphic to  $\mathbb{Z}^m$  with some  $m \leq \text{rank } H^1(\mathcal{X}, \mathbb{Z})$ . This implies that the Abelian group  $\mathcal{O}^\times(\mathcal{X})$  admits the desired direct decomposition.  $\square$

**Examples 3.7** (*The groups of units on the balanced spaces*). (a) The discriminant  $D_n$  is the restriction to  $\mathcal{C}^n = \mathbb{C}_{(z)}^n \setminus \{z \mid d_n(z) = 0\}$  of the irreducible discriminant polynomial  $d_n$ . Since  $\mathcal{C}^n = \mathcal{C}_{\text{blc}}^{n-1} \times \mathbb{C}$ , the group  $H^1(\mathcal{C}_{\text{blc}}^{n-1}, \mathbb{Z}) = H^1(\mathcal{C}^n, \mathbb{Z}) \cong \mathbb{Z}$  is generated by the cohomology class of  $D_n$ . Any element of  $\mathcal{O}^\times(\mathcal{C}_{\text{blc}}^{n-1})$  is of the form  $sD_n^k$  with  $s \in \mathbb{C}^*$  and  $k \in \mathbb{Z}$ . Hence  $\mathcal{O}^\times(\mathcal{C}_{\text{blc}}^{n-1}) \cong \mathbb{C}^* \times \mathbb{Z}$ .

(b) The projection  $D_n: \mathcal{C}_{\text{blc}}^{n-1} \rightarrow \mathbb{C}^*$ ,  $Q \mapsto D_n(Q)$ , is a locally trivial fiber bundle with fibers isomorphic to  $\mathcal{SC}_{\text{blc}}^{n-2}$ . Since  $\pi_2(\mathbb{C}^*) = 0$ , the final segment of the corresponding long exact sequence of homotopy groups looks as follows:

$$1 \rightarrow \pi_1(\mathcal{SC}_{\text{blc}}^{n-2}) \rightarrow \pi_1(\mathcal{C}_{\text{blc}}^{n-1}) \rightarrow \pi_1(\mathbb{C}^*) \rightarrow 1.$$

Now,  $\pi_1(\mathcal{C}_{\text{blc}}^{n-1})$  is the Artin braid group  $\mathbf{A}_{n-1}$ , and we can rewrite this sequence as

$$1 \rightarrow \pi_1(\mathcal{SC}_{\text{blc}}^{n-2}) \rightarrow \mathbf{A}_{n-1} \rightarrow \mathbb{Z} \rightarrow 1,$$

so that the commutator subgroup  $\mathbf{A}'_{n-1}$  is contained in  $\pi_1(\mathcal{SC}_{\text{blc}}^{n-2})$ . Since  $\mathbf{A}_{n-1}/\mathbf{A}'_{n-1} \cong \mathbb{Z}$  and the torsion of any nontrivial quotient group of  $\mathbb{Z}$  is nontrivial, it follows that  $\pi_1(\mathcal{SC}_{\text{blc}}^{n-2}) \cong \mathbf{A}'_{n-1}$ . By [20, Lemma 2.2],  $\mathbf{A}''_{n-1} = \mathbf{A}'_{n-1}$  for any  $n > 4$ , and so  $\text{Hom}(\mathbf{A}'_{n-1}, \mathbb{Z}) = 0$ . Finally,  $H^1(\mathcal{SC}_{\text{blc}}^{n-2}, \mathbb{Z}) \cong \text{Hom}(\pi_1(\mathcal{SC}_{\text{blc}}^{n-2}), \mathbb{Z}) = \text{Hom}(\mathbf{A}'_{n-1}, \mathbb{Z}) = 0$  and  $\mathcal{O}^\times(\mathcal{SC}_{\text{blc}}^{n-2}) \cong \mathbb{C}^*$ .

(c) The discriminant  $d_n$  and its restriction  $d_n|_{z_1=0}$  to the hyperplane  $z_1 = 0$  are quasi-homogeneous. So, the zero level sets  $\Sigma^{n-1} = \{d_n = 0\}$  and  $\Sigma_{\text{blc}}^{n-2} = \Sigma^{n-1} \cap \{z_1 = 0\}$  are contractible. Hence  $H^1(\Sigma_{\text{blc}}^{n-2}, \mathbb{Z}) = 0$ . By Lemma 3.6,  $\mathcal{O}^\times(\Sigma_{\text{blc}}^{n-2}) \cong \mathbb{C}^*$ .

### 3.4. The neutral component $\text{Aut}_0(\mathcal{X} \times \mathbb{C})$ .

**Definition 3.8.** The *neutral component*  $G_0$  of an ind-group  $G = \bigcup_{i \in \mathbb{N}} G_i$  is defined as the union of those connected components of the  $G_i$  that contain the unity  $e_G$  of  $G$ . In other words,  $G_0$  is the union of all connected algebraic subvarieties of  $G$  passing through  $e_G$ . Recall that a subset  $V \subset G$  is an algebraic subvariety if it is a subvariety of some  $G_i$ . Clearly,  $G_0$  is a normal ind-subgroup of  $G$ .

For an irreducible affine variety  $\mathcal{X}$ ,  $\text{Aut}_0 \mathcal{X}$  is as well the neutral component of  $\text{Aut } \mathcal{X}$  in the sense of [47].

From Corollary 2.4, Lemma 3.6, and decomposition (21) we derive the following.

**Theorem 3.9.** *For a tight cylinder  $\mathcal{X} \times \mathbb{C}$  we have*

$$\mathrm{Aut}_0(\mathcal{X} \times \mathbb{C}) \cong \mathcal{O}_+(\mathcal{X}) \rtimes (\mathbb{C}^* \times \mathrm{Aut}_0 \mathcal{X}). \quad (35)$$

*Proof.* For a semi-direct product of two ind-groups  $H$  and  $H'$  we have  $(H \rtimes H')_0 = H_0 \rtimes H'_0$ . Thus, from (21) we get a decomposition

$$\mathrm{Aut}_0(\mathcal{X} \times \mathbb{C}) \cong (\mathcal{O}_+(\mathcal{X}) \rtimes \mathbb{C}^*) \rtimes \mathrm{Aut}_0 \mathcal{X}.$$

It suffices to show that the factors  $\mathbb{C}^*$  and  $\mathrm{Aut}_0(\mathcal{X})$  in this decomposition commute, i.e., that  $FF' = F'F$  for any two automorphisms  $F, F' \in \mathrm{Aut}_0(\mathcal{X} \times \mathbb{C})$  of the form  $F : (x, y) \mapsto (x, ty)$  and  $F' : (x, y) \mapsto (Sx, y)$ , where  $S \in \mathrm{Aut} \mathcal{X}$  and  $t \in \mathbb{C}^*$ , see (18). However, the latter equality is evident.  $\square$

**Remark 3.10** (*The unipotent radical of  $\mathrm{Aut}_0(\mathcal{X} \times \mathbb{C})$* ). Due to Proposition 2.8(a), the base  $\mathcal{X}$  of a tight cylinder  $\mathcal{X} \times \mathbb{C}$  (in particular, any rigid variety  $\mathcal{X}$ ) does not admit any non-trivial action of a unipotent linear algebraic group. Thus, any such subgroup of  $\mathrm{Aut}_0(\mathcal{X} \times \mathbb{C})$  is contained in the subgroup  $\mathrm{SAut}(\mathcal{X} \times \mathbb{C}) \cong \mathcal{O}_+(\mathcal{X})$ , see (35), and so, is Abelian. Due to Proposition 2.14, the normal Abelian subgroup  $\mathrm{SAut}(\mathcal{X} \times \mathbb{C})$  can be regarded as the unipotent radical of  $\mathrm{Aut}_0(\mathcal{X} \times \mathbb{C})$ . Note that  $\mathrm{SAut}(\mathcal{X} \times \mathbb{C})$  is a union of an increasing sequence of connected algebraic subgroups, see Example 3.5 (a). We need the following more precise statement.

**Lemma 3.11.** *Let  $\mathcal{X} \times \mathbb{C}$  be a tight cylinder. Then the special automorphism group  $\mathrm{SAut}(\mathcal{X} \times \mathbb{C}) \subset \mathrm{Aut}_0(\mathcal{X} \times \mathbb{C})$  is a countable increasing union of connected unipotent algebraic subgroups  $U_i \subset \mathrm{SAut}(\mathcal{X} \times \mathbb{C})$ , which are normal in  $\mathrm{Aut}_0(\mathcal{X} \times \mathbb{C})$ .*

*Proof.* The action of  $\mathrm{Aut}_0 \mathcal{X}$  on the normal subgroup  $\mathcal{O}_+(\mathcal{X}) \triangleleft \mathrm{Aut}_0(\mathcal{X} \times \mathbb{C})$  in (35) is given by  $b \mapsto b \circ S$  for  $b \in \mathcal{O}_+(\mathcal{X})$  and  $S \in \mathrm{Aut}_0 \mathcal{X}$ , cf. the proof of Theorem 3.9. The  $\mathbb{C}^*$ -subgroup in (35) acts on  $\mathcal{O}_+(\mathcal{X})$  via homotheties  $b \mapsto t^{-1}b$ , where  $b \in \mathcal{O}_+(\mathcal{X})$  and  $t \in \mathbb{C}^*$ . Therefore, the linear representation of the product  $\mathbb{C}^* \times \mathrm{Aut}_0 \mathcal{X}$  on  $\mathcal{O}_+(\mathcal{X})$  is locally finite. In particular, the finite dimensional subspace  $G_i = \{f \in \mathcal{O}_+(\mathcal{X}) \mid \deg f \leq i\}$  as in (34) is contained in another finite dimensional subspace, say  $U_i$ , which is stable under the action of  $\mathbb{C}^* \times \mathrm{Aut}_0 \mathcal{X}$ , hence is normal when regarded as a subgroup of  $\mathrm{Aut}_0(\mathcal{X} \times \mathbb{C})$ . Since the sequence  $(G_i)_{i \in \mathbb{N}}$  is increasing, we can choose the sequence  $(U_i)_{i \in \mathbb{N}}$  being also increasing.  $\square$

**Corollary 3.12.** *Let  $\mathcal{X} \times \mathbb{C}$  be a tight cylinder over an affine variety  $\mathcal{X}$ . Suppose that  $\mathrm{Aut}_0 \mathcal{X}$  is an algebraic group.<sup>9</sup> Then  $\mathrm{Aut}_0(\mathcal{X} \times \mathbb{C}) = \bigcup_{i \in \mathbb{N}} B_i$ , where  $(B_i)_{i \in \mathbb{N}}$  is an increasing sequence of connected algebraic subgroups.*

*Proof.* It is enough to let  $B_i = U_i \rtimes (\mathbb{C}^* \times \mathrm{Aut}_0 \mathcal{X})$ .  $\square$

<sup>9</sup>The latter assumption holds if  $\bar{k}(\mathrm{reg} \mathcal{X}) \geq 0$ , where  $\bar{k}$  stands for the logarithmic Kodaira dimension. If this is the case, then  $\mathrm{Aut}_0 \mathcal{X}$  is an algebraic torus ([25, Proposition 5]).

**3.5. Algebraic subgroups of  $\text{Aut}_0(\mathcal{X} \times \mathbb{C})$ .** In this subsection we keep the assumptions of Corollary 3.12. By this corollary the group  $\text{Aut}_0(\mathcal{X} \times \mathbb{C})$  is a union of connected affine algebraic subgroups. The notions of semisimple and unipotent elements, and as well of the Jordan decomposition, are well defined in  $\text{Aut}_0(\mathcal{X} \times \mathbb{C})$  due to their invariance. Moreover, by virtue of Remark 3.10 for any connected affine algebraic subgroup  $G$  of  $\text{Aut}_0(\mathcal{X} \times \mathbb{C})$ , the unipotent radical of  $G$  equals  $G \cap \text{SAut}(\mathcal{X} \times \mathbb{C})$ . So  $\text{SAut}(\mathcal{X} \times \mathbb{C})$  is the set of all unipotent elements of  $\text{Aut}_0(\mathcal{X} \times \mathbb{C})$ . The next result shows that, under the assumptions of Corollary 3.12, decomposition (35) can be viewed as an analog of the Mostow decomposition for algebraic groups. Recall that Mostow's version of the Levi-Malcev Theorem [43] (see also [22] or [46, Ch. II, §1, Theorem 3]) states that any connected algebraic group over a field of characteristic zero admits a decomposition into a semi-direct product of its unipotent radical and a maximal reductive subgroup. Any two such maximal reductive subgroups are conjugated via an element of the unipotent radical.

**Theorem 3.13.** *Let  $\mathcal{X}$  be an affine variety of dimension  $> 1$  such that  $\text{Aut}_0 \mathcal{X}$  is an algebraic group. Then the following hold.*

- (a) *The group  $\text{Aut}_0 \mathcal{X}$  is isomorphic to an algebraic torus  $(\mathbb{C}^*)^r$ .*
- (b) *The group  $\text{Aut}_0(\mathcal{X} \times \mathbb{C}) \cong \mathcal{O}_+(\mathcal{X}) \rtimes (\mathbb{C}^*)^{r+1}$  is metabelian.*
- (c) *Any connected algebraic subgroup  $G$  of  $\text{Aut}(\mathcal{X} \times \mathbb{C})$  is either Abelian or metabelian of rank  $\leq r + 1$ .*
- (d) *Any algebraic torus in  $\text{Aut}_0(\mathcal{X} \times \mathbb{C})$  is contained in a maximal torus. Any maximal torus is of rank  $r + 1$ , and two such tori are conjugated via an element of  $\text{SAut}(\mathcal{X} \times \mathbb{C})$ .*
- (e) *Any semisimple element of  $\text{Aut}_0(\mathcal{X} \times \mathbb{C})$  is contained in a maximal torus. Any finite subgroup of  $\text{Aut}_0(\mathcal{X} \times \mathbb{C})$  is Abelian and contained in a maximal torus.*

*Proof.* By our assumptions  $\text{Aut}_0 \mathcal{X}$  is a connected linear algebraic group without any unipotent subgroup. Indeed, assuming that there is such a subgroup  $U$ , and taking it to be one-dimensional, we have  $U = \exp(\mathbb{C}\partial)$ , where  $\partial \in \text{LND}(\mathcal{O}(\mathcal{X}))$ . Then  $\mathcal{U} = \exp((\ker \partial)\partial)$  is an infinite dimensional subgroup of  $\text{Aut}_0 \mathcal{X}$ , a contradiction.

Hence by Theorem 2.18 the cylinder  $\mathcal{X} \times \mathbb{C}$  is tight, and so,  $\text{Aut}_0 \mathcal{X}$  is an algebraic torus by [25, Lemma 3]. This proves (a).

By virtue of (35) and (a) we have

$$\text{Aut}_0(\mathcal{X} \times \mathbb{C}) \cong \mathcal{O}_+(\mathcal{X}) \rtimes (\mathbb{C}^*)^{r+1}. \quad (36)$$

This proves (b).

By Corollary 3.12 the group  $\text{Aut}_0(\mathcal{X} \times \mathbb{C})$  is covered by an increasing sequence of connected algebraic subgroups  $(B_i)_{i \in \mathbb{N}}$ , where  $B_i = U_i \rtimes (\mathbb{C}^*)^{r+1}$  is metabelian.. Any

algebraic subgroup  $G \subset \text{Aut}_0(\mathcal{X} \times \mathbb{C})$  is contained in one of them, say  $G \subset B_i$ . This proves (c).

Now (d) follows by the classical Mostow Theorem applied to an appropriate subgroup  $B_i$ , which contains the tori under consideration.

The same argument proves (e). Indeed, both assertions of (e) hold for connected solvable affine algebraic groups due to [23, Ch. VII, Proposition 19.4(a)].  $\square$

**Remark 3.14.** The assumption that  $\text{Aut}_0(\mathcal{X})$  is an algebraic group is important. For instance, the group  $\text{Aut}_0(S_n)$  of the Danielewski surface  $S_n$  is non-algebraic, and (d) does not hold for the cylinder  $S_n \times \mathbb{C}$ , see Example 2.9. Indeed, the group  $\text{Aut}(S_n \times \mathbb{C})$  contains a sequence of pairwise non-conjugate algebraic two-tori ([10, Thm. 2]).

**3.6. Semisimple and torsion elements.** We let  $\mathcal{T}$  denote the maximal torus in  $\text{Aut}_0(\mathcal{X} \times \mathbb{C})$  which corresponds to the factor  $\mathbb{C}^* \times \text{Aut}_0 \mathcal{X} \cong (\mathbb{C}^*)^{r+1}$  under the isomorphisms as in (35) and (36). From Theorem 3.13 ((d) and (e)) we deduce the following corollary.

**Corollary 3.15.** *Under the assumptions of Theorem 3.13 any semisimple (in particular, any torsion) element of the group  $\text{Aut}_0(\mathcal{X} \times \mathbb{C})$  is conjugate to an element of the maximal torus  $\mathcal{T}$  via an element of the unipotent radical  $\text{SAut}(\mathcal{X} \times \mathbb{C})$ . The same conclusion holds for any finite subgroup of  $\text{Aut}_0(\mathcal{X} \times \mathbb{C})$ .*

Using Corollary 3.15 we arrive at the following description of all semisimple and torsion elements in the automorphism groups of tight cylinders.

**Proposition 3.16.** *Under the assumptions of Theorem 3.13 an element  $F \in \text{Aut}_0(\mathcal{X} \times \mathbb{C})$  is semisimple if and only if it can be written as*

$$F: (x, y) \mapsto (Sx, ty + tb(x) - b(Sx)), \quad \text{where } (x, y) \in \mathcal{X} \times \mathbb{C}, \quad (37)$$

for some triplet  $(S, t, b)$  with  $S \in \text{Aut}_0 \mathcal{X}$ ,  $t \in \mathbb{C}^*$ , and  $b \in \mathcal{O}(\mathcal{X})$ . Such an element  $F$  is torsion with  $F^m = \text{id}$  if and only if  $S^m = \text{id}$  and  $t^m = 1$ .

**3.7. The Lie algebra of  $\text{Aut}_0(\mathcal{X} \times \mathbb{C})$ .** The Lie algebra of an ind-group is defined in [49], see also [30]. For an ind-group  $G$  of type  $G = \varinjlim_i G_i$ , where  $(G_i)_{i \in \mathbb{N}}$  is an increasing sequence of connected algebraic subgroups of  $G$ , the Lie algebra  $\text{Lie}(G)$  coincides with the inductive limit  $\varinjlim_i \text{Lie}(G_i)$ . From Corollary 3.12 and decomposition (35) we deduce the following presentation.

**Theorem 3.17.** *Under the assumptions of Theorem 3.13 we have*

$$\text{Lie}(\text{Aut}_0(\mathcal{X} \times \mathbb{C})) = I \rtimes L. \quad (38)$$

Here

$$I \stackrel{\text{def}}{=} \{b(x)\partial/\partial y \mid b \in \mathcal{O}_+(\mathcal{X})\} = \mathcal{O}_+(\mathcal{X})\partial/\partial y \quad (39)$$

is the Abelian ideal consisting of all locally nilpotent derivations of the algebra  $\mathcal{O}(\mathcal{X} \times \mathbb{C})$ , and

$$L \cong \text{Lie}(\mathbb{C}^* \times \text{Aut}_0 \mathcal{X}) \quad (40)$$

is the Cartan Lie subalgebra of  $\text{Lie}(\text{Aut}_0(\mathcal{X} \times \mathbb{C}))$  corresponding to the second factor in (35), i.e., a maximal Abelian subalgebra consisting of semisimple elements. Furthermore, we have

$$\text{Lie}(\text{Aut}_0(\mathcal{X} \times \mathbb{C})) = \langle b(x)\partial/\partial y, y\partial/\partial y, \partial \mid b \in \mathcal{O}_+(\mathcal{X}), \partial \in \text{Lie}(\text{Aut}_0 \mathcal{X}) \rangle \quad (41)$$

with relations

$$[\partial, y\partial/\partial y] = 0, \quad [\partial, b\partial/\partial y] = (\partial b)\partial/\partial y, \quad \text{and} \quad [b\partial/\partial y, y\partial/\partial y] = b\partial/\partial y \quad (42)$$

for any  $b \in \mathcal{O}_+(\mathcal{X})$  and any  $\partial \in \text{Lie}(\text{Aut}_0 \mathcal{X})$ .

*Proof.* Decomposition (38) is a direct consequence of (35), and (41) follows from (35) and (38). The first relation in (42) follows from the fact that the factors  $\mathbb{C}^*$  and  $\text{Aut}_0 \mathcal{X}$  in (35) commute. To show the other two relations it suffices to verify these on the functions of the form  $f(x)y^k \in \mathcal{O}(\mathcal{X} \times \mathbb{C}) = \mathcal{O}(\mathcal{X})[y]$ , where  $k \geq 0$ . The latter computation is easy, and so we omit it.  $\square$

#### 4. AUTOMORPHISMS OF CONFIGURATION SPACES AND DISCRIMINANT LEVELS

**4.1. Automorphisms of balanced spaces.** In view of Corollary 2.22, to compute the automorphism groups of the varieties  $\mathcal{C}^n$ ,  $\mathcal{SC}^{n-1}$ , and  $\Sigma^{n-1}$  we need to know the automorphism groups of the corresponding balanced spaces  $\mathcal{C}_{\text{blc}}^{n-1}$ ,  $\mathcal{SC}_{\text{blc}}^{n-2}$ , and  $\Sigma_{\text{blc}}^{n-2}$ . The latter groups have been already described in the literature. We formulate the corresponding results and provide necessary references. Then we give a short argument for (a) based on Tame Map Theorem. The proof of (c) will be done in Section 7.

**Theorem 4.1.** *For any natural  $n > 2$  the following holds.*

(a)  $\text{Aut } \mathcal{C}_{\text{blc}}^{n-1} \cong \mathbb{C}^*$ . Any automorphism  $S \in \text{Aut } \mathcal{C}_{\text{blc}}^{n-1}$  is of the form  $Q^\circ \mapsto sQ^\circ$ , where  $Q^\circ \in \mathcal{C}_{\text{blc}}^{n-1}$  and  $s \in \mathbb{C}^*$ . While  $\text{Aut } \mathcal{C}_{\text{blc}}^1 = \text{Aut } \mathbb{C}^* \cong \mathbb{C}^* \rtimes (\mathbb{Z}/2\mathbb{Z})$ .

(b)  $\text{Aut } \mathcal{SC}_{\text{blc}}^{n-2} \cong \mathbb{Z}/n(n-1)\mathbb{Z}$ . Any automorphism  $S \in \text{Aut } \mathcal{SC}_{\text{blc}}^{n-2}$  is of the form  $Q^\circ \mapsto sQ^\circ$ , where  $Q^\circ \in \mathcal{SC}_{\text{blc}}^{n-2}$ ,  $s \in \mathbb{C}^*$ , and  $s^{n(n-1)} = 1$ .

(c)  $\text{Aut } \Sigma_{\text{blc}}^{n-2} \cong \mathbb{C}^*$ . Any automorphism  $S \in \text{Aut } \Sigma_{\text{blc}}^{n-2}$  is of the form  $Q^\circ \mapsto sQ^\circ$ , where  $s \in \mathbb{C}^*$  and every point  $Q^\circ \in \Sigma_{\text{blc}}^{n-2}$  is considered as an unordered multiset  $Q^\circ = \{q_1, \dots, q_n\} \subset \mathbb{C}$  with at least one repetition.

For  $n > 4$  statement (a) is a simple consequence of Tame Map Theorem, see [32] and [33, Sec. 8.2.1]; we reproduce a short argument. In Theorem 10.3(c) we provide a more general result in the analytic setting. See also Section 8 for an alternative proof avoiding the reference to Tame Map Theorem and including the cases  $n = 3, 4$ .

A proof of (b) is sketched in [29]. Actually, the theorem of Kaliman ([29, Theorem]) says that *for  $n \neq 4$  every non-constant holomorphic endomorphism of  $\mathcal{SC}_{\text{blc}}^{n-2}$  is a biregular automorphism of the above form*. A complete proof of this result can be found in [37, Theorem 12.13]. This proof exploits the following property of the Artin braid group  $\mathbf{A}_{n-1}$  (see [34, Theorem 7.7] or [36, Theorem 8.9]): *For  $n > 4$ , the intersection  $\mathbf{A}'_{n-1} \cap \mathbf{PA}_{n-1}$  of the commutator subgroup  $\mathbf{A}'_{n-1}$  of  $\mathbf{A}_{n-1}$  with the pure braid group  $\mathbf{PA}_{n-1}$  is invariant under any endomorphism of  $\mathbf{A}'_{n-1}$* . This is no longer true for  $n = 3, 4$ , see Example 9.4. In case  $n = 3$ ,  $\mathcal{SC}_{\text{blc}}^1$  is a smooth affine elliptic curve with  $j = 0$ , and, once again, its automorphism group is as in (b). In case  $n = 4$  we extend Kaliman's Theorem for automorphisms using a different approach, see Theorem 9.1. Note that in the case of endomorphisms, the original Kaliman's Theorem does not hold if  $n = 4$ ; see Example 9.3.

Our proof of (c) is based on a part of Tame Map Theorem due to Zinde ([51, Theorems 7 and 8]), which describes the automorphisms of the configuration space  $\mathcal{C}^n(\mathbb{C}^*)$ . Since by Lemma 2.20  $\text{reg } \Sigma_{\text{blc}}^{n-2} \cong \mathcal{C}^{n-2}(\mathbb{C}^*)$ , from results in [loc. cit.] it follows that for  $n > 4$

$$\text{Aut}(\text{reg } \Sigma_{\text{blc}}^{n-2}) \cong \text{Aut } \mathcal{C}^{n-2}(\mathbb{C}^*) \cong (\text{Aut } \mathbb{C}^*) \times \mathbb{Z} \cong (\mathbb{C}^* \times \mathbb{Z}) \rtimes (\mathbb{Z}/2\mathbb{Z}).$$

In Theorem 7.1 we show that only the elements of the connected component  $\mathbb{C}^*$  of  $\text{Aut}(\text{reg } \Sigma_{\text{blc}}^{n-2})$  can be extended to automorphisms of the whole variety  $\Sigma_{\text{blc}}^{n-2}$ . This implies both assertions in (c) for  $n > 4$ . In Section 6 we provide an alternative proof of Zinde's Theorem in our particular setting, which does not address Tame Map Theorem. We extend the description of the group  $\text{Aut } \mathcal{C}^n(\mathbb{C}^*)$  given in Zinde's Theorem to the cases where  $n = 4$  and  $n = 3$ . The theorem does not hold any longer for  $n = 2$ ; this case is treated in Theorem 6.26. Using the description of the group  $\text{Aut } \mathcal{C}^2(\mathbb{C}^*)$  from Theorem 6.26 we complete the proof of (c) in cases  $n = 4$  and  $n = 3$ .

