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# SELF-ADJOINT EXTENSIONS OF DIFFERENTIAL OPERATORS ON RIEMANNIAN MANIFOLDS

OGNJEN MILATOVIC, FRANÇOISE TRUC

ABSTRACT. We study  $H = D^*D + V$ , where  $D$  is a first order elliptic differential operator acting on sections of a Hermitian vector bundle over a Riemannian manifold  $M$ , and  $V$  is a Hermitian bundle endomorphism. In the case when  $M$  is geodesically complete, we establish the essential self-adjointness of positive integer powers of  $H$ . In the case when  $M$  is not necessarily geodesically complete, we give a sufficient condition for the essential self-adjointness of  $H$ , expressed in terms of the behavior of  $V$  relative to the Cauchy boundary of  $M$ .

## 1. INTRODUCTION

As a fundamental problem in mathematical physics, self-adjointness of Schrödinger operators has attracted the attention of researchers over many years now, resulting in numerous sufficient conditions for this property in  $L^2(\mathbb{R}^n)$ . For reviews of the corresponding results, see, for instance, the books [14, 29].

The study of the corresponding problem in the context of a non-compact Riemannian manifold was initiated by Gaffney [15, 16] with the proof of the essential self-adjointness of the Laplacian on differential forms. About two decades later, Cordes (see Theorem 3 in [11]) proved the essential self-adjointness of positive integer powers of the operator

$$\Delta_{M,\mu} := -\frac{1}{\kappa} \left( \frac{\partial}{\partial x^i} \left( \kappa g^{ij} \frac{\partial}{\partial x^j} \right) \right) \quad (1.1)$$

on an  $n$ -dimensional geodesically complete Riemannian manifold  $M$  equipped with a (smooth) metric  $g = (g_{ij})$  (here,  $(g^{ij}) = ((g_{ij})^{-1})$ ) and a positive smooth measure  $d\mu$  (i.e. in any local coordinates  $x^1, x^2, \dots, x^n$  there exists a strictly positive  $C^\infty$ -density  $\kappa(x)$  such that  $d\mu = \kappa(x) dx^1 dx^2 \dots dx^n$ ). Theorem 1 of our paper extends this result to the operator  $(D^*D + V)^k$  for all  $k \in \mathbb{Z}_+$ , where  $D$  is a first order elliptic differential operator acting on sections of a Hermitian vector bundle over a geodesically complete Riemannian manifold,  $D^*$  is the formal adjoint of  $D$ , and  $V$  is a self-adjoint Hermitian bundle endomorphism; see Section 2.3 for details.

In the context of a general Riemannian manifold (not necessarily geodesically complete), Cordes (see Theorem IV.1.1 in [12] and Theorem 4 in [11]) proved the essential self-adjointness of  $P^k$  for all  $k \in \mathbb{Z}_+$ , where

$$Pu := \Delta_{M,\mu}u + qu, \quad u \in C^\infty(M), \quad (1.2)$$

and  $q \in C^\infty(M)$  is real-valued. Thanks to a Roelcke-type estimate (see Lemma 3.1 below), the technique of Cordes [12] can be applied to the operator  $(D^*D + V)^k$  acting on sections of Hermitian vector bundles over a general Riemannian manifold. To make our exposition shorter, in Theorem 1 we consider the geodesically complete case. Our Theorem 2 concerns  $(\nabla^*\nabla + V)^k$ , where  $\nabla$  is a metric connection on a Hermitian vector bundle over a non-compact geodesically complete Riemannian manifold. This result extends Theorem 1.1 of [13] where Cordes showed that if  $(M, g)$  is non-compact and geodesically complete and  $P$  is semi-bounded from below on  $C_c^\infty(M)$ , then  $P^k$  is essentially self-adjoint on  $C_c^\infty(M)$ , for all  $k \in \mathbb{Z}_+$ .

For the remainder of the introduction, the notation  $D^*D + V$  is used in the same sense as described earlier in this section. In the setting of geodesically complete Riemannian manifolds, the essential self-adjointness of  $D^*D + V$  with  $V \in L_{\text{loc}}^\infty$  was established in [21], providing a generalization of the results in [3, 27, 28, 32] concerning Schrödinger operators on functions (or differential forms). Subsequently, the operator  $D^*D + V$  with a singular potential  $V$  was considered in [5]. Recently, in the case  $V \in L_{\text{loc}}^\infty$ , the authors of [4] extended the main result of [5] to the operator  $D^*D + V$  acting on sections of infinite-dimensional bundles whose fibers are modules of finite type over a von Neumann algebra.

In the context of an incomplete Riemannian manifold, the authors of [17, 22, 23] studied the so-called Gaffney Laplacian, a self-adjoint realization of the scalar Laplacian generally different from the closure of  $\Delta_{M, d\mu}|_{C_c^\infty(M)}$ . For a study of Gaffney Laplacian on differential forms, see [24].

Our Theorem 3 gives a condition on the behavior of  $V$  relative to the Cauchy boundary of  $M$  that will guarantee the essential self-adjointness of  $D^*D + V$ ; for details see Section 2.4 below. Related results can be found in [6, 25, 26] in the context of (magnetic) Schrödinger operators on domains in  $\mathbb{R}^n$ , and in [10] concerning the magnetic Laplacian on domains in  $\mathbb{R}^n$  and certain types of Riemannian manifolds.

Finally, let us mention that Chernoff [7] used the hyperbolic equation approach to establish the essential self-adjointness of positive integer powers of Laplace–Beltrami operator on differential forms. This approach was also applied in [2, 8, 9, 18, 19, 31] to prove essential self-adjointness of second-order operators (acting on scalar functions or sections of Hermitian vector bundles) on Riemannian manifolds. Additionally, the authors of [18, 19] used path integral techniques.

The paper is organized as follows. The main results are stated in Section 2, a preliminary lemma is proven in Section 3, and the main results are proven in Sections 4–6.

## 2. MAIN RESULTS

**2.1. The setting.** Let  $M$  be an  $n$ -dimensional smooth, connected Riemannian manifold without boundary. We denote the Riemannian metric on  $M$  by  $g^{TM}$ . We assume that  $M$  is equipped with a positive smooth measure  $d\mu$ , i.e. in any local coordinates  $x^1, x^2, \dots, x^n$  there exists a strictly positive  $C^\infty$ -density  $\kappa(x)$  such that  $d\mu = \kappa(x) dx^1 dx^2 \dots dx^n$ . Let  $E$  be a Hermitian vector bundle over  $M$  and let  $L^2(E)$  denote the Hilbert space of square integrable sections of  $E$  with respect to the inner product

$$(u, v) = \int_M \langle u(x), v(x) \rangle_{E_x} d\mu(x), \quad (2.1)$$

where  $\langle \cdot, \cdot \rangle_{E_x}$  is the fiberwise inner product. The corresponding norm in  $L^2(E)$  is denoted by  $\| \cdot \|$ . In Sobolev space notations  $W_{\text{loc}}^{k,2}(E)$  used in this paper, the superscript  $k \in \mathbb{Z}_+$  indicates the order of the highest derivative. The corresponding dual space is denoted by  $W_{\text{loc}}^{-k,2}(E)$ .

Let  $F$  be another Hermitian vector bundle on  $M$ . We consider a first order differential operator  $D: C_c^\infty(E) \rightarrow C_c^\infty(F)$ , where  $C_c^\infty$  stands for the space of smooth compactly supported sections. In the sequel, by  $\sigma(D)$  we denote the principal symbol of  $D$ .

**Assumption (A0)** Assume that  $D$  is elliptic. Additionally, assume that there exists a constant  $\lambda_0 > 0$  such that

$$|\sigma(D)(x, \xi)| \leq \lambda_0 |\xi|, \quad \text{for all } x \in M, \xi \in T_x^*M, \quad (2.2)$$

where  $|\xi|$  is the length of  $\xi$  induced by the metric  $g^{TM}$  and  $|\sigma(D)(x, \xi)|$  is the operator norm of  $\sigma(D)(x, \xi): E_x \rightarrow F_x$ .

