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Ognjen Milatovic, Francoise Truc

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SELF-ADJOINT EXTENSIONS OF DIFFERENTIAL OPERATORS ON RIEMANNIAN MANIFOLDS

OGNJEN MILATOVIC, FRANÇOISE TRUC

ABSTRACT. We study $H = D^*D + V$, where D is a first order elliptic differential operator acting on sections of a Hermitian vector bundle over a Riemannian manifold M , and V is a Hermitian bundle endomorphism. In the case when M is geodesically complete, we establish the essential self-adjointness of positive integer powers of H . In the case when M is not necessarily geodesically complete, we give a sufficient condition for the essential self-adjointness of H , expressed in terms of the behavior of V relative to the Cauchy boundary of M .

1. INTRODUCTION

As a fundamental problem in mathematical physics, self-adjointness of Schrödinger operators has attracted the attention of researchers over many years now, resulting in numerous sufficient conditions for this property in $L^2(\mathbb{R}^n)$. For reviews of the corresponding results, see, for instance, the books [14, 29].

The study of the corresponding problem in the context of a non-compact Riemannian manifold was initiated by Gaffney [15, 16] with the proof of the essential self-adjointness of the Laplacian on differential forms. About two decades later, Cordes (see Theorem 3 in [11]) proved the essential self-adjointness of positive integer powers of the operator

$$\Delta_{M,\mu} := -\frac{1}{\kappa} \left(\frac{\partial}{\partial x^i} \left(\kappa g^{ij} \frac{\partial}{\partial x^j} \right) \right) \quad (1.1)$$

on an n -dimensional geodesically complete Riemannian manifold M equipped with a (smooth) metric $g = (g_{ij})$ (here, $(g^{ij}) = ((g_{ij})^{-1})$) and a positive smooth measure $d\mu$ (i.e. in any local coordinates x^1, x^2, \dots, x^n there exists a strictly positive C^∞ -density $\kappa(x)$ such that $d\mu = \kappa(x) dx^1 dx^2 \dots dx^n$). Theorem 1 of our paper extends this result to the operator $(D^*D + V)^k$ for all $k \in \mathbb{Z}_+$, where D is a first order elliptic differential operator acting on sections of a Hermitian vector bundle over a geodesically complete Riemannian manifold, D^* is the formal adjoint of D , and V is a self-adjoint Hermitian bundle endomorphism; see Section 2.3 for details.

In the context of a general Riemannian manifold (not necessarily geodesically complete), Cordes (see Theorem IV.1.1 in [12] and Theorem 4 in [11]) proved the essential self-adjointness of P^k for all $k \in \mathbb{Z}_+$, where

$$Pu := \Delta_{M,\mu}u + qu, \quad u \in C^\infty(M), \quad (1.2)$$

and $q \in C^\infty(M)$ is real-valued. Thanks to a Roelcke-type estimate (see Lemma 3.1 below), the technique of Cordes [12] can be applied to the operator $(D^*D + V)^k$ acting on sections of Hermitian vector bundles over a general Riemannian manifold. To make our exposition shorter, in Theorem 1 we consider the geodesically complete case. Our Theorem 2 concerns $(\nabla^*\nabla + V)^k$, where ∇ is a metric connection on a Hermitian vector bundle over a non-compact geodesically complete Riemannian manifold. This result extends Theorem 1.1 of [13] where Cordes showed that if (M, g) is non-compact and geodesically complete and P is semi-bounded from below on $C_c^\infty(M)$, then P^k is essentially self-adjoint on $C_c^\infty(M)$, for all $k \in \mathbb{Z}_+$.

For the remainder of the introduction, the notation $D^*D + V$ is used in the same sense as described earlier in this section. In the setting of geodesically complete Riemannian manifolds, the essential self-adjointness of $D^*D + V$ with $V \in L_{\text{loc}}^\infty$ was established in [21], providing a generalization of the results in [3, 27, 28, 32] concerning Schrödinger operators on functions (or differential forms). Subsequently, the operator $D^*D + V$ with a singular potential V was considered in [5]. Recently, in the case $V \in L_{\text{loc}}^\infty$, the authors of [4] extended the main result of [5] to the operator $D^*D + V$ acting on sections of infinite-dimensional bundles whose fibers are modules of finite type over a von Neumann algebra.

In the context of an incomplete Riemannian manifold, the authors of [17, 22, 23] studied the so-called Gaffney Laplacian, a self-adjoint realization of the scalar Laplacian generally different from the closure of $\Delta_{M, d\mu}|_{C_c^\infty(M)}$. For a study of Gaffney Laplacian on differential forms, see [24].

Our Theorem 3 gives a condition on the behavior of V relative to the Cauchy boundary of M that will guarantee the essential self-adjointness of $D^*D + V$; for details see Section 2.4 below. Related results can be found in [6, 25, 26] in the context of (magnetic) Schrödinger operators on domains in \mathbb{R}^n , and in [10] concerning the magnetic Laplacian on domains in \mathbb{R}^n and certain types of Riemannian manifolds.

Finally, let us mention that Chernoff [7] used the hyperbolic equation approach to establish the essential self-adjointness of positive integer powers of Laplace–Beltrami operator on differential forms. This approach was also applied in [2, 8, 9, 18, 19, 31] to prove essential self-adjointness of second-order operators (acting on scalar functions or sections of Hermitian vector bundles) on Riemannian manifolds. Additionally, the authors of [18, 19] used path integral techniques.