*Proof of (a) for  $n > 4$ .* The extension  $F$  of  $S$  to  $\mathcal{C}^n$  defined by  $F(Q) = S(Q - \text{bc}(Q))$  for all  $Q \in \mathcal{C}^n$  is a non-Abelian endomorphism of  $\mathcal{C}^n$  such that  $F(\mathcal{C}^n) \subset \mathcal{C}_{\text{blc}}^{n-1}$ . By Tame Map Theorem and Remark 1.3 (b), there is a unique morphism  $T: \mathcal{C}^n \rightarrow \text{Aff } \mathbb{C}$  such that  $F = F_T$ . Since  $F$  preserves  $\mathcal{C}_{\text{blc}}^{n-1}$ , we have  $T(Q)(0) = 0$  for any  $Q \in \mathcal{C}_{\text{blc}}^{n-1}$  and hence also for any  $Q \in \mathcal{C}^n$ . So,  $T(Q)\zeta = a(Q)\zeta$  for all  $\zeta \in \mathbb{C}$  and  $Q \in \mathcal{C}^n$ , where  $a \in \mathcal{O}^\times(\mathcal{C}^n)$ . According to Example 3.7 (a),  $a = sD_n^k$  for some  $s \in \mathbb{C}^*$  and  $k \in \mathbb{Z}$ , so that  $S(Q) = sD_n^k(Q) \cdot Q$  on  $\mathcal{C}_{\text{blc}}^{n-1}$ . Similarly, for the inverse automorphism  $S^{-1}$  we obtain that  $S^{-1}(Q) = tD_n^l \cdot Q$  on  $\mathcal{C}_{\text{blc}}^{n-1}$  with some  $t \in \mathbb{C}^*$  and  $l \in \mathbb{Z}$ . Since  $D_n$  is a homogeneous function on  $\mathcal{C}^n$  (namely,  $D_n(sQ) = s^{n(n-1)}Q$  for all  $Q \in \mathcal{C}^n$  and  $s \in \mathbb{C}^*$ ), from the identity  $S \circ S^{-1} = \text{id}$  we deduce that  $k = l = 0$  and  $t = s^{-1}$ , as required.  $\square$

By Theorem 4.1 in all three cases the automorphism groups of the corresponding balanced spaces are algebraic groups. Hence Theorem 3.13 applies and leads to the following corollary.



**Corollary 4.2.** *For any  $n > 2$  the conclusions (b)-(e) of Theorem 3.13 hold with  $r = 1$  for the groups  $\text{Aut}_0 \mathcal{C}^n$  and  $\text{Aut}_0 \Sigma^{n-1}$ , and with  $r = 0$  for the group  $\text{Aut}_0 \mathcal{SC}^{n-1}$ , when these varieties are viewed as the cylinders in (27).*

**Remark 4.3.** Recall that  $\text{Sym}^n \mathbb{C}$  regarded as the space of all unordered multisets  $Q = \{q_1, \dots, q_n\} \subset \mathbb{C}$  is a disjoint union of  $\mathcal{C}^n$  and  $\Sigma^{n-1}$ . The tautological  $(\text{Aff } \mathbb{C})$ -action on  $\mathbb{C}$  induces the diagonal  $(\text{Aff } \mathbb{C})$ -action on  $\text{Sym}^n \mathbb{C}$ ; both  $\mathcal{C}^n$  and  $\Sigma^{n-1}$  are invariant under the latter action. It follows from Tame Map Theorem and Remark 1.3 (d) that for  $n > 2$  the  $(\text{Aut } \mathcal{C}^n)$ -orbits coincide with the orbits of the diagonal  $(\text{Aff } \mathbb{C})$ -action on  $\mathcal{C}^n$  (see [38, Section 2.2]). As follows from Corollaries 3.1, 4.2 and Theorems 3.13, 4.1, for  $n > 2$  the  $(\text{Aut } \Sigma^{n-1})$ -orbits coincide with the orbits of the above diagonal  $(\text{Aff } \mathbb{C})$ -action on  $\Sigma^{n-1}$ . For  $n > 2$  the  $(\text{Aut } \mathcal{SC}^{n-1})$ -orbits coincide with the orbits of the subgroup  $\mathbb{C} \rtimes (\mathbb{Z}/n(n-1)\mathbb{Z}) \subset \text{Aff } \mathbb{C}$  acting on  $\mathcal{SC}^{n-1}$ .

**4.2. The groups  $\text{Aut } \mathcal{C}^n$ ,  $\text{Aut } \mathcal{SC}^{n-1}$ , and  $\text{Aut } \Sigma^{n-1}$ .** For our favorite varieties  $\mathcal{C}^n$ ,  $\mathcal{SC}^{n-1}$ , and  $\Sigma^{n-1}$  we dispose at present all necessary ingredients in decomposition (21). Gathering this information we obtain the following description.

**Theorem 4.4.** *If  $n > 2$ , then*

$$\text{Aut } \mathcal{C}^n \cong (\mathcal{O}_+(\mathcal{C}_{\text{blc}}^{n-1}) \rtimes (\mathbb{C}^*)^2) \rtimes \mathbb{Z}, \quad (43)$$

while for  $n = 2$ ,

$$\text{Aut } \mathcal{C}^2 \cong (\mathcal{O}_+(\mathbb{C}^*) \rtimes (\mathbb{C}^*)^2) \rtimes (\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})), \quad (44)$$

where  $\mathcal{O}_+(\mathbb{C}^*) = \mathbb{C}[t, t^{-1}]$ . Furthermore, for  $n > 2$

$$\text{Aut } \mathcal{SC}^{n-1} \cong \mathcal{O}_+(\mathcal{SC}_{\text{blc}}^{n-2}) \rtimes (\mathbb{C}^* \times (\mathbb{Z}/n(n-1)\mathbb{Z})) \quad (45)$$

and

$$\text{Aut } \Sigma^{n-1} \cong \mathcal{O}_+(\Sigma_{\text{blc}}^{n-1}) \rtimes (\mathbb{C}^*)^2. \quad (46)$$

All these groups are solvable, and  $\text{Aut } \mathcal{SC}^{n-1}$  and  $\text{Aut } \Sigma^{n-1}$  are metabelian. In addition, any finite subgroup of one of the groups in (43), (45), and (46) is Abelian.

*Proof.* Likewise this is done in the proof of Theorem 3.9, one can show that the factor  $\mathbb{C}^*$  of the group of units on the corresponding balanced space commutes with the last factor in (21). Taking this into account, the isomorphisms in (43)-(46) are obtained after substitution of the factors in (21) using Examples 3.7 and Theorem 4.1.

For the connected group  $\text{Aut } \Sigma^{n-1}$  in (46) the last assertion holds due to Theorem 3.13. The same argument applies in the case of  $\text{Aut } \mathcal{C}^n$ . Indeed, the decomposition in (43) provides a surjection  $\eta : \text{Aut } \mathcal{C}^n \rightarrow \mathbb{Z}$ , and any finite subgroup of  $\text{Aut } \mathcal{C}^n$  is contained in the kernel  $\ker \eta = \text{Aut}_0 \mathcal{C}^n$ .

The isomorphism in (45) yields a surjection  $\text{Aut } \mathcal{SC}^{n-1} \rightarrow \mathbb{C}^* \times (\mathbb{Z}/n(n-1)\mathbb{Z})$  with a torsion free kernel  $\text{SAut } \mathcal{SC}^{n-1} \cong \mathcal{O}_+(\mathcal{SC}_{\text{blc}}^{n-2})$ . Since any finite subgroup of  $\text{Aut } \mathcal{SC}^{n-1}$

meets this kernel just in the neutral element, it injects into the Abelian group  $\mathbb{C}^* \times (\mathbb{Z}/n(n-1)\mathbb{Z})$  and so is Abelian.  $\square$

**4.3. Automorphisms of  $\mathcal{C}^n$ ,  $\mathcal{SC}^{n-1}$ , and  $\Sigma^{n-1}$ .** These varieties can be viewed as subvarieties of the  $n$ th symmetric power  $\text{Sym}^n \mathbb{C} = \mathbb{C}_{(q)}^n / \mathbf{S}(n) \cong \mathbb{C}_{(z)}^n$ . The elements of the first two are  $n$ -point configurations in  $\mathbb{C}$ , while the discriminant variety  $\Sigma^{n-1} = (\text{Sym}^n \mathbb{C}) \setminus \mathcal{C}^n$  consists of all unordered multisets  $Q = \{q_1, \dots, q_n\} \subset \mathbb{C}$  with at least one repetition (see Section 1). As before, we let  $D_n$  be the discriminant viewed as a regular function on  $\mathcal{C}^n$ .

**Theorem 4.5.** *Let  $n > 2$ , let  $\mathcal{Z}$  be one of the varieties  $\mathcal{C}^n$ ,  $\mathcal{SC}^{n-1}$ , and  $\Sigma^{n-1}$ , and  $\mathcal{Z}_{\text{blc}}$  be the corresponding balanced space (see (25)). A map  $F: \mathcal{Z} \rightarrow \mathcal{Z}$  is an automorphism if and only if*

$$F(Q) = sQ^\circ + a(Q^\circ) \text{bc}(Q) + b(Q^\circ) \quad \text{for all } Q \in \mathcal{Z}, \quad (47)$$

where  $Q^\circ = Q - \text{bc}(Q) \in \mathcal{Z}_{\text{blc}}$ ,  $s \in \mathbb{C}^*$ ,  $b \in \mathcal{O}_+(\mathcal{Z}_{\text{blc}})$ , and

- (a)  $a = tD_n^k$  with  $t \in \mathbb{C}^*$  and  $k \in \mathbb{Z}$ , if  $\mathcal{Z} = \mathcal{C}^n$ ;
- (b)  $a \in \mathbb{C}^*$  and  $s^{n(n-1)} = 1$ , if  $\mathcal{Z} = \mathcal{SC}^{n-1}$ ;
- (c)  $a \in \mathbb{C}^*$ , if  $\mathcal{Z} = \Sigma^{n-1}$ .

*Proof.* Let  $F$  be an automorphism of the cylinder  $\mathcal{Z} = \mathcal{Z}_{\text{blc}} \times \mathbb{C}$  (cf. (27)). According to Corollary 2.22, Theorem 4.1, and Example 3.7,  $F$  is triangular of the form

$$F(Q) = (sQ^\circ, a(Q^\circ) \text{bc}(Q) + b(Q^\circ)) = sQ^\circ + a(Q^\circ) \text{bc}(Q) + b(Q^\circ),$$

where in each of the cases (a), (b), (c) the triplet  $(s, a, b)$  is as before. Conversely, such an  $F$  is a (triangular) automorphism of  $\mathcal{Z}_{\text{blc}} \times \mathbb{C} = \mathcal{Z}$  corresponding to the automorphism  $S: Q^\circ \mapsto sQ^\circ$  of  $\mathcal{Z}_{\text{blc}}$  and the morphism  $A: \mathcal{Z}_{\text{blc}} \rightarrow \text{Aff } \mathbb{C}$ ,  $A(Q^\circ): \zeta \mapsto a\zeta + b$  for all  $Q^\circ \in \mathcal{Z}_{\text{blc}}$  and  $\zeta \in \mathbb{C}$ .  $\square$

**Remarks 4.6.** (a) Consider the algebraic torus  $\mathcal{T}$  of rank 2 consisting of all transformations

$$\nu(s, t): Q \mapsto s \cdot (Q - \text{bc}(Q)) + t \text{bc}(Q) \quad \text{for any } (s, t) \in (\mathbb{C}^*)^2 \text{ and } Q \in \text{Sym}^n \mathbb{C}. \quad (48)$$

Both subvarieties  $\mathcal{C}^n$  and  $\Sigma^{n-1}$  in  $\text{Sym}^n \mathbb{C}$  are invariant under this action. In fact,  $\mathcal{T}$  is a maximal torus in each of the automorphism groups  $\text{Aut } \mathcal{C}^n$  and  $\text{Aut } \Sigma^{n-1}$ . The subgroup of  $\mathcal{T}$  given by  $s^{n(n-1)} = 1$  and isomorphic to  $(\mathbb{Z}/n(n-1)\mathbb{Z}) \times \mathbb{C}^*$  acts on  $\mathcal{SC}^{n-1}$ .

(b) Let  $\mathcal{Z}$  be again one of the varieties  $\mathcal{C}^n$ ,  $\mathcal{SC}^{n-1}$ , and  $\Sigma^{n-1}$ , where  $n > 2$ , and let  $\mathcal{Z}_{\text{blc}}$  be the corresponding balanced space. Using Proposition 2.12 one can deduce the following: *Any  $\mathbb{C}_+$ -action on  $\mathcal{Z}$  is of the form*

$$Q \mapsto Q + \lambda b(Q - \text{bc}(Q)), \quad \text{where } Q \in \mathcal{Z}, \lambda \in \mathbb{C}_+, b \in \mathcal{O}(\mathcal{Z}_{\text{blc}}). \quad (49)$$

The case  $b = 1$  corresponds to the  $\tau$ -action on  $\mathcal{Z}$ .

(c) It follows from (47) that for any  $F$  in one of the above groups, one has  $F = F_T$  with  $T$  as in (12).

**4.4. The group  $\text{Aut}(\mathbb{C}^n, \Sigma^{n-1})$ .** The space  $\text{Sym}^n \mathbb{C} \cong \mathcal{C}^n \cup \Sigma^{n-1}$  of all unordered  $n$ -multisets  $Q = \{q_1, \dots, q_n\} \subset \mathbb{C}$  can be identified with the space  $\mathbb{C}_{(z)}^n \cong \mathbb{C}^n$  of all polynomials (1). The corresponding balanced space  $\mathcal{C}_{\text{blc}}^{n-1} \cup \Sigma_{\text{blc}}^{n-2} \cong \mathbb{C}^{n-1}$  consists of all polynomials  $\lambda^n + z_2 \lambda^{n-2} + \dots + z_n$ . An automorphism  $F$  of  $\mathcal{C}^n$  as in (47) extends to an endomorphism of the ambient affine space  $\mathbb{C}^n$  if and only if the rational functions  $a(Q - \text{bc}(Q))$  and  $b(Q - \text{bc}(Q))$  on  $\mathbb{C}^n$  in (47) are regular, i.e.  $a, b \in \mathcal{O}(\mathbb{C}_{\text{blc}}^n) \cong \mathbb{C}[z_2, \dots, z_n]$ . Such an endomorphism  $F$  admits an inverse, say  $F'$ , on  $\mathbb{C}^n$  if and only if the corresponding functions  $a'$  and  $b'$  are also regular i.e.  $a', b' \in \mathcal{O}(\mathbb{C}_{\text{blc}}^n)$ . In particular  $a = \text{const} \in \mathbb{C}^*$ . This leads to the following description.

**Theorem 4.7.** *For any  $n > 2$  we have*

$$\text{Aut}(\mathbb{C}^n, \Sigma^{n-1}) \cong \mathbb{C}[z_2, \dots, z_n] \rtimes (\mathbb{C}^*)^2,$$

where the 2-torus  $(\mathbb{C}^*)^2$  and the group  $\mathbb{C}[z_2, \dots, z_n] \cong \mathcal{O}_+(\mathbb{C}^{n-1})$  act on  $\mathbb{C}^n \cong \text{Sym}^n \mathbb{C}$  via (48) and (49) with  $\lambda = 1$ , respectively.  $\square$

**4.5. The Lie algebras  $\text{Lie}(\text{Aut}_0 \mathcal{C}^n)$ ,  $\text{Lie}(\text{Aut}_0 \mathcal{S}\mathcal{C}^{n-1})$ , and  $\text{Lie}(\text{Aut}_0 \Sigma^{n-1})$ .**

**4.5.1. The Lie algebra  $\text{Lie}(\text{Aut}_0 \mathcal{C}^n)$ .** Let  $\partial_\tau \in \text{LND}(\mathcal{O}(\mathcal{C}^n))$  be the infinitesimal generator of the  $\mathbb{C}_+$ -action  $\tau$  on  $\mathcal{C}^n \subset \mathbb{C}_{(z)}^n$ . By (38) and Remark 4.6 (b), for  $n > 2$  there is the Levi-Malcev-Mostow decomposition

$$\text{Lie}(\text{Aut}_0 \mathcal{C}^n) = I \oplus \text{Lie } \mathcal{T}$$

with Abelian summands, where  $I = \mathcal{O}(\mathcal{C}_{\text{blc}}^{n-1})\partial_\tau$  is as in (39) and the 2-torus  $\mathcal{T} \subset \text{Aut}_0 \mathcal{C}^n$  as in Remark 4.6 (b) consists of all transformations  $\nu(s, t)$  as in (48) with  $(s, t) \in (\mathbb{C}^*)^2$  and  $Q \in \mathcal{C}^n$ . Thus  $\mathcal{T}$  is the direct product of two its 1-subtori with infinitesimal generators, say  $\partial_s$  and  $\partial_t$ , respectively. These derivations are locally finite and locally bounded on  $\mathcal{O}(\mathcal{C}^n)$ , and their sum  $\partial_s + \partial_t$  is the infinitesimal generator of the  $\mathbb{C}^*$ -action  $Q \mapsto \lambda Q$  ( $\lambda \in \mathbb{C}^*$ ) on  $\mathcal{C}^n$ . With this notation we have the following description.

**Proposition 4.8.** *For  $n > 2$  the Lie algebra*

$$\text{Lie}(\text{Aut}_0 \mathcal{C}^n) = \langle I, \partial_s, \partial_t \rangle, \quad \text{where } I = \mathcal{O}(\mathcal{C}_{\text{blc}}^{n-1})\partial_\tau, \quad (50)$$

*is uniquely determined by the commutator relations*

$$[\partial_s, \partial_t] = 0, \quad [\partial_s, b\partial_\tau] = (\partial_s b)\partial_\tau, \quad \text{and} \quad [b\partial_\tau, \partial_t] = b\partial_\tau, \quad (51)$$

where  $b$  runs over  $\mathcal{O}(\mathbb{C}_{\text{blc}}^{n-1})$ . Furthermore, in the coordinates  $z_1, \dots, z_n$  in  $\mathbb{C}^n = \mathbb{C}_{(z)}^n$  the derivations  $\partial_\tau$ ,  $\partial_t$ , and  $\partial_s$  are given by

$$\partial_\tau = \sum_{i=1}^n (n-i+1)z_{i-1} \frac{\partial}{\partial z_i}, \quad \partial_t = (-z_1/n)\partial_\tau, \quad \text{and} \quad \partial_s = \sum_{k=1}^n kz_k \frac{\partial}{\partial z_k} - \partial_t, \quad (52)$$

where  $z_0 \stackrel{\text{def}}{=} 1$ .

*Proof.* From (41) and (42) in Theorem 3.17 we obtain (50) and (51), respectively. The diagonal  $\mathbb{C}_+$ -action  $(q_1, \dots, q_n) \mapsto (q_1 + \lambda, \dots, q_n + \lambda)$ ,  $\lambda \in \mathbb{C}_+$ , on the affine space  $\mathbb{C}_{(q)}^n$  has for infinitesimal generator the derivation

$$\partial^{(n)} = \sum_{i=1}^n \frac{\partial}{\partial q_i} \in \text{LND}(\mathbb{C}[q_1, \dots, q_n]).$$

This  $\mathbb{C}_+$ -action on  $\mathbb{C}_{(q)}^n$  descends to the  $\mathbb{C}_+$ -action  $\tau$  on the base of the Vieta covering

$$p: \mathbb{C}_{(q)}^n \rightarrow \mathbb{C}_{(z)}^n / \mathbf{S}(n) = \mathbb{C}_{(z)}^n, \quad (q_1, \dots, q_n) \mapsto (z_1, \dots, z_n),$$

where  $z_i = (-1)^i \sigma_i(q_1, \dots, q_n)$  and  $\sigma_i$  is the elementary symmetric polynomial of degree  $i$ . We have  $\partial^{(n)}(\sigma_i) = (n-i+1)\sigma_{i-1}$ . Hence in the coordinates  $z_1, \dots, z_n$  on  $\mathbb{C}_{(z)}^n$  the infinitesimal generator  $\partial_\tau$  of  $\tau$  is given by the first equality in (52).

The derivations  $\partial_s$ ,  $\partial_t$ , and  $\partial_\tau$  preserve the subring  $\mathbb{C}[z_1, \dots, z_n] \subset \mathcal{O}(\mathbb{C}^n)$  and admit natural extensions from  $\mathbb{C}[z_1, \dots, z_n]$  to  $\mathbb{C}[q_1, \dots, q_n]$  denoted by the same symbols, where

$$\partial_\tau: q_i \mapsto 1, \quad \partial_t: q_i \mapsto \frac{1}{n} \sum_{k=1}^n q_k, \quad \text{and} \quad \partial_s: q_i \mapsto q_i - \frac{1}{n} \sum_{k=1}^n q_k, \quad i = 1, \dots, n.$$

It follows that  $\partial_t = (-z_1/n)\partial_\tau$ , which yields the second equality in (52). Applying these derivations to the coordinate functions  $z_i = (-1)^i \sigma_i(q_1, \dots, q_n)$  the last equality in (52) follows as well.  $\square$

4.5.2. *The Lie algebra*  $\text{Lie}(\text{Aut}_0 \Sigma^{n-1})$ . We have seen in the proof of Proposition 4.8 that the  $\mathbb{C}_+$ -action  $\tau$  and the action of the 2-torus  $\mathcal{T}$  on  $\mathbb{C}^n$  extend regularly to the ambient affine space  $\mathbb{C}_{(z)}^n$ , along with the derivations  $\partial_\tau$ ,  $\partial_t$ , and  $\partial_s$  given by (52). The discriminant  $d_n$  on  $\mathbb{C}_{(z)}^n$  is invariant under  $\tau$ . Hence  $\partial_\tau d_n = 0$ , and so, the complete vector field  $\partial_\tau$  is tangent along the level hypersurfaces of  $d_n$ , in particular, along  $\mathcal{S}\mathbb{C}^{n-1} = \{d_n = 1\}$  and  $\Sigma^{n-1} = \{d_n = 0\}$ . The induced locally nilpotent derivations of the structure rings  $\mathcal{O}(\mathcal{S}\mathbb{C}^{n-1})$  and  $\mathcal{O}(\Sigma^{n-1})$  will be still denoted by  $\partial_\tau$ .

The action of the 2-torus  $\mathcal{T}$  on  $\mathbb{C}_{(z)}^n$  stabilizes  $\Sigma^{n-1}$ . Hence  $\partial_t$  and  $\partial_s$  generate commuting semisimple derivations of  $\mathcal{O}(\Sigma^{n-1})$  denoted by the same symbols. Using these observations and notation we can deduce from Theorem 3.17 and Corollary 4.2 the following description (cf. [40]).

**Proposition 4.9.** *For  $n > 2$  the Lie algebra*

$$\mathrm{Lie}(\mathrm{Aut}_0 \Sigma^{n-1}) = \langle I, \partial_s, \partial_t \rangle, \quad \text{where } I = \mathcal{O}_+(\Sigma_{\mathrm{blc}}^{n-2})\partial_\tau,$$

*is uniquely determined by relations (51), where  $b$  runs over  $\mathcal{O}_+(\Sigma_{\mathrm{blc}}^{n-2})$ .*

*Proof.* The proof goes along the same lines as that of Proposition 4.8, and so we leave it to the reader.  $\square$

4.5.3. *The Lie algebra  $\mathrm{Lie}(\mathrm{Aut}_0 \mathcal{SC}^{n-1})$ .* Since  $\partial_\tau d_n = 0$ , for the derivation  $\partial_t = (-z_1/n)\partial_\tau$  (see (52)) we have  $\partial_t d_n = 0$ . Hence the vector field  $\partial_t$  is tangent as well to each of the level hypersurfaces of  $d_n$ . In particular,  $\partial_t$  induces a semisimple derivation of  $\mathcal{O}(\mathcal{SC}^{n-1})$  (denoted again by  $\partial_t$ ) and generates a  $\mathbb{C}^*$ -action  $T$  on  $\mathcal{SC}^{n-1}$ . So we arrive at the following description.

**Proposition 4.10.** *For  $n > 2$  the Lie algebra*

$$\mathrm{Lie}(\mathrm{Aut}_0 \mathcal{SC}^{n-1}) = \langle I, \partial_t \rangle, \quad \text{where } I = \mathcal{O}_+(\mathcal{SC}_{\mathrm{blc}}^{n-2})\partial_\tau,$$

*is uniquely determined by the relations  $[b\partial_\tau, \partial_t] = b\partial_\tau$ , where  $b$  runs over  $\mathcal{O}_+(\mathcal{SC}_{\mathrm{blc}}^{n-2})$ .*

*Proof.* This follows from Theorem 3.17 and Corollary 4.2 in the same way as before. We leave the details to the reader.  $\square$

## 5. MORE ON THE GROUP $\mathrm{Aut}(\mathcal{X} \times \mathbb{C})$

5.1. **The center of  $\mathrm{Aut}(\mathcal{X} \times \mathbb{C})$ .** The following lemma provides a formula for the commutator of two triangular automorphisms of a product  $\mathcal{X} \times \mathbb{C}$ . We let

$$F = F(S, A): (x, y) \mapsto (Sx, A(x)y),$$

where  $(x, y) \in \mathcal{X} \times \mathbb{C}$ ,  $S \in \mathrm{Aut} \mathcal{X}$ , and  $A: \mathcal{X} \rightarrow \mathrm{Aff} \mathbb{C}$  (cf. (30)).

**Lemma 5.1.** *Suppose that the group  $\mathrm{Aut} \mathcal{X}$  is Abelian. Then for any  $F = F(S, A)$  and  $F' = F(S', A')$  in  $\mathrm{Aut}_\Delta(\mathcal{X} \times \mathbb{C})$  and any  $(x, y) \in \mathcal{X} \times \mathbb{C}$  we have*

$$[F', F](x, y) = (x, (A'(x))^{-1}(A(S'x))^{-1}A'(Sx)A(x)y). \quad (53)$$

*Consequently,  $F$  and  $F'$  commute if and only if*

$$A(S'x)A'(x) = A'(Sx)A(x) \quad \text{for any } x \in \mathcal{X}. \quad (54)$$

*Proof.* The proof is straightforward.  $\square$

Applying this lemma to general cylinders we deduce the following facts.