*Remark 2.2.* Assumption (A0) is satisfied if  $D = \nabla$ , where  $\nabla: C^\infty(E) \rightarrow C^\infty(T^*M \otimes E)$  is a covariant derivative corresponding to a metric connection on a Hermitian vector bundle  $E$  over  $M$ .

**2.3. Schrödinger-type Operator.** Let  $D^*: C_c^\infty(F) \rightarrow C_c^\infty(E)$  be the formal adjoint of  $D$  with respect to the inner product (2.1). We consider the operator

$$H = D^*D + V, \quad (2.3)$$

where  $V \in L_{\text{loc}}^\infty(\text{End } E)$  is a linear self-adjoint bundle endomorphism. In other words, for all  $x \in M$ , the operator  $V(x): E_x \rightarrow E_x$  is self-adjoint and  $|V(x)| \in L_{\text{loc}}^\infty(M)$ , where  $|V(x)|$  is the norm of the operator  $V(x): E_x \rightarrow E_x$ .

#### 2.4. Statements of Results.

**Theorem 1.** *Let  $M$ ,  $g^{TM}$ , and  $d\mu$  be as in Section 2.1. Assume that  $(M, g^{TM})$  is geodesically complete. Let  $E$  and  $F$  be Hermitian vector bundles over  $M$ , and let  $D: C_c^\infty(E) \rightarrow C_c^\infty(F)$  be a first order differential operator satisfying the assumption (A0). Assume that  $V \in C^\infty(\text{End } E)$  and*

$$V(x) \geq C, \quad \text{for all } x \in M,$$

*where  $C$  is a constant, and the inequality is understood in operator sense. Then  $H^k$  is essentially self-adjoint on  $C_c^\infty(E)$ , for all  $k \in \mathbb{Z}_+$ .*

*Remark 2.5.* In the case  $V = 0$ , the following result related to Theorem 1 can be deduced from Chernoff (see Theorem 2.2 in [7]):

*Assume that  $(M, g)$  is a geodesically complete Riemannian manifold with metric  $g$ . Let  $D$  be as in Theorem 1, and define*

$$c(x) := \sup\{|\sigma(D)(x, \xi)|: |\xi|_{T_x^*M} = 1\}.$$

*Fix  $x_0 \in M$  and define*

$$c(r) := \sup_{x \in B(x_0, r)} c(x),$$

where  $r > 0$  and  $B(x_0, r) := \{x \in M : d_g(x_0, x) < r\}$ . Assume that

$$\int_0^\infty \frac{1}{c(r)} dr = \infty. \quad (2.4)$$

Then the operator  $(D^*D)^k$  is essentially self-adjoint on  $C_c^\infty(E)$  for all  $k \in \mathbb{Z}_+$ .

At the end of this section we give an example of an operator for which Theorem 1 guarantees the essential self-adjointness of  $(D^*D)^k$ , whereas Chernoff's result cannot be applied.

The next theorem is concerned with operators whose potential  $V$  is not necessarily semi-bounded from below.

**Theorem 2.** Let  $M$ ,  $g^{TM}$ , and  $d\mu$  be as in Section 2.1. Assume that  $(M, g^{TM})$  is noncompact and geodesically complete. Let  $E$  be a Hermitian vector bundle over  $M$  and let  $\nabla$  be a Hermitian connection on  $E$ . Assume that  $V \in C^\infty(\text{End } E)$  and

$$V(x) \geq q(x), \quad \text{for all } x \in M, \quad (2.5)$$

where  $q \in C^\infty(M)$  and the inequality is understood in the sense of operators  $E_x \rightarrow E_x$ . Additionally, assume that

$$((\Delta_{M,\mu} + q)u, u) \geq C\|u\|^2, \quad \text{for all } u \in C_c^\infty(M), \quad (2.6)$$

where  $C \in \mathbb{R}$  and  $\Delta_{M,\mu}$  is as in (1.1) with  $g$  replaced by  $g^{TM}$ . Then the operator  $(\nabla^*\nabla + V)^k$  is essentially self-adjoint on  $C_c^\infty(E)$ , for all  $k \in \mathbb{Z}_+$ .

*Remark 2.6.* Let us stress that non-compactness is required in the proof to ensure the existence of a positive smooth solution of an equation involving  $\Delta_{M,\mu} + q$ . In the case of a compact manifold, such a solution exists under an additional assumption; see Theorem III.6.3 in [12].

In our last result we will need the notion of Cauchy boundary. Let  $d_{g^{TM}}$  be the distance function corresponding to the metric  $g^{TM}$ . Let  $(\widehat{M}, \widehat{d}_{g^{TM}})$  be the metric completion of  $(M, d_{g^{TM}})$ . We define the *Cauchy boundary*  $\partial_C M$  as follows:  $\partial_C M := \widehat{M} \setminus M$ . Note that  $(M, d_{g^{TM}})$  is metrically complete if and only if  $\partial_C M$  is empty. For  $x \in M$  we define

$$r(x) := \inf_{z \in \partial_C M} \widehat{d}_{g^{TM}}(x, z). \quad (2.7)$$

We will also need the following assumption:

**Assumption (A1)** Assume that  $\widehat{M}$  is a smooth manifold and that the metric  $g^{TM}$  extends to  $\partial_C M$ .

*Remark 2.7.* Let  $N$  be a (smooth)  $n$ -dimensional Riemannian manifold without boundary. Denote the metric on  $N$  by  $g^{TN}$  and assume that  $(N, g^{TN})$  is geodesically complete. Let  $\Sigma$  be a  $k$ -dimensional closed sub-manifold of  $N$  with  $k < n$ . Then  $M := N \setminus \Sigma$  has the properties  $\widehat{M} = N$  and  $\partial_C M = \Sigma$ . Thus, assumption (A1) is satisfied.

**Theorem 3.** Let  $M$ ,  $g^{TM}$ , and  $d\mu$  be as in Section 2.1. Assume that (A1) is satisfied. Let  $E$  and  $F$  be Hermitian vector bundles over  $M$ , and let  $D: C_c^\infty(E) \rightarrow C_c^\infty(F)$  be a first order differential operator satisfying the assumption (A0). Assume that  $V \in L_{\text{loc}}^\infty(\text{End } E)$  and there exists a constant  $C$  such that

$$V(x) \geq \left( \frac{\lambda_0}{r(x)} \right)^2 - C, \quad \text{for all } x \in M, \quad (2.8)$$

where  $\lambda_0$  is as in (2.2), the distance  $r(x)$  is as in (2.7), and the inequality is understood in the sense of linear operators  $E_x \rightarrow E_x$ . Then  $H$  is essentially self-adjoint on  $C_c^\infty(E)$ .

In order to describe the example mentioned in Remark 2.5, we need the following

*Remark 2.8.* As explained in [5], we can use a first-order elliptic operator  $D: C_c^\infty(E) \rightarrow C_c^\infty(F)$  to define a metric on  $M$ . For  $\xi, \eta \in T_x^*M$ , define

$$\langle \xi, \eta \rangle = \frac{1}{m} \text{Re Tr} ((\sigma(D)(x, \xi))^* \sigma(D)(x, \eta)), \quad m = \dim E_x, \quad (2.9)$$

where  $\text{Tr}$  denotes the usual trace of a linear operator. Since  $D$  is an elliptic first-order differential operator and  $\sigma(D)(x, \xi)$  is linear in  $\xi$ , it is easily checked that (2.9) defines an inner product on  $T_x^*M$ . Its dual defines a Riemannian metric on  $M$ . Denoting this metric by  $g^{TM}$  and using elementary linear algebra, it follows that (2.2) is satisfied with  $\lambda_0 = \sqrt{m}$ .