The paper is organized as follows. The main results are stated in Section 2, a preliminary lemma is proven in Section 3, and the main results are proven in Sections 4–6.

2. MAIN RESULTS

2.1. The setting. Let M be an n -dimensional smooth, connected Riemannian manifold without boundary. We denote the Riemannian metric on M by g^{TM} . We assume that M is equipped with a positive smooth measure $d\mu$, i.e. in any local coordinates x^1, x^2, \dots, x^n there exists a strictly positive C^∞ -density $\kappa(x)$ such that $d\mu = \kappa(x) dx^1 dx^2 \dots dx^n$. Let E be a Hermitian vector bundle over M and let $L^2(E)$ denote the Hilbert space of square integrable sections of E with respect to the inner product

$$(u, v) = \int_M \langle u(x), v(x) \rangle_{E_x} d\mu(x), \quad (2.1)$$

where $\langle \cdot, \cdot \rangle_{E_x}$ is the fiberwise inner product. The corresponding norm in $L^2(E)$ is denoted by $\| \cdot \|$. In Sobolev space notations $W_{\text{loc}}^{k,2}(E)$ used in this paper, the superscript $k \in \mathbb{Z}_+$ indicates the order of the highest derivative. The corresponding dual space is denoted by $W_{\text{loc}}^{-k,2}(E)$.

Let F be another Hermitian vector bundle on M . We consider a first order differential operator $D: C_c^\infty(E) \rightarrow C_c^\infty(F)$, where C_c^∞ stands for the space of smooth compactly supported sections. In the sequel, by $\sigma(D)$ we denote the principal symbol of D .

Assumption (A0) Assume that D is elliptic. Additionally, assume that there exists a constant $\lambda_0 > 0$ such that

$$|\sigma(D)(x, \xi)| \leq \lambda_0 |\xi|, \quad \text{for all } x \in M, \xi \in T_x^*M, \quad (2.2)$$

where $|\xi|$ is the length of ξ induced by the metric g^{TM} and $|\sigma(D)(x, \xi)|$ is the operator norm of $\sigma(D)(x, \xi): E_x \rightarrow F_x$.

Remark 2.2. Assumption (A0) is satisfied if $D = \nabla$, where $\nabla: C^\infty(E) \rightarrow C^\infty(T^*M \otimes E)$ is a covariant derivative corresponding to a metric connection on a Hermitian vector bundle E over M .

2.3. Schrödinger-type Operator. Let $D^*: C_c^\infty(F) \rightarrow C_c^\infty(E)$ be the formal adjoint of D with respect to the inner product (2.1). We consider the operator

$$H = D^*D + V, \quad (2.3)$$

where $V \in L_{\text{loc}}^\infty(\text{End } E)$ is a linear self-adjoint bundle endomorphism. In other words, for all $x \in M$, the operator $V(x): E_x \rightarrow E_x$ is self-adjoint and $|V(x)| \in L_{\text{loc}}^\infty(M)$, where $|V(x)|$ is the norm of the operator $V(x): E_x \rightarrow E_x$.

2.4. Statements of Results.

Theorem 1. *Let M , g^{TM} , and $d\mu$ be as in Section 2.1. Assume that (M, g^{TM}) is geodesically complete. Let E and F be Hermitian vector bundles over M , and let $D: C_c^\infty(E) \rightarrow C_c^\infty(F)$ be a first order differential operator satisfying the assumption (A0). Assume that $V \in C^\infty(\text{End } E)$ and*

$$V(x) \geq C, \quad \text{for all } x \in M,$$

where C is a constant, and the inequality is understood in operator sense. Then H^k is essentially self-adjoint on $C_c^\infty(E)$, for all $k \in \mathbb{Z}_+$.

Remark 2.5. In the case $V = 0$, the following result related to Theorem 1 can be deduced from Chernoff (see Theorem 2.2 in [7]):

Assume that (M, g) is a geodesically complete Riemannian manifold with metric g . Let D be as in Theorem 1, and define

$$c(x) := \sup\{|\sigma(D)(x, \xi)|: |\xi|_{T_x^*M} = 1\}.$$

Fix $x_0 \in M$ and define

$$c(r) := \sup_{x \in B(x_0, r)} c(x),$$

where $r > 0$ and $B(x_0, r) := \{x \in M : d_g(x_0, x) < r\}$. Assume that

$$\int_0^\infty \frac{1}{c(r)} dr = \infty. \quad (2.4)$$

Then the operator $(D^*D)^k$ is essentially self-adjoint on $C_c^\infty(E)$ for all $k \in \mathbb{Z}_+$.

At the end of this section we give an example of an operator for which Theorem 1 guarantees the essential self-adjointness of $(D^*D)^k$, whereas Chernoff's result cannot be applied.

The next theorem is concerned with operators whose potential V is not necessarily semi-bounded from below.