**Proposition 5.2.** *Let  $\mathcal{X}$  be an affine variety. If the group  $\mathrm{Aut} \mathcal{X}$  is Abelian then the center of the group  $\mathrm{Aut}_\Delta(\mathcal{X} \times \mathbb{C})$  is trivial. The same conclusion holds for the groups  $\mathrm{Aut} \mathcal{C}^n$ ,  $\mathrm{Aut} \mathcal{SC}^{n-1}$ , and  $\mathrm{Aut} \Sigma^{n-1}$ , where  $n > 2$ .*

*Proof.* Consider two elements  $F = F(S, A)$  and  $F' = F(S', A')$  in  $\text{Aut}_\Delta(\mathcal{X} \times \mathbb{C})$ , where  $S, S' \in \text{Aut } \mathcal{X}$  and  $A: y \mapsto ay + b$ ,  $A': y \mapsto a'y + b'$  with  $a, a' \in \mathcal{O}^\times(\mathcal{X})$  and  $b, b' \in \mathcal{O}_+(\mathcal{X})$ . If  $F$  and  $F'$  commute then (54) is equivalent to the system

$$(a \circ S') \cdot a' = a \cdot (a' \circ S) \quad \text{and} \quad (a' \circ S) \cdot b + b' \circ S = (a \circ S') \cdot b' + b \circ S'. \quad (55)$$

Assume that  $F$  is a central element i.e. (55) holds for any  $F'$ . Letting in the second relation  $S' = \text{id}$ ,  $a' = 2$ , and  $b' = 0$  yields  $b = 0$ . Now this relation reduces to

$$b' \circ S = (a \circ S') \cdot b'.$$

Letting  $b' = 1$  yields  $a = 1$  and so  $A = \text{id}$  and  $b' \circ S = b'$  for any  $b' \in \mathcal{O}_+(\mathcal{X})$ . If  $S \neq \text{id}$  this leads to a contradiction, provided that  $b'$  is non-constant on an  $S$ -orbit in  $\mathcal{X}$ . Hence  $S = \text{id}$ , and so,  $F = \text{id}$ , as claimed.

The last assertion follows now from Corollary 2.4, since the bases of the cylinders  $\mathcal{C}^n$ ,  $\mathcal{S}\mathcal{C}^{n-1}$ , and  $\Sigma^{n-1}$  are rigid varieties with Abelian automorphism groups, see (27), Proposition 2.21, and Theorem 4.1.  $\square$

**5.2. Commutator series.** Let us introduce the following notation.

**Notation.** Let  $\mathcal{X}$  be an affine variety with a tight cylinder  $\mathcal{X} \times \mathbb{C}$ , and let  $\mathcal{D} \subseteq \text{Aut}(\mathcal{X} \times \mathbb{C})$  be the subgroup consisting of all automorphisms of the form  $F = F(\text{id}, A)$ , where  $A: y \mapsto ty + b$  with  $t \in \mathbb{C}^*$  and  $b \in \mathcal{O}_+(\mathcal{X})$ . It is easily seen that

$$\text{SAut}(\mathcal{X} \times \mathbb{C}) \triangleleft \mathcal{D} \triangleleft \text{Aut}_0(\mathcal{X} \times \mathbb{C}).$$

Furthermore,  $\mathcal{D} \cong \mathcal{O}_+(\mathcal{X}) \rtimes \mathbb{C}^*$  under the isomorphism in (35), with quotient group  $\text{Aut}_0(\mathcal{X} \times \mathbb{C})/\mathcal{D} \cong \text{Aut}_0 \mathcal{X}$ . In particular, for  $\mathcal{X} \times \mathbb{C} \cong \mathcal{S}\mathcal{C}^{n-1}$ ,  $n > 2$ , we have  $\mathcal{D} = \text{Aut}_0(\mathcal{X} \times \mathbb{C})$ , see Corollary 2.4 and Theorem 4.4.

It is known ([23, Ch. VII, Theorem 19.3(a)]) that for any connected solvable affine algebraic group  $G$  the commutator subgroup  $[G, G]$  is contained in the unipotent radical  $G_u$  of  $G$ . In our setting a similar result holds.

**Theorem 5.3.** *Let  $\mathcal{X} \not\cong \mathbb{C}$  be an affine variety such that  $\text{Aut}_0 \mathcal{X}$  is an algebraic group. Then*

$$[\text{Aut}_0(\mathcal{X} \times \mathbb{C}), \text{Aut}_0(\mathcal{X} \times \mathbb{C})] = [\mathcal{D}, \mathcal{D}] = \text{SAut}(\mathcal{X} \times \mathbb{C}). \quad (56)$$

Consequently, the commutator series of the group  $\text{Aut}_0(\mathcal{X} \times \mathbb{C})$  is

$$1 \triangleleft \text{SAut}(\mathcal{X} \times \mathbb{C}) \triangleleft \text{Aut}_0(\mathcal{X} \times \mathbb{C}). \quad (57)$$

The same conclusions hold for the groups  $\text{Aut}_0 \mathcal{C}^n$ ,  $\text{Aut}_0 \mathcal{S}\mathcal{C}^{n-1}$ , and  $\text{Aut} \Sigma^{n-1} = \text{Aut}_0 \Sigma^{n-1}$ , where  $n > 2$ .

*Proof.* Since the group  $\text{Aut}_0 \mathcal{X}$  is algebraic and  $\mathcal{X} \not\cong \mathbb{C}$ , this group does not contain any unipotent one-parameter subgroup. Hence by [25, Lemma 3]  $\text{Aut}_0 \mathcal{X}$  is an algebraic

torus, hence is Abelian. By (35),  $\text{SAut}(\mathcal{X} \times \mathbb{C}) \triangleleft \text{Aut}_0(\mathcal{X} \times \mathbb{C})$  is a normal subgroup with the Abelian quotient

$$\text{Aut}_0(\mathcal{X} \times \mathbb{C}) / \text{SAut}(\mathcal{X} \times \mathbb{C}) \cong \mathbb{C}^* \times \text{Aut}_0 \mathcal{X}.$$

Hence  $[\text{Aut}_0(\mathcal{X} \times \mathbb{C}), \text{Aut}_0(\mathcal{X} \times \mathbb{C})] \subset \text{SAut}(\mathcal{X} \times \mathbb{C})$ . To show (56) it suffices to establish the inclusion  $\text{SAut}(\mathcal{X} \times \mathbb{C}) \subset [\mathcal{D}, \mathcal{D}]$ . However, by virtue of (53) any  $F = F(\text{id}, A) \in \text{SAut}(\mathcal{X} \times \mathbb{C})$ , where  $A : y \mapsto y + b$  with  $b \in \mathcal{O}_+(\mathcal{X})$ , can be written as commutator  $F = [F', F'']$ , where  $F' = F(\text{id}, A')$  and  $F'' = F(\text{id}, A'')$  with  $A' : y \mapsto -y - b/2$  and  $A'' : y \mapsto y + b/2$  (in fact,  $A = [A', A'']$ ).

Now (57) follows from (56), since the group  $\text{SAut}(\mathcal{X} \times \mathbb{C}) \cong \mathcal{O}_+(\mathcal{X})$  is Abelian.

By Theorem 4.1, for  $n > 2$  the groups  $\text{Aut}_0 \mathcal{C}^n$ ,  $\text{Aut}_0 \mathcal{SC}^{n-1}$ , and  $\text{Aut} \Sigma^{n-1} = \text{Aut}_0 \Sigma^{n-1}$  satisfy our assumptions. So, the conclusions hold also for these groups.  $\square$

For the group  $\text{Aut} \mathcal{SC}^{n-1}$  the following hold.

**Theorem 5.4.** *For  $n > 2$  we have  $[\text{Aut} \mathcal{SC}^{n-1}, \text{Aut} \mathcal{SC}^{n-1}] = \text{SAut} \mathcal{SC}^{n-1}$ . Hence the commutator series of  $\text{Aut} \mathcal{SC}^{n-1}$  is*

$$1 \triangleleft \text{SAut} \mathcal{SC}^{n-1} \triangleleft \text{Aut} \mathcal{SC}^{n-1}$$

with the Abelian normal subgroup  $\text{SAut} \mathcal{SC}^{n-1} \cong \mathcal{O}_+(\mathcal{SC}_{\text{blc}}^{n-2})$  and the Abelian quotient group

$$\text{Aut} \mathcal{SC}^{n-1} / \text{SAut} \mathcal{SC}^{n-1} \cong \mathbb{C}^* \times (\mathbb{Z}/n(n-1)\mathbb{Z}). \quad (58)$$

*Proof.* Theorem 5.3 yields the inclusion  $\text{SAut} \mathcal{SC}^{n-1} \subset [\text{Aut} \mathcal{SC}^{n-1}, \text{Aut} \mathcal{SC}^{n-1}]$ . The opposite inclusion follows from (58), which is in turn a consequence of Theorem 4.4. Hence the assertions follow.  $\square$

Consider further the group  $\text{Aut} \mathcal{C}^n$ . Note that the quotient groups

$$(\text{Aut} \mathcal{C}^n) / \mathcal{D} \cong \mathbb{C}^* \times \mathbb{Z} \quad \text{and} \quad \mathcal{D} / \text{SAut} \mathcal{C}^n \cong \mathbb{C}^* \quad (59)$$

are Abelian, see Theorem 4.4. Hence

$$[\text{Aut} \mathcal{C}^n, \text{Aut} \mathcal{C}^n] \subseteq \mathcal{D}, \quad \text{where} \quad [\mathcal{D}, \mathcal{D}] = \text{SAut} \mathcal{C}^n, \quad (60)$$

see Theorem 5.3. More precisely, the following holds.

**Theorem 5.5.** *For  $n > 2$  we have  $[\text{Aut} \mathcal{C}^n, \text{Aut} \mathcal{C}^n] = \mathcal{D}$ . Hence the commutator series of the group  $\text{Aut} \mathcal{C}^n$  is*

$$1 \triangleleft \text{SAut} \mathcal{C}^n \triangleleft \mathcal{D} \triangleleft \text{Aut} \mathcal{C}^n$$

with Abelian quotient groups, see (59).

*Proof.* By virtue of (60) to establish the first equality it suffices to prove the inclusion  $\mathcal{D} \subseteq [\text{Aut } \mathcal{C}^n, \text{Aut } \mathcal{C}^n]$ . We show that, moreover, any element  $F_0 \in \mathcal{D}$  is a product of two commutators in  $\text{Aut } \mathcal{C}^n$ .

Indeed, choosing as before  $F$  and  $F'$  in  $\mathcal{D}$  such that  $[F', F]: Q \rightarrow Q + b(Q)$  and replacing  $F_0$  by  $[F', F]^{-1}F_0$  we may suppose that  $F_0 = F(\text{id}, A_0)$ , where  $A_0: y \mapsto ty$  with  $t \in \mathbb{C}^*$ .

Let  $\tilde{F} = F(S, A)$  and  $\tilde{F}' = F(S', A')$ , where

$S: Q^\circ \mapsto sQ^\circ$ ,  $S': Q^\circ \mapsto s'Q^\circ$ , and  $A(Q^\circ): y \mapsto sD_n^k(Q^\circ)y$ ,  $A'(Q^\circ): y \mapsto s'D_n^{k'}(Q^\circ)y$  for  $Q^\circ \in \mathcal{C}_{\text{blc}}^{n-1}$  and  $y \in \mathbb{C}$ . By (53) we obtain

$$[\tilde{F}', \tilde{F}](Q) = F(\text{id}, A''), \quad \text{where } A'': y \mapsto (s^{k'} s'^{-k})^{n(n-1)}y$$

does not depend on  $Q^\circ \in \mathcal{C}_{\text{blc}}^{n-1}$ . Given  $t \in \mathbb{C}^*$  we can find  $s, s' \in \mathbb{C}^*$  and  $k, k' \in \mathbb{Z}$  such that  $(s^{k'} s'^{-k})^{n(n-1)} = t$ . With this choice,  $F_0 = [\tilde{F}', \tilde{F}]$  and we are done.  $\square$

**5.3. Torsion in  $\text{Aut}_0(\mathcal{Z}_{\text{blc}} \times \mathbb{C})$ .** Let  $\mathcal{Z}$  be one of the varieties  $\mathcal{C}^n$ ,  $\mathcal{S}\mathcal{C}^{n-1}$ , or  $\Sigma^{n-1}$ , where  $n > 2$ , and let  $\mathcal{Z}_{\text{blc}}$  be the corresponding balanced variety. Denote by  $G_{\mathcal{Z}}$  one of the groups  $\text{Aut } \mathcal{C}^n$ ,  $\text{Aut}_0 \mathcal{S}\mathcal{C}^{n-1}$ , and  $\text{Aut } \Sigma^{n-1}$ . With this notation we have the following results.

**Theorem 5.6.** *The semisimple elements of the group  $G_{\mathcal{Z}}$  are precisely the automorphisms of the form*

$$F: Q \mapsto sQ^\circ + t \cdot \text{bc}(Q) + t \cdot b(Q^\circ) - b(sQ^\circ) \quad \text{for all } Q \in \mathcal{Z}, \quad (61)$$

where  $Q^\circ = Q - \text{bc}(Q) \in \mathcal{Z}_{\text{blc}}$ ,  $b \in \mathcal{O}_+(\mathcal{Z}_{\text{blc}})$ , and  $s, t \in \mathbb{C}^*$ , with  $s^{n(n-1)} = 1$  when  $\mathcal{Z} = \mathcal{S}\mathcal{C}^{n-1}$ . Such an element  $F$  is torsion with  $F^m = \text{id}$  if and only if, in addition,  $s^m = t^m = 1$ .

*Proof.* Since  $\text{Aut } \mathcal{C}^n / \text{Aut}_0 \mathcal{C}^n \cong \mathbb{Z}$ , the torsion elements of  $\text{Aut } \mathcal{C}^n$  are that of the neutral component  $\text{Aut}_0 \mathcal{C}^n$ . Taking into account that the group  $\text{Aut } \Sigma^{n-1}$  ( $n > 2$ ) is connected, in all three cases Proposition 3.16 applies and yields the result after a simple calculation using the description in (48) and (49).  $\square$

In Example 5.9 we construct some particular torsion elements of the group  $\text{Aut } \mathcal{C}^n$ . We show that for any  $b \in \mathcal{O}_+(\mathcal{C}_{\text{blc}}^{n-1})$  there is an element  $F \in \text{Tors}(\text{Aut } \mathcal{C}^n)$  of the form

$$F: Q \mapsto sQ^\circ + t \text{bc}(Q) + b(Q^\circ), \quad \text{where } Q^\circ = Q - \text{bc}(Q). \quad (62)$$

We use the following lemma. Its proof proceeds by induction on  $m$ ; we leave the details to the reader.



**Lemma 5.7.** *Let  $n > 2$ , and let  $F \in \text{Aut}_0 \mathcal{C}^n$  be given by (62). Letting  $Q^\circ = Q - \text{bc}(Q)$  for any  $m \in \mathbb{N}$  we have*

$$F^m(Q) = s^m Q^\circ + t^m \text{bc}(Q) + \sum_{j=0}^{m-1} t^{m-j-1} b(s^j Q^\circ).$$

Consequently,  $F^m = \text{id}$  if and only if  $s^m = t^m = 1$  and the function  $b$  satisfies the equation

$$\sum_{j=0}^{m-1} t^{m-j-1} b(s^j Q^\circ) = 0 \quad \text{for any } Q^\circ \in \mathcal{C}_{\text{blc}}^{n-1}. \quad (63)$$

**Remark 5.8.** It follows from Lemma 5.7 and Theorem 5.6 that for  $m \geq 2$  a function  $b \in \mathcal{O}(\mathcal{C}_{\text{blc}}^{n-1})$  satisfies (63) for a given pair  $(s, t)$  of  $m$ th roots of unity if and only if it can be written as  $b(Q^\circ) = \tilde{t}b(Q^\circ) - \tilde{b}(sQ^\circ)$  for some  $\tilde{b} \in \mathcal{O}(\mathcal{C}_{\text{blc}}^{n-1})$ . The inversion formula

$$\tilde{b}(Q^\circ) = \sum_{j=0}^{m-1} \frac{m-j}{m} t^{m-j} b(s^{j-1} Q^\circ) \quad \text{for any } Q^\circ \in \mathcal{C}_{\text{blc}}^{n-1} \quad (64)$$

allows to find such a function  $\tilde{b} \in \mathcal{O}(\mathcal{C}_{\text{blc}}^{n-1})$  for a given solution  $b \in \mathcal{O}(\mathcal{C}_{\text{blc}}^{n-1})$  of (63).

**Examples 5.9** (*Automorphisms of  $\mathcal{C}^n$  of finite order*). (a) For  $m > 1$ , pick any  $b \in \mathcal{O}(\mathcal{C}_{\text{blc}}^{n-1})$  and any  $m$ th root of unity  $t \neq 1$ . Then the automorphism  $F: Q \mapsto (Q - \text{bc}(Q)) + t \text{bc}(Q) + b(Q - \text{bc}(Q))$  satisfies  $F \neq \text{id}$  and  $F^m = \text{id}$ .

(b) Let  $b \in \mathcal{O}(\mathcal{C}^n)$  be invariant under the diagonal ( $\text{Aff } \mathbb{C}$ )-action on  $\mathcal{C}^n$ . For instance,  $b(Q) = cD_n^{-k}(Q) \sum_{q', q'' \in Q} (q' - q'')^{kn(n-1)}$  is such a function for any  $k \in \mathbb{N}$  and  $c \in \mathbb{C}$ .

Take any  $m > 2$ , and let  $s$  and  $t$  be two distinct  $m$ th roots of unity, where  $t \neq 1$ . Then the automorphism  $F$  as in (62) satisfies  $F \neq \text{id}$  and  $F^m = \text{id}$ .

(c) Any automorphism  $F$  of  $\mathcal{C}^n$  of the form  $F: Q \mapsto -Q + b(Q - \text{bc}(Q))$  with  $b \in \mathcal{O}(\mathcal{C}_{\text{blc}}^{n-1})$  is an involution. For instance, one can take

$$b(Q) = cD_n^r(Q) \sum_{\{q', q''\} \subset Q} (q' - q'')^{2m} \quad \text{with } r, m \in \mathbb{Z}, \quad |r| + |m| > 0, \quad \text{and } c \in \mathbb{C}.$$

## 6. THE GROUP $\text{Aut } \mathcal{C}^n(\mathbb{C}^*)$ : ZINDE'S THEOREM

In this section we provide a description of the group  $\text{Aut } \mathcal{C}^n(\mathbb{C}^*)$  according to Zinde [51]. This description will be used in the next section. For the convenience of the reader, we give an alternative proof in our setting. The original Zinde's Theorem contains a description of the biholomorphic automorphisms of  $\mathcal{C}^n(\mathbb{C}^*)$  for  $n > 4$ , and the structure of the group of all such automorphisms. The proof in [51] is based on Zinde's part of Tame Map Theorem (see the Introduction), which says that for  $n > 4$

any non-Abelian holomorphic endomorphism of  $\mathcal{C}^n(\mathbb{C}^*)$  is tame (see [51, Theorem 7]). Our approach is quite different. Indeed, we deal with biregular automorphisms only. In this particular case, for any  $n > 2$  we provide a proof which does not refer to Tame Map Theorem. However, the starting point of both proofs is the same, see Corollary 6.3.

To formulate our extension of Zinde’s Theorem, we need a portion of notation. Consider the function  $h_n \in \mathcal{O}^\times(\mathcal{C}^n(\mathbb{C}^*))$ , where  $h_n(Q) = D_n(Q)/(q_1 \cdot \dots \cdot q_n)^{n-1}$ . It is invariant under the diagonal action of  $\text{Aut } \mathbb{C}^*$  on  $\mathcal{C}^n(\mathbb{C}^*)$ . For  $\varepsilon \in \{1, -1\}$  and  $Q = \{q_1, \dots, q_n\} \in \mathcal{C}^n(\mathbb{C}^*)$  we let  $Q^\varepsilon = \{q_1^\varepsilon, \dots, q_n^\varepsilon\}$ . With this notation, we have the following description of the group  $\text{Aut } \mathcal{C}^n(\mathbb{C}^*)$ .

**Zinde’s Theorem** ([51, Theorem 8]). *Let  $n > 2$ . A map  $F: \mathcal{C}^n(\mathbb{C}^*) \rightarrow \mathcal{C}^n(\mathbb{C}^*)$  is an automorphism if and only if there exist  $\varepsilon \in \{1, -1\}$ ,  $s \in \mathbb{C}^*$ , and  $k \in \mathbb{Z}$  such that*

$$F(Q) = sh_n^k(Q)Q^\varepsilon \text{ for all } Q \in \mathcal{C}^n(\mathbb{C}^*). \quad (65)$$

Furthermore, for  $n > 2$

$$\text{Aut } \mathcal{C}^n(\mathbb{C}^*) \cong (\mathbb{C}^* \times \mathbb{Z}) \rtimes (\mathbb{Z}/2\mathbb{Z}),$$

where the factors  $\mathbb{Z}$  and  $\mathbb{Z}/2\mathbb{Z}$  commute, while for  $n = 2$

$$\text{Aut } \mathcal{C}^2(\mathbb{C}^*) \cong ((\mathbb{C}^* \times \mathbb{Z}) \rtimes (\mathbb{Z}/2\mathbb{Z})) \rtimes (\mathbb{Z}/2\mathbb{Z}).$$

Thus, the case  $n = 2$  is exceptional: the automorphisms as in (65) form a subgroup of index 2 in the group  $\text{Aut } \mathcal{C}^2(\mathbb{C}^*)$ ; see Theorem 6.26 for a more precise description. The proof of Zinde’s Theorem for  $n > 2$  is done in 6.1-6.23.

The following lemma is a generalization to the series B of the classical Artin Theorem ([2, Theorem 3]), which says that the pure braid group  $\mathbf{PA}_{n-1}$  is a characteristic subgroup of the Artin braid group  $\mathbf{A}_{n-1}$ . Recall that a subgroup  $N$  of a group  $G$  is called *characteristic* if  $N$  is stable under all automorphisms of  $G$ .

**Lemma 6.1.** *Let  $\mathbf{B}_n = \pi_1(\mathcal{C}^n(\mathbb{C}^*))$  be the Artin-Brieskorn braid group of type  $B_n$ , and let  $\mathbf{PB}_n \subset \mathbf{B}_n$  be the kernel of the natural surjection  $\mathbf{B}_n \rightarrow \mathbf{S}(n)$ . Then  $\mathbf{PB}_n$  is a characteristic subgroup of  $\mathbf{B}_n$  for any  $n > 2$ .*

*Proof.* For  $n \geq 5$  the assertion follows from a more general result due to Zinde ([50, Theorem 5]), which says that  $\mathbf{PB}_n$  is invariant under any endomorphism of  $\mathbf{B}_n$  with non-Abelian image. Alternatively, the assertion also follows for  $n = 3$  and any  $n \geq 5$  from a result of Ivanov ([26, Theorem 2]), which generalizes Artin’s Theorem to the braid groups of general surfaces. The result for  $n = 4$  is as well announced in [26] (a remark after Theorem 2), however, the proof of this was never published. We provide therefore yet another proof, which works for all  $n \geq 3$ .

Our approach uses the following result. We let  $\mathbf{WB}_n$  be the finite Coxeter group of type  $B_n$ , and  $\mathbf{CB}_n$  (the ‘colored braid group’) be the kernel of the natural surjection  $\omega_n: \mathbf{B}_n \rightarrow \mathbf{WB}_n$ . By a result of Cohen and Paris ([9, Propositions 2.4 and 3.9]),  $\mathbf{CB}_n$  is a characteristic subgroup of  $\mathbf{B}_n$  for any  $n \in \mathbb{N}$ .

Let further  $\mathcal{E}_n \cong (\mathbb{Z}/2\mathbb{Z})^n$  be the subgroup of  $\mathbf{WB}_n$  generated by the orthogonal reflexions in  $\mathbb{R}^n$  with mirrors being the coordinate hyperplanes. We have  $\mathbf{WB}_n = \mathcal{E}_n \rtimes \mathbf{S}(n)$ . If  $\pi_n: \mathbf{WB}_n \rightarrow \mathbf{S}(n)$  is the surjection with kernel  $\mathcal{E}_n$ , then  $\mathbf{PB}_n = \ker(\pi_n \circ \omega_n) \supset \ker \omega_n = \mathbf{CB}_n$ .

Let  $\alpha \in \text{Aut } \mathbf{B}_n$ . Since  $\mathbf{CB}_n \subset \mathbf{B}_n$  is a characteristic subgroup, we have  $\alpha(\mathbf{CB}_n) = \mathbf{CB}_n$ . Hence  $\alpha$  induces an automorphism  $\bar{\alpha} \in \text{Aut } \mathbf{WB}_n$  of the quotient group. By a lemma of Franzsen ([18, Lemma 2.6]), for any  $n \geq 3$  the subgroup  $\mathcal{E}_n \subset \mathbf{WB}_n$  is characteristic. Thus,  $\bar{\alpha}(\mathcal{E}_n) = \mathcal{E}_n$ . Since  $\mathbf{PB}_n = \omega_n^{-1}(\ker \pi_n) = \omega_n^{-1}(\mathcal{E}_n)$ , then  $\alpha(\mathbf{PB}_n) = \mathbf{PB}_n$ , and so,  $\mathbf{PB}_n \subset \mathbf{B}_n$  is a characteristic subgroup too.  $\square$

**Remarks 6.2.** 1. Lemma 6.1 does not hold any longer for  $n = 2$ . Indeed, recall that the group  $\mathbf{B}_2$  admits a presentation

$$\mathbf{B}_2 = \langle \sigma_1, \sigma_2 \mid \sigma_1 \sigma_2 \sigma_1 \sigma_2 = \sigma_2 \sigma_1 \sigma_2 \sigma_1 \rangle$$

with standard generators  $\sigma_1$  and  $\sigma_2$ . Let  $\rho: \mathbf{B}_2 \rightarrow \mathbf{S}(2)$  be the standard surjection sending  $\sigma_1$  to  $\sigma = (1, 2)$  and  $\sigma_2$  to  $\text{id}$ . Thus, we have  $\mathbf{PB}_2 = \ker \rho = \langle \sigma_1^2, \sigma_2 \rangle$ . Let  $\alpha \in \text{Aut } \mathbf{B}_2$  be the involution of  $\mathbf{B}_2$  interchanging  $\sigma_1$  and  $\sigma_2$ . It is called a *graph involution* in [18], since it is induced by an involution of the Dynkin graph of type  $B_2$  (the only graph of series  $B$ , which admits a nontrivial automorphism). Then  $\alpha(\mathbf{PB}_2) = \langle \sigma_1, \sigma_2^2 \rangle \neq \mathbf{PB}_2$ . Therefore, the subgroup  $\mathbf{PB}_2 \subset \mathbf{B}_2$  is not characteristic.

2. Likewise, Franzsen’s Lemma cited above does not hold for  $n = 2$ . Indeed, we have

$$\mathbf{WB}_2 = \mathcal{E}_2 \rtimes \langle \sigma \rangle = \{\text{id}, \varepsilon_1, \varepsilon_2, w_0, \sigma, w_0 \sigma, \varepsilon_1 \sigma, \varepsilon_2 \sigma\},$$

where  $\sigma = (1, 2)$  acts naturally on  $\mathbb{R}^2$  as an orthogonal reflexion,  $\mathcal{E}_2 = \langle \varepsilon_1, \varepsilon_2 \rangle \cong (\mathbb{Z}/2\mathbb{Z})^2$  with  $\varepsilon_i$  being the reflexion in  $\mathbb{R}^2$  of sign change of the  $i$ th coordinate,  $i = 1, 2$ , and  $w_0 = \varepsilon_1 \varepsilon_2 = -\text{id}$ .<sup>10</sup> It is easy to check that the involution

$$\bar{\alpha} \in \text{Aut } \mathbf{WB}_2, \quad \text{id} \mapsto \text{id}, w_0 \mapsto w_0, \varepsilon_1 \leftrightarrow \sigma, \varepsilon_2 \leftrightarrow w_0 \sigma, \varepsilon_1 \sigma \leftrightarrow \varepsilon_2 \sigma$$

does not stabilize the subgroup  $\mathcal{E}_2$ . Hence this subgroup is not characteristic. (In fact,  $\bar{\alpha}$  is induced by  $\alpha$  as above via the natural surjection  $\text{Aut } \mathbf{B}_2 \rightarrow \text{Aut } \mathbf{WB}_2$ .)