**Example 2.9.** Let  $M = \mathbb{R}^2$  with the standard metric and measure, and  $V = 0$ . Denoting respectively by  $C_c^\infty(\mathbb{R}^2; \mathbb{R})$  and  $C_c^\infty(\mathbb{R}^2; \mathbb{R}^2)$  the spaces of smooth compactly supported functions  $f: \mathbb{R}^2 \rightarrow \mathbb{R}$  and  $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ , we define the operator  $D: C_c^\infty(\mathbb{R}^2; \mathbb{R}) \rightarrow C_c^\infty(\mathbb{R}^2; \mathbb{R}^2)$  by

$$D = \begin{pmatrix} a(x, y) \frac{\partial}{\partial x} \\ b(x, y) \frac{\partial}{\partial y} \end{pmatrix},$$

where

$$\begin{aligned} a(x, y) &= (1 - \cos(2\pi e^x))x^2 + 1; \\ b(x, y) &= (1 - \sin(2\pi e^y))y^2 + 1. \end{aligned}$$

Since  $a, b$  are smooth real-valued nowhere vanishing functions in  $\mathbb{R}^2$ , it follows that the operator  $D$  is elliptic. We are interested in the operator

$$H := D^*D = -\frac{\partial}{\partial x} \left( a^2 \frac{\partial}{\partial x} \right) - \frac{\partial}{\partial y} \left( b^2 \frac{\partial}{\partial y} \right).$$

The matrix of the inner product on  $T^*M$  defined by  $D$  via (2.9) is  $\text{diag}(a^2/2, b^2/2)$ . The matrix of the corresponding Riemannian metric  $g^{TM}$  on  $M$  is  $\text{diag}(2a^{-2}, 2b^{-2})$ , so the metric itself is  $ds^2 = 2a^{-2}dx^2 + 2b^{-2}dy^2$  and it is geodesically complete (see Example 3.1 of [5]). Moreover, thanks to Remark 2.8, assumption (A0) is satisfied. Thus, by Theorem 1 the operator  $(D^*D)^k$  is essentially self-adjoint for all  $k \in \mathbb{Z}_+$ . Furthermore, in Example 3.1 of [5] it was shown that for the considered operator  $D$  the condition (2.4) is not satisfied. Thus, the result stated in Remark 2.5 does not apply.

### 3. ROELCKE-TYPE INEQUALITY

Let  $M$ ,  $d\mu$ ,  $D$ , and  $\sigma(D)$  be as in Section 2.1. Set  $\widehat{D} := -i\sigma(D)$ , where  $i = \sqrt{-1}$ . Then for any Lipschitz function  $\psi: M \rightarrow \mathbb{R}$  and  $u \in W_{\text{loc}}^{1,2}(E)$  we have

$$D(\psi u) = \widehat{D}(d\psi)u + \psi Du, \quad (3.1)$$

where we have suppressed  $x$  for simplicity. We also note that  $\widehat{D}^*(\xi) = -(\widehat{D}(\xi))^*$ , for all  $\xi \in T_x^*M$ .

For a compact set  $K \subset M$ , and  $u, v \in W_{\text{loc}}^{1,2}(E)$ , we define

$$(u, v)_K := \int_K \langle u(x), v(x) \rangle d\mu(x), \quad (Du, Dv)_K := \int_K \langle Du(x), Dv(x) \rangle d\mu(x). \quad (3.2)$$

In order to prove Theorem 1 we need the following important lemma, which is an extension of Lemma 2.1 in [12] to operator (2.3). In the context of the scalar Laplacian on a Riemannian manifold, this kind of result is originally due to Roelcke [30].

**Lemma 3.1.** *Let  $M$ ,  $g^{TM}$ , and  $d\mu$  be as in Section 2.1. Let  $E$  and  $F$  be Hermitian vector bundles over  $M$ , and let  $D: C_c^\infty(E) \rightarrow C_c^\infty(F)$  be a first order differential operator satisfying the assumption (A0). Let  $\rho: M \rightarrow [0, \infty)$  be a function satisfying the following properties:*

- (i)  $\rho(x)$  is Lipschitz continuous with respect to the distance induced by the metric  $g^{TM}$ ;
- (ii)  $\rho(x_0) = 0$ , for some fixed  $x_0 \in M$ ;
- (iii) the set  $B_T := \{x \in M: \rho(x) \leq T\}$  is compact, for some  $T > 0$ .

Then the following inequality holds for all  $u \in W_{\text{loc}}^{2,2}(E)$  and  $v \in W_{\text{loc}}^{2,2}(E)$ :

$$\int_0^T |(Du, Dv)_{B_t} - (D^*Du, v)_{B_t}| dt \leq \lambda_0 \int_{B_T} |d\rho(x)| |Du(x)| |v(x)| d\mu(x), \quad (3.3)$$

where  $B_t$  is as in (iii) (with  $t$  instead of  $T$ ), the constant  $\lambda_0$  is as in (2.2), and  $|d\rho(x)|$  is the length of  $d\rho(x) \in T_x^*M$  induced by  $g^{TM}$ .

*Proof.* For  $\varepsilon > 0$  and  $t \in (0, T)$ , we define a continuous piecewise linear function  $F_{\varepsilon,t}$  as follows:

$$F_{\varepsilon,t}(s) = \begin{cases} 1 & \text{for } s < t - \varepsilon \\ (t - s)/\varepsilon & \text{for } t - \varepsilon \leq s < t \\ 0 & \text{for } s \geq t \end{cases}$$

The function  $f_{\varepsilon,t}(x) := F_{\varepsilon,t}(\rho(x))$ , is Lipschitz continuous with respect to the distance induced by the metric  $g^{TM}$ , and  $d(f_{\varepsilon,t}(\rho(x))) = (F'_{\varepsilon,t}(\rho(x)))d\rho(x)$ . Moreover we have  $f_{\varepsilon,t}v \in W_{\text{loc}}^{1,2}(E)$  for all  $v \in W_{\text{loc}}^{1,2}(E)$ , since

$$D(f_{\varepsilon,t}v) = \widehat{D}(df_{\varepsilon,t})v + f_{\varepsilon,t}Dv.$$

It follows from the compactness of  $B_T$  that  $B_t$  is compact for all  $t \in (0, T)$ . Using integration by parts (see Lemma 8.8 in [5]), for all  $u \in W_{\text{loc}}^{2,2}(E)$  and  $v \in W_{\text{loc}}^{2,2}(E)$  we have

$$(D^*Du, v f_{\varepsilon,t})_{B_t} = (Du, D(v f_{\varepsilon,t}))_{B_t} = (Du, f_{\varepsilon,t}Dv)_{B_t} + (Du, \widehat{D}(df_{\varepsilon,t})v)_{B_t},$$

which, together with (2.2), gives

$$\begin{aligned}
& |(Du, f_{\varepsilon,t}Dv)_{B_t} - (D^*Du, v f_{\varepsilon,t})_{B_t}| = |(Du, \widehat{D}(df_{\varepsilon,t})v)_{B_t}| \\
& \leq \int_{B_t} |Du(x)| |\widehat{D}(df_{\varepsilon,t}(x))v(x)| d\mu(x) \leq \lambda_0 \int_{B_t} |Du(x)| |df_{\varepsilon,t}(x)| |v(x)| d\mu(x) \\
& = \lambda_0 \int_{B_t} |Du(x)| |F'_{\varepsilon,t}(\rho(x))| |d\rho(x)| |v(x)| d\mu(x) \\
& \leq \lambda_0 \int_{B_T} |Du(x)| |F'_{\varepsilon,t}(\rho(x))| |d\rho(x)| |v(x)| d\mu(x), \tag{3.4}
\end{aligned}$$

where  $|df_{\varepsilon,t}(x)|$  and  $|d\rho(x)|$  are the norms of  $df_{\varepsilon,t}(x) \in T_x^*M$  and  $d\rho(x) \in T_x^*M$  induced by  $g^{TM}$ .