Theorem 2. Let M , g^{TM} , and $d\mu$ be as in Section 2.1. Assume that (M, g^{TM}) is noncompact and geodesically complete. Let E be a Hermitian vector bundle over M and let ∇ be a Hermitian connection on E . Assume that $V \in C^\infty(\text{End } E)$ and

$$V(x) \geq q(x), \quad \text{for all } x \in M, \quad (2.5)$$

where $q \in C^\infty(M)$ and the inequality is understood in the sense of operators $E_x \rightarrow E_x$. Additionally, assume that

$$((\Delta_{M,\mu} + q)u, u) \geq C\|u\|^2, \quad \text{for all } u \in C_c^\infty(M), \quad (2.6)$$

where $C \in \mathbb{R}$ and $\Delta_{M,\mu}$ is as in (1.1) with g replaced by g^{TM} . Then the operator $(\nabla^*\nabla + V)^k$ is essentially self-adjoint on $C_c^\infty(E)$, for all $k \in \mathbb{Z}_+$.

Remark 2.6. Let us stress that non-compactness is required in the proof to ensure the existence of a positive smooth solution of an equation involving $\Delta_{M,\mu} + q$. In the case of a compact manifold, such a solution exists under an additional assumption; see Theorem III.6.3 in [12].

In our last result we will need the notion of Cauchy boundary. Let $d_{g^{TM}}$ be the distance function corresponding to the metric g^{TM} . Let $(\widehat{M}, \widehat{d}_{g^{TM}})$ be the metric completion of $(M, d_{g^{TM}})$. We define the *Cauchy boundary* $\partial_C M$ as follows: $\partial_C M := \widehat{M} \setminus M$. Note that $(M, d_{g^{TM}})$ is metrically complete if and only if $\partial_C M$ is empty. For $x \in M$ we define

$$r(x) := \inf_{z \in \partial_C M} \widehat{d}_{g^{TM}}(x, z). \quad (2.7)$$

We will also need the following assumption:

Assumption (A1) Assume that \widehat{M} is a smooth manifold and that the metric g^{TM} extends to $\partial_C M$.

Remark 2.7. Let N be a (smooth) n -dimensional Riemannian manifold without boundary. Denote the metric on N by g^{TN} and assume that (N, g^{TN}) is geodesically complete. Let Σ be a k -dimensional closed sub-manifold of N with $k < n$. Then $M := N \setminus \Sigma$ has the properties $\widehat{M} = N$ and $\partial_C M = \Sigma$. Thus, assumption (A1) is satisfied.

Theorem 3. Let M , g^{TM} , and $d\mu$ be as in Section 2.1. Assume that (A1) is satisfied. Let E and F be Hermitian vector bundles over M , and let $D: C_c^\infty(E) \rightarrow C_c^\infty(F)$ be a first order differential operator satisfying the assumption (A0). Assume that $V \in L_{\text{loc}}^\infty(\text{End } E)$ and there exists a constant C such that

$$V(x) \geq \left(\frac{\lambda_0}{r(x)} \right)^2 - C, \quad \text{for all } x \in M, \quad (2.8)$$

where λ_0 is as in (2.2), the distance $r(x)$ is as in (2.7), and the inequality is understood in the sense of linear operators $E_x \rightarrow E_x$. Then H is essentially self-adjoint on $C_c^\infty(E)$.

In order to describe the example mentioned in Remark 2.5, we need the following

Remark 2.8. As explained in [5], we can use a first-order elliptic operator $D: C_c^\infty(E) \rightarrow C_c^\infty(F)$ to define a metric on M . For $\xi, \eta \in T_x^*M$, define

$$\langle \xi, \eta \rangle = \frac{1}{m} \text{Re Tr}((\sigma(D)(x, \xi))^* \sigma(D)(x, \eta)), \quad m = \dim E_x, \quad (2.9)$$

where Tr denotes the usual trace of a linear operator. Since D is an elliptic first-order differential operator and $\sigma(D)(x, \xi)$ is linear in ξ , it is easily checked that (2.9) defines an inner product on T_x^*M . Its dual defines a Riemannian metric on M . Denoting this metric by g^{TM} and using elementary linear algebra, it follows that (2.2) is satisfied with $\lambda_0 = \sqrt{m}$.

Example 2.9. Let $M = \mathbb{R}^2$ with the standard metric and measure, and $V = 0$. Denoting respectively by $C_c^\infty(\mathbb{R}^2; \mathbb{R})$ and $C_c^\infty(\mathbb{R}^2; \mathbb{R}^2)$ the spaces of smooth compactly supported functions $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ and $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$, we define the operator $D: C_c^\infty(\mathbb{R}^2; \mathbb{R}) \rightarrow C_c^\infty(\mathbb{R}^2; \mathbb{R}^2)$ by

$$D = \begin{pmatrix} a(x, y) \frac{\partial}{\partial x} \\ b(x, y) \frac{\partial}{\partial y} \end{pmatrix},$$

where

$$\begin{aligned} a(x, y) &= (1 - \cos(2\pi e^x))x^2 + 1; \\ b(x, y) &= (1 - \sin(2\pi e^y))y^2 + 1. \end{aligned}$$

Since a, b are smooth real-valued nowhere vanishing functions in \mathbb{R}^2 , it follows that the operator D is elliptic. We are interested in the operator

$$H := D^*D = -\frac{\partial}{\partial x} \left(a^2 \frac{\partial}{\partial x} \right) - \frac{\partial}{\partial y} \left(b^2 \frac{\partial}{\partial y} \right).$$

The matrix of the inner product on T^*M defined by D via (2.9) is $\text{diag}(a^2/2, b^2/2)$. The matrix of the corresponding Riemannian metric g^{TM} on M is $\text{diag}(2a^{-2}, 2b^{-2})$, so the metric itself is $ds^2 = 2a^{-2}dx^2 + 2b^{-2}dy^2$ and it is geodesically complete (see Example 3.1 of [5]). Moreover, thanks to Remark 2.8, assumption (A0) is satisfied. Thus, by Theorem 1 the operator $(D^*D)^k$ is essentially self-adjoint for all $k \in \mathbb{Z}_+$. Furthermore, in Example 3.1 of [5] it was shown that for the considered operator D the condition (2.4) is not satisfied. Thus, the result stated in Remark 2.5 does not apply.