The next corollary follows from a more general result of Zinde if  $n > 4$ , see [51, §2].

**Corollary 6.3.** *For  $n > 2$  any automorphism  $F$  of  $\mathcal{C}^n(\mathbb{C}^*)$  can be lifted to an automorphism  $\tilde{F}$  of  $\mathcal{C}_{\text{ord}}^n(\mathbb{C}^*)$ .*

<sup>10</sup>Note that the natural surjection  $\mathbf{B}_2 \rightarrow \mathbf{WB}_2$  sends the standard generators  $\sigma_1$  and  $\sigma_2$  of  $\mathbf{B}_2$  to  $\sigma$  and  $\varepsilon_1$ , respectively, and sends the generator  $(\sigma_1 \sigma_2)^2$  of the center  $Z(\mathbf{B}_2) \cong \mathbb{Z}$  to  $\varepsilon_1 \varepsilon_2$ .

*Proof.* By the monodromy theorem, a continuous selfmap  $F$  of the base of the Galois covering  $\pi: \mathcal{C}_{\text{ord}}^n(\mathbb{C}^*) \rightarrow \mathcal{C}^n(\mathbb{C}^*) = \mathcal{C}_{\text{ord}}^n(\mathbb{C}^*)/\mathbf{S}(n)$  admits a lift to the cover if and only if the induced endomorphism  $F_*$  of the fundamental group  $\pi_1(\mathcal{C}^n(\mathbb{C}^*)) = \mathbf{B}_n$  stabilizes the subgroup  $\pi_*(\pi_1(\mathcal{C}_{\text{ord}}^n(\mathbb{C}^*))) = \mathbf{PB}_n$ . Note that this endomorphism is well defined only up to conjugation, i.e., up to the left multiplication by an inner automorphism. However, the property to stabilize the normal subgroup  $\mathbf{PB}_n$  does not depend of this ambiguity in the definition of  $F_*$ . If  $F$  is an automorphism, then  $F_*(\mathbf{PB}_n) = \mathbf{PB}_n$  by Lemma 6.1. Hence  $F$  admits a lift  $\tilde{F}$  to an automorphism of  $\mathcal{C}_{\text{ord}}^n(\mathbb{C}^*)$ .  $\square$

Note that this corollary does not hold for  $n = 2$ ; see Example 6.25.

The following lemma should be well known. For the lack of a reference, we provide a short argument.

**Lemma 6.4.** *Let  $\pi: \tilde{\mathcal{X}} \rightarrow \mathcal{X}$  be a Galois covering with a Galois group  $G$ . Suppose that any automorphism  $\alpha \in \text{Aut } \mathcal{X}$  admits a lift to an automorphism  $\tilde{\alpha} \in \text{Aut } \tilde{\mathcal{X}}$ . Then the set of all possible such lifts coincides with the normalizer  $N$  of  $G$  in  $\text{Aut } \tilde{\mathcal{X}}$ , and  $\text{Aut } \mathcal{X} \cong N/G$ .*

*Proof.* An automorphism  $\gamma \in \text{Aut } \tilde{\mathcal{X}}$  is a lift of some  $\alpha \in \text{Aut } \mathcal{X}$  if and only if  $\gamma$  sends the fibers of  $\pi$  into fibers. The  $\pi$ -fibers are the  $G$ -orbits, hence  $\gamma$  acts on the family of the  $G$ -orbits via

$$\gamma(G.\tilde{x}) = G.\gamma(\tilde{x}) \quad \forall \tilde{x} \in \tilde{\mathcal{X}}. \quad (66)$$

Letting  $\tilde{x}' = \gamma(\tilde{x})$  we can write (66) in the form

$$\gamma \circ G \circ \gamma^{-1}(\tilde{x}') = G.\tilde{x}' \quad \forall \tilde{x}' \in \tilde{\mathcal{X}}. \quad (67)$$

Any  $\gamma \in N$  satisfies (67). Conversely, let  $\gamma \in \text{Aut } \tilde{\mathcal{X}}$  satisfies (67). Then  $\gamma \circ G \circ \gamma^{-1}$  consists of deck transformations. However, the deck transformations form the Galois group  $G \simeq \pi_1(\mathcal{X})/\pi_1(\tilde{\mathcal{X}})$ . Hence  $\gamma \circ G \circ \gamma^{-1} = G$ , these groups being of the same cardinality. Thus,  $\gamma \in N$ . This proves the first assertion. The proof of the second is easy and is left to the reader.  $\square$

**Notation 6.5.** Given a group  $G$  and a subgroup  $S$  of  $G$ , we let  $\text{Norm}_G(S)$  denote the normalizer of  $S$  in  $G$ . In the sequel  $N = \text{Norm}_{\text{Aut } \mathcal{C}_{\text{ord}}^n(\mathbb{C}^*)}(\mathbf{S}(n))$  stands for the normalizer of  $\mathbf{S}(n)$  in  $\text{Aut } \mathcal{C}_{\text{ord}}^n(\mathbb{C}^*)$ .

**Corollary 6.6.** *Consider the Galois covering  $\mathcal{C}_{\text{ord}}^n(\mathbb{C}^*) \rightarrow \mathcal{C}^n(\mathbb{C}^*)$  with the Galois group  $\mathbf{S}(n)$  acting on  $\mathcal{C}_{\text{ord}}^n(\mathbb{C}^*)$  via permutations of coordinates. Then  $\text{Aut } \mathcal{C}^n(\mathbb{C}^*) \cong N/\mathbf{S}(n)$ .*

*Proof.* Indeed, the assumptions of Lemma 6.4 are fulfilled in this example due to Lemma 6.3. Applying Lemma 6.4 yields the result.  $\square$

Thus, to prove Zinde's Theorem we have to determine the normalizer  $N$ . This is done below.

To describe the automorphism group  $\text{Aut } \mathcal{C}_{\text{ord}}^n(\mathbb{C}^*)$  we use the following direct product decomposition.

**Notation 6.7.** We let as usual  $\mathbb{C}^{**} = \mathbb{C}^* \setminus \{1\}$ . For any  $n \geq 1$  there is an isomorphism

$$\mathcal{C}_{\text{ord}}^n(\mathbb{C}^*) \xrightarrow{\cong} \mathcal{C}_{\text{ord}}^{n-1}(\mathbb{C}^{**}) \times \mathbb{C}^*, \quad Q = (q_1, \dots, q_n) \mapsto (Q', y) = \left( \left( \frac{q_1}{q_n}, \dots, \frac{q_{n-1}}{q_n} \right), q_n \right) \quad (68)$$

with inverse  $\eta: (Q', y) \mapsto (yQ', y)$ , where  $Q' \in \mathcal{C}_{\text{ord}}^{n-1}(\mathbb{C}^{**})$  and  $y \in \mathbb{C}^*$ . It is equivariant with respect to the  $\mathbb{C}^*$ -actions on  $\mathcal{C}_{\text{ord}}^n(\mathbb{C}^*)$  via  $Q \mapsto sQ$  and on the product  $\mathcal{C}_{\text{ord}}^{n-1}(\mathbb{C}^{**}) \times \mathbb{C}^*$  via  $(Q', y) \mapsto (Q', sy)$ , where  $s \in \mathbb{C}^*$ . The first projection  $\text{pr}_1: \mathcal{C}_{\text{ord}}^n(\mathbb{C}^*) \rightarrow \mathcal{C}_{\text{ord}}^{n-1}(\mathbb{C}^{**})$ ,  $Q \mapsto Q'$ , is the orbit map of the  $\mathbb{C}^*$ -actions on  $\mathcal{C}_{\text{ord}}^n(\mathbb{C}^*)$ .

The following simple lemma is a crucial point of our approach.

**Lemma 6.8.** *For  $n \geq 1$  the image of any regular map  $f: \mathbb{C}^* \rightarrow \mathcal{C}_{\text{ord}}^n(\mathbb{C}^*)$  is contained in an orbit of the  $\mathbb{C}^*$ -action on  $\mathcal{C}_{\text{ord}}^n(\mathbb{C}^*)$ .*

*Proof.* Since any morphism  $\mathbb{C}^* \rightarrow \mathbb{C}^{**}$  is constant, also  $\text{pr}_1 \circ f: \mathbb{C}^* \rightarrow \mathcal{C}_{\text{ord}}^{n-1}(\mathbb{C}^{**}) \subset (\mathbb{C}^{**})^{n-1}$  is. This yields the assertion.  $\square$

The next corollary is straightforward.

**Corollary 6.9.** *For  $n \geq 1$  any automorphism  $F \in \text{Aut } \mathcal{C}_{\text{ord}}^n(\mathbb{C}^*)$  preserves the family of the  $\mathbb{C}^*$ -orbits in  $\mathcal{C}_{\text{ord}}^n(\mathbb{C}^*)$  and induces an automorphism  $\varphi \in \text{Aut } \mathcal{C}_{\text{ord}}^{n-1}(\mathbb{C}^{**})$  such that  $\text{pr}_1 \circ F = \varphi \circ \text{pr}_1$ . The correspondence  $F \mapsto \varphi$  defines a surjective homomorphism  $\rho: \text{Aut } \mathcal{C}_{\text{ord}}^n(\mathbb{C}^*) \rightarrow \text{Aut } \mathcal{C}_{\text{ord}}^{n-1}(\mathbb{C}^{**})$ .*

Due to the following proposition, the automorphisms of  $\mathcal{C}_{\text{ord}}^n(\mathbb{C}^*)$  are “triangular” in the sense close to that of Section 2; cf. Corollary 2.22.

**Proposition 6.10.** (a) *For  $n \geq 1$  any automorphism  $F \in \text{Aut}(\mathcal{C}_{\text{ord}}^{n-1}(\mathbb{C}^{**}) \times \mathbb{C}^*) \cong \text{Aut } \mathcal{C}_{\text{ord}}^n(\mathbb{C}^*)$  can be written as  $F(x, y) = (Sx, A(x)y)$  for some  $A \in \text{Mor}(\mathcal{C}_{\text{ord}}^{n-1}(\mathbb{C}^{**}), \text{Aut } \mathbb{C}^*)$  and  $S \in \text{Aut } \mathcal{C}_{\text{ord}}^{n-1}(\mathbb{C}^{**})$ , for all  $(x, y) \in \mathcal{C}_{\text{ord}}^{n-1}(\mathbb{C}^{**}) \times \mathbb{C}^*$ . Vice versa, for any such pair  $(S, A)$  the above formula defines an automorphism  $F = F(S, A)$  of the direct product  $\mathcal{C}_{\text{ord}}^{n-1}(\mathbb{C}^{**}) \times \mathbb{C}^*$ .*

(b) *There is a decomposition*

$$\text{Aut } \mathcal{C}_{\text{ord}}^n(\mathbb{C}^*) \cong \text{Mor}(\mathcal{C}_{\text{ord}}^{n-1}(\mathbb{C}^{**}), \text{Aut } \mathbb{C}^*) \rtimes \text{Aut } \mathcal{C}_{\text{ord}}^{n-1}(\mathbb{C}^{**}). \quad (69)$$

*Proof.* The exact sequence

$$0 \rightarrow \ker \rho \rightarrow \text{Aut } \mathcal{C}_{\text{ord}}^n(\mathbb{C}^*) \xrightarrow{\rho} \text{Aut } \mathcal{C}_{\text{ord}}^{n-1}(\mathbb{C}^{**}) \rightarrow 0 \quad (70)$$

splits, with splitting homomorphism

$$\text{Aut } \mathcal{C}_{\text{ord}}^{n-1}(\mathbb{C}^{**}) \ni \varphi \mapsto (\varphi, \text{id}) \in \text{Aut}(\mathcal{C}_{\text{ord}}^{n-1}(\mathbb{C}^{**}) \times \mathbb{C}^*) \cong \text{Aut } \mathcal{C}_{\text{ord}}^n(\mathbb{C}^*).$$

Hence  $\text{Aut } \mathcal{C}_{\text{ord}}^n(\mathbb{C}^*) \cong \ker \rho \rtimes \text{Aut } \mathcal{C}_{\text{ord}}^{n-1}(\mathbb{C}^{**})$ . It remains to show that  $\ker \rho \cong \text{Mor}(\mathcal{C}_{\text{ord}}^{n-1}(\mathbb{C}^{**}), \text{Aut } \mathbb{C}^*)$ .

Indeed, any  $F \in \ker \rho$  preserves every  $\mathbb{C}^*$ -orbit and induces an automorphism of this orbit. Identifying  $\mathcal{C}_{\text{ord}}^n(\mathbb{C}^*)$  with the direct product  $\mathcal{C}_{\text{ord}}^{n-1}(\mathbb{C}^{**}) \times \mathbb{C}^*$  via isomorphism (68), we obtain a morphism  $A: \mathcal{C}_{\text{ord}}^{n-1}(\mathbb{C}^{**}) \rightarrow \text{Aut } \mathbb{C}^*$  such that  $F(x, y) = (x, A(x)y)$ , where  $x \in \mathcal{C}_{\text{ord}}^{n-1}(\mathbb{C}^{**})$  and  $y \in \mathbb{C}^*$ . Vice versa, for every morphism  $A: \mathcal{C}_{\text{ord}}^{n-1}(\mathbb{C}^{**}) \rightarrow \text{Aut } \mathbb{C}^*$  this formula gives an automorphism  $F \in \ker \rho$ .  $\square$

Our next aim is to describe the factors in (69). We start with the factor  $\text{Aut } \mathcal{C}_{\text{ord}}^{n-1}(\mathbb{C}^{**})$ , see Lemma 6.11. This lemma follows from a more general result of Kaliman [29]; see [38, Theorem 16] for a proof.

**Lemma 6.11.** (S. Kaliman [29]) *For  $n > 1$  we have  $\text{Aut } \mathcal{C}_{\text{ord}}^{n-1}(\mathbb{C}^{**}) \cong \mathbf{S}(n+2)$ .*

**Remark 6.12.** A hint for the proof is as follows; see [38, §3.2 and §4.10]. The diagonal action of the group  $\mathbf{PSL}(2, \mathbb{C})$  on the configuration space  $\mathcal{C}_{\text{ord}}^{n+2}(\mathbb{P}^1)$  commutes with the  $\mathbf{S}(n+2)$ -action via permutations of coordinates. There is an isomorphism  $\mathcal{C}_{\text{ord}}^{n-1}(\mathbb{C}^{**}) \cong \mathcal{C}_{\text{ord}}^{n+2}(\mathbb{P}^1)/\mathbf{PSL}(2, \mathbb{C})$ . The  $\mathbf{S}(n+2)$ -action on  $\mathcal{C}_{\text{ord}}^{n+2}(\mathbb{P}^1)$  descends to an effective  $\mathbf{S}(n+2)$ -action on the quotient  $\mathcal{C}_{\text{ord}}^{n-1}(\mathbb{C}^{**})$ . Using an explicit description of all non-constant morphisms  $\mathcal{C}_{\text{ord}}^{n-1}(\mathbb{C}^{**}) \rightarrow \mathbb{C}^{**}$ , it was shown in [ibid] that the image of  $\mathbf{S}(n+2)$  exhausts the whole automorphism group  $\text{Aut } \mathcal{C}_{\text{ord}}^{n-1}(\mathbb{C}^{**})$ .

From Proposition 6.10(b) and Lemma 6.11 we obtain such a corollary.

**Corollary 6.13.** *For  $n \geq 2$  we have*

$$\text{Aut } \mathcal{C}_{\text{ord}}^n(\mathbb{C}^*) \cong (\mathcal{O}^\times(\mathcal{C}_{\text{ord}}^{n-1}(\mathbb{C}^{**})) \rtimes (\mathbb{Z}/2\mathbb{Z})) \rtimes \mathbf{S}(n+2), \quad (71)$$

where

$$\mathcal{O}^\times(\mathcal{C}_{\text{ord}}^{n-1}(\mathbb{C}^{**})) \cong \mathcal{O}^\times(\mathcal{C}_{\text{ord}}^n(\mathbb{C}^*))^{\mathbb{C}^*}. \quad (72)$$

In the following lemma we describe the group of units on the configuration space  $\mathcal{C}^n(\mathbb{C}^*)$  and its  $\mathbb{C}^*$ -stable part.

**Lemma 6.14.** (a) *For  $n \geq 2$  we have  $\mathcal{O}^\times(\mathcal{C}^n(\mathbb{C}^*)) = \mathbb{C}^* \rtimes \mathbb{Z}^2$ , where the factor  $\mathbb{Z}^2$  is freely generated by the invertible functions  $D_n$  and  $z_n$  on  $\mathcal{C}^n(\mathbb{C}^*) \subset \mathbb{C}_{(z)}^n$ .*

(b) *Furthermore, a function  $f \in \mathcal{O}^\times(\mathcal{C}^n(\mathbb{C}^*))$  is  $\mathbb{C}^*$ -invariant if and only if  $f = sh_n^k$ , where  $s \in \mathbb{C}^*$ ,  $h_n = D_n z_n^{1-n}$ , and  $k \in \mathbb{Z}$ . Hence*

$$\mathcal{O}^\times(\mathcal{C}^n(\mathbb{C}^*))^{\mathbb{C}^*} \cong \mathbb{C}^* \times \mathbb{Z}.$$

*Proof.* The configuration space  $\mathcal{C}^n(\mathbb{C}^*)$  can be realized as the complement in the affine space  $\mathbb{C}_{(z)}^n$  to the union of the coordinate hyperplane  $z_n = 0$  and the discriminant hypersurface  $d_n(z) = 0$ . Hence the discriminant  $D_n = d_n|_{\mathcal{C}^n(\mathbb{C}^*)}$  and the function  $z_n(Q) = (-1)^n \prod_{q \in Q} q$  freely generate the quotient group  $\mathcal{O}^\times(\mathcal{C}^n(\mathbb{C}^*))/\mathbb{C}^* \cong H^1(\mathcal{C}^n(\mathbb{C}^*); \mathbb{Z}) \cong \mathbb{Z}^2$ .

This proves (a). Now (b) follows readily, because  $D_n(\lambda Q) = \lambda^{n(n-1)}Q$  and  $z_n(\lambda Q) = \lambda^n Q$  for any  $Q \in \mathcal{C}^n(\mathbb{C}^*)$  and  $\lambda \in \mathbb{C}^*$ .  $\square$

We need the following elementary lemma.

**Lemma 6.15.** *Let  $\rho: G \rightarrow H$  be a homomorphism of groups, and let  $S$  be a subgroup of  $G$ . If  $N = \text{Norm}_G(S)$  and  $\bar{N} = \text{Norm}_H(\rho(S))$ , then  $N \subset \rho^{-1}(\bar{N})$ .*

**6.16.** By Lemma 6.14(b), the function  $\tilde{h}_n = h_n \circ \pi$  generates the group  $\mathcal{O}^\times(\mathcal{C}_{\text{ord}}^n(\mathbb{C}^*))^{\mathbb{C}^* \times \mathbf{S}(n)}/\mathbb{C}^*$ . This function participates in the next lemma.

**Lemma 6.17.** *Let  $n \geq 2$ . (a) For any  $\varepsilon \in \{1, -1\}$ ,  $s \in \mathbb{C}^*$ , and  $k \in \mathbb{Z}$ , the formula<sup>11</sup>*

$$F: Q \mapsto s\tilde{h}_n^k(Q)Q^\varepsilon, \quad \text{where } Q \in \mathcal{C}_{\text{ord}}^n(\mathbb{C}^*), \quad (73)$$

*defines an automorphism  $F \in \text{Aut } \mathcal{C}_{\text{ord}}^n(\mathbb{C}^*)$ . All these automorphisms form a subgroup  $H_1 \subset \text{Aut } \mathcal{C}_{\text{ord}}^n(\mathbb{C}^*)$  such that  $H_1 \cong (\mathbb{C}^* \times \mathbb{Z}) \rtimes (\mathbb{Z}/2\mathbb{Z})$ , where the factors  $\mathbb{Z}$  and  $\mathbb{Z}/2\mathbb{Z}$  commute.*

*(b) The subgroup  $H_1$  commutes with the Galois group  $H_2 = \mathbf{S}(n)$  of the covering  $\pi: \mathcal{C}_{\text{ord}}^n(\mathbb{C}^*) \rightarrow \mathcal{C}^n(\mathbb{C}^*)$ . If  $H \subset \text{Aut } \mathcal{C}_{\text{ord}}^n(\mathbb{C}^*)$  is the subgroup generated by  $H_1$  and  $H_2$ , then*

$$H \cong H_1 \times H_2 \cong ((\mathbb{C}^* \times \mathbb{Z}) \rtimes (\mathbb{Z}/2\mathbb{Z})) \times \mathbf{S}(n).$$

*(c) We have  $H \subset N = \text{Norm}_{\text{Aut } \mathcal{C}_{\text{ord}}^n(\mathbb{C}^*)}(\mathbf{S}(n))$ .*

*Proof.* The inclusion  $H \subset N$  follows from the fact that the groups  $H_1$  and  $\mathbf{S}(n)$  commute. The rest of the proof is easy, so we leave it to the reader.  $\square$

**6.18.** In Proposition 6.23 we show that, in fact,  $N = H$  for any  $n > 2$ . Then by Corollary 6.6,  $\text{Aut } \mathcal{C}^n(\mathbb{C}^*) = H/\mathbf{S}(n) \simeq (\mathbb{C}^* \times \mathbb{Z}) \rtimes (\mathbb{Z}/2\mathbb{Z})$ , with the factors defined as in Zinde's Theorem. This gives a proof of Zinde's Theorem.

In the sequel we use the following portion of notation.

**Notation 6.19.** Recall that by Corollary 6.13, for  $n > 1$

$$\text{Aut } \mathcal{C}_{\text{ord}}^n(\mathbb{C}^*) \cong \left( \mathcal{O}^\times(\mathcal{C}_{\text{ord}}^n(\mathbb{C}^*))^{\mathbb{C}^*} \rtimes (\mathbb{Z}/2\mathbb{Z}) \right) \rtimes \mathbf{S}(n+2) = \tilde{H}_1 \times \tilde{H}_2, \quad (74)$$

where  $\tilde{H}_1 = \mathcal{O}^\times(\mathcal{C}_{\text{ord}}^n(\mathbb{C}^*))^{\mathbb{C}^*} \rtimes (\mathbb{Z}/2\mathbb{Z})$  and  $\tilde{H}_2 = \mathbf{S}(n+2)$ . Letting  $\tilde{N}_2 = \text{Norm}_{\tilde{H}_2}(\rho(\mathbf{S}(n)))$ , where  $\rho: \tilde{H}_1 \times \tilde{H}_2 \rightarrow \tilde{H}_2$  is the quotient morphism, by Lemma 6.15 we obtain  $\rho(N) \subset \tilde{N}_2$ . In Lemma 6.21 we describe the subgroup  $\tilde{N}_2 \subset \tilde{H}_2$ .

<sup>11</sup>Cf. (65).

**Notation 6.20.** In the sequel we deal with the following involutions in  $\text{Aut } \mathcal{C}_{\text{ord}}^n(\mathbb{C}^*)$ . We let  $\sigma_i = (i, i+1) \in \mathbf{S}(n)$ ,  $i = 1, \dots, n-1$ , where  $\mathbf{S}(n)$  acts on  $\mathcal{C}_{\text{ord}}^n(\mathbb{C}^*)$  via permutations of coordinates. Clearly,  $\langle \sigma_1, \dots, \sigma_k \rangle = \mathbf{S}(k+1)$  for any  $k = 1, \dots, n-1$ . For  $Q = (q_1, \dots, q_n) \in \mathcal{C}_{\text{ord}}^n(\mathbb{C}^*)$  we let

$$\iota: Q \mapsto Q^{-1}, \quad \tau: Q \mapsto q_n^{-2}Q, \quad \nu: Q \mapsto q_n^2Q^{-1}, \quad \text{and} \quad \sigma' = q_{n-1}^{-1}q_n\sigma_{n-1}(Q). \quad (75)$$

We have  $\nu = \tau \circ \iota = \iota \circ \tau$ . In fact,  $\nu$  and  $\sigma'$  represent the commuting transpositions  $(n, n+2) \in \mathbf{S}(n+2)$  and  $(n-1, n+1) \in \mathbf{S}(n+2)$ , respectively, under the  $\mathbf{S}(n+2)$ -action on  $\mathcal{C}_{\text{ord}}^n(\mathbb{C}^*)$  (see Remark 6.12).

In the next lemma we describe the normalizer  $\tilde{N}_2 = \text{Norm}_{\tilde{H}_2}(\rho(\mathbf{S}(n)))$ , where  $\tilde{H}_2 = \mathbf{S}(n+2) \cong \text{Aut } \mathcal{C}_{\text{ord}}^{n-1}(\mathbb{C}^{**})$ .

**Lemma 6.21.** (a) *In the notation of 6.20 we have*

$$\rho(\mathbf{S}(n)) = \langle \sigma_1, \dots, \sigma_{n-2}, \sigma' \rangle = \text{Stab}_{\mathbf{S}(n+2)}(n) \cap \text{Stab}_{\mathbf{S}(n+2)}(n+2) \cong \mathbf{S}(n).$$

(b) *Furthermore,*

$$\tilde{N}_2 = \langle \sigma_1, \dots, \sigma_{n-2}, \sigma', \nu \rangle = \rho(\mathbf{S}(n)) \times \langle \nu \rangle \cong \mathbf{S}(n) \times (\mathbb{Z}/2\mathbb{Z}).$$

*Proof.* (a) It is easily seen that  $\rho(\sigma_i) = \sigma_i$ ,  $i = 1, \dots, n-2$ , while  $\rho(\sigma_{n-1}) = \sigma' \notin \mathbf{S}(n)$ . Hence the intersection of  $\mathbf{S}(n) \subset \text{Aut } \mathcal{C}_{\text{ord}}^n(\mathbb{C}^*)$  with the subgroup  $\mathbf{S}(n+2)$  as in (74) coincides with  $\mathbf{S}(n-1) = \langle \sigma_1, \dots, \sigma_{n-2} \rangle = \text{Stab}_{\mathbf{S}(n)}(n) \subset \mathbf{S}(n)$ . In particular, the restriction  $\rho|_{\mathbf{S}(n-1)}$  is identical. Now (a) follows.