Fixing  $\varepsilon > 0$ , integrating the leftmost and the rightmost side of (3.4) from  $t = 0$  to  $t = T$ , and noting that  $F'_{\varepsilon,t}(\rho(x))$  is the only term on the rightmost side depending on  $t$ , we obtain

$$\begin{aligned}
& \int_0^T |(Du, f_{\varepsilon,t}Dv)_{B_t} - (D^*Du, v f_{\varepsilon,t})_{B_t}| dt \\
& \leq \lambda_0 \int_{B_T} |Du(x)| |d\rho(x)| |v(x)| I_\varepsilon(x) d\mu(x), \tag{3.5}
\end{aligned}$$

where

$$I_\varepsilon(x) := \int_0^T |F'_{\varepsilon,t}(\rho(x))| dt.$$

We now let  $\varepsilon \rightarrow 0+$  in (3.5). On the left-hand side of (3.5), as  $\varepsilon \rightarrow 0+$ , we have  $f_{\varepsilon,t}(x) \rightarrow \chi_{B_t}(x)$  almost everywhere, where  $\chi_{B_t}(x)$  is the characteristic function of the set  $B_t$ . Additionally,  $|f_{\varepsilon,t}(x)| \leq 1$  for all  $x \in B_t$  and all  $t \in (0, T)$ ; thus, by dominated convergence theorem, as  $\varepsilon \rightarrow 0+$  the left-hand side of (3.5) converges to the left-hand side of (3.3). On the right-hand side of (3.5) an easy calculation shows that  $I_\varepsilon(x) \rightarrow 1$ , as  $\varepsilon \rightarrow 0+$ . Additionally, we have  $|I_\varepsilon(x)| \leq 1$ , a.e. on  $B_T$ ; hence, by the dominated convergence theorem, as  $\varepsilon \rightarrow 0+$  the right-hand side of (3.5) converges to the right-hand side of (3.3). This establishes the inequality (3.3).  $\square$

#### 4. PROOF OF THEOREM 1

We first give the definitions of minimal and maximal operators associated with the expression  $H$  in (2.3).

**4.1. Minimal and Maximal Operators.** We define  $H_{\min}u := Hu$ , with  $\text{Dom}(H_{\min}) := C_c^\infty(E)$ , and  $H_{\max} := (H_{\min})^*$ , where  $T^*$  denotes the adjoint of operator  $T$ . Denoting  $\mathcal{D}_{\max} := \{u \in L^2(E) : Hu \in L^2(E)\}$ , we recall the following well-known property:  $\text{Dom}(H_{\max}) = \mathcal{D}_{\max}$  and  $H_{\max}u = Hu$  for all  $u \in \mathcal{D}_{\max}$ .

From now on, throughout this section, we assume that the hypotheses of Theorem 1 are satisfied. Let  $x_0 \in M$ , and define  $\rho(x) := d_{g^{TM}}(x_0, x)$ , where  $d_{g^{TM}}$  is the distance function corresponding to the metric  $g^{TM}$ . By the definition of  $\rho(x)$  and the geodesic completeness of  $(M, g^{TM})$ , it follows that  $\rho(x)$  satisfies all hypotheses of Lemma 3.1. Using Lemma 3.1 and Proposition 4.2 below, we are able to apply the method of Cordes [11, 12] to our context. As we



will see, Cordes's technique reduces our problem to a system of ordinary differential inequalities of the same type as in Section IV.3 of [12].

**Proposition 4.2.** *Let  $A$  be a densely defined operator with domain  $\mathcal{D}$  in a Hilbert space  $\mathcal{H}$ . Assume that  $A$  is semi-bounded from below, that  $A\mathcal{D} \subseteq \mathcal{D}$ , and that there exists  $c_0 \in \mathbb{R}$  such that the following two properties hold:*

- (i)  $((A + c_0 I)u, u)_{\mathcal{H}} \geq \|u\|_{\mathcal{H}}^2$ , for all  $u \in \mathcal{D}$ , where  $I$  denotes the identity operator in  $\mathcal{H}$ ;
- (ii)  $(A + c_0 I)^k$  is essentially self-adjoint on  $\mathcal{D}$ , for some  $k \in \mathbb{Z}_+$ .

*Then,  $(A + cI)^j$  is essentially self-adjoint on  $\mathcal{D}$ , for all  $j = 1, 2, \dots, k$  and all  $c \in \mathbb{R}$ .*

*Remark 4.3.* To prove Proposition 4.2, one may mimick the proof of Proposition 1.4 in [12], which was carried out for the operator  $P$  defined in (1.2) with  $\mathcal{D} = C_c^\infty(M)$ , since only abstract functional analysis facts and the property  $P\mathcal{D} \subseteq \mathcal{D}$  were used.

We start the proof of Theorem 1 by noticing that the operator  $H_{\min}$  is essentially self-adjoint on  $C_c^\infty(E)$ ; see Corollary 2.9 in [5]. Thanks to Proposition 4.2, without any loss of generality we can change  $V(x)$  to  $V(x) + C \text{Id}(x)$ , where  $C$  is a sufficiently large constant in order to have

$$V(x) \geq (\lambda_0^2 + 1)\text{Id}(x), \quad \text{for all } x \in M, \quad (4.1)$$

where  $\lambda_0$  is as in (2.2) and  $\text{Id}(x)$  is the identity endomorphism of  $E_x$ . Using non-negativity of  $D^*D$  and (4.1) we have

$$(H_{\min}u, u) \geq \|u\|^2, \quad \text{for all } u \in C_c^\infty(E), \quad (4.2)$$

which leads to

$$\|u\|^2 \leq (Hu, u) \leq \|Hu\|\|u\|, \quad \text{for all } u \in C_c^\infty(E),$$

and, hence,  $\|Hu\| \geq \|u\|$ , for all  $u \in C_c^\infty(E)$ . Therefore,

$$(H^2u, u) = (Hu, Hu) = \|Hu\|^2 \geq \|u\|^2, \quad \text{for all } u \in C_c^\infty(E), \quad (4.3)$$

and

$$(H^3u, u) = (HHu, Hu) \geq \|Hu\|^2 \geq \|u\|^2, \quad \text{for all } u \in C_c^\infty(E).$$

By (4.3) we have

$$\|u\|^2 \leq (H^2u, u) \leq \|H^2u\|\|u\|, \quad \text{for all } u \in C_c^\infty(E),$$

and, hence,  $\|H^2u\| \geq \|u\|$ , for all  $u \in C_c^\infty(E)$ . This, in turn, leads to

$$(H^4u, u) = (H^2u, H^2u) = \|H^2u\|^2 \geq \|u\|^2, \quad \text{for all } u \in C_c^\infty(E).$$

Continuing like this, we obtain  $(H^k u, u) \geq \|u\|^2$ , for all  $u \in C_c^\infty(E)$  and all  $k \in \mathbb{Z}_+$ . In this case, by an abstract fact (see Theorem X.26 in [29]), the essential self-adjointness of  $H^k$  on  $C_c^\infty(E)$  is equivalent to the following statement: if  $u \in L^2(E)$  satisfies  $H^k u = 0$ , then  $u = 0$ .