3. ROELCKE-TYPE INEQUALITY

Let M , $d\mu$, D , and $\sigma(D)$ be as in Section 2.1. Set $\widehat{D} := -i\sigma(D)$, where $i = \sqrt{-1}$. Then for any Lipschitz function $\psi: M \rightarrow \mathbb{R}$ and $u \in W_{\text{loc}}^{1,2}(E)$ we have

$$D(\psi u) = \widehat{D}(d\psi)u + \psi Du, \quad (3.1)$$

where we have suppressed x for simplicity. We also note that $\widehat{D}^*(\xi) = -(\widehat{D}(\xi))^*$, for all $\xi \in T_x^*M$.

For a compact set $K \subset M$, and $u, v \in W_{\text{loc}}^{1,2}(E)$, we define

$$(u, v)_K := \int_K \langle u(x), v(x) \rangle d\mu(x), \quad (Du, Dv)_K := \int_K \langle Du(x), Dv(x) \rangle d\mu(x). \quad (3.2)$$

In order to prove Theorem 1 we need the following important lemma, which is an extension of Lemma 2.1 in [12] to operator (2.3). In the context of the scalar Laplacian on a Riemannian manifold, this kind of result is originally due to Roelcke [30].

Lemma 3.1. *Let M , g^{TM} , and $d\mu$ be as in Section 2.1. Let E and F be Hermitian vector bundles over M , and let $D: C_c^\infty(E) \rightarrow C_c^\infty(F)$ be a first order differential operator satisfying the assumption (A0). Let $\rho: M \rightarrow [0, \infty)$ be a function satisfying the following properties:*

- (i) $\rho(x)$ is Lipschitz continuous with respect to the distance induced by the metric g^{TM} ;
- (ii) $\rho(x_0) = 0$, for some fixed $x_0 \in M$;
- (iii) the set $B_T := \{x \in M: \rho(x) \leq T\}$ is compact, for some $T > 0$.

Then the following inequality holds for all $u \in W_{\text{loc}}^{2,2}(E)$ and $v \in W_{\text{loc}}^{2,2}(E)$:

$$\int_0^T |(Du, Dv)_{B_t} - (D^*Du, v)_{B_t}| dt \leq \lambda_0 \int_{B_T} |d\rho(x)| |Du(x)| |v(x)| d\mu(x), \quad (3.3)$$

where B_t is as in (iii) (with t instead of T), the constant λ_0 is as in (2.2), and $|d\rho(x)|$ is the length of $d\rho(x) \in T_x^*M$ induced by g^{TM} .

Proof. For $\varepsilon > 0$ and $t \in (0, T)$, we define a continuous piecewise linear function $F_{\varepsilon,t}$ as follows:

$$F_{\varepsilon,t}(s) = \begin{cases} 1 & \text{for } s < t - \varepsilon \\ (t - s)/\varepsilon & \text{for } t - \varepsilon \leq s < t \\ 0 & \text{for } s \geq t \end{cases}$$

The function $f_{\varepsilon,t}(x) := F_{\varepsilon,t}(\rho(x))$, is Lipschitz continuous with respect to the distance induced by the metric g^{TM} , and $d(f_{\varepsilon,t}(\rho(x))) = (F'_{\varepsilon,t}(\rho(x)))d\rho(x)$. Moreover we have $f_{\varepsilon,t}v \in W_{\text{loc}}^{1,2}(E)$ for all $v \in W_{\text{loc}}^{1,2}(E)$, since

$$D(f_{\varepsilon,t}v) = \widehat{D}(df_{\varepsilon,t})v + f_{\varepsilon,t}Dv.$$

It follows from the compactness of B_T that B_t is compact for all $t \in (0, T)$. Using integration by parts (see Lemma 8.8 in [5]), for all $u \in W_{\text{loc}}^{2,2}(E)$ and $v \in W_{\text{loc}}^{2,2}(E)$ we have

$$(D^*Du, v f_{\varepsilon,t})_{B_t} = (Du, D(v f_{\varepsilon,t}))_{B_t} = (Du, f_{\varepsilon,t}Dv)_{B_t} + (Du, \widehat{D}(df_{\varepsilon,t})v)_{B_t},$$

which, together with (2.2), gives

$$\begin{aligned}
& |(Du, f_{\varepsilon,t}Dv)_{B_t} - (D^*Du, v f_{\varepsilon,t})_{B_t}| = |(Du, \widehat{D}(df_{\varepsilon,t})v)_{B_t}| \\
& \leq \int_{B_t} |Du(x)| |\widehat{D}(df_{\varepsilon,t}(x))v(x)| d\mu(x) \leq \lambda_0 \int_{B_t} |Du(x)| |df_{\varepsilon,t}(x)| |v(x)| d\mu(x) \\
& = \lambda_0 \int_{B_t} |Du(x)| |F'_{\varepsilon,t}(\rho(x))| |d\rho(x)| |v(x)| d\mu(x) \\
& \leq \lambda_0 \int_{B_T} |Du(x)| |F'_{\varepsilon,t}(\rho(x))| |d\rho(x)| |v(x)| d\mu(x), \tag{3.4}
\end{aligned}$$

where $|df_{\varepsilon,t}(x)|$ and $|d\rho(x)|$ are the norms of $df_{\varepsilon,t}(x) \in T_x^*M$ and $d\rho(x) \in T_x^*M$ induced by g^{TM} .