(b) Let  $\sigma \in \tilde{N}_2 \subset \mathbf{S}(n+2)$ , i.e.,  $\sigma$  normalizes the subgroup  $\rho(\mathbf{S}(n)) \subset \mathbf{S}(n+2)$ . Then  $\sigma$  respects the orbit structure of  $\rho(\mathbf{S}(n))$  acting on  $\{1, \dots, n+2\}$ . Hence it preserves the orbit  $\{1, \dots, n-1, n+1\}$  and sends the fixed points  $n$  and  $n+2$  of  $\rho(\mathbf{S}(n))$  into fixed points. This implies (b).  $\square$

**6.22.** By Lemma 6.15 we have  $\rho(N) \subset \tilde{N}_2$ . Then by Lemma 6.21(b) and decomposition (6.19), any automorphism  $F \in N$  admits one of the following presentations:

- (i)  $F: Q \mapsto f(Q) \cdot \rho(\sigma)(Q)$ ,
- (ii)  $F: Q \mapsto f(Q) \cdot (\rho(\sigma) \circ \nu)(Q)$ ,
- (iii)  $F: Q \mapsto f(Q) \cdot (\tau \circ \rho(\sigma))(Q)$ ,
- (vi)  $F: Q \mapsto f(Q) \cdot (\tau \circ \rho(\sigma) \circ \nu)(Q)$ ,

where  $Q \in \mathcal{C}_{\text{ord}}^n(\mathbb{C}^*)$ ,  $f \in \mathcal{O}^\times(\mathcal{C}_{\text{ord}}^n(\mathbb{C}^*))^{\mathbb{C}^*}$ ,  $\sigma \in \mathbf{S}(n)$ , and the involutions  $\tau$  and  $\nu$  are defined in (75). Note that  $\tau$  generates the factor  $\mathbb{Z}/2\mathbb{Z}$  in (74).

In the following proposition we prove the equality  $N = H$ , thus finishing the proof of Zinde's Theorem, see 6.18.



**Proposition 6.23.** *Let  $n > 2$ . Then for  $F \in N$  the cases (ii) and (iii) never happen. Furthermore, any  $F \in N$  as in (i) and (iv), respectively, can be given via*

$$F: Q \mapsto s\tilde{h}_n^k(Q) \cdot \sigma(Q) \quad \text{and} \quad F: Q \mapsto s\tilde{h}_n^k(Q) \cdot (\iota \circ \sigma)(Q),$$

respectively, where  $Q \in \mathcal{C}_{\text{ord}}^n(\mathbb{C}^*)$ ,  $s \in \mathbb{C}^*$ ,  $k \in \mathbb{Z}$ ,  $\sigma \in \mathbf{S}(n)$ , and  $\iota$  is defined in (75). Consequently,  $N = H$ .

*Proof.* We begin with the following observations. Note first that the multiplication by a function  $f \in \mathcal{O}^\times(\mathcal{C}_{\text{ord}}^n(\mathbb{C}^*))$  and the  $\mathbf{S}(n)$ -action on  $\mathcal{C}_{\text{ord}}^n(\mathbb{C}^*)$  commute. Furthermore, any  $\sigma \in \mathbf{S}(n)$  can be written as a word, say,  $w(\sigma_1, \dots, \sigma_{n-1})$ . Hence  $\rho(\sigma) = w(\sigma_1, \dots, \sigma_{n-2}, \sigma')$ , where  $\sigma' \in \mathbf{S}(n+2)$  and  $\sigma' = \rho(\sigma_{n-1}) = q_{n-1}^{-1}q_n\sigma_{n-1}$ , see (75) and the proof of Lemma 6.21(a). Since  $q_{n-1}^{-1}q_n \in \mathcal{O}^\times(\mathcal{C}_{\text{ord}}^n(\mathbb{C}^*))^{\mathbb{C}^*}$ , it follows that  $\rho(\sigma) = g \cdot w(\sigma_1, \dots, \sigma_{n-1})$  for a certain function  $g \in \mathcal{O}^\times(\mathcal{C}_{\text{ord}}^n(\mathbb{C}^*))^{\mathbb{C}^*}$ .

For  $F$  in (i), the latter yields a presentation  $F: Q \mapsto \tilde{f}(Q) \cdot \sigma(Q)$ , where  $\tilde{f} \in \mathcal{O}^\times(\mathcal{C}_{\text{ord}}^n(\mathbb{C}^*))^{\mathbb{C}^*}$ . If  $F \in N$ , then also  $F \circ \sigma^{-1} \in N$ , because  $\sigma \in \mathbf{S}(n) \subset N$ . However,  $F \circ \sigma^{-1}: Q \mapsto \tilde{f}(Q)Q$  normalizes the subgroup  $\mathbf{S}(n) \subset \text{Aut } \mathcal{C}_{\text{ord}}^n(\mathbb{C}^*)$  if and only if  $\tilde{f} \in \mathcal{O}^\times(\mathcal{C}_{\text{ord}}^n(\mathbb{C}^*))^{\mathbb{C}^*}$  is  $\mathbf{S}(n)$ -invariant, i.e., if and only if  $\tilde{f} = s\tilde{h}_n^k$  for some  $s \in \mathbb{C}^*$  and  $k \in \mathbb{Z}$ , see 6.16. Thus  $F \in H$ .

The same argument allows to write  $F$  in (iv) as

$$F: Q \mapsto \tilde{f}(Q) \cdot (\tau \circ \sigma \circ \nu)(Q) = \tilde{f}(Q) \cdot (\tau \circ \nu \circ \sigma)(Q) = \tilde{f}(Q) \cdot (\iota \circ \sigma)(Q),$$

where  $\tilde{f} \in \mathcal{O}^\times(\mathcal{C}_{\text{ord}}^n(\mathbb{C}^*))^{\mathbb{C}^*}$ . We use here the equality  $\iota = \tau \circ \nu$  (see 6.20) and the fact that  $\nu$  and  $\sigma$  commute. Assume further that  $F \in N$ . Since both  $\sigma$  and  $\iota$  belong to  $N$ , then  $Q \mapsto \tilde{f}(Q)Q$  is in  $N$ , too. This implies as before that  $\tilde{f} = s\tilde{h}_n^k$  for some  $s \in \mathbb{C}^*$  and  $k \in \mathbb{Z}$ , and so,  $F \in H$ .

It remains to eliminate the possibility that some  $F$  in (ii) or in (iii) belongs to  $N$ . With the same reasoning as before, any  $F$  in (ii) can be presented as

$$F: Q \mapsto \tilde{f}(Q) \cdot (\sigma \circ \nu)(Q) = \tilde{f}(Q) \cdot (\iota \circ \sigma \circ \tau)(Q) = q_n^{-2}\tilde{f}(Q)(\iota \circ \sigma)(Q),$$

where  $\tilde{f} \in \mathcal{O}^\times(\mathcal{C}_{\text{ord}}^n(\mathbb{C}^*))^{\mathbb{C}^*}$ . Suppose to the contrary that  $F \in N$ . Then also  $\tilde{F} = F \circ (\sigma^{-1} \circ \iota) \in N$ , where  $\tilde{F}: Q \mapsto q_n^{-2}\tilde{f}(Q)Q$ . Hence for any  $\sigma \in \mathbf{S}(n)$  there exists  $\sigma'' \in \mathbf{S}(n)$  such that  $\tilde{F} \circ \sigma = \sigma'' \circ \tilde{F}$ . Thus, for any  $Q = (q_1, \dots, q_n) \in \mathcal{C}_{\text{ord}}^n(\mathbb{C}^*)$ ,

$$q_{\sigma(n)}^{-2}\tilde{f}(\sigma(Q))\sigma(Q) = q_n^{-2}\tilde{f}(Q)\sigma''(Q). \quad (76)$$

Let  $Q$  be chosen so that  $|q_i| < |q_j|$  for  $i < j$ . Then both  $\sigma$  and  $\sigma''$  are uniquely determined by the images  $\sigma(Q)$  and  $\sigma''(Q)$ . Since by (76) these two sequences are proportional, we have  $\sigma = \sigma''$ . So, (76) is equivalent to

$$q_{\sigma(n)}^{-2}f(\sigma(Q)) = q_n^{-2}f(Q) \quad \forall \sigma \in \mathbf{S}(n), \quad \forall Q \in \mathcal{C}_{\text{ord}}^n(\mathbb{C}^*). \quad (77)$$

We claim that there is no function  $\tilde{f} \in \mathcal{O}^\times(\mathcal{C}_{\text{ord}}^n(\mathbb{C}^*))^{\mathbb{C}^*}$  satisfying (77). Indeed, these equalities mean that the function  $\tilde{f}/q_n^2$  is  $\mathbf{S}(n)$ -invariant. Hence  $\tilde{f}/q_n^2$  descends to a

function, say,  $g \in \mathcal{O}^\times(\mathcal{C}^n(\mathbb{C}^*))$ . By Lemma 6.14(a) we have  $g = sD_n^k z_n^l$  for some  $s \in \mathbb{C}^*$  and  $k, l \in \mathbb{Z}$ . Therefore,  $\tilde{f} = g \circ \pi = sq_n^2 D_n^k z_n^l$ . Since  $\tilde{f}$  is  $\mathbb{C}^*$ -invariant, we obtain  $n(n-1)k + nl + 2 = 0$ . The latter equality is impossible whatever are  $k, l \in \mathbb{Z}$ , because by our assumption  $n > 2$ , in particular,  $n \not\equiv 2$ . This proves our claim.

The possibility that some  $F$  in (iii) belongs to  $N$  can be ruled out in a similar way. We leave the details to the reader.  $\square$

**6.24.** Next we turn to the case  $n = 2$ . Example 6.25 shows that Zinde's Theorem does not hold any more for  $n = 2$  (however, it is evidently true for  $n = 1$ ). Note that for any  $n \geq 1$ , the automorphisms of  $\mathcal{C}_{\text{ord}}^n(\mathbb{C}^*)$  as in (65) form a subgroup of the group  $\text{Aut } \mathcal{C}^n(\mathbb{C}^*)$ . We let  $\text{Aut}_{\text{Zin}} \mathcal{C}^n(\mathbb{C}^*)$  denote this subgroup. Our proof of Zinde's Theorem shows actually that for any  $n \geq 1$ , an automorphism  $F \in \text{Aut } \mathcal{C}^n(\mathbb{C}^*)$  admits a lift to  $\tilde{F} \in \text{Aut } \mathcal{C}_{\text{ord}}^n(\mathbb{C}^*)$  if and only if  $F \in \text{Aut}_{\text{Zin}} \mathcal{C}^n(\mathbb{C}^*)$ . In the following example we consider an automorphism  $U \in \text{Aut } \mathcal{C}^2(\mathbb{C}^*)$ , which does not admit a lift to  $\mathcal{C}_{\text{ord}}^2(\mathbb{C}^*)$ . Hence  $\text{Aut}_{\text{Zin}} \mathcal{C}^2(\mathbb{C}^*) \neq \text{Aut } \mathcal{C}^2(\mathbb{C}^*)$ . In particular, Corollary 6.3 does not hold any longer for  $n = 2$ .

**Example 6.25.** The affine surface  $\mathcal{C}^2(\mathbb{C}^*)$  is isomorphic to the complement  $\mathbb{C}_z^2 \setminus (\Gamma_1 \cup \Gamma_2)$ , where the plane affine curves  $\Gamma_1$  and  $\Gamma_2$  are given in the coordinates  $(z_1, z_2)$  by equations  $z_1^2 - 4z_2 = 0$  and  $z_2 = 0$ , respectively. Consider the involution  $U \in \text{Aut } \mathcal{C}^2(\mathbb{C}^*)$  given in these coordinates as

$$U: Q = (z_1, z_2) \mapsto U(Q) = (z_1, z_1^2/4 - z_2).$$

Then  $U$  extends to a triangular automorphism of the plane  $\mathbb{C}_z^2$  interchanging  $\Gamma_1$  and  $\Gamma_2$ . Choose vanishing loops  $\sigma_1$  and  $\sigma_2$  of  $\Gamma_1$  and  $\Gamma_2$ , respectively, with the base point  $(4, 2) \in \mathbb{C}_z^2$  fixed by  $U$ . Then these loops are interchanged by  $U$ . According to Brieskorn ([6]) the classes of these loops (denoted by the same letters) are the standard generators of the Artin-Brieskorn braid group  $\mathbf{B}_2 = \pi_1(\mathcal{C}^2(\mathbb{C}^*)) = \pi_1(\mathbb{C}_z^2 \setminus (\Gamma_1 \cup \Gamma_2))$ . The induced graph automorphism  $U_*$  of  $\mathbf{B}_2$  interchanges  $\sigma_1$  and  $\sigma_2$ , see Remark 6.2.1. It does not stabilize the pure braid group  $\mathbf{PB}_2 = \langle \sigma_1^2, \sigma_2 \rangle \subset \mathbf{B}_2$ ; indeed,  $U_*(\mathbf{PB}_2) = \langle \sigma_1, \sigma_2^2 \rangle \neq \mathbf{PB}_2$ . Hence  $U$  cannot be lifted to  $\mathcal{C}_{\text{ord}}^2(\mathbb{C}^*)$ ; see the proof of Corollary 6.3.

Alternatively, the latter can be seen as follows. Assuming that there exists a lift  $\tilde{U}$  of  $U$  to  $\mathcal{C}_{\text{ord}}^2(\mathbb{C}^*)$ , it must be given for  $Q \in \mathcal{C}_{\text{ord}}^2(\mathbb{C}^*)$  by

$$\tilde{U}: Q = (q_1, q_2) \mapsto \left( \frac{q_1 + q_2}{2} \pm \sqrt{q_1 q_2}, \frac{q_1 + q_2}{2} \mp \sqrt{q_1 q_2} \right).$$

However, due to ramifications this formula does not define a morphism of  $\mathcal{C}_{\text{ord}}^2(\mathbb{C}^*)$ .

Thus,  $U \in \text{Aut } \mathcal{C}^2(\mathbb{C}^*) \setminus \text{Aut}_{\text{Zin}} \mathcal{C}^2(\mathbb{C}^*)$ , i.e.,  $U$  is not one of the automorphisms as in (65). So, Zinde's Theorem does not hold any longer for  $n = 2$ .

The next result completes the description of the groups  $\text{Aut } \mathcal{C}^n(\mathbb{C}^*)$ ,  $n \geq 2$ .

**Theorem 6.26.** *We have  $\text{Aut } \mathcal{C}^2(\mathbb{C}^*) = \text{Aut}_{\text{Zin}} \mathcal{C}^2(\mathbb{C}^*) \rtimes \langle U \rangle$ , where  $U \in \text{Aut } \mathcal{C}^2(\mathbb{C}^*)$  is the involution as in Example 6.25.*

In the proof we use the following simple lemma. Consider the graph automorphism  $\bar{\alpha} \in \text{Aut } \mathbf{WB}_2$  as in Remark 6.2.2. Given a group  $G$ , we let  $\text{Inn } G$  stand for the group of inner automorphisms of  $G$ .

**Lemma 6.27.** (Franzsen, [18, p. 21]) *We have  $\text{Aut } \mathbf{WB}_2 = (\text{Inn } \mathbf{WB}_2) \rtimes \langle \bar{\alpha} \rangle$ . In particular, the outer automorphism group  $\text{Out } \mathbf{WB}_2$  is a cyclic group of order 2 generated by the image of the graph automorphism  $\bar{\alpha}$ .*

Since the proof is not given in [18], we provide a short argument.

*Proof.* In the notation of Remark 6.2.2 the conjugacy classes of  $\mathbf{WB}_2$  are

$$C_0 = \{e\}, C_1 = \{\varepsilon_1 \varepsilon_2\}, C_2 = \{\varepsilon_1 \sigma, \varepsilon_2 \sigma\}, C_3 = \{\varepsilon_1, \varepsilon_2\}, C_4 = \{\sigma, \varepsilon_1 \varepsilon_2 \sigma\}. \quad (78)$$

Any automorphism  $\beta \in \text{Aut } \mathbf{WB}_2$  induces a permutation of the set  $(C_0, C_1, C_2, C_3, C_4)$ . It is easily seen that  $\beta$  preserves the classes  $C_0, C_1, C_2$  and either preserves or interchanges  $C_3$  and  $C_4$ . Thus, the induced action of  $\beta$  on the conjugacy classes yields a surjection  $\mu: \text{Aut } \mathbf{WB}_2 \rightarrow \mathbf{S}(2)$ . Clearly,  $\bar{\alpha} \notin \ker \mu$ , whereas  $\text{Inn } \mathbf{WB}_2 \subset \ker \mu$ . It suffices to show that  $\text{Inn } \mathbf{WB}_2 = \ker \mu$ . The latter holds indeed, since  $C_2 \cdot C_3 = C_4$ , and so, there are just 4 different possibilities for the action of an element  $\beta \in \ker \mu$  on  $\mathbf{WB}_2$ . However, it is easy to check that all these possibilities can be realized by inner automorphisms. Finally, our argument shows that

$$\text{Out } \mathbf{WB}_2 = \text{Aut } \mathbf{WB}_2 / \ker \mu = \mathbf{S}(2) \cong \mathbb{Z}/2\mathbb{Z}$$

is generated by the image of  $\bar{\alpha}$ . □

**Remarks 6.28.** 1. Let  $X$  be a connected and locally linearly connected topological space, and  $p \in X$  a base point. Any continuous selfmap  $f: X \rightarrow X$  induces an endomorphism  $f_*: \pi_1(X, p) \rightarrow \pi_1(X, p)$ , which is well defined up to a conjugation. That is, the coset  $[f_*]$  of  $f_*$  in  $\text{End } \pi_1(X, p)$  modulo the group  $\text{Inn } \pi_1(X, p)$  is well defined. Moreover, the correspondence  $f \mapsto [f_*]$  defines a homomorphism of semigroups  $\text{End } X \rightarrow \text{End } \pi_1(X, p) / \text{Inn } \pi_1(X, p)$ . which restricts further to a homomorphism

$$\text{Aut } X \rightarrow \text{Out } \pi_1(X, p) = \text{Aut } \pi_1(X, p) / \text{Inn } \pi_1(X, p),$$

where  $\text{End } X$  and  $\text{Aut } X$  denote (in this context) the semigroup of continuous selfmaps of  $X$  and the group of homeomorphisms of  $X$ , respectively.

2. Furthermore, the action of  $f_*$  is well defined on the set of conjugacy classes in  $\pi_1(X, p)$ . For  $f \in \text{Aut } X$  we let  $f_{\dagger}$  be the induced permutation of the conjugacy classes. The correspondence  $f \rightarrow f_{\dagger}$  yields a homomorphism  $\text{Aut } X \rightarrow \text{Bij } C(\pi_1(X, p))$  to the permutation group of the set  $C(\pi_1(X, p))$  of conjugacy classes of  $\pi_1(X, p)$ .

3. Let  $G$  be a group and  $H \subset G$  be a subgroup. If  $H$  is characteristic in  $G$ , then clearly  $\text{Aut } G$  and  $\text{Inn } G$  act on the quotient  $G/N$ , and there is a natural homomorphism  $\text{Out } G \rightarrow \text{Out } (G/N)$ .

*Proof of Theorem 6.26.* By the Cohen-Paris' Theorem [9, Proposition 2.4], the kernel  $\mathbf{CB}_2 := \ker(\omega_2: \mathbf{B}_2 \rightarrow \mathbf{WB}_2)$  is a characteristic subgroup of the group  $\mathbf{B}_2$ , that is, it is stable under the action of  $\text{Aut } \mathbf{B}_2$ . By virtue of Remarks 6.28 and Lemma 6.27 this leads to the natural homomorphism

$$\widehat{\mu}: \text{Aut } \mathcal{C}^2(\mathbb{C}^*) \rightarrow \text{Out } \mathbf{B}_2 \rightarrow \text{Out } (\mathbf{B}_2/\mathbf{CB}_2) = \text{Out } \mathbf{WB}_2 \cong \mathbb{Z}/2\mathbb{Z} \quad (79)$$

in the realization  $\mathbf{B}_2 = \pi_1(\mathcal{C}^2(\mathbb{C}^*))$ . The involution  $U \in \text{Aut } \mathcal{C}^2(\mathbb{C}^*)$  as in Example 6.25 induces the involution  $U_* = \alpha \in \text{Aut } \mathbf{B}_2$  of Remark 6.2.1, which descends to the graph involution  $\bar{\alpha} \in \text{Aut } \mathbf{WB}_2$  as in Lemma 6.27, see also Remark 6.2.2. Hence the image  $\widehat{\mu}(U)$  generates the cyclic group  $\mathbb{Z}/2\mathbb{Z}$  in (79).

We claim that  $\ker \widehat{\mu} = \text{Aut}_{\text{Zin}} \mathcal{C}^2(\mathbb{C}^*)$ . Indeed, in the notation as in (78) we have  $\mathcal{E}_2 = \langle \varepsilon_1, \varepsilon_2 \rangle = C_0 \cup C_1 \cup C_3 \subset \mathbf{WB}_2$ , see Remark 6.2.2. It follows from the proof of Lemma 6.27 that  $\text{Stab}_{\text{Aut } \mathbf{WB}_2}(\mathcal{E}_2) = \ker \mu = \text{Inn } \mathbf{WB}_2$ , and so,  $\text{Stab}_{\text{Out } \mathbf{WB}_2}(\mathcal{E}_2) = \{\text{id}\}$ .

Furthermore,  $F \in \text{Aut } \mathcal{C}^2(\mathbb{C}^*)$  belongs to  $\text{Aut}_{\text{Zin}} \mathcal{C}^2(\mathbb{C}^*)$  if and only if the induced automorphism  $F_* \in \text{Aut } \mathbf{B}_2$  (whatever is its realization) stabilizes the subgroup  $\mathbf{PB}_2 \subset \mathbf{B}_2$ , see the discussion in 6.24, if and only if the induced automorphism of  $\mathbf{WB}_2$  stabilizes the subgroup  $\mathcal{E}_2 \subset \mathbf{WB}_2$ , i.e., belongs to  $\text{Inn } \mathbf{WB}_2$ , if and only if  $F \in \ker \widehat{\mu}$ , as claimed.

Therefore,  $\text{Aut}_{\text{Zin}} \mathcal{C}^2(\mathbb{C}^*) \subset \text{Aut } \mathcal{C}^2(\mathbb{C}^*)$  is a subgroup of index 2 which does not contain  $U$ . Now theorem follows.  $\square$

## 7. THE GROUP $\text{Aut } \Sigma_{\text{blc}}^{n-2}$

In this section we prove part (c) of Theorem 4.1. Let us recall this assertion.

**Theorem 7.1.** *For  $n > 2$  we have*

$$\text{Aut } \Sigma_{\text{blc}}^{n-2} \cong \mathbb{C}^*,$$

where  $s \in \mathbb{C}^*$  acts on  $Q \in \Sigma_{\text{blc}}^{n-2}$  via  $Q \mapsto sQ$ .

For the proof we need some preparation. Recall (see e.g. [1], [44]) that for  $n \geq 4$  the singular locus  $\text{sing } \Sigma^{n-1} = \Sigma^{n-1} \setminus \text{reg } \Sigma^{n-1}$  of  $\Sigma^{n-1}$  is the union<sup>12</sup> of the *Maxwell stratum*  $\Sigma_{\text{Maxw}}^{n-2}$  and the *Arnold caustic*  $\Sigma_{\text{cau}}^{n-2}$  defined by

$$\Sigma_{\text{Maxw}}^{n-2} = p(\{q_{n-2} = q_{n-1} = q_n\}) \quad \text{and} \quad \Sigma_{\text{cau}}^{n-2} = p(\{q_{n-3} = q_{n-2}, q_{n-1} = q_n\}), \quad (80)$$

<sup>12</sup>This is not a stratification of  $\text{sing } \Sigma^{n-1}$  since  $\Sigma_{\text{Maxw}}^{n-2} \cap \Sigma_{\text{cau}}^{n-2} \neq \emptyset$ .

where  $p$  is the projection (24). So,  $\Sigma_{\text{Maxw}}^{n-2}$  and  $\Sigma_{\text{cau}}^{n-2}$  consist, respectively, of all unordered  $n$ -multisets  $Q \subset \mathbb{C}$  that can be written as  $Q = \{q_1, \dots, q_{n-3}, u, u, u\}$  and  $Q = \{q_1, \dots, q_{n-4}, u, u, v, v\}$ .

*Proof of Theorem 7.1.* We start with the case where  $n > 4$ . Consider the isomorphism

$$\varphi: \text{reg } \Sigma_{\text{blc}}^{n-2} \xrightarrow{\cong} \mathcal{C}^{n-2}(\mathbb{C}^*)$$

as in (28) of Lemma 2.20 with inverse  $\psi = \varphi^{-1}$  given by (29). We can associate to any  $F \in \text{Aut } \mathcal{C}^{n-2}(\mathbb{C}^*)$  an automorphism  $\tilde{F} = \varphi^{-1} \circ F \circ \varphi \in \text{Aut}(\text{reg } \Sigma_{\text{blc}}^{n-2})$ . By Zinde's Theorem, for  $n > 4$  the automorphism  $F$  is given by (65). We have to show that, for  $F$  as in (65), the automorphism  $\tilde{F}$  extends to an automorphism of  $\Sigma_{\text{blc}}^{n-2}$  if and only if  $k = 0$  and  $\varepsilon = 1$ , that is, if and only if  $F \in \text{Aut}(\mathcal{C}^{n-2}(\mathbb{C}^*))$  belongs to the identity component  $\text{Aut}_0(\mathcal{C}^{n-2}(\mathbb{C}^*)) \cong \mathbb{C}^*$ .

Note that the  $\mathbb{C}^*$ -action  $Q \mapsto sQ$  ( $s \in \mathbb{C}^*$ ,  $Q \in \mathcal{C}^{n-2}(\mathbb{C}^*)$ ) on  $\mathcal{C}^{n-2}(\mathbb{C}^*)$  induces a  $\mathbb{C}^*$ -action on  $\text{reg } \Sigma_{\text{blc}}^{n-2}$  given again by  $Q \mapsto sQ$  ( $s \in \mathbb{C}^*$ ). The latter  $\mathbb{C}^*$ -action extends to  $\Sigma_{\text{blc}}^{n-2}$  so that the origin  $\bar{0} \in \Sigma_{\text{blc}}^{n-2}$  is a unique fixed point. This fixed point lies in the closure of any one-dimensional  $\mathbb{C}^*$ -orbit.

Thus, without loss of generality we may restrict to the case, where  $s = 1$  in (65), and so,  $F = F_{k,\varepsilon}: Q \mapsto h_{n-2}^k Q^\varepsilon$ .

The  $(\text{Aut } \mathbb{C}^*)$ -invariant function  $h_{n-2} \in \mathcal{O}^\times(\mathcal{C}^{n-2}(\mathbb{C}^*))$  yields an invertible regular function  $g \stackrel{\text{def}}{=} h_{n-2} \circ \varphi$  on  $\text{reg } \Sigma_{\text{blc}}^{n-2}$ . An automorphism  $F_k: Q \mapsto h_{n-2}^k Q$  of  $\mathcal{C}^{n-2}(\mathbb{C}^*)$  ( $k \in \mathbb{Z}$ ) induces the automorphism  $\tilde{F}_k: Q \mapsto g^k Q$  of  $\text{reg } \Sigma_{\text{blc}}^{n-2}$ .