Let  $u \in L^2(E)$  satisfy  $H^k u = 0$ . Since  $V \in C^\infty(E)$ , by local elliptic regularity it follows that  $u \in C^\infty(E) \cap L^2(E)$ . Define

$$f_j := H^{k-j}u, \quad j = 0, \pm 1, \pm 2, \dots \quad (4.4)$$

Here, in the case  $k - j < 0$ , the definition (4.4) is interpreted as  $((H_{\max})^{-1})^{j-k}$ . We already noted that  $H_{\min}$  is essentially self-adjoint and positive. Furthermore, it is well known that the self-adjoint closure of  $H_{\min}$  coincides with  $H_{\max}$ . Therefore  $H_{\max}$  is a positive self-adjoint operator, and  $(H_{\max})^{-1}: L^2(E) \rightarrow L^2(E)$  is bounded. This, together with  $f_k = u \in L^2(E)$  explains the following property:  $f_j \in L^2(E)$ , for all  $j \geq k$ . Additionally, observe that  $f_j = 0$  for all  $j \leq 0$  because  $f_0 = 0$ . Furthermore, we note that  $f_j \in C^\infty(E)$ , for all  $j \in \mathbb{Z}$ . The last assertion is obvious for  $j \leq k$ , and for  $j > k$  it can be seen by showing that  $H^j f_j = 0$  in distributional sense and using  $f_j \in L^2(E)$  together with local elliptic regularity. To see this, let  $v \in C_c^\infty(E)$  be arbitrary, and note that

$$(f_j, H^j v) = (H^{k-j} u, H^j v) = (u, H^k v) = (H^k u, v) = 0.$$

Finally, observe that

$$H^l f_j = f_{j-l}, \quad \text{for all } j \in \mathbb{Z} \text{ and } l \in \mathbb{Z}_+ \cup \{0\}. \quad (4.5)$$

With  $f_j$  as in (4.4), define the functions  $\alpha_j$  and  $\beta_j$  on the interval  $0 \leq T < \infty$  by the formulas

$$\alpha_j(T) := \lambda_0^2 \int_0^T (f_j, f_j)_{B_t} dt, \quad \beta_j(T) := \int_0^T (Df_j, Df_j)_{B_t} dt, \quad (4.6)$$

where  $\lambda_0$  is as in (4.1) and  $(\cdot, \cdot)_{B_t}$  is as in (3.2).

In the sequel, to simplify the notations, the functions  $\alpha_j(T)$  and  $\beta_j(T)$ , the inner products  $(\cdot, \cdot)_{B_t}$ , and the corresponding norms  $\|\cdot\|_{B_t}$  appearing in (4.6) will be denoted by  $\alpha_j$ ,  $\beta_j$ ,  $(\cdot, \cdot)_t$ , and  $\|\cdot\|_t$ , respectively.

Note that  $\alpha_j$  and  $\beta_j$  are absolutely continuous on  $[0, \infty)$ . Furthermore,  $\alpha_j$  and  $\beta_j$  have a left first derivative and a right first derivative at each point. Additionally,  $\alpha_j$  and  $\beta_j$  are differentiable, except at (at most) countably many points. In the sequel, to simplify notations, we shall denote the right first derivatives of  $\alpha_j$  and  $\beta_j$  by  $\alpha'_j$  and  $\beta'_j$ . Note that  $\alpha_j$ ,  $\beta_j$ ,  $\alpha'_j$  and  $\beta'_j$  are non-decreasing and non-negative functions. Note also that  $\alpha_j$  and  $\beta_j$  are convex functions. Furthermore, since  $f_j = 0$  for all  $j \leq 0$ , it follows that  $\alpha_j \equiv 0$  and  $\beta_j \equiv 0$  for all  $j \leq 0$ . Finally, using (4.1) and the property  $f_j \in L^2(E) \cap C^\infty(E)$  for all  $j \geq k$ , observe that

$$\lambda_0^2 (f_j, f_j) + (Df_j, Df_j) \leq (Vf_j, f_j) + (Df_j, Df_j) = (f_j, Hf_j) = (f_j, f_{j-1}) < \infty,$$

for all  $j > k$ . Here, “integration by parts” in the first equality is justified because  $H_{\min}$  is essentially self-adjoint (i.e.  $C_c^\infty(E)$  is an operator core of  $H_{\max}$ ). Hence,  $\alpha'_j$  and  $\beta'_j$  are bounded for all  $j > k$ . It turns out that  $\alpha_j$  and  $\beta_j$  satisfy a system of differential inequalities, as seen in the next proposition.

**Proposition 4.4.** *Let  $\alpha_j$  and  $\beta_j$  be as in (4.6). Then, for all  $j \geq 1$  and all  $T \geq 0$  we have*

$$\alpha_j + \beta_j \leq \sqrt{\alpha'_j \beta'_j} + \sum_{l=0}^{\infty} \left( \sqrt{\alpha'_{j+l+1} \beta'_{j-l-1}} + \sqrt{\alpha'_{j-l-1} \beta'_{j+l+1}} \right) \quad (4.7)$$

and

$$\alpha_j \leq \lambda_0^2 \left( \sum_{l=0}^{\infty} \left( \sqrt{\alpha'_{j+l+1} \beta'_{j-l-1}} + \sqrt{\alpha'_{j-l-1} \beta'_{j+l+1}} \right) \right), \quad (4.8)$$

where  $\lambda_0$  is as in (4.1) and  $\alpha'_i, \beta'_i$  denote the right-hand derivatives.

*Remark 4.5.* Note that the sums in (4.7) and (4.8) are finite since  $\alpha_i \equiv 0$  and  $\beta_i \equiv 0$  for  $i \leq 0$ . As our goal is to show that  $f_k = u = 0$ , we will only use the first  $k$  inequalities in (4.7) and the first  $k$  inequalities in (4.8).

**Proof of Proposition 4.4.** From (4.6) and (4.1) it follows that

$$\alpha_j + \beta_j \leq \int_0^T ((f_j, Vf_j)_t + (Df_j, Df_j)_t) dt. \quad (4.9)$$

We start from (4.9), use (3.3), Cauchy–Schwarz inequality, and (4.5) to obtain

$$\begin{aligned} \alpha_j + \beta_j &\leq \int_0^T ((f_j, Vf_j)_t + (Df_j, Df_j)_t) dt \\ &= \int_0^T |(f_j, Hf_j)_t - (f_j, D^*Df_j)_t + (Df_j, Df_j)_t| dt \\ &\leq \lambda_0 \int_{B_T} |Df_j(x)| |f_j(x)| d\mu(x) + \int_0^T |(f_j, Hf_j)_t| dt \\ &\leq \sqrt{\alpha'_j \beta'_j} + \int_0^T |(Hf_{j+1}, f_{j-1})_t| dt. \end{aligned}$$

We continue the process as follows:

$$\begin{aligned} \alpha_j + \beta_j &\leq \sqrt{\alpha'_j \beta'_j} + \int_0^T |(Hf_{j+1}, f_{j-1})_t| dt \\ &= \sqrt{\alpha'_j \beta'_j} + \int_0^T |(D^*Df_{j+1}, f_{j-1})_t + (f_{j+1}, Vf_{j-1})_t| dt \\ &\leq \sqrt{\alpha'_j \beta'_j} + \int_0^T |(D^*Df_{j+1}, f_{j-1})_t - (Df_{j+1}, Df_{j-1})_t| dt \\ &\quad + \int_0^T |(Df_{j+1}, Df_{j-1})_t - (f_{j+1}, D^*Df_{j-1})_t| dt + \int_0^T |(f_{j+1}, Hf_{j-1})_t| dt \\ &\leq \sqrt{\alpha'_j \beta'_j} + \sqrt{\alpha'_{j+1} \beta'_{j-1}} + \sqrt{\alpha'_{j-1} \beta'_{j+1}} + \int_0^T |(Hf_{j+2}, f_{j-2})_t| dt, \end{aligned}$$

where we used triangle inequality, (3.3), Cauchy–Schwarz inequality, and (4.5). We continue like this until the last term reaches the subscript  $j - l \leq 0$ , which makes the last term equal zero by properties of  $f_i$  discussed above. This establishes (4.7).