Fixing $\varepsilon > 0$, integrating the leftmost and the rightmost side of (3.4) from $t = 0$ to $t = T$, and noting that $F'_{\varepsilon,t}(\rho(x))$ is the only term on the rightmost side depending on t , we obtain

$$\begin{aligned}
& \int_0^T |(Du, f_{\varepsilon,t}Dv)_{B_t} - (D^*Du, v f_{\varepsilon,t})_{B_t}| dt \\
& \leq \lambda_0 \int_{B_T} |Du(x)| |d\rho(x)| |v(x)| I_\varepsilon(x) d\mu(x), \tag{3.5}
\end{aligned}$$

where

$$I_\varepsilon(x) := \int_0^T |F'_{\varepsilon,t}(\rho(x))| dt.$$

We now let $\varepsilon \rightarrow 0+$ in (3.5). On the left-hand side of (3.5), as $\varepsilon \rightarrow 0+$, we have $f_{\varepsilon,t}(x) \rightarrow \chi_{B_t}(x)$ almost everywhere, where $\chi_{B_t}(x)$ is the characteristic function of the set B_t . Additionally, $|f_{\varepsilon,t}(x)| \leq 1$ for all $x \in B_t$ and all $t \in (0, T)$; thus, by dominated convergence theorem, as $\varepsilon \rightarrow 0+$ the left-hand side of (3.5) converges to the left-hand side of (3.3). On the right-hand side of (3.5) an easy calculation shows that $I_\varepsilon(x) \rightarrow 1$, as $\varepsilon \rightarrow 0+$. Additionally, we have $|I_\varepsilon(x)| \leq 1$, a.e. on B_T ; hence, by the dominated convergence theorem, as $\varepsilon \rightarrow 0+$ the right-hand side of (3.5) converges to the right-hand side of (3.3). This establishes the inequality (3.3). \square

4. PROOF OF THEOREM 1

We first give the definitions of minimal and maximal operators associated with the expression H in (2.3).

4.1. Minimal and Maximal Operators. We define $H_{\min}u := Hu$, with $\text{Dom}(H_{\min}) := C_c^\infty(E)$, and $H_{\max} := (H_{\min})^*$, where T^* denotes the adjoint of operator T . Denoting $\mathcal{D}_{\max} := \{u \in L^2(E) : Hu \in L^2(E)\}$, we recall the following well-known property: $\text{Dom}(H_{\max}) = \mathcal{D}_{\max}$ and $H_{\max}u = Hu$ for all $u \in \mathcal{D}_{\max}$.

From now on, throughout this section, we assume that the hypotheses of Theorem 1 are satisfied. Let $x_0 \in M$, and define $\rho(x) := d_{g^{TM}}(x_0, x)$, where $d_{g^{TM}}$ is the distance function corresponding to the metric g^{TM} . By the definition of $\rho(x)$ and the geodesic completeness of (M, g^{TM}) , it follows that $\rho(x)$ satisfies all hypotheses of Lemma 3.1. Using Lemma 3.1 and Proposition 4.2 below, we are able to apply the method of Cordes [11, 12] to our context. As we

will see, Cordes's technique reduces our problem to a system of ordinary differential inequalities of the same type as in Section IV.3 of [12].

Proposition 4.2. *Let A be a densely defined operator with domain \mathcal{D} in a Hilbert space \mathcal{H} . Assume that A is semi-bounded from below, that $A\mathcal{D} \subseteq \mathcal{D}$, and that there exists $c_0 \in \mathbb{R}$ such that the following two properties hold:*

- (i) $((A + c_0 I)u, u)_{\mathcal{H}} \geq \|u\|_{\mathcal{H}}^2$, for all $u \in \mathcal{D}$, where I denotes the identity operator in \mathcal{H} ;
- (ii) $(A + c_0 I)^k$ is essentially self-adjoint on \mathcal{D} , for some $k \in \mathbb{Z}_+$.

Then, $(A + cI)^j$ is essentially self-adjoint on \mathcal{D} , for all $j = 1, 2, \dots, k$ and all $c \in \mathbb{R}$.

Remark 4.3. To prove Proposition 4.2, one may mimick the proof of Proposition 1.4 in [12], which was carried out for the operator P defined in (1.2) with $\mathcal{D} = C_c^\infty(M)$, since only abstract functional analysis facts and the property $P\mathcal{D} \subseteq \mathcal{D}$ were used.