The subgroup  $\text{Aut}_0(\mathcal{C}^{n-2}(\mathbb{C}^*)) \cong \mathbb{C}^*$  of  $\text{Aut}(\mathcal{C}^{n-2}(\mathbb{C}^*))$  being normal, an automorphism  $F \in \text{Aut}(\mathcal{C}^{n-2}(\mathbb{C}^*))$  sends any  $\mathbb{C}^*$ -orbit in  $\mathcal{C}^{n-2}(\mathbb{C}^*)$  into a  $\mathbb{C}^*$ -orbit of the same dimension. Since the function  $h_{n-2}$  is constant along the  $\mathbb{C}^*$ -orbits, the multiplication  $Q \mapsto h_{n-2}^k Q$  preserves each  $\mathbb{C}^*$ -orbit. Hence the automorphism  $F_{k,\varepsilon}: Q \mapsto h_{n-2}^k Q^\varepsilon$  sends the  $\mathbb{C}^*$ -orbits in  $\mathcal{C}^{n-2}(\mathbb{C}^*)$  into  $\mathbb{C}^*$ -orbits. It follows that  $\tilde{F}_{k,\varepsilon}$  also sends the  $\mathbb{C}^*$ -orbits in  $\text{reg } \Sigma_{\text{blc}}^{n-2}$  into  $\mathbb{C}^*$ -orbits.

The involution  $Q \mapsto Q^{-1}$  on  $\mathcal{C}^{n-2}(\mathbb{C}^*)$  sends any  $\mathbb{C}^*$ -orbit into another such orbit interchanging the punctures, while the multiplication  $Q \mapsto h_{n-2}^k Q$  preserves the punctures. Hence  $\tilde{F}_{k,\varepsilon}$  interchanges the punctures of the  $\mathbb{C}^*$ -orbits in  $\text{reg } \Sigma_{\text{blc}}^{n-2}$  if and only if  $\varepsilon = -1$ .

On the other hand, if  $\tilde{F} \in \text{Aut}(\text{reg } \Sigma_{\text{blc}}^{n-2})$  admits an extension, say  $\bar{F}$ , to an automorphism of  $\Sigma_{\text{blc}}^{n-2}$ , then  $\bar{F}$  should fix the origin. Indeed,  $\bar{F}$  normalizes the  $\mathbb{C}^*$ -action on  $\Sigma_{\text{blc}}^{n-2}$ , hence it preserves the unique  $\mathbb{C}^*$ -fixed point  $0 \in \Sigma_{\text{blc}}^{n-2}$ . This point is a unique common point of the  $\mathbb{C}^*$ -orbit closures. Hence  $\tilde{F}$  cannot interchange the punctures of the  $\mathbb{C}^*$ -orbits in  $\text{reg } \Sigma_{\text{blc}}^{n-2}$ . This proves that  $\varepsilon = 1$  for such an extendable  $\tilde{F}$ .

The function  $h_{n-2} \in \mathcal{O}^\times(\mathcal{C}^{n-2}(\mathbb{C}^*))$  can be regarded as the rational function  $d_{n-2}(z)/z_{n-2}^{n-3}$  on  $\mathbb{C}_{(z)}^{n-2}$ , where  $z_{n-2} = (-1)^{n-2} \prod_{i=1}^{n-2} q_i$ . It has pole along the coordinate

hyperplane  $z_{n-2} = 0$ , and  $h_{n-2}^{-1}$  has pole along the discriminant hypersurface  $\Sigma^{n-3} = \{d_{n-2} = 0\}$ . It follows by (28) that  $g$  regarded as a rational function on  $\Sigma_{\text{blc}}^{n-2}$  has pole along the caustic  $\Sigma_{\text{cau,blc}}^{n-3} = \Sigma_{\text{cau}}^{n-2} \cap \Sigma_{\text{blc}}^{n-2}$ , and  $g^{-1}$  has pole along the Maxwell stratum  $\Sigma_{\text{Maxw,blc}}^{n-3} = \Sigma_{\text{Maxw}}^{n-2} \cap \Sigma_{\text{blc}}^{n-2}$ , see (80). Anyway, the automorphism  $\tilde{F}_k: Q \mapsto g_n^k Q$  of  $\text{reg } \Sigma_{\text{blc}}^{n-2}$  does not admit an extension to an automorphism of  $\Sigma_{\text{blc}}^{n-2}$  unless  $k = 0$  in (65). Thus,  $\tilde{F}_{k,\varepsilon} \in \text{Aut}(\text{reg } \Sigma_{\text{blc}}^{n-2})$  admits an extension to an automorphism of  $\Sigma_{\text{blc}}^{n-2}$  if and only if  $k = 0$  and  $\varepsilon = 1$ , as stated. This ends the proof in the case  $n > 4$ .

Let now  $n = 4$ . The automorphism group  $\text{Aut } \mathcal{C}^2(\mathbb{C}^*)$  is described in Theorem 6.26. Due to this theorem, any automorphism  $F \in \text{Aut } \mathcal{C}^2(\mathbb{C}^*)$  can be written either as  $F = F' \circ U$  or as  $F = F'$ , where  $F': Q \mapsto sg_2^k Q^\varepsilon$  is as in (65), and  $U \in \text{Aut } \mathcal{C}^2(\mathbb{C}^*)$  is the involution as in Example 6.25. In the second case we have as before that  $\tilde{F} = \varphi^{-1} \circ F \circ \varphi \in \text{Aut}(\text{reg } \Sigma_{\text{blc}}^2)$  admits an extension to an automorphism of  $\Sigma_{\text{blc}}^{n-2}$  if and only if  $\varepsilon = 1$ ,  $k = 0$ , and so,  $F \in \mathbb{C}^*$ .

Assume further that  $F = F' \circ U$ . The identity component  $\mathbb{C}^*$  of  $\text{Aut } \mathcal{C}^2(\mathbb{C}^*)$  being normal,  $F$  preserves the family of  $\mathbb{C}^*$ -orbits in  $\mathcal{C}^2(\mathbb{C}^*)$ . So does  $\tilde{F} = \varphi^{-1} \circ F \circ \varphi \in \text{Aut}(\text{reg } \Sigma_{\text{blc}}^2)$  as well, since  $\varphi: \text{reg } \Sigma_{\text{blc}}^2 \xrightarrow{\cong} \mathcal{C}^2(\mathbb{C}^*)$  is  $\mathbb{C}^*$ -equivariant.

Likewise in Example 6.25, we realize  $\mathcal{C}^2(\mathbb{C}^*)$  as  $\mathbb{C}_{(z)}^2 \setminus (C_1 \cup C_2)$  with  $C_1 = \{z_2 = 0\}$  and  $C_2 = \{z_1^2 - 4z_2 = 0\}$ . Since  $U$  extends to an automorphism of the ambient affine plane  $\mathbb{C}^2$  (denoted by the same letter), it sends any  $\mathbb{C}^*$ -orbits in  $\mathcal{C}^2(\mathbb{C}^*)$  into another one without interchanging the punctures. The same arguments as before prove that  $\tilde{F}$  does not extend to an automorphism of  $\Sigma_{\text{blc}}^2$  unless  $\varepsilon = 1$  and  $k = 0$ . Applying a suitable element of the  $\mathbb{C}^*$ -action on  $\Sigma_{\text{blc}}^2$ , we may consider that also  $s = 1$ , and so,  $F' = \text{id}$  and  $F = U$ .

We claim that  $\tilde{F} = \tilde{U} = \varphi^{-1} \circ U \circ \varphi \in \text{Aut}(\text{reg } \Sigma_{\text{blc}}^2)$  cannot be extended to an automorphism of  $\Sigma_{\text{blc}}^2$ , and so,  $\text{Aut } \Sigma_{\text{blc}}^2$  reduces to its identity component  $\mathbb{C}^*$ , as required. Indeed, suppose that  $\tilde{U}$  does extend to an automorphism of  $\Sigma_{\text{blc}}^2$ , which will be denoted by the same symbol  $\tilde{U}$ . Observe that the morphism  $\psi = \varphi^{-1}: \mathcal{C}^2(\mathbb{C}^*) \rightarrow \text{reg } \Sigma_{\text{blc}}^2$  as in (29) extends naturally to a birational morphism  $\bar{\psi}: \mathbb{C}^2 \rightarrow \Sigma_{\text{blc}}^2$ . The latter morphism fits in the commutative diagram

$$\begin{array}{ccc} \mathbb{C}^2 & \xrightarrow{U} & \mathbb{C}^2 \\ \bar{\psi} \downarrow & & \downarrow \bar{\psi} \\ \Sigma_{\text{blc}}^2 & \xrightarrow{\tilde{U}} & \Sigma_{\text{blc}}^2 \end{array}$$

The morphism  $\bar{\psi}$  is surjective and sends  $C_1$  to the Maxwell stratum  $\Sigma_{\text{Maxw,blc}}^1$  and  $C_2$  to the Arnold caustic  $\Sigma_{\text{cau,blc}}^1$ . Furthermore, the restriction  $\psi|_{C_1}: C_1 \rightarrow \Sigma_{\text{Maxw,blc}}^1$  is a bijection, while  $\psi|_{C_2}: C_2 \rightarrow \Sigma_{\text{cau,blc}}^1$  has degree 2. However, the existence of a diagram of morphisms with these properties contradicts the fact that the involution  $U$  interchanges the curves  $C_1$  and  $C_2$ . This completes the proof in the case  $n = 4$ .

In the remaining case  $n = 3$  the plane affine curve  $\Sigma_{\text{blc}}^1$  is a standard cuspidal cubic, and so,  $\text{Aut } \Sigma_{\text{blc}}^1 = \mathbb{C}^*$ , as desired.  $\square$

## 8. THE GROUP $\text{Aut } \mathcal{C}_{\text{blc}}^{n-1}(\mathbb{C})$ REVISITED

In this section we give an alternative proof of Theorem 4.1(a) following the lines of the proof of Zinde's Theorem in Section 6. For the reader's convenience we recall this statement. To make a link to Zinde's Theorem, it will be convenient to replace  $n$  by  $n + 1$  in Theorem 4.1(a).

**Theorem 8.1.** *For any  $n > 1$  we have  $\text{Aut } \mathcal{C}_{\text{blc}}^n(\mathbb{C}) \cong \mathbb{C}^*$ . Any automorphism  $F \in \text{Aut } \mathcal{C}_{\text{blc}}^{n-1}(\mathbb{C})$  is of the form  $Q \mapsto sQ$ , where  $s \in \mathbb{C}^*$  and  $Q \in \mathcal{C}_{\text{blc}}^{n-1}(\mathbb{C})$ .*

The reader can find two different proofs of this result in Sections 6 and 10. Both of them refer to Tame Map Theorem. The alternative proof given below avoids addressing Tame Map Theorem. In turn, by Proposition 3.3, Tame Map Theorem in the particular case of biregular automorphisms of the configuration spaces  $\mathcal{C}_{\text{blc}}^n(\mathbb{C})$  and  $\mathcal{C}^n(\mathbb{C}^*)$  can be derived from Theorem 8.1 and Zinde's Theorem proven in Section 6, respectively.

**Notation 8.2.** For  $Q = (q_1, \dots, q_n) \in \mathcal{C}_{\text{ord}}^n(\mathbb{C}^*)$  we let as before  $\text{bc}(Q) = \frac{1}{n} \sum_{i=1}^n q_i$ . Consider the map

$$\tilde{\varphi}: \mathcal{C}_{\text{ord}}^n(\mathbb{C}^*) \rightarrow \mathcal{C}_{\text{blc,ord}}^n(\mathbb{C}), \quad Q \mapsto \left( Q - \frac{n}{n+1} \text{bc}(Q), -\frac{n}{n+1} \text{bc}(Q) \right). \quad (81)$$

This map is an isomorphism with inverse

$$\tilde{\varphi}^{-1}: \mathcal{C}_{\text{blc,ord}}^n(\mathbb{C}) \rightarrow \mathcal{C}_{\text{ord}}^n(\mathbb{C}^*), \quad (q_1, \dots, q_{n+1}) \mapsto (q_1 - q_{n+1}, \dots, q_n - q_{n+1}). \quad (82)$$

**Proposition 8.3.** *For  $n > 1$  the isomorphism  $\tilde{\varphi}$  as in (81) fits in the commutative diagram*

$$\begin{array}{ccccc} \mathcal{C}_{\text{ord}}^{n-1}(\mathbb{C}^{**}) \times \mathbb{C}^* & \xrightarrow{\eta} & \mathcal{C}_{\text{ord}}^n(\mathbb{C}^*) & \xrightarrow{\tilde{\varphi}} & \mathcal{C}_{\text{blc,ord}}^n(\mathbb{C}) \\ & & \downarrow / \mathbf{S}(n) & & \downarrow / \mathbf{S}(n+1) \\ & & \mathcal{C}^n(\mathbb{C}^*) & \xrightarrow{\tilde{\psi}} & \mathcal{C}_{\text{blc}}^n(\mathbb{C}) \end{array} \quad (83)$$

where

- (i) the vertical columns are Galois coverings with the Galois groups  $\mathbf{S}(n)$  and  $\mathbf{S}(n+1)$ , respectively, acting by permutations of coordinates. Furthermore,  $\eta$  is an isomorphism as in (68), and  $\tilde{\psi}$  is an unramified  $n$ -sheeted covering;
- (ii)  $\tilde{\varphi}$  conjugates  $\mathbf{S}(n)$  with the stabilizer  $\text{Stab}_{\mathbf{S}(n+1)}(n+1) \subset \mathbf{S}(n+1)$ ;
- (iii) the morphisms in (83) are equivariant with respect to the free  $\mathbb{C}^*$ -actions  $Q \mapsto sQ$ ,  $s \in \mathbb{C}^*$ , on each of the four corners of the square, and the standard  $\mathbb{C}^*$ -action on the second factor of  $\mathcal{C}_{\text{ord}}^{n-1}(\mathbb{C}^{**}) \times \mathbb{C}^*$  identical on the first factor;
- (iv) the automorphisms of each of the varieties in (83) preserve the family of the  $\mathbb{C}^*$ -orbits on this variety.

*Proof.* The assertions (i)–(iii) can be verified without difficulty; (iv) follows from (iii) due to Corollary 6.9.  $\square$

**8.4.** The proof of Theorem 8.1 starts as follows. Given  $F \in \text{Aut } \mathcal{C}_{\text{blc}}^n(\mathbb{C})$ , we lift it to an automorphism  $\tilde{F}$  of  $\mathcal{C}_{\text{blc,ord}}^n(\mathbb{C})$ , and then conjugate with an automorphism  $\tilde{F}'$  of  $\mathcal{C}_{\text{ord}}^{n-1}(\mathbb{C}^{**}) \times \mathbb{C}^*$ . Due to Proposition 6.10(a), the resulting automorphism  $\tilde{F}'$  has a triangular form  $\tilde{F}': (Q', y) \mapsto (SQ', A(Q')y)$ , see Notation 6.7. The next lemma makes this first step possible.

**Lemma 8.5.** *Any automorphism  $F$  of  $\mathcal{C}_{\text{blc}}^n(\mathbb{C})$  admits a lift to an automorphism  $\tilde{F}$  of  $\mathcal{C}_{\text{blc,ord}}^n(\mathbb{C})$ .*

*Proof.* By virtue of (27) we have  $\mathcal{C}_{\text{blc}}^n(\mathbb{C}) \times \mathbb{C} \cong \mathcal{C}^{n+1}(\mathbb{C})$ . Hence  $\pi_1(\mathcal{C}_{\text{blc}}^n(\mathbb{C})) \cong \pi_1(\mathcal{C}^{n+1}(\mathbb{C})) = \mathbf{A}_n$  is the Artin braid group with  $n + 1$  strands. Similarly, the isomorphism  $\mathcal{C}_{\text{blc,ord}}^n(\mathbb{C}) \times \mathbb{C} \cong \mathcal{C}_{\text{ord}}^{n+1}(\mathbb{C})$  yields an isomorphism  $\pi_1(\mathcal{C}_{\text{blc,ord}}^n(\mathbb{C})) \cong \pi_1(\mathcal{C}_{\text{ord}}^{n+1}(\mathbb{C})) = \mathbf{PA}_n$ , where as before  $\mathbf{PA}_n \subset \mathbf{A}_n$  is the pure braid group on  $n + 1$  strands. By virtue of Artin's Theorem ([2, Theorem 3])<sup>13</sup>,  $\mathbf{PA}_n$  is a characteristic subgroup of  $\mathbf{A}_n$ . Therefore, the induced automorphism  $F_*$  of  $\mathbf{A}_n$  (which is well defined modulo an inner automorphism of  $\mathbf{A}_n$ ) preserves the subgroup  $\mathbf{PA}_n$ . Now the assertion follows by the monodromy theorem.  $\square$

**Notation 8.6.** Consider the subgroup

$$\tilde{\mathbf{S}} = \tilde{\varphi}^{-1} \mathbf{S}(n+1) \tilde{\varphi} \subset \text{Aut } \mathcal{C}_{\text{ord}}^n(\mathbb{C}^*)$$

conjugated with  $\mathbf{S}(n+1) \subset \text{Aut } \mathcal{C}_{\text{blc,ord}}^n(\mathbb{C})$  via  $\tilde{\varphi}$ . We let  $\tilde{N} = \text{Norm}_{\text{Aut } \mathcal{C}_{\text{ord}}^n(\mathbb{C}^*)}(\tilde{\mathbf{S}})$  be the normalizer of  $\tilde{\mathbf{S}}$  in  $\text{Aut } \mathcal{C}_{\text{ord}}^n(\mathbb{C}^*)$ .

By Lemma 6.4,  $\tilde{N}$  is isomorphic to the subgroup in  $\text{Aut } \mathcal{C}_{\text{blc,ord}}^n(\mathbb{C})$  of all lifts of the automorphisms from  $\text{Aut } \mathcal{C}_{\text{blc}}^n(\mathbb{C})$ . The next corollary of Lemma 6.4 is immediate.

**Corollary 8.7.** *We have  $\text{Aut } \mathcal{C}_{\text{blc}}^n(\mathbb{C}) \cong \tilde{N}/\mathbf{S}(n+1)$ .*

**8.8.** In Proposition 8.13 we show that  $\tilde{N} = \mathbb{C}^* \times \mathbf{S}(n+1)$ , and so, by Corollary 8.7,  $\text{Aut } \mathcal{C}_{\text{blc}}^n(\mathbb{C}) \cong \mathbb{C}^*$ , where in both cases we mean the standard diagonal action of  $\mathbb{C}^*$  on  $\mathcal{C}_{\text{blc,ord}}^n(\mathbb{C})$  and on  $\mathcal{C}_{\text{blc}}^n(\mathbb{C})$ , respectively. This proves Theorem 8.1.

**Remark 8.9.** We let as before  $\sigma_i = (i, i+1) \in \mathbf{S}(k)$  for  $k > i$ . Note that for any  $i = 1, \dots, n-1$  the natural actions of the transposition  $\sigma_i$  on the varieties in the upper line of (83) are mutually conjugated via the isomorphisms  $\eta$  and  $\tilde{\varphi}$ . In other words, the natural  $\mathbf{S}(n)$ -action on  $\mathcal{C}_{\text{ord}}^n(\mathbb{C}^*)$  is conjugated with the  $\text{Stab}_{\mathbf{S}(n+1)}(n+1)$ -action on  $\mathcal{C}_{\text{blc,ord}}^n(\mathbb{C})$  and the  $(\text{Stab}_{\mathbf{S}(n+2)}(n+1) \cap \text{Stab}_{\mathbf{S}(n+2)}(n+2))$ -action on  $\mathcal{C}_{\text{ord}}^{n-1}(\mathbb{C}^{**}) \times \mathbb{C}^*$ ,

<sup>13</sup>See [38, Theorem 10] for a more general result.



where  $\mathbf{S}(n+2)$  is identified with  $\text{Aut } \mathcal{C}_{\text{ord}}^{n-1}(\mathbb{C}^{**}) \times \{\text{id}\} \subset \text{Aut}(\mathcal{C}_{\text{ord}}^{n-1}(\mathbb{C}^{**}) \times \mathbb{C}^*)$ , see Lemma 6.11.

The transposition  $\sigma_n = (n, n+1) \in \mathbf{S}(n+2)$  acts on  $\mathcal{C}_{\text{ord}}^{n-1}(\mathbb{C}^{**})$  via

$$Q = (q_1, \dots, q_{n-1}) \mapsto \alpha(Q) = (\alpha(q_1), \dots, \alpha(q_{n-1})),$$

where  $Q \in \mathcal{C}_{\text{ord}}^{n-1}(\mathbb{C}^{**})$  and  $\alpha(z) = 1 - z$  for  $z \in \mathbb{C}$ . The action of the transposition  $\sigma_n \in \mathbf{S}(n+1) \subset \text{Aut } \mathcal{C}_{\text{blc,ord}}^n(\mathbb{C})$  on  $\mathcal{C}_{\text{blc,ord}}^n(\mathbb{C})$  gives rise to the involution, say,  $\varrho$  on  $\mathcal{C}_{\text{ord}}^n(\mathbb{C}^*)$ , where

$$\varrho = \tilde{\varphi}^{-1} \circ \sigma_n \circ \tilde{\varphi} \in \tilde{\mathbf{S}}, \quad Q = (q_1, \dots, q_n) \mapsto (q_1 - q_n, \dots, q_{n-1} - q_n, -q_n). \quad (84)$$

In turn,  $\varrho$  gives rise to the involution  $(\sigma_n, \delta)$  on  $\mathcal{C}_{\text{ord}}^{n-1}(\mathbb{C}^{**}) \times \mathbb{C}^*$ , where  $\delta(z) = -z$  for  $z \in \mathbb{C}^*$ . Thus, in the notation of 6.19,

$$\rho(\varrho) = \rho(\tilde{\varphi}^{-1} \circ \sigma_n \circ \tilde{\varphi}) = \rho(\sigma_n, \delta) = \sigma_n \in \mathbf{S}(n+2) = \tilde{H}_2, \quad (85)$$

where  $\rho: \text{Aut } \mathcal{C}_{\text{ord}}^n(\mathbb{C}^*) = \tilde{H}_1 \times \tilde{H}_2 \rightarrow \tilde{H}_2$  is the natural surjection.

These observations lead to the following lemma.

**Lemma 8.10.** *Letting  $\tilde{N}_2 = \text{Norm}_{\mathbf{S}(n+2)}(\rho(\tilde{\mathbf{S}})) \subset \tilde{H}_2 = \mathbf{S}(n+2)$  we have*

$$\tilde{N}_2 = \rho(\tilde{\mathbf{S}}) = \text{Stab}_{\mathbf{S}(n+2)}(n+2) \subset \tilde{N}. \quad (86)$$

*Proof.* According to Remark 8.9, the isomorphism  $\tilde{\varphi}$  conjugates  $\sigma_i \in \mathbf{S}(n+1) \subset \text{Aut } \mathcal{C}_{\text{blc,ord}}^n(\mathbb{C})$  to  $\sigma_i \in \mathbf{S}(n) \subset \text{Aut } \mathcal{C}_{\text{ord}}^n(\mathbb{C}^*)$  for  $i = 1, \dots, n-1$ , and  $\sigma_n$  to  $\delta \circ \sigma_n \in \text{Aut } \mathcal{C}_{\text{ord}}^n(\mathbb{C}^*)$ , where  $\delta \in \mathbb{C}^* \subset \tilde{N}$ ,  $\delta: Q \mapsto -Q$ , see (85). The latter projects further to  $\sigma_n \in \mathbf{S}(n+2)$  under  $\rho$ , see again (85). Hence  $\rho(\tilde{\mathbf{S}}) = \text{Stab}_{\mathbf{S}(n+2)}(n+2) \subset \mathbf{S}(n+2)$ . The normalizer of the subgroup  $\text{Stab}_{\mathbf{S}(n+2)}(n+2) \cong \mathbf{S}(n+1)$  in  $\mathbf{S}(n+2)$  coincides with this subgroup. These observations yield the equalities in (86).

We have  $\tilde{\mathbf{S}} = \tilde{\varphi}^{-1} \mathbf{S}(n+1) \tilde{\varphi} \not\subset \mathbf{S}(n+2) = \tilde{H}_2$ . Indeed,  $\tilde{\varphi}^{-1} \sigma_n \tilde{\varphi} = \delta \sigma_n$  by (85). Nevertheless,  $\rho(\tilde{\mathbf{S}}) = \text{Stab}_{\mathbf{S}(n+2)}(n+2) \subset \tilde{N}$ , since  $\mathbb{C}^* \subset \tilde{N}$  and  $\tilde{\mathbf{S}} \subset \tilde{N}$  by definition of  $\tilde{N}$ , see 8.6. Hence  $\mathbb{C}^* \cdot \tilde{\mathbf{S}} \subset \tilde{N}$ , and, in particular,  $\sigma_n = \rho(\delta \sigma_n) = \delta(\delta \sigma_n) \in \tilde{N}$ , because  $\delta \sigma_n \in \tilde{\mathbf{S}} \subset \tilde{N}$ . It remains to note that also  $\mathbf{S}(n-1) \subset \tilde{\mathbf{S}} \subset \tilde{N}$ , see Remark 8.9, and so,  $\rho(\tilde{\mathbf{S}}) = \langle \mathbf{S}(n-1), \rho(\delta \sigma_n) \rangle \subset \tilde{N}$ .  $\square$

**8.11.** By Lemma 6.15 we have  $\rho(\tilde{N}) \subset \tilde{N}_2$ . Due to Lemma 8.10 and decomposition (74), any automorphism  $F \in \tilde{N}$  admits one of the following presentations:

$$(i) \quad F: Q \mapsto f(Q) \cdot \sigma(Q) \quad \text{and} \quad (ii) \quad F: Q \mapsto f(Q) \cdot (\tau \circ \sigma)(Q), \quad (87)$$

where  $Q \in \mathcal{C}_{\text{ord}}^n(\mathbb{C}^*)$ ,  $f \in \mathcal{O}^\times(\mathcal{C}_{\text{ord}}^n(\mathbb{C}^*))^{\mathbb{C}^*}$ ,  $\sigma \in \text{Stab}_{\mathbf{S}(n+2)}(n+2) = \mathbf{S}(n+1) \subset \mathbf{S}(n+2)$ , and  $\tau: Q \mapsto q_n^{-2}Q$  as in (75) generates the factor  $\mathbb{Z}/2\mathbb{Z}$  in (74).