To show (4.8), we start from the definition of  $\alpha_j$ , use (3.3), Cauchy–Schwarz inequality, and (4.5) to obtain

$$\begin{aligned}
\alpha_j &= \lambda_0^2 \int_0^T (f_j, f_j)_t dt = \lambda_0^2 \int_0^T |(f_j, H f_{j+1})_t| dt \\
&= \lambda_0^2 \int_0^T |(f_j, D^* D f_{j+1})_t + (V f_j, f_{j+1})_t| dt \\
&\leq \lambda_0^2 \int_0^T |(f_j, D^* D f_{j+1})_t - (D f_j, D f_{j+1})_t| dt \\
&\quad + \lambda_0^2 \int_0^T |(D f_j, D f_{j+1})_t - (D^* D f_j, f_{j+1})_t| dt + \lambda_0^2 \int_0^T |(H f_j, f_{j+1})_t| dt \\
&\leq \lambda_0^2 \left( \sqrt{\alpha'_{j+1} \beta'_j} + \sqrt{\alpha'_j \beta'_{j+1}} \right) + \lambda_0^2 \int_0^T |(f_{j-1}, f_{j+1})_t| dt.
\end{aligned}$$

We continue like this until the last term reaches the subscript  $j - l \leq 0$ , which makes the last term equal zero by properties of  $f_i$  discussed above. This establishes (4.8).  $\square$

**End of the proof of Theorem 1.** We will now transform the system (4.7)–(4.8) by introducing new variables:

$$\omega_j(T) := \alpha_j(T) + \beta_j(T), \quad \theta_j(T) := \alpha_j(T) - \beta_j(T) \quad T \in [0, \infty). \quad (4.10)$$

To carry out the transformation, observe that Cauchy–Schwarz inequality applied to vectors  $\langle \sqrt{\alpha'_i}, \sqrt{\beta'_i} \rangle$  and  $\langle \sqrt{\beta'_p}, \sqrt{\alpha'_p} \rangle$  in  $\mathbb{R}^2$  gives

$$\sqrt{\alpha'_i \beta'_p} + \sqrt{\alpha'_p \beta'_i} \leq \sqrt{\omega'_i \omega'_p},$$

which, together with (4.7)–(4.8) leads to

$$\omega_j \leq \frac{1}{2} \sqrt{(\omega'_j)^2 - (\theta'_j)^2} + \sum_{l=0}^{\infty} \sqrt{\omega'_{j+l+1} \omega'_{j-l-1}} \quad (4.11)$$

and

$$\frac{1}{2}(\omega_j + \theta_j) \leq \lambda_0^2 \left( \sum_{l=0}^{\infty} \sqrt{\omega'_{j+l+1} \omega'_{j-l-1}} \right), \quad (4.12)$$

where  $\lambda_0$  is as in (4.1) and  $\omega'_i, \theta'_i$  denote the right-hand derivatives.

The functions  $\omega_j$  and  $\theta_j$  satisfy the following properties: (i)  $\omega_j$  and  $\theta_j$  are absolutely continuous on  $[0, \infty)$ , and the right-hand derivatives  $\omega'_j$  and  $\theta'_j$  exist everywhere; (ii)  $\omega_j$  and  $\omega'_j$  are non-negative and non-increasing; (iii)  $\omega_j$  is convex; (iv)  $\omega'_j$  is bounded for all  $j \geq k$ ; (v)  $\omega_j(0) = \theta_j(0) = 0$ ; and (vi)  $|\theta_j(T)| \leq \omega_j(T)$  and  $|\theta'_j(T)| \leq \omega'_j(T)$  for all  $T \in [0, \infty)$ .

In Section IV.3 of [12] it was shown that if  $\omega_j$  and  $\theta_j$  are functions satisfying the above described properties (i)–(vi) and the system (4.11)–(4.12), then  $\omega_j \equiv 0$  for all  $j = 1, 2, \dots, k$ . In particular, we have  $\omega_k(T) = 0$ , for all  $T \in [0, \infty)$ , and hence  $f_k = 0$ . Going back to (4.4), we get  $u = 0$ , and this concludes the proof of essential self-adjointness of  $H^k$  on  $C_c^\infty(E)$ . The essential self-adjointness of  $H^2, H^3, \dots$ , and  $H^{k-1}$  on  $C_c^\infty(E)$  follows by Proposition 4.2.  $\square$

## 5. PROOF OF THEOREM 2

We adapt the proof of Theorem 1.1 in [13] to our type of operator. By assumption (2.6) it follows that

$$((\Delta_{M,\mu} + q - C + 1)u, u) \geq \|u\|^2, \quad \text{for all } u \in C_c^\infty(M). \quad (5.1)$$

Since (5.1) is satisfied and since  $M$  is non-compact and  $g^{TM}$  is geodesically complete, a result of Agmon [1] (see also Proposition III.6.2 in [12]) guarantees the existence of a function  $\gamma \in C^\infty(M)$  such that  $\gamma(x) > 0$  for all  $x \in M$ , and

$$(\Delta_{M,\mu} + q - C + 1)\gamma = \gamma. \quad (5.2)$$

We now use the function  $\gamma$  to transform the operator  $H = \nabla^* \nabla + V$ . Let  $L_{\mu_1}^2(E)$  be the space of square integrable sections of  $E$  with inner product  $(\cdot, \cdot)_{\mu_1}$  as in (2.1), where  $d\mu$  is replaced by  $d\mu_1 := \gamma^2 d\mu$ . For clarity, we denote  $L^2(E)$  from Section 2.1 by  $L_\mu^2(E)$ . In what follows, the formal adjoints of  $\nabla$  with respect to inner products  $(\cdot, \cdot)_\mu$  and  $(\cdot, \cdot)_{\mu_1}$  will be denoted by  $\nabla^{*,\mu}$  and  $\nabla^{*,\mu_1}$ , respectively. It is easy to check that the map  $T_\gamma: L_\mu^2(E) \rightarrow L_{\mu_1}^2(E)$  defined by  $Tu := \gamma^{-1}u$  is unitary. Furthermore, under the change of variables  $u \mapsto \gamma^{-1}u$ , the differential expression  $H = \nabla^{*,\mu} \nabla + V$  gets transformed into  $H_1 := \gamma^{-1}H\gamma$ . Since  $T$  is unitary, the essential self-adjointness of  $H^k|_{C_c^\infty(E)}$  in  $L_\mu^2(E)$  is equivalent to essential self-adjointness of  $(H_1)^k|_{C_c^\infty(E)}$  in  $L_{\mu_1}^2(E)$ .

In the sequel, we will show that  $H_1$  has the following form:

$$H_1 = \nabla^{*,\mu_1} \nabla + \tilde{V}, \quad (5.3)$$

with

$$\tilde{V}(x) := \frac{\Delta_{M,\mu}\gamma}{\gamma} \text{Id}(x) + V(x).$$

To see this, let  $w, z \in C_c^\infty(E)$  and consider

$$\begin{aligned} (H_1 w, z)_{\mu_1} &= \int_M \langle \gamma^{-1} H(\gamma w), z \rangle \gamma^2 d\mu = \int_M \langle H(\gamma w), \gamma z \rangle d\mu = (H(\gamma w), \gamma z)_\mu \\ &= (\nabla(\gamma w), \nabla(\gamma z))_\mu + (V\gamma w, \gamma z)_\mu = (\gamma^2 \nabla w, \nabla z)_\mu + (d\gamma \otimes w, d\gamma \otimes z)_{L_\mu^2(T^*M \otimes E)} \\ &\quad + (\gamma \nabla w, d\gamma \otimes z)_{L_\mu^2(T^*M \otimes E)} + (d\gamma \otimes w, \gamma \nabla z)_{L_\mu^2(T^*M \otimes E)} + (V\gamma w, \gamma z)_\mu. \end{aligned} \quad (5.4)$$

Setting  $\xi := d(\gamma^2/2) \in T^*M$  and using equation (1.34) in Appendix C of [33] we have

$$(\gamma \nabla w, d\gamma \otimes z)_{L_\mu^2(T^*M \otimes E)} = (\nabla w, \xi \otimes z)_{L_\mu^2(T^*M \otimes E)} = (\nabla_X w, z)_\mu, \quad (5.5)$$

where  $X$  is the vector field associated with  $\xi \in T^*M$  via the metric  $g^{TM}$ .