We start the proof of Theorem 1 by noticing that the operator H_{\min} is essentially self-adjoint on $C_c^\infty(E)$; see Corollary 2.9 in [5]. Thanks to Proposition 4.2, without any loss of generality we can change $V(x)$ to $V(x) + C \text{Id}(x)$, where C is a sufficiently large constant in order to have

$$V(x) \geq (\lambda_0^2 + 1)\text{Id}(x), \quad \text{for all } x \in M, \quad (4.1)$$

where λ_0 is as in (2.2) and $\text{Id}(x)$ is the identity endomorphism of E_x . Using non-negativity of D^*D and (4.1) we have

$$(H_{\min}u, u) \geq \|u\|^2, \quad \text{for all } u \in C_c^\infty(E), \quad (4.2)$$

which leads to

$$\|u\|^2 \leq (Hu, u) \leq \|Hu\|\|u\|, \quad \text{for all } u \in C_c^\infty(E),$$

and, hence, $\|Hu\| \geq \|u\|$, for all $u \in C_c^\infty(E)$. Therefore,

$$(H^2u, u) = (Hu, Hu) = \|Hu\|^2 \geq \|u\|^2, \quad \text{for all } u \in C_c^\infty(E), \quad (4.3)$$

and

$$(H^3u, u) = (HHu, Hu) \geq \|Hu\|^2 \geq \|u\|^2, \quad \text{for all } u \in C_c^\infty(E).$$

By (4.3) we have

$$\|u\|^2 \leq (H^2u, u) \leq \|H^2u\|\|u\|, \quad \text{for all } u \in C_c^\infty(E),$$

and, hence, $\|H^2u\| \geq \|u\|$, for all $u \in C_c^\infty(E)$. This, in turn, leads to

$$(H^4u, u) = (H^2u, H^2u) = \|H^2u\|^2 \geq \|u\|^2, \quad \text{for all } u \in C_c^\infty(E).$$

Continuing like this, we obtain $(H^k u, u) \geq \|u\|^2$, for all $u \in C_c^\infty(E)$ and all $k \in \mathbb{Z}_+$. In this case, by an abstract fact (see Theorem X.26 in [29]), the essential self-adjointness of H^k on $C_c^\infty(E)$ is equivalent to the following statement: if $u \in L^2(E)$ satisfies $H^k u = 0$, then $u = 0$.

Let $u \in L^2(E)$ satisfy $H^k u = 0$. Since $V \in C^\infty(E)$, by local elliptic regularity it follows that $u \in C^\infty(E) \cap L^2(E)$. Define

$$f_j := H^{k-j}u, \quad j = 0, \pm 1, \pm 2, \dots \quad (4.4)$$

Here, in the case $k - j < 0$, the definition (4.4) is interpreted as $((H_{\max})^{-1})^{j-k}$. We already noted that H_{\min} is essentially self-adjoint and positive. Furthermore, it is well known that the self-adjoint closure of H_{\min} coincides with H_{\max} . Therefore H_{\max} is a positive self-adjoint operator, and $(H_{\max})^{-1}: L^2(E) \rightarrow L^2(E)$ is bounded. This, together with $f_k = u \in L^2(E)$ explains the following property: $f_j \in L^2(E)$, for all $j \geq k$. Additionally, observe that $f_j = 0$ for all $j \leq 0$ because $f_0 = 0$. Furthermore, we note that $f_j \in C^\infty(E)$, for all $j \in \mathbb{Z}$. The last assertion is obvious for $j \leq k$, and for $j > k$ it can be seen by showing that $H^j f_j = 0$ in distributional sense and using $f_j \in L^2(E)$ together with local elliptic regularity. To see this, let $v \in C_c^\infty(E)$ be arbitrary, and note that

$$(f_j, H^j v) = (H^{k-j} u, H^j v) = (u, H^k v) = (H^k u, v) = 0.$$

Finally, observe that

$$H^l f_j = f_{j-l}, \quad \text{for all } j \in \mathbb{Z} \text{ and } l \in \mathbb{Z}_+ \cup \{0\}. \quad (4.5)$$

With f_j as in (4.4), define the functions α_j and β_j on the interval $0 \leq T < \infty$ by the formulas

$$\alpha_j(T) := \lambda_0^2 \int_0^T (f_j, f_j)_{B_t} dt, \quad \beta_j(T) := \int_0^T (Df_j, Df_j)_{B_t} dt, \quad (4.6)$$

where λ_0 is as in (4.1) and $(\cdot, \cdot)_{B_t}$ is as in (3.2).

In the sequel, to simplify the notations, the functions $\alpha_j(T)$ and $\beta_j(T)$, the inner products $(\cdot, \cdot)_{B_t}$, and the corresponding norms $\|\cdot\|_{B_t}$ appearing in (4.6) will be denoted by α_j , β_j , $(\cdot, \cdot)_t$, and $\|\cdot\|_t$, respectively.

Note that α_j and β_j are absolutely continuous on $[0, \infty)$. Furthermore, α_j and β_j have a left first derivative and a right first derivative at each point. Additionally, α_j and β_j are differentiable, except at (at most) countably many points. In the sequel, to simplify notations, we shall denote the right first derivatives of α_j and β_j by α'_j and β'_j . Note that α_j , β_j , α'_j and β'_j are non-decreasing and non-negative functions. Note also that α_j and β_j are convex functions. Furthermore, since $f_j = 0$ for all $j \leq 0$, it follows that $\alpha_j \equiv 0$ and $\beta_j \equiv 0$ for all $j \leq 0$. Finally, using (4.1) and the property $f_j \in L^2(E) \cap C^\infty(E)$ for all $j \geq k$, observe that

$$\lambda_0^2 (f_j, f_j) + (Df_j, Df_j) \leq (Vf_j, f_j) + (Df_j, Df_j) = (f_j, Hf_j) = (f_j, f_{j-1}) < \infty,$$

for all $j > k$. Here, ‘‘integration by parts’’ in the first equality is justified because H_{\min} is essentially self-adjoint (i.e. $C_c^\infty(E)$ is an operator core of H_{\max}). Hence, α'_j and β'_j are bounded for all $j > k$. It turns out that α_j and β_j satisfy a system of differential inequalities, as seen in the next proposition.