**Lemma 8.12.** *Let  $F \in \tilde{N}$  be as in (i) (as in (ii), respectively). Then the function  $f$  ( $q_n^{-2}f$ , respectively) is  $\tilde{\mathbf{S}}$ -invariant.*

*Proof.* We have:  $F \in \tilde{N}$  if and only if  $F \circ \sigma^{-1} \in \tilde{N}$ . Indeed,  $\sigma \in \rho(\tilde{\mathbf{S}}) \subset \tilde{N}$  by Lemma 8.10. Thus, it suffices to prove the assertions for the automorphisms  $F \in \tilde{N}$  of the forms

$$(i') \quad F: Q \mapsto f(Q)Q \quad \text{and} \quad (ii') \quad F: Q \mapsto q_n^{-2}f(Q)Q,$$

where  $Q \in \mathcal{C}_{\text{ord}}^n(\mathbb{C}^*)$  and  $f \in \mathcal{O}^\times(\mathcal{C}_{\text{ord}}^n(\mathbb{C}^*))^{\mathbb{C}^*}$ .

To this end, consider the quotient morphism

$$\theta: \mathcal{C}_{\text{ord}}^{n+2}(\mathbb{P}^1) \xrightarrow{\mathbf{PSL}(2, \mathbb{C})} \mathcal{C}_{\text{ord}}^{n-1}(\mathbb{C}^{**})$$

with respect to the natural diagonal  $\mathbf{PSL}(2, \mathbb{C})$ -action on  $\mathcal{C}_{\text{ord}}^{n+2}(\mathbb{P}^1)$  (see Remark 6.12). It admits a section

$$\mathcal{C}_{\text{ord}}^{n-1}(\mathbb{C}^{**}) \ni Q = (q_1, \dots, q_{n-1}) \mapsto (q_1, \dots, q_{n-1}, 0, 1, \infty) \in \mathcal{C}_{\text{ord}}^{n+2}(\mathbb{P}^1).$$

The latter leads to a  $\mathbf{PSL}(2, \mathbb{C})$ -equivariant factorization

$$\theta: \mathcal{C}_{\text{ord}}^{n+2}(\mathbb{P}^1) \xrightarrow{\cong} \mathcal{C}_{\text{ord}}^{n-1}(\mathbb{C}^{**}) \times \mathbf{PSL}(2, \mathbb{C}) \xrightarrow{\text{pr}_1} \mathcal{C}_{\text{ord}}^{n-1}(\mathbb{C}^{**}).$$

The  $\mathbf{S}(n+2)$ -action on  $\mathcal{C}_{\text{ord}}^{n-1}(\mathbb{C}^{**})$  lifts naturally to the direct product, and then also to  $\mathcal{C}_{\text{ord}}^{n+2}(\mathbb{P}^1)$ , where it acts via permutations of coordinates, see Remark 6.12. Consider natural isomorphisms

$$\begin{aligned} \mathcal{O}^\times(\mathcal{C}_{\text{ord}}^n(\mathbb{C}^*))^{\mathbb{C}^*} &\cong \mathcal{O}^\times(\mathcal{C}_{\text{ord}}^{n-1}(\mathbb{C}^{**}) \times \mathbb{C}^*)^{\mathbb{C}^*} \cong \mathcal{O}^\times(\mathcal{C}_{\text{ord}}^{n-1}(\mathbb{C}^{**})) \\ &\cong \theta^*(\mathcal{O}^\times(\mathcal{C}_{\text{ord}}^{n-1}(\mathbb{C}^{**}))) = \mathcal{O}^\times(\mathcal{C}_{\text{ord}}^{n+2}(\mathbb{P}^1))^{\mathbf{PSL}(2, \mathbb{C})}. \end{aligned}$$

Any function  $f \in \mathcal{O}^\times(\mathcal{C}_{\text{ord}}^n(\mathbb{C}^*))^{\mathbb{C}^*}$  lifts through this chain of isomorphisms to a function  $\tilde{f} \in \mathcal{O}^\times(\mathcal{C}_{\text{ord}}^{n+2}(\mathbb{P}^1))^{\mathbf{PSL}(2, \mathbb{C})}$ . Respectively, an automorphism  $F \in \tilde{N}$  as in (i') lifts first to a triangular automorphism

$$F(\text{id}, A) \in \text{Aut}(\mathcal{C}_{\text{ord}}^{n-1}(\mathbb{C}^{**}) \times \mathbb{C}^*), \quad Q = (Q', q) \mapsto (Q', f(Q')q) \text{ for all } Q \in \mathcal{C}_{\text{ord}}^{n-1}(\mathbb{C}^{**}) \times \mathbb{C}^*,$$

and then to

$$\tilde{F} \in \text{Aut}(\mathcal{C}_{\text{ord}}^{n+2}(\mathbb{P}^1) \times \mathbb{C}^*), \quad Q = (Q', q) \mapsto (Q', \tilde{f}(Q')q) \text{ for all } Q \in \mathcal{C}_{\text{ord}}^{n+2}(\mathbb{P}^1) \times \mathbb{C}^*,$$

where, as before,  $\tilde{f} \in \mathcal{O}^\times(\mathcal{C}_{\text{ord}}^{n+2}(\mathbb{P}^1))^{\mathbf{PSL}(2, \mathbb{C})}$  is the lift of  $f$ . Since  $F \in \tilde{N}$ , the automorphism  $\tilde{F}$  normalizes the subgroup  $\mathbf{S}(n+1) = \text{Stab}_{\mathbf{S}(n+2)}(n+2) \subset \mathbf{S}(n+2)$  acting naturally on  $\mathcal{C}_{\text{ord}}^{n+2}(\mathbb{P}^1) \times \mathbb{C}^*$  identically on the second factor. Hence for any  $\sigma \in \mathbf{S}(n+1)$  there exists  $\sigma' \in \mathbf{S}(n+1)$  such that  $\tilde{F} \circ \sigma = \sigma' \circ \tilde{F}$ , where for any  $Q = (Q', q) \in \mathcal{C}_{\text{ord}}^{n+2}(\mathbb{P}^1) \times \mathbb{C}^*$  we have

$$\tilde{F} \circ \sigma: Q \mapsto (\sigma(Q'), \tilde{f}(\sigma(Q'))q) \quad \text{and} \quad \sigma' \circ \tilde{F}: Q \mapsto (\sigma'(Q'), \tilde{f}(Q')q).$$

Thus the equality  $\tilde{F} \circ \sigma = \sigma' \circ \tilde{F}$  holds if and only if  $\sigma = \sigma'$  and  $\tilde{f} \circ \sigma = \tilde{f}$ . Hence  $\tilde{f}$  is  $\mathbf{S}(n+1)$ -invariant, and  $\tilde{F}$  commutes with the  $\mathbf{S}(n+1)$ -action on  $\mathcal{C}_{\text{ord}}^{n+2}(\mathbb{P}^1) \times \mathbb{C}^*$ .

Since  $\tilde{\mathbf{S}} = \eta \mathbf{S}(n+1) \eta^{-1} \subset \text{Aut } \mathcal{C}_{\text{ord}}^n(\mathbb{C}^*)$  (see (83)), it follows that the function  $f \in \mathcal{O}^\times(\mathcal{C}_{\text{ord}}^n(\mathbb{C}^*))^{\mathbb{C}^*}$  is  $\tilde{\mathbf{S}}$ -invariant.

For an automorphism  $F \in \tilde{N}$  as in (ii'), the same argument shows that the function  $q_n^{-2} f$  is  $\tilde{\mathbf{S}}$ -invariant, as required.  $\square$

The next proposition ends the proof of Theorem 8.1.

**Proposition 8.13.** *In the notation as in 8.8–8.10 we have  $\tilde{N} = \mathbb{C}^* \times \mathbf{S}(n+1)$ , where the subgroup  $\mathbb{C}^* \subset \tilde{N}$  acts on  $\mathcal{C}_{\text{ord}}^n(\mathbb{C}^*)$  via  $(s, Q) \mapsto sQ$  for  $s \in \mathbb{C}^*$  and  $Q \in \mathcal{C}_{\text{ord}}^n(\mathbb{C}^*)$ .*

*Proof.* Consider first  $F \in \tilde{N}$  as in (87(i)). By Lemma 8.12 the function  $f$  in (87(i)) is  $\tilde{\mathbf{S}}$ -invariant. Hence  $f$  is  $\mathbf{S}(n)$ -invariant, where  $\mathbf{S}(n) \subset \tilde{\mathbf{S}}$  acts on  $\mathcal{C}_{\text{ord}}^n(\mathbb{C}^*)$  via permutations of coordinates. It follows that  $f = s\tilde{h}_n^k$  for some  $s \in \mathbb{C}^*$  and  $k \in \mathbb{Z}$ , see 6.16. Furthermore,  $f$  is  $\sigma_n$ -invariant, where  $\sigma_n \in \mathbf{S}(n+1)$  acts on  $\mathcal{C}_{\text{ord}}^n(\mathbb{C}^*)$  via the involution  $\varrho$  in (84). The latter invariance translates as the identity

$$s\tilde{h}_n^k(Q) = sh_n^k(\varrho(Q)),$$

which is definitely wrong unless  $k = 0$ , and so,  $f = s \in \mathbb{C}^*$ . Thus, in case (i) we have  $F \in \mathbb{C}^*$ .

It remains to eliminate the possibility that some  $F$  as in (87(ii)) belongs to  $\tilde{N}$ . By Lemma 8.12, in this case the function  $q_n^{-2} f \in \mathcal{O}^\times(\mathcal{C}_{\text{ord}}^n(\mathbb{C}^*))$  is  $\tilde{\mathbf{S}}$ -invariant, and, in particular,  $\mathbf{S}(n)$ -invariant, while  $f$  is  $\mathbb{C}^*$ -invariant. However, for  $n > 2$  these lead to a contradiction in the same way as in the proof of Proposition 6.23, case (ii). Indeed, an argument in this proof shows that for  $n = 2$  the condition  $q_n^{-2} f \in \mathcal{O}^\times(\mathcal{C}_{\text{ord}}^n(\mathbb{C}^*))^{\mathbb{C}^* \times \mathbf{S}(n)}$  implies the equality  $q_n^{-2} f = sd_2^k/z_2^{k+1}$  for some  $s \in \mathbb{C}^*$  and  $k \in \mathbb{Z}$ . The latter function is also  $\varrho$ -invariant. Using (84), this can be translated as the identity

$$(-1)^k q_1^{3k+1} = (q_1 - q_2)^{3k+1} \quad \text{for all } (q_1, q_2) \in \mathcal{C}_{\text{ord}}^2(\mathbb{C}^*),$$

which is definitely wrong whatever is  $k \in \mathbb{Z}$ .  $\square$

## 9. THE GROUP $\text{Aut } \mathcal{SC}_{\text{blc}}^2$

In this section we prove Kaliman's Theorem 4.1(b) in the remaining case  $n = 4$ . Let us repeat this statement.

**Theorem 9.1.** *We have  $\text{Aut } \mathcal{SC}_{\text{blc}}^2 \cong \mathbb{Z}/12\mathbb{Z}$ , where  $\zeta \in \mathbb{Z}/12\mathbb{Z}$  acts on  $\mathcal{SC}_{\text{blc}}^2$  via  $Q \mapsto \zeta Q$  for  $Q \in \mathcal{SC}_{\text{blc}}^2$ .*

The proof is done in Subsection 9.2. In Subsection 9.1 we construct an elliptic fibration  $\mathcal{SC}_{\text{blc}}^2 \rightarrow \mathcal{SC}_{\text{blc}}^1$  over an elliptic curve  $\mathcal{SC}_{\text{blc}}^1$ . Its fibers and the base curve are smooth affine plane cubics with one place at infinity. The group  $\mu_{12}$  of the roots of unity of order 12 acts naturally on  $\mathcal{SC}_{\text{blc}}^2$  preserving the fibration. The action of  $-1 \in \mu_{12}$  yields the hyperelliptic involution on each fiber. We show in Subsection 9.2

that any automorphism  $F \in \text{Aut } \mathcal{SC}_{\text{blc}}^2$  acts as an element of  $\mu_{12}$ . This gives a proof of Theorem 9.1.

9.1. **Elliptic fibration of  $\mathcal{SC}_{\text{blc}}^2$ .** Given a quartic polynomial

$$f(X) = X^4 + z_2X^2 + z_3X + z_4 \quad (88)$$

we consider its cubic resolvent<sup>14</sup>

$$R_3(X) = X^3 + v_1X^2 + v_2X + v_3,$$

where

$$v_1 = -z_2, \quad v_2 = -4z_4, \quad \text{and} \quad v_3 = 4z_2z_4 - z_3^2.$$

If  $q_1, \dots, q_4$  are the roots of  $f$ , then, up to reordering, the roots of  $R_3$  are

$$\lambda_1 = q_1q_2 + q_3q_4, \quad \lambda_2 = q_1q_3 + q_2q_4, \quad \text{and} \quad \lambda_3 = q_1q_4 + q_2q_3.$$

We have  $\text{discr } R_3 = \text{discr } f$ .

The Tschirnhausen transformation gives the cubic polynomial

$$g(Y) = R_3(Y + z_2/3) = Y^3 + u_2Y + u_3, \quad (89)$$

where

$$u_2 = -z_2^2/3 - 4z_4 \quad \text{and} \quad u_3 = 8z_2z_4/3 - 2z_2^3/27 - z_3^2. \quad (90)$$

Once again, we have  $\text{discr } g = \text{discr } R_3 = \text{discr } f$ .

The balanced special configuration space  $\mathcal{SC}_{\text{blc}}^2$  can be realized as the surface in  $\mathbb{C}^3$  with coordinates  $(z_2, z_3, z_4)$  given by equation  $\text{discr } f = 1$ , and the balanced special configuration space  $\mathcal{SC}_{\text{blc}}^1$  as the curve in  $\mathbb{C}^2$  with coordinates  $(u_2, u_3)$  given by equation

$$\text{discr } g = -(4u_2^3 + 27u_3^2) = 1, \quad (91)$$

where  $f$  and  $g$  are as in (88) and (89) respectively. Clearly,  $\mathcal{SC}_{\text{blc}}^2$  is a smooth affine surface and  $\mathcal{SC}_{\text{blc}}^1$  is a smooth affine elliptic cubic curve with one place at infinity and zero  $j$ -invariant. Since  $\text{discr } g = \text{discr } f$ , the correspondence

$$\pi: \mathcal{SC}_{\text{blc}}^2 \rightarrow \mathcal{SC}_{\text{blc}}^1, \quad f \mapsto g,$$

yields a surjective morphism given by formulas (90).

**Lemma 9.2.** *In the notation as before, the morphism  $\pi: \mathcal{SC}_{\text{blc}}^2 \rightarrow \mathcal{SC}_{\text{blc}}^1$  yields an elliptic fibration on  $\mathcal{SC}_{\text{blc}}^2$ . The fiber  $E(P) = \pi^*(P)$  over a point  $P = (u_2, u_3) \in \mathcal{SC}_{\text{blc}}^1$  is a smooth, reduced elliptic cubic with one place at infinity and with  $j(E(P)) = 2^8 3^3 u_2^3$ .*

<sup>14</sup>In fact, the choice of a cubic resolvent is irrelevant for our purposes.

*Proof.* Since  $\text{discr } g = \text{discr } f$ , plugging the expressions for  $u_2$  and  $u_3$  from (90) into (91) gives the following equation of  $\mathcal{SC}_{\text{blc}}^2$  in  $\mathbb{C}_{(z)}^3$ , where  $z = (z_2, z_3, z_4)$ :

$$4(-z_2^2/3 - 4z_4)^3 + 27(8z_2z_4/3 - 2z_2^3/27 - z_3^2)^2 = -1. \quad (92)$$

We can equally realize  $\mathcal{SC}_{\text{blc}}^2$  in  $\mathbb{C}_{(z,u)}^5$ , where  $u = (u_2, u_3)$ , as intersection of three hypersurfaces given by equations (90) and (91). It is convenient however to simplify the latter system by eliminating the variable  $z_4$  from (90). In this way we arrive at the relation

$$z_3^2 = Z_2^3 + g_2Z_2 + g_3, \quad (93)$$

where

$$Z_2 = -2z_2/3, \quad g_2 = u_2, \quad \text{and} \quad g_3 = -u_3.$$

This yields an embedding of  $\mathcal{SC}_{\text{blc}}^2$  in  $\mathbb{C}^4$  with coordinates  $(z_2, z_3, u_2, u_3)$  onto the complete intersection of two hypersurfaces given by (91) and (93). Then the morphism  $\pi: \mathcal{SC}_{\text{blc}}^2 \rightarrow \mathcal{SC}_{\text{blc}}^1$  coincides with the restriction to  $\mathcal{SC}_{\text{blc}}^2$  of the standard projection  $(z_2, z_3, u_2, u_3) \mapsto (u_2, u_3)$ .

For a general point  $(u_2, u_3) \in \mathbb{C}^2$  equation (93) defines a smooth elliptic cubic curve  $E = E(u_2, u_3)$  in  $\mathbb{C}_{(z_2, z_3)}^2$  with one place at infinity. The member of this family of elliptic curves in  $\mathbb{C}^4$  over a point  $(u_2, u_3) \in \mathbb{C}^2$  is singular if and only if the polynomial  $h(Z) = Z^3 + g_2Z + g_3$  has a multiple root, if and only if its discriminant vanishes, i.e.,

$$\text{discr } h = -(4g_2^3 + 27g_3^2) = -(4u_2^3 + 27u_3^2) = 0. \quad (94)$$

Any nonsingular member is a reduced and irreducible elliptic cubic with one place at infinity. Since (91) and (94) are incompatible, the induced elliptic fibration on the surface  $\mathcal{SC}_{\text{blc}}^2 \subset \mathbb{C}^4$  has no degenerate fiber, as stated.

Finally, the formula for the  $j$ -invariant in the lemma is the classical one, where the denominator disappears because of the equalities  $\text{discr } h = \text{discr } g = 1$ .  $\square$

**9.2. Proof of Theorem 9.1.** In this subsection we give a proof of Theorem 9.1.

We let  $\mu_n$  stand for the group of  $n$ th roots of unity acting on  $\mathbb{C}$  in a natural way. The group  $\mu_3 \times \mu_2 = \mu_6 \cong \mathbb{Z}/6\mathbb{Z}$  acts on  $\mathcal{SC}_{\text{blc}}^1$  via

$$(u_2, u_3) \mapsto (\zeta u_2, \xi u_3), \quad \text{where} \quad \zeta \in \mu_3, \quad \xi \in \mu_2,$$

or, in other terms, via

$$(u_2, u_3) \mapsto (\theta^2 u_2, \theta^3 u_3), \quad \text{where} \quad \theta \in \mu_6. \quad (95)$$

Recall that  $j(\mathcal{SC}_{\text{blc}}^1) = 0$  and  $\text{Aut } \mathcal{SC}_{\text{blc}}^1 = \mu_6$  with the action of  $\mu_6$  on  $\mathcal{SC}_{\text{blc}}^1$  as in (95); see, e.g., [24, Ch. 12, Remark 4.8] or [21, Ch. IV, Corollary 4.7].

We claim that any automorphism  $F \in \text{Aut } \mathcal{SC}_{\text{blc}}^2$  preserves the elliptic fibration  $\pi: \mathcal{SC}_{\text{blc}}^2 \rightarrow \mathcal{SC}_{\text{blc}}^1$  as in Lemma 9.2. Indeed, our family  $E(P)$ ,  $P \in \mathcal{SC}_{\text{blc}}^1$ , is not isotrivial, i.e., the  $j$ -invariant  $j(E(P)) = 2^8 3^3 u_2^3$  is a non-constant function of  $P \in$

$\mathcal{SC}_{\text{blc}}^1$ . If  $P = P_0^\pm = (0, \pm u_3) \in \mathcal{SC}_{\text{blc}}^1$ , where  $u_3 = i\sqrt{3}/9$ , then  $j(E(P)) = 0$ , and if  $P = (u_2, 0) \in \mathcal{SC}_{\text{blc}}^1$ , where  $u_2^3 = -1/4 \in \mathbb{R}$ , then  $j(E(P)) = 12^3 = 1728$ . For a general point  $P \in \mathcal{SC}_{\text{blc}}^1$ , the  $j$ -invariant  $j(E(P))$  is different from 0 and 1728.

Assume that there is an automorphism  $F \in \text{Aut } \mathcal{SC}_{\text{blc}}^2$  that does not preserve the fibration  $\pi$ . Then for a general fiber  $E(P)$ , the morphism  $\pi \circ F|_{E(P)}: E(P) \rightarrow \mathcal{SC}_{\text{blc}}^1$  is non-constant. It extends to the projectivizations  $\overline{E(P)}$  and  $\overline{\mathcal{SC}_{\text{blc}}^1}$  of  $E(P)$  and  $\mathcal{SC}_{\text{blc}}^1$ , respectively, sending the point at infinity of  $E(P)$  to the point at infinity of  $\mathcal{SC}_{\text{blc}}^1$ . Thus  $\pi \circ F|_{E(P)}: E(P) \rightarrow \mathcal{SC}_{\text{blc}}^1$  is an isogeny. However, the set of all elliptic curves isogeneous to  $\mathcal{SC}_{\text{blc}}^1$  is countable, see [21, Ch. IV, Exercise 4.9.b]. This yields a contradiction.

Thus for any  $P \in \mathcal{SC}_{\text{blc}}^1$  we have  $F(E(P)) = E(P')$  for some point  $P' \in \mathcal{SC}_{\text{blc}}^1$ . The correspondence  $\varphi: P \mapsto P'$  defines an automorphism of  $\mathcal{SC}_{\text{blc}}^1$ . This gives a homomorphism  $\rho: \text{Aut } \mathcal{SC}_{\text{blc}}^2 \rightarrow \text{Aut } \mathcal{SC}_{\text{blc}}^1$ ,  $F \mapsto \varphi$ , where  $\varphi \in \text{Aut } \mathcal{SC}_{\text{blc}}^1 = \mu_6$  acts as in (95) for a certain  $\theta \in \mu_6$ .

Consider the action on  $\mathcal{SC}_{\text{blc}}^2$  of the cyclic group  $\mu_{12} \cong \mathbb{Z}/12\mathbb{Z}$  via

$$(z_2, z_3, u_2, u_3) \mapsto (\zeta^2 z_2, \zeta^3 z_3, \zeta^4 u_2, \zeta^6 u_3), \quad \text{where } \zeta \in \mu_{12}. \quad (96)$$

The projection  $\pi: \mathcal{SC}_{\text{blc}}^2 \rightarrow \mathcal{SC}_{\text{blc}}^1$  induces the surjection  $\mu_{12} \rightarrow \mu_6$ ,  $\zeta \mapsto \zeta^2$ . It follows that  $\rho: \text{Aut } \mathcal{SC}_{\text{blc}}^2 \rightarrow \mu_6$  is a surjection. Theorem 9.1 claims that in fact  $\text{Aut } \mathcal{SC}_{\text{blc}}^2 = \mu_{12}$ . To confirm this claim, it suffices to establish the equality  $\ker \rho = \mu_2$ , where  $\mu_2$  acts on  $\mathcal{SC}_{\text{blc}}^2$  via (96) with  $\zeta \in \{1, -1\}$ . This action restricts to the hyperelliptic involution on any fiber of  $\pi$ .

Suppose that  $F \in \ker \rho$ , i.e.,  $\varphi = \text{id}$ , and so,  $F$  preserves each fiber of  $\pi$ . For a fiber  $E = E(P)$ , the automorphism  $\alpha = F|_E \in \text{Aut } E$  extends to an automorphism  $\bar{\alpha}$  of the projectivization  $\bar{E}$  of  $E$ . The extended automorphism  $\bar{\alpha}$  fixes the unique point of  $\bar{E}$  at infinity. This point is a flex of the cubic  $\bar{E}$  that can be chosen for zero of the group law on  $\bar{E}$ . Any automorphism of  $\bar{E}$  that fixes the zero point is a group automorphism. Hence also  $\bar{\alpha}$  is.

For a general point  $P \in \mathcal{SC}_{\text{blc}}^1$ , the value  $j(E(P))$  is different from 0 and 1728. So, the hyperelliptic involution of  $E(P)$  is a unique non-identical automorphism preserving the point at infinity, see [21, Ch. IV, Corollary 4.7]. Thus  $\alpha = F|_E \in \text{Aut } E$  is either identical or the hyperelliptic involution. Hence the kernel  $\ker \rho = \mu_2$  is contained in  $\mu_{12}$ . It follows that the group  $\text{Aut } \mathcal{SC}_{\text{blc}}^2$  coincides with  $\mu_{12}$  acting on  $\mathcal{SC}_{\text{blc}}^2$  via (96), where the latter action is induced by the action  $Q \mapsto \zeta Q$  for  $\zeta \in \mu_{12}$  and  $Q \in \mathcal{SC}_{\text{blc}}^2$ . This proves Theorem 9.1.  $\square$

The following example is essentially borrowed in [33, §8, 2.2] (cf. [37, §14.1]). It shows that Kaliman's theorem on endomorphisms does not hold any longer for  $n = 4$ , although it does hold for automorphisms in this case as well, see Theorem 9.1.

**Example 9.3.** Consider the endomorphism

$$F: \mathcal{SC}_{\text{blc}}^2 \xrightarrow{\pi} \mathcal{SC}_{\text{blc}}^1 \xrightarrow{\varphi} E(P_0^+) \hookrightarrow \mathcal{SC}_{\text{blc}}^2, \quad (97)$$

where  $\varphi$  is an isomorphism of  $\mathcal{SC}_{\text{blc}}^1$  onto the fiber  $E(P_0^+)$  over the point  $P_0^+ = (0, i\sqrt{3}/9) \in \mathcal{SC}_{\text{blc}}^1$ . Such an isomorphism exists since  $j(E(P_0^+)) = j(\mathcal{SC}_{\text{blc}}^1) = 0$ . In contrast with the case  $n \neq 4$  in Kaliman's theorem,  $F$  is not an automorphism.

More explicitly, an endomorphism  $F$  as in (97) can be given as follows. Note that the curve  $E(P_0^+)$  on the surface  $\mathcal{SC}_{\text{blc}}^2 \subset \mathbb{C}_{(z)}^3$  is contained in the complete intersection given by equations (92) of  $\mathcal{SC}_{\text{blc}}^2$  and  $u_2 = 0$ . The latter intersection is a disjoint union of the fibers  $E(P_0^\pm)$  of  $\pi$  with  $j = 0$ . The equation  $u_2 = 0$  is equivalent to  $12z_4 + z_2^2 = 0$ , see (90). Consider the family  $(F_{a,b})$  of endomorphisms of  $\mathbb{C}_{(z)}^3$  defined by

$$F_{a,b}: (z_2, z_3, z_4) \mapsto (u_2, u_3) \mapsto \left( au_2, bu_3, -\frac{(au_2)^2}{12} \right),$$

where  $a, b \in \mathbb{C}$  and  $u_2, u_3$  are given by formulas (90). In other terms,

$$F_{a,b}: f = X^4 + z_2X^2 + z_3X + z_4 \mapsto F_{a,b}(f) = X^4 + au_2X^2 + bu_3X - \frac{(au_2)^2}{12}.$$

A simple computation shows that  $\text{discr}(F_{a,b}(f)) = -(1/27)(8a^3u_2^3 + 27b^2u_3^2)^2$ . Choosing the constants  $a$  and  $b$  such that  $a^3 = \frac{i3\sqrt{3}}{2}$  and  $b^2 = i3\sqrt{3}$  we obtain the equality

$$-(1/27)(8a^3u_2^3 + 27b^2u_3^2)^2 = (4u_2^3 + 27u_3^2)^2,$$

and so,

$$\text{discr}(F_{a,b}(f)) = (\text{discr}(g))^2 = (\text{discr}(f))^2,$$

see (91). With this choice of constants,  $F = F_{a,b}$  yields an endomorphism with one-dimensional fibers of every one of the spaces  $\mathcal{C}_{\text{blc}}^3$ ,  $\mathcal{SC}_{\text{blc}}^2$ , and  $\Sigma_{\text{blc}}^2 \subset \mathbb{C}_{(z)}^3$ . Moreover, we have  $F(\mathcal{SC}_{\text{blc}}^2) \subset E(P_0^+) \cup E(P_0^-)$ . Acting with the subgroup  $\mu_2 \times \mu_3 \subset \mu_{12} = \text{Aut } \mathcal{SC}_{\text{blc}}^2$  (see (96)) we can achieve in addition that  $F(\mathcal{SC}_{\text{blc}}^2) = E(P_0^+)$ , and so,  $F$  fits in (97).