Furthermore, by equation (1.35) in Appendix C of [33] we have

$$\begin{aligned} (d\gamma \otimes w, \gamma \nabla z)_{L_\mu^2(T^*M \otimes E)} &= (\xi \otimes w, \nabla z)_{L_\mu^2(T^*M \otimes E)} = (\nabla^{*,\mu}(\xi \otimes w), z)_\mu \\ &= -(\text{div}_\mu(X)w, z)_\mu - (\nabla_X w, z)_\mu, \end{aligned} \quad (5.6)$$

where, in local coordinates  $x^1, x^2, \dots, x^n$ , for  $X = X^j \frac{\partial}{\partial x^j}$ , with Einstein summation convention,

$$\text{div}_\mu(X) := \frac{1}{\kappa} \left( \frac{\partial}{\partial x^j} (\kappa X^j) \right).$$

(Recall that  $d\mu = \kappa(x) dx^1 dx^2 \dots dx^n$ , where  $\kappa(x)$  is a positive  $C^\infty$ -density.) Since  $X^j = (g^{TM})^{jl} \left( \gamma \frac{\partial \gamma}{\partial x^j} \right)$ , we have

$$\operatorname{div}_\mu(X) = |d\gamma|^2 - \gamma(\Delta_{M,\mu}\gamma), \quad (5.7)$$

where  $|d\gamma(x)|$  is the norm of  $d\gamma(x) \in T_x^*M$  induced by  $g^{TM}$ , and  $\Delta_{M,\mu}$  is as in (1.1) with metric  $g^{TM}$ . Combining (5.4)–(5.7) and noting that

$$(d\gamma \otimes w, d\gamma \otimes z)_{L_\mu^2(T^*M \otimes E)} = \int_M |d\gamma|^2 \langle w, z \rangle d\mu,$$

we obtain

$$\begin{aligned} (H_1 w, z)_{\mu_1} &= \int_M \langle \nabla w, \nabla z \rangle \gamma^2 d\mu + \int_M \langle Vw, z \rangle \gamma^2 d\mu + \int_M \gamma(\Delta_{M,\mu}\gamma) \langle w, z \rangle d\mu \\ &= (\nabla w, \nabla z)_{L_{\mu_1}^2(T^*M \otimes E)} + (Vw, z)_{\mu_1} + (\gamma^{-1}(\Delta_{M,\mu}\gamma)w, z)_{\mu_1} \\ &= (\nabla^{*,\mu_1} \nabla w, z)_{\mu_1} + (Vw, z)_{\mu_1} + (\gamma^{-1}(\Delta_{M,\mu}\gamma)w, z)_{\mu_1}, \end{aligned} \quad (5.8)$$

which shows (5.3).

By (2.5) and (5.2) it follows that

$$\tilde{V}(x) = \frac{\Delta_{M,\mu}\gamma}{\gamma} \operatorname{Id}(x) + V(x) \geq (C-1)\operatorname{Id}(x), \quad \text{for all } x \in M,$$

where  $C$  is as in (2.6). Thus, by Theorem 1 the operator  $(H_1)^k|_{C_c^\infty(E)}$  is essentially self-adjoint in  $L_{\mu_1}^2(E)$  for all  $k \in \mathbb{Z}_+$ .  $\square$

## 6. PROOF OF THEOREM 3

Throughout the section, we assume that the hypotheses of Theorem 3 are satisfied. In subsequent discussion, the notation  $\widehat{D}$  is as in (3.1) and the operators  $H_{\min}$  and  $H_{\max}$  are as in Section 4.1. We begin with the following lemma, whose proof is a direct consequence of the definition of  $H_{\max}$  and local elliptic regularity.

**Lemma 6.1.** *Under the assumption  $V \in L_{\operatorname{loc}}^\infty(\operatorname{End} E)$ , we have the following inclusion:  $\operatorname{Dom}(H_{\max}) \subset W_{\operatorname{loc}}^{2,2}(E)$ .*

The proof of the next lemma is given in Lemma 8.10 of [5].

**Lemma 6.2.** *For any  $u \in \operatorname{Dom}(H_{\max})$  and any Lipschitz function with compact support  $\psi: M \rightarrow \mathbb{R}$ , we have:*

$$(D(\psi u), D(\psi u)) + (V\psi u, \psi u) = \operatorname{Re}(\psi H u, \psi u) + \|\widehat{D}(d\psi)u\|^2. \quad (6.1)$$

**Corollary 6.3.** *Let  $H$  be as in (2.3), let  $u \in L^2(E)$  be a weak solution of  $Hu = 0$ , and let  $\psi: M \rightarrow \mathbb{R}$  be a Lipschitz function with compact support. Then*

$$(\psi u, H(\psi u)) = \|\widehat{D}(d\psi)u\|^2, \quad (6.2)$$

where  $(\cdot, \cdot)$  on the left-hand side denotes the duality between  $W_{\operatorname{loc}}^{1,2}(E)$  and  $W_{\operatorname{comp}}^{-1,2}(E)$ .

*Proof.* Since  $u \in L^2(E)$  and  $Hu = 0$ , we have  $u \in \text{Dom}(H_{\max}) \subset W_{\text{loc}}^{2,2}(E) \subset W_{\text{loc}}^{1,2}(E)$ , where the first inclusion follows by Lemma 6.1. Since  $\psi$  is a Lipschitz compactly supported function, we get  $\psi u \in W_{\text{comp}}^{1,2}(E)$  and, hence,  $H(\psi u) \in W_{\text{comp}}^{-1,2}(E)$ . Now the equality (6.2) follows from (6.1), the assumption  $Hu = 0$ , and

$$(\psi u, H(\psi u)) = (\psi u, D^* D(\psi u)) + (V\psi u, \psi u) = (D(\psi u), D(\psi u)) + (V\psi u, \psi u),$$

where in the second equality we used integration by parts; see Lemma 8.8 in [5]. Here, the two leftmost symbols  $(\cdot, \cdot)$  denote the duality between  $W_{\text{comp}}^{1,2}(E)$  and  $W_{\text{loc}}^{-1,2}(E)$ , while the remaining ones stand for  $L^2$ -inner products.  $\square$

The key ingredient in the proof of Theorem 3 is the Agmon-type estimate given in the next lemma, whose proof, inspired by an idea of [25], is based on the technique developed in [10] for magnetic Laplacians on an open set with compact boundary in  $\mathbb{R}^n$ .

**Lemma 6.4.** *Let  $\lambda \in \mathbb{R}$  and let  $v \in L^2(E)$  be a weak solution of  $(H - \lambda)v = 0$ . Assume that there exists a constant  $c_1 > 0$  such that, for all  $u \in W_{\text{comp}}^{1,2}(E)$ ,*

$$(u, (H - \lambda)u) \geq \lambda_0^2 \int_M \max\left(\frac{1}{r(x)^2}, 1\right) |u(x)|^2 d\mu(x) + c_1 \|u\|^2, \quad (6.3)$$

where  $r(x)$  is as in (2.7),  $\lambda_0$  is as in (2.2), the symbol  $(\cdot, \cdot)$  on the left-hand side denotes the duality between  $W_{\text{comp}}^{1,2}(E)$  and  $W_{\text{loc}}^{-1,2}(E)$ , and  $|\cdot|$  is the norm in the fiber  $E_x$ .

Then, the following equality holds:  $v = 0$ .