Proposition 4.4. *Let α_j and β_j be as in (4.6). Then, for all $j \geq 1$ and all $T \geq 0$ we have*

$$\alpha_j + \beta_j \leq \sqrt{\alpha'_j \beta'_j} + \sum_{l=0}^{\infty} \left(\sqrt{\alpha'_{j+l+1} \beta'_{j-l-1}} + \sqrt{\alpha'_{j-l-1} \beta'_{j+l+1}} \right) \quad (4.7)$$

and

$$\alpha_j \leq \lambda_0^2 \left(\sum_{l=0}^{\infty} \left(\sqrt{\alpha'_{j+l+1} \beta'_{j-l-1}} + \sqrt{\alpha'_{j-l-1} \beta'_{j+l+1}} \right) \right), \quad (4.8)$$

where λ_0 is as in (4.1) and α'_i, β'_i denote the right-hand derivatives.

Remark 4.5. Note that the sums in (4.7) and (4.8) are finite since $\alpha_i \equiv 0$ and $\beta_i \equiv 0$ for $i \leq 0$. As our goal is to show that $f_k = u = 0$, we will only use the first k inequalities in (4.7) and the first k inequalities in (4.8).

Proof of Proposition 4.4. From (4.6) and (4.1) it follows that

$$\alpha_j + \beta_j \leq \int_0^T ((f_j, Vf_j)_t + (Df_j, Df_j)_t) dt. \quad (4.9)$$

We start from (4.9), use (3.3), Cauchy–Schwarz inequality, and (4.5) to obtain

$$\begin{aligned} \alpha_j + \beta_j &\leq \int_0^T ((f_j, Vf_j)_t + (Df_j, Df_j)_t) dt \\ &= \int_0^T |(f_j, Hf_j)_t - (f_j, D^*Df_j)_t + (Df_j, Df_j)_t| dt \\ &\leq \lambda_0 \int_{B_T} |Df_j(x)| |f_j(x)| d\mu(x) + \int_0^T |(f_j, Hf_j)_t| dt \\ &\leq \sqrt{\alpha'_j \beta'_j} + \int_0^T |(Hf_{j+1}, f_{j-1})_t| dt. \end{aligned}$$

We continue the process as follows:

$$\begin{aligned} \alpha_j + \beta_j &\leq \sqrt{\alpha'_j \beta'_j} + \int_0^T |(Hf_{j+1}, f_{j-1})_t| dt \\ &= \sqrt{\alpha'_j \beta'_j} + \int_0^T |(D^*Df_{j+1}, f_{j-1})_t + (f_{j+1}, Vf_{j-1})_t| dt \\ &\leq \sqrt{\alpha'_j \beta'_j} + \int_0^T |(D^*Df_{j+1}, f_{j-1})_t - (Df_{j+1}, Df_{j-1})_t| dt \\ &\quad + \int_0^T |(Df_{j+1}, Df_{j-1})_t - (f_{j+1}, D^*Df_{j-1})_t| dt + \int_0^T |(f_{j+1}, Hf_{j-1})_t| dt \\ &\leq \sqrt{\alpha'_j \beta'_j} + \sqrt{\alpha'_{j+1} \beta'_{j-1}} + \sqrt{\alpha'_{j-1} \beta'_{j+1}} + \int_0^T |(Hf_{j+2}, f_{j-2})_t| dt, \end{aligned}$$

where we used triangle inequality, (3.3), Cauchy–Schwarz inequality, and (4.5). We continue like this until the last term reaches the subscript $j - l \leq 0$, which makes the last term equal zero by properties of f_i discussed above. This establishes (4.7).

To show (4.8), we start from the definition of α_j , use (3.3), Cauchy–Schwarz inequality, and (4.5) to obtain

$$\begin{aligned}
\alpha_j &= \lambda_0^2 \int_0^T (f_j, f_j)_t dt = \lambda_0^2 \int_0^T |(f_j, Hf_{j+1})_t| dt \\
&= \lambda_0^2 \int_0^T |(f_j, D^*Df_{j+1})_t + (Vf_j, f_{j+1})_t| dt \\
&\leq \lambda_0^2 \int_0^T |(f_j, D^*Df_{j+1})_t - (Df_j, Df_{j+1})_t| dt \\
&\quad + \lambda_0^2 \int_0^T |(Df_j, Df_{j+1})_t - (D^*Df_j, f_{j+1})_t| dt + \lambda_0^2 \int_0^T |(Hf_j, f_{j+1})_t| dt \\
&\leq \lambda_0^2 \left(\sqrt{\alpha'_{j+1}\beta'_j} + \sqrt{\alpha'_j\beta'_{j+1}} \right) + \lambda_0^2 \int_0^T |(f_{j-1}, f_{j+1})_t| dt.
\end{aligned}$$