The next example is an algebraic counterpart of the previous one.

**Example 9.4.** Recall that for  $n > 4$ , the subgroup  $\mathbf{A}'_{n-1} \cap \mathbf{PA}_{n-1} \subset \mathbf{A}'_{n-1}$  is stable under any endomorphism of  $\mathbf{A}'_{n-1}$ , see the discussion following Theorem 4.1. This does not hold any longer for  $n = 4$ . Indeed, recall the presentation ([20, Theorem 2.1])

$$\mathbf{A}'_3 = \langle s, t, u, v \mid usu^{-1} = t, utu^{-1} = t^2s^{-1}t, vsv^{-1} = s^{-1}t, vtv^{-1} = (s^{-1}t)^3s^{-2}t \rangle,$$

where

$$s = \sigma_3\sigma_1^{-1}, \quad t = \sigma_2\sigma_3\sigma_1^{-1}\sigma_2^{-1}, \quad u = \sigma_2\sigma_1^{-1}, \quad v = \sigma_1\sigma_2\sigma_1^{-2}.$$

We have  $\mathbf{A}'_3 = T \rtimes V$ , where  $T = \langle s, t \rangle \cong \mathbb{F}_2$  is a normal subgroup of  $\mathbf{A}'_3$  and  $V = \langle u, v \rangle \cong \mathbb{F}_2$ . Consider the composition  $f = i \circ p \in \text{End } \mathbf{A}'_3$ , where  $p: \mathbf{A}'_3 \rightarrow V = \mathbf{A}'_3/T$

is the quotient morphism and  $i: V \xrightarrow{\cong} T \hookrightarrow \mathbf{A}'_3$  an isomorphism onto the subgroup  $T$ . We have

$$f: s \mapsto 1, t \mapsto 1, u \mapsto s, v \mapsto t.$$

We claim that  $uv \in \mathbf{PA}_3$ , whereas  $f(uv) = st \notin \mathbf{PA}_3$ . Indeed, the images of  $u$  and  $v$  in the alternating group  $\mathbf{A}(4) \subset \mathbf{S}(4)$  are mutually inverse three-cycles, while the images of  $s$  and  $t$  are products of independent transpositions, which generate the Klein four-group  $K \subset \mathbf{A}(4)$ . Thus, the subgroup  $\mathbf{A}'_3 \cap \mathbf{PA}_3$  of  $\mathbf{A}'_3$  is not stable under  $f$ .

We claim that, likewise  $f$ , the endomorphism  $F_* \in \text{End } \pi_1(\mathcal{SC}_{\text{blc}}^2) = \text{End } \mathbf{A}'_3$  does not preserve the intersection  $\mathbf{A}'_3 \cap \mathbf{PA}_3$ . Implicitly, this follows from the proof of Kaliman's Theorem in [37, Theorem 12.13]. Indeed, this proof shows that, if  $H$  is an endomorphism of  $\mathcal{SC}_{\text{blc}}^2$  such that  $H_*$  preserves the subgroup  $\mathbf{A}'_3 \cap \mathbf{PA}_3$ , then  $H \in \text{Aut } \mathcal{SC}_{\text{blc}}^2$ . Let us give a direct proof of our claim, which uses some ideas from [20].

The homotopy exact sequence of the fiber bundle  $\pi: \mathcal{SC}_{\text{blc}}^2 \rightarrow \mathcal{SC}_{\text{blc}}^1$  with general fiber  $E = E(P)$  is

$$1 \rightarrow \pi_1(E) \rightarrow \pi_1(\mathcal{SC}_{\text{blc}}^2) \xrightarrow{\pi_*} \pi_1(\mathcal{SC}_{\text{blc}}^1) \rightarrow 1.$$

This leads to a semi-direct product decomposition  $\mathbf{A}'_3 \cong \pi_1(\mathcal{SC}_{\text{blc}}^2) = B \rtimes A$ , where  $A = \pi_1(\mathcal{SC}_{\text{blc}}^1) \cong \mathbb{F}_2$  and  $B = \pi_1(E) \cong \mathbb{F}_2$ , cf. [20, Corollary 2.7] (cf. also [37, §14.1]). Let us show that  $T = B$ .

Indeed,  $T$  is the intersection of the members of the lower central series of the group  $\mathbf{A}'_3$ , see [20, Theorem 2.10.a]. Hence the image of  $T$  in the quotient group  $\mathbf{A}'_3/B \cong A$  is the intersection of the members of the lower central series of the group  $A \cong \mathbb{F}_2$ . The latter intersection is trivial due to a theorem of Magnus ([41]; see also [31, Ch. IX, §36]). Thus  $T \subset B$ . Hence  $V = \mathbf{A}'_3/T \cong (B/T) \rtimes A$ . Due to another theorem of Magnus ([41, §5, VIII]; see also [45, Theorem 41.52]), a free group of finite rank is Hopfian, i.e., it does not admit an isomorphic proper quotient group. Since  $A \cong V \cong \mathbb{F}_2$ , this implies that  $B/T = 1$ , and so,  $T = B$ .

It follows from our constructions that the endomorphisms  $F_*$  and  $f$  of  $\mathbf{A}'_3$  with the same image  $T = B$  and the same kernel  $T = B$  differ by an automorphism, say,  $\alpha$  of  $T$ . The image of  $T$  in the alternating group  $\mathbf{A}(4)$  is the Klein four-group  $K$ . The images of  $\alpha(s) = F_*(u)$  and  $\alpha(t) = F_*(v)$  generate  $K$ . Hence the image of  $F_*(uv) = \alpha(st)$  in  $K$  is different from 1. Once again, we have  $uv \in \mathbf{PA}_3$ , while  $F_*(uv) \notin \mathbf{PA}_3$ . Thus, the subgroup  $\mathbf{A}'_3 \cap \mathbf{PA}_3$  of  $\mathbf{A}'_3$  is not stable under the endomorphism  $F_* \in \text{End } \mathbf{A}'_3$ .

## 10. HOLOMORPHIC ENDOMORPHISMS OF THE BALANCED CONFIGURATION SPACE

In this section,  $\mathcal{O}_{\text{hol}}^\times(\mathcal{Z})$  stands for the multiplicative group of the algebra  $\mathcal{O}_{\text{hol}}(\mathcal{Z})$  of all holomorphic functions on a complex space  $\mathcal{Z}$ , and  $\mathcal{O}_{\text{hol},+}(\mathcal{Z})$  for its additive group.



By (27), any holomorphic endomorphism  $f$  of  $\mathcal{C}_{\text{blc}}^{n-1}$  extends to a holomorphic endomorphism of  $\mathcal{C}^n$ . Such an extension is non-Abelian whenever  $f$  is non-Abelian<sup>15</sup>. The *minimal* extension  $F$  given by  $F(Q) = f(Q - \text{bc}(Q))$  for all  $Q \in \mathcal{C}^n$  maps  $\mathcal{C}^n$  to  $\mathcal{C}_{\text{blc}}^{n-1} \subset \mathcal{C}^n$ , see (26).

Among the affine transformations of  $\mathbb{C}$  acting diagonally on  $\mathcal{C}^n$ , only the elements of the multiplicative subgroup  $\mathbb{C}^* \subset \text{Aff } \mathbb{C}$  fixing the origin  $0 \in \mathbb{C}$  preserve the balanced configuration space  $\mathcal{C}_{\text{blc}}^{n-1} \subset \mathcal{C}^n$ . Let  $\mathcal{S}$  denote this  $\mathbb{C}^*$ -action on each of the spaces  $\mathcal{C}^n$  and  $\mathcal{C}_{\text{blc}}^{n-1}$ , and let  $\mathcal{O}_{\text{hol}}^{\mathcal{S}}(\mathcal{C}_{\text{blc}}^{n-1})$  be the subalgebra of  $\mathcal{O}_{\text{hol}}(\mathcal{C}_{\text{blc}}^{n-1})$  of all  $\mathcal{S}$ -invariant functions.

For any configuration  $Q^\circ \in \mathcal{C}_{\text{blc}}^{n-1}$  its  $\mathbb{C}^*$ -stabilizer  $\text{St}_{\mathbb{C}^*}(Q^\circ) = \{\zeta \in \mathbb{C}^* \mid \zeta \cdot Q^\circ = Q^\circ\}$  is a cyclic rotation subgroup in  $\mathbb{C}^*$  of order  $\leq n$  permuting elements of  $Q^\circ$ . If  $n \geq 3$ , it follows that the set  $\{Q^\circ \in \mathcal{C}_{\text{blc}}^{n-1} \mid \text{St}_{\mathbb{C}^*}(Q^\circ) \neq \{1\}\}$  is a Zariski closed subset in  $\mathcal{C}_{\text{blc}}^{n-1}$  of dimension 1 and  $\{Q^\circ \in \mathcal{C}_{\text{blc}}^{n-1} \mid \text{St}_{\mathbb{C}^*}(Q^\circ) = \{1\}\}$  is a Zariski open dense subset of  $\mathcal{C}_{\text{blc}}^{n-1}$ .

**Definition 10.1.** We say that a holomorphic self-map  $f$  of  $\mathcal{C}_{\text{blc}}^{n-1}$  is  $\mathbb{C}^*$ -*tame*, if there is a holomorphic function  $h: \mathcal{C}_{\text{blc}}^{n-1} \rightarrow \mathbb{C}^*$  such that  $f(Q^\circ) = h(Q^\circ) \cdot Q^\circ$  for all  $Q^\circ \in \mathcal{C}_{\text{blc}}^{n-1}$ .

Note that the cohomology group  $H^1(\mathcal{C}_{\text{blc}}^{n-1}, \mathbb{Z}) \cong \mathbb{Z}$  of the Stein manifold  $\mathcal{C}_{\text{blc}}^{n-1}$  is generated by the cohomology class of the discriminant  $D_n|_{\mathcal{C}_{\text{blc}}^{n-1}}$  (see (2)) restricted to  $\mathcal{C}_{\text{blc}}^{n-1}$ . Hence any function  $h \in \mathcal{O}_{\text{hol}}^\times(\mathcal{C}_{\text{blc}}^{n-1})$  can be written as  $h = e^\chi D_n^m$  with some  $\chi \in \mathcal{O}_{\text{hol}}(\mathcal{C}_{\text{blc}}^{n-1})$  and  $m \in \mathbb{Z}$ .

The results below, stated in [32] and [33, Sec. 8.2.1], are simple consequences of the analytic counterpart of Tame Map Theorem (see [37] or [38] for the proof) and the facts mentioned above.

**Theorem 10.2.** *For  $n > 4$  every non-Abelian holomorphic self-map  $f$  of  $\mathcal{C}_{\text{blc}}^{n-1}$  is  $\mathbb{C}^*$ -tame, i.e., it can be given by*

$$f(Q^\circ) = \mathcal{S}_{e^{\chi(Q^\circ)} D_n^m(Q^\circ)} Q^\circ = e^{\chi(Q^\circ)} D_n^m(Q^\circ) \cdot Q^\circ \quad \text{for all } Q^\circ \in \mathcal{C}_{\text{blc}}^{n-1}, \quad (98)$$

where  $\chi \in \mathcal{O}_{\text{hol}}(\mathcal{C}_{\text{blc}}^{n-1})$  and  $m \in \mathbb{Z}$ .

*Proof.* The map  $f$  admits a holomorphic non-Abelian extension  $F: \mathcal{C}^n \rightarrow \mathcal{C}_{\text{blc}}^{n-1} \subset \mathcal{C}^n$  defined by  $F(Q) = f(Q - \text{bc}(Q))$  for all  $Q \in \mathcal{C}^n$ . By the analytic version of Tame Map Theorem,  $F(Q) = A(Q)Q + B(Q)$  for all  $Q \in \mathcal{C}^n$  with  $A \in \mathcal{O}_{\text{hol}}^\times(\mathcal{C}^n)$  and  $B \in \mathcal{O}_{\text{hol}}(\mathcal{C}^n)$ . Since  $\mathcal{C}_{\text{blc}}^{n-1} \subset \mathcal{C}^n$  and  $\text{bc}(Q^\circ) = 0$  for any  $Q^\circ \in \mathcal{C}_{\text{blc}}^{n-1}$ , we see that

$$f(Q^\circ) = a(Q^\circ)Q^\circ + b(Q^\circ) \quad \text{for all } Q^\circ \in \mathcal{C}_{\text{blc}}^{n-1}, \quad \text{where } a = A|_{\mathcal{C}_{\text{blc}}^{n-1}} \text{ and } b = B|_{\mathcal{C}_{\text{blc}}^{n-1}}.$$

Moreover,  $b = 0$ . Indeed, the condition  $\text{bc}(f(Q^\circ)) = \text{bc}(Q^\circ) = 0$  implies that

$$b(Q^\circ) = a(Q^\circ) \text{bc}(Q^\circ) + b(Q^\circ) = \text{bc}(a(Q^\circ)Q^\circ + b(Q^\circ)) = \text{bc}(f(Q^\circ)) = 0$$

<sup>15</sup>The latter means that the image of the induced endomorphism of the corresponding fundamental group is non-Abelian, see the Introduction.

for all  $Q^\circ \in \mathcal{C}_{\text{blc}}^{n-1}$  and

$$a = e^\chi D_n^m \text{ for some } m \in \mathbb{Z} \text{ and } \chi \in \mathcal{O}_{\text{hol}}(\mathcal{C}_{\text{blc}}^{n-1}).$$

This proves (98).  $\square$

**Theorem 10.3.** *Let  $n \geq 3$ , and let  $f = f_{\chi,m}: \mathcal{C}_{\text{blc}}^{n-1} \rightarrow \mathcal{C}_{\text{blc}}^{n-1}$  be a holomorphic map as in (98). Then the following hold.*

(a) *The map  $f$  is surjective<sup>16</sup>, and the set  $f^{-1}(Q^\circ)$  is discrete for any  $Q^\circ \in \mathcal{C}_{\text{blc}}^{n-1}$ . This set consists of all points  $\omega \cdot Q^\circ$ , where  $\omega \in \mathbb{C}^*$  is any root of the system of equations*

$$\omega^{mn(n-1)+1} e^{\chi(\omega \cdot Q^\circ)} D_n^m(Q^\circ) \cdot Q^\circ = Q^\circ, \quad (99)$$

*which always has solutions.*

(b) *The map  $f$  is proper (in the complex topology) if and only if  $\chi \in \mathcal{O}_{\text{hol}}^S(\mathcal{C}_{\text{blc}}^{n-1})$ . In this case  $f: \mathcal{C}_{\text{blc}}^{n-1} \rightarrow \mathcal{C}_{\text{blc}}^{n-1}$  is a finite unramified cyclic holomorphic covering of degree  $N = mn(n-1)+1$ . The corresponding normal subgroup  $f_*(\pi_1(\mathcal{C}_{\text{blc}}^{n-1}))$  of index  $N$  in the Artin braid group  $\mathbf{A}_{n-1} = \pi_1(\mathcal{C}_{\text{blc}}^{n-1})$  consists of all the elements  $g = \sigma_{i_1}^{m_1} \cdots \sigma_{i_q}^{m_q} \in \mathbf{A}_{n-1}$  such that  $N$  divides  $m_1 + \cdots + m_q$ , where  $\{\sigma_1, \dots, \sigma_{n-1}\}$  is the standard system of generators of  $\mathbf{A}_{n-1}$ . Every two such coverings of the same degree are equivalent.*

(c) *The map  $f$  is a biholomorphic automorphism of  $\mathcal{C}_{\text{blc}}^{n-1}$  if and only if it is of the form  $f(Q^\circ) = e^{\chi(Q^\circ)} \cdot Q^\circ$  for any  $Q^\circ \in \mathcal{C}_{\text{blc}}^{n-1}$  and some  $\chi \in \mathcal{O}_{\text{hol}}^S(\mathcal{C}_{\text{blc}}^{n-1})$ . Every automorphism is isotopic to the identity and  $\text{Aut}_{\text{hol}} \mathcal{C}_{\text{blc}}^{n-1} \cong \mathcal{O}_{\text{hol},+}^S(\mathcal{C}_{\text{blc}}^{n-1})/2\pi i\mathbb{Z}$ .*

(d) *If  $f$  is regular, then  $\chi = \text{const}$  and so  $f(Q^\circ) = cD_n^m(Q^\circ) \cdot Q^\circ$  for all  $Q^\circ \in \mathcal{C}_{\text{blc}}^{n-1}$ , where  $c \in \mathbb{C}^*$  and  $m \in \mathbb{Z}$ . Every biregular automorphism  $f$  of  $\mathcal{C}_{\text{blc}}^{n-1}$  is of the form  $f(Q^\circ) = s \cdot Q^\circ$ ,  $Q^\circ \in \mathcal{C}_{\text{blc}}^{n-1}$ , where  $s \in \mathbb{C}^*$ . In particular, the group of all biregular automorphisms  $\text{Aut} \mathcal{C}_{\text{blc}}^{n-1}$  is isomorphic to  $\mathbb{C}^*$ .*

*Proof.* (a) Given a configuration  $Q^\circ \in \mathcal{C}_{\text{blc}}^{n-1}$ , we set

$$\psi_{Q^\circ}(\omega) \stackrel{\text{def}}{=} \omega^{mn(n-1)+1} e^{\chi(\omega \cdot Q^\circ)} D_n^m(Q^\circ) \text{ for any } \omega \in \mathbb{C}^*. \quad (100)$$

Clearly  $\psi_{Q^\circ} \in \mathcal{O}_{\text{hol}}^X(\mathcal{C}_{\text{blc}}^{n-1})$  and  $\psi_{Q^\circ} \neq \text{const}$ , since  $mn(n-1)+1 \neq 0$  and  $e^{\chi(\omega \cdot Q^\circ)}$  cannot be a non-constant rational function of  $\omega \in \mathbb{C}^*$ . Hence, by the Picard theorem,  $\psi_{Q^\circ}(\mathbb{C}^*) = \mathbb{C}^*$ . According to (98), we have

$$f(\omega \cdot Q^\circ) = e^{\chi(\omega \cdot Q^\circ)} D_n^m(\omega \cdot Q^\circ) \cdot \omega \cdot Q^\circ = \omega^{mn(n-1)+1} e^{\chi(\omega \cdot Q^\circ)} D_n^m(Q^\circ) \cdot Q^\circ = \psi_{Q^\circ}(\omega) \cdot Q^\circ.$$

Thus, taking  $\omega \in \mathbb{C}^*$  such that  $\psi_{Q^\circ}(\omega) = 1$ , we see that  $Q^\circ \in f(\mathcal{C}_{\text{blc}}^{n-1})$ . Hence  $f$  is surjective. Furthermore, all such  $\omega$  satisfy the system of equations (99). Since the stabilizer  $\text{St}_{\mathbb{C}^*}(Q^\circ)$  is finite, all solutions  $\omega$  of (99) form a finite union of countable discrete subsets of  $\mathbb{C}^*$ . Thus the set  $f^{-1}(Q^\circ)$  is countable and discrete.

<sup>16</sup>In view of Theorem 10.2, for  $n > 4$  any non-Abelian holomorphic endomorphism of  $\mathcal{C}_{\text{blc}}^{n-1}$  is surjective.

(b) If  $f$  as in (98) is proper then  $f^{-1}(Q^\circ)$  is finite for any  $Q^\circ \in \mathcal{C}_{\text{blc}}^{n-1}$ . This is possible only when the exponent  $\chi(\omega \cdot Q^\circ)$  in (99) and (100) does not depend on  $\omega \in \mathbb{C}^*$ , i.e., the function  $\chi$  is  $\mathcal{S}$ -invariant. Then, for any fixed  $Q^\circ$ , the function (100) takes the form

$$\psi_{Q^\circ}(\omega) = \tilde{\psi}_{Q^\circ}(\omega) \stackrel{\text{def}}{=} \omega^{mn(n-1)+1} e^{\chi(Q^\circ)} D_n^m(Q^\circ).$$

The latter function is homogeneous of degree  $N = mn(n-1) + 1$ , and the equation  $\tilde{\psi}_{Q^\circ}(\omega) = 1$  has precisely  $N$  distinct roots  $\omega_1, \dots, \omega_N$ . If the stabilizer  $\text{St}_{\mathbb{C}^*}(Q^\circ)$  is trivial, then  $f^{-1}(Q^\circ)$  consists on  $N$  distinct points  $\omega_1 Q^\circ, \dots, \omega_N Q^\circ$ . If  $\text{St}_{\mathbb{C}^*}(Q^\circ) \neq \{1\}$ , then, to find the preimage  $f^{-1}(Q^\circ)$ , we have to solve the inclusion  $\tilde{\psi}_{Q^\circ}(\omega) \in \text{St}_{\mathbb{C}^*}(Q^\circ)$ . Fix some  $\omega_0$  such that  $\tilde{\psi}_{Q^\circ}(\omega_0) = 1$ , take any solution  $\omega \in \mathbb{C}^*$  of the above inclusion, and set  $\lambda = \omega/\omega_0$ . Then

$$\lambda^N = \left( \frac{\omega}{\omega_0} \right)^N = \frac{\tilde{\psi}_{Q^\circ}(\omega)}{\tilde{\psi}_{Q^\circ}(\omega_0)} = \tilde{\psi}_{Q^\circ}(\omega) \in \text{St}_{\mathbb{C}^*}(Q^\circ). \quad (101)$$

The preimage  $f^{-1}(Q^\circ)$  of  $Q^\circ$  consists of all configurations  $\omega Q^\circ = \omega_0 \lambda Q^\circ$ , where  $\lambda$  runs over all solutions of the inclusion (101). All such configurations  $\omega_0 \lambda Q^\circ$  form a periodic sequence  $\omega_0 \lambda^k Q^\circ$ ,  $k \in \mathbb{Z}_{\geq 0}$ , with period  $N$ ; therefore, this sequence contains precisely  $N$  distinct elements. It follows easily from these facts that  $f: \mathcal{C}_{\text{blc}}^{n-1} \rightarrow \mathcal{C}_{\text{blc}}^{n-1}$  is an unramified cyclic covering of degree  $N$ .

The proof of the other assertions in (b) is easy, and we leave it to the reader.

(c) For a given  $\chi \in \mathcal{O}_{\text{hol}}^{\mathcal{S}}(\mathcal{C}_{\text{blc}}^{n-1})$  we let  $f_1(Q^\circ) = e^{\chi(Q^\circ)} Q^\circ$  and  $f_2(Q^\circ) = e^{-\chi(Q^\circ)} Q^\circ$ . It follows from the  $\mathcal{S}$ -invariance of  $\chi$  that  $f_1(f_2(Q^\circ)) = f_2(f_1(Q^\circ)) = Q^\circ$  for every  $Q^\circ \in \mathcal{C}_{\text{blc}}^{n-1}$ . Thus  $f_1$  and  $f_2$  are mutually inverse biholomorphic automorphisms of  $\mathcal{C}_{\text{blc}}^{n-1}$ . To prove the converse note that any automorphism is a proper map. According to Theorem 10.2 (formula (98)) and part (b), such a map is of the form  $Q^\circ \mapsto e^{\chi(Q^\circ)} Q^\circ$  with  $\chi \in \mathcal{O}_{\text{hol}}^{\mathcal{S}}(\mathcal{C}_{\text{blc}}^{n-1})$ . The other two assertions of part (c) are clear.

(d) A map as in (98) is regular if and only if  $\chi = \text{const}$ , i.e.,

$$f(Q^\circ) = s D_n^m(Q^\circ) \cdot Q^\circ \quad \text{for all } Q^\circ \in \mathcal{C}_{\text{blc}}^{n-1}, \quad \text{where } s \in \mathbb{C}^* \text{ and } m \in \mathbb{Z}.$$

It is a biregular automorphism of  $\mathcal{C}_{\text{blc}}^{n-1}$  if and only if  $m = 0$ . Hence  $\text{Aut } \mathcal{C}_{\text{blc}}^{n-1} \cong \mathbb{C}^*$ .  $\square$

**Remark 10.4** (*Dimension of the image*). In what follows we assume that  $n > 4$ . According to [38, Theorem 14], for  $X = \mathbb{C}$  or  $\mathbb{P}^1$  and any non-Abelian holomorphic endomorphism  $F$  of  $\mathcal{C}^n(X)$  we have  $\dim_{\mathbb{C}} F(\mathcal{C}^n(X)) \geq n - \dim_{\mathbb{C}}(\text{Aut } X) + 1$ . Moreover, by [38, Remark 7] or theorems 10.2 and 10.3(a) above, for  $X = \mathbb{C}$  the composition  $\pi \circ F$  of any non-Abelian holomorphic endomorphism  $F$  of  $\mathcal{C}^n$  with the projection  $\pi: \mathcal{C}^n \rightarrow \mathcal{C}_{\text{blc}}^{n-1}$  is surjective, so that  $\dim_{\mathbb{C}} F(\mathcal{C}^n) \geq n - 1$ . Clearly, the latter bound cannot be improved. Seemingly, for  $X = \mathbb{P}^1$  no example of  $F$  with  $\dim_{\mathbb{C}} F(\mathcal{C}^n(\mathbb{P}^1)) < n$  is known. Zinde ([51]) proved that for  $X = \mathbb{C}^*$  any non-Abelian holomorphic endomorphism of  $\mathcal{C}^n(\mathbb{C}^*)$  is surjective.

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TECHNION-ISRAEL INSTITUTE OF TECHNOLOGY, HAIFA 32000, ISRAEL  
*E-mail address:* vlin@tx.technion.ac.il

UNIVERSITÉ GRENOBLE I, INSTITUT FOURIER, UMR 5582 CNRS-UJF, BP 74, 38402 SAINT  
MARTIN D'HÈRES CÉDEX, FRANCE  
*E-mail address:* Mikhail.Zaidenberg@ujf-grenoble.fr