*Proof.* Let  $\rho$  and  $R$  be numbers satisfying  $0 < \rho < 1/2$  and  $1 < R < +\infty$ . For any  $\varepsilon > 0$ , we define the function  $f_\varepsilon: M \rightarrow \mathbb{R}$  by  $f_\varepsilon(x) = F_\varepsilon(r(x))$ , where  $r(x)$  is as in (2.7) and  $F_\varepsilon: [0, \infty) \rightarrow \mathbb{R}$  is the continuous piecewise affine function defined by

$$F_\varepsilon(s) = \begin{cases} 0 & \text{for } s \leq \varepsilon \\ \rho(s - \varepsilon)/(\rho - \varepsilon) & \text{for } \varepsilon \leq s \leq \rho \\ s & \text{for } \rho \leq s \leq 1 \\ 1 & \text{for } 1 \leq s \leq R \\ R + 1 - s & \text{for } R \leq s \leq R + 1 \\ 0 & \text{for } s \geq R + 1. \end{cases}$$

Let us fix  $x_0 \in M$ . For any  $\alpha > 0$ , we define the function  $p_\alpha: M \rightarrow \mathbb{R}$  by

$$p_\alpha(x) = P_\alpha(d_{g_{TM}}(x_0, x)),$$

where  $P_\alpha: [0, \infty) \rightarrow \mathbb{R}$  is the continuous piecewise affine function defined by

$$P_\alpha(s) = \begin{cases} 1 & \text{for } s \leq 1/\alpha \\ -\alpha s + 2 & \text{for } 1/\alpha \leq s \leq 2/\alpha \\ 0 & \text{for } s \geq 2/\alpha. \end{cases}$$

Since  $\widehat{d}_{g_{TM}}(x_0, x) \leq d_{g_{TM}}(x_0, x)$ , it follows that the support of  $f_\varepsilon p_\alpha$  is contained in the set  $B_\alpha := \{x \in M: \widehat{d}_{g_{TM}}(x_0, x) \leq 2/\alpha\}$ . By assumption (A1) we know that  $\widehat{M}$  is a geodesically complete Riemannian manifold. Hence, by Hopf–Rinow Theorem the set  $B_\alpha$  is compact. Therefore, the

support of  $f_\varepsilon p_\alpha$  is compact. Additionally, note that  $f_\varepsilon p_\alpha$  is a  $\beta$ -Lipschitz function (with respect to the distance corresponding to the metric  $g^{TM}$ ) with  $\beta = \frac{\rho}{\rho - \varepsilon} + \alpha$ .

Since  $v \in L^2(E)$  and  $(H - \lambda)v = 0$ , we have  $v \in \text{Dom}(H_{\max}) \subset W_{\text{loc}}^{2,2}(E) \subset W_{\text{loc}}^{1,2}(E)$ , where the first inclusion follows by Lemma 6.1. Since  $f_\varepsilon p_\alpha$  is a Lipschitz compactly supported function, we get  $f_\varepsilon p_\alpha v \in W_{\text{comp}}^{1,2}(E)$  and, hence,  $((H - \lambda)(f_\varepsilon p_\alpha v)) \in W_{\text{comp}}^{-1,2}(E)$ .

Using (2.2) we have

$$\|\widehat{D}(d(f_\varepsilon p_\alpha))v\|^2 \leq \lambda_0^2 \int_M |d(f_\varepsilon p_\alpha)(x)|^2 |v(x)|^2 d\mu(x), \quad (6.4)$$

where  $|d(f_\varepsilon p_\alpha)(x)|$  is the norm of  $d(f_\varepsilon p_\alpha)(x) \in T_x^*M$  induced by  $g^{TM}$ .

By Corollary 6.3 with  $H - \lambda$  in place of  $H$  and the inequality (6.4), we get

$$(f_\varepsilon p_\alpha v, (H - \lambda)(f_\varepsilon p_\alpha v)) \leq \lambda_0^2 \left( \frac{\rho}{\rho - \varepsilon} + \alpha \right)^2 \|v\|^2. \quad (6.5)$$

On the other hand, using the definitions of  $f_\varepsilon$  and  $p_\alpha$  and the assumption (6.3) we have

$$(f_\varepsilon p_\alpha v, (H - \lambda)(f_\varepsilon p_\alpha v)) \geq \lambda_0^2 \int_{S_{\rho,R,\alpha}} |v(x)|^2 d\mu(x) + c_1 \|f_\varepsilon p_\alpha v\|^2, \quad (6.6)$$

where

$$S_{\rho,R,\alpha} := \{x \in M : \rho \leq r(x) \leq R \text{ and } d_{g^{TM}}(x_0, x) \leq 1/\alpha\}.$$

In (6.6) and (6.5), the symbol  $(\cdot, \cdot)$  stands for the duality between  $W_{\text{comp}}^{1,2}(E)$  and  $W_{\text{loc}}^{-1,2}(E)$ . We now combine (6.6) and (6.5) to get

$$\lambda_0^2 \int_{S_{\rho,R,\alpha}} |v(x)|^2 d\mu(x) + c_1 \|f_\varepsilon p_\alpha v\|^2 \leq \lambda_0^2 \left( \frac{\rho}{\rho - \varepsilon} + \alpha \right)^2 \|v\|^2.$$

We fix  $\rho$ ,  $R$ , and  $\varepsilon$ , and let  $\alpha \rightarrow 0+$ . After that we let  $\varepsilon \rightarrow 0+$ . The last step is to do  $\rho \rightarrow 0+$  and  $R \rightarrow +\infty$ . As a result, we get  $v = 0$ .  $\square$

**End of the proof of Theorem 3.** Using integration by parts (see Lemma 8.8 in [5]), we have

$$(u, Hu) = (u, D^*Du) + (Vu, u) = (Du, Du) + (Vu, u) \geq (Vu, u), \quad \text{for all } u \in W_{\text{comp}}^{1,2}(E),$$

where the two leftmost symbols  $(\cdot, \cdot)$  denote the duality between  $W_{\text{comp}}^{1,2}(E)$  and  $W_{\text{loc}}^{-1,2}(E)$ , while the remaining ones stand for  $L^2$ -inner products. Hence, by assumption (2.8) we get:

$$\begin{aligned} (u, (H - \lambda)u) &\geq \lambda_0^2 \int_M \frac{1}{r(x)^2} |u(x)|^2 d\mu(x) - (\lambda + C) \|u\|^2 \\ &\geq \lambda_0^2 \int_M \max\left(\frac{1}{r(x)^2}, 1\right) |u(x)|^2 d\mu(x) - (\lambda + C + 1) \|u\|^2. \end{aligned} \quad (6.7)$$

Choosing, for instance,  $\lambda = -C - 2$  in (6.7) we get the inequality (6.3) with  $c_1 = 1$ .

Thus,  $H_{\min} - \lambda$  with  $\lambda = -C - 2$  is a symmetric operator satisfying  $(u, (H_{\min} - \lambda)u) \geq \|u\|^2$ , for all  $u \in C_c^\infty(E)$ . In this case, it is known (see Theorem X.26 in [29]) that the essential self-adjointness of  $H_{\min} - \lambda$  is equivalent to the following statement: if  $v \in L^2(E)$  satisfies



$(H - \lambda)v = 0$ , then  $v = 0$ . Thus, by Lemma 6.4, the operator  $(H_{\min} - \lambda)$  is essentially self-adjoint. Hence,  $H_{\min}$  is essentially self-adjoint.  $\square$

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DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF NORTH FLORIDA, JACKSONVILLE, FL 32224, USA.

*E-mail address:* `omilatov@unf.edu`

GRENOBLE UNIVERSITY, INSTITUT FOURIER, UNITÉ MIXTE DE RECHERCHE CNRS-UJF 5582, BP 74, 38402-SAINT MARTIN D'HÈRES CEDEX, FRANCE.

*E-mail address:* `francoise.truc@ujf-grenoble.fr`