We continue like this until the last term reaches the subscript $j-l \leq 0$, which makes the last term equal zero by properties of f_i discussed above. This establishes (4.8). \square

End of the proof of Theorem 1. We will now transform the system (4.7)–(4.8) by introducing new variables:

$$\omega_j(T) := \alpha_j(T) + \beta_j(T), \quad \theta_j(T) := \alpha_j(T) - \beta_j(T) \quad T \in [0, \infty). \quad (4.10)$$

To carry out the transformation, observe that Cauchy–Schwarz inequality applied to vectors $\langle \sqrt{\alpha'_i}, \sqrt{\beta'_i} \rangle$ and $\langle \sqrt{\beta'_p}, \sqrt{\alpha'_p} \rangle$ in \mathbb{R}^2 gives

$$\sqrt{\alpha'_i\beta'_p} + \sqrt{\alpha'_p\beta'_i} \leq \sqrt{\omega'_i\omega'_p},$$

which, together with (4.7)–(4.8) leads to

$$\omega_j \leq \frac{1}{2} \sqrt{(\omega'_j)^2 - (\theta'_j)^2} + \sum_{l=0}^{\infty} \sqrt{\omega'_{j+l+1}\omega'_{j-l-1}} \quad (4.11)$$

and

$$\frac{1}{2}(\omega_j + \theta_j) \leq \lambda_0^2 \left(\sum_{l=0}^{\infty} \sqrt{\omega'_{j+l+1}\omega'_{j-l-1}} \right), \quad (4.12)$$

where λ_0 is as in (4.1) and ω'_i, θ'_i denote the right-hand derivatives.

The functions ω_j and θ_j satisfy the following properties: (i) ω_j and θ_j are absolutely continuous on $[0, \infty)$, and the right-hand derivatives ω'_j and θ'_j exist everywhere; (ii) ω_j and ω'_j are non-negative and non-increasing; (iii) ω_j is convex; (iv) ω'_j is bounded for all $j \geq k$; (v) $\omega_j(0) = \theta_j(0) = 0$; and (vi) $|\theta_j(T)| \leq \omega_j(T)$ and $|\theta'_j(T)| \leq \omega'_j(T)$ for all $T \in [0, \infty)$.

In Section IV.3 of [12] it was shown that if ω_j and θ_j are functions satisfying the above described properties (i)–(vi) and the system (4.11)–(4.12), then $\omega_j \equiv 0$ for all $j = 1, 2, \dots, k$. In particular, we have $\omega_k(T) = 0$, for all $T \in [0, \infty)$, and hence $f_k = 0$. Going back to (4.4), we get $u = 0$, and this concludes the proof of essential self-adjointness of H^k on $C_c^\infty(E)$. The essential self-adjointness of H^2, H^3, \dots , and H^{k-1} on $C_c^\infty(E)$ follows by Proposition 4.2. \square

5. PROOF OF THEOREM 2

We adapt the proof of Theorem 1.1 in [13] to our type of operator. By assumption (2.6) it follows that

$$((\Delta_{M,\mu} + q - C + 1)u, u) \geq \|u\|^2, \quad \text{for all } u \in C_c^\infty(M). \quad (5.1)$$

Since (5.1) is satisfied and since M is non-compact and g^{TM} is geodesically complete, a result of Agmon [1] (see also Proposition III.6.2 in [12]) guarantees the existence of a function $\gamma \in C^\infty(M)$ such that $\gamma(x) > 0$ for all $x \in M$, and

$$(\Delta_{M,\mu} + q - C + 1)\gamma = \gamma. \quad (5.2)$$

We now use the function γ to transform the operator $H = \nabla^* \nabla + V$. Let $L_{\mu_1}^2(E)$ be the space of square integrable sections of E with inner product $(\cdot, \cdot)_{\mu_1}$ as in (2.1), where $d\mu$ is replaced by $d\mu_1 := \gamma^2 d\mu$. For clarity, we denote $L^2(E)$ from Section 2.1 by $L_\mu^2(E)$. In what follows, the formal adjoints of ∇ with respect to inner products $(\cdot, \cdot)_\mu$ and $(\cdot, \cdot)_{\mu_1}$ will be denoted by $\nabla^{*,\mu}$ and ∇^{*,μ_1} , respectively. It is easy to check that the map $T_\gamma: L_\mu^2(E) \rightarrow L_{\mu_1}^2(E)$ defined by $T_\gamma u := \gamma^{-1}u$ is unitary. Furthermore, under the change of variables $u \mapsto \gamma^{-1}u$, the differential expression $H = \nabla^{*,\mu} \nabla + V$ gets transformed into $H_1 := \gamma^{-1}H\gamma$. Since T is unitary, the essential self-adjointness of $H^k|_{C_c^\infty(E)}$ in $L_\mu^2(E)$ is equivalent to essential self-adjointness of $(H_1)^k|_{C_c^\infty(E)}$ in $L_{\mu_1}^2(E)$.

In the sequel, we will show that H_1 has the following form:

$$H_1 = \nabla^{*,\mu_1} \nabla + \tilde{V}, \quad (5.3)$$

with

$$\tilde{V}(x) := \frac{\Delta_{M,\mu}\gamma}{\gamma} \text{Id}(x) + V(x).$$

To see this, let $w, z \in C_c^\infty(E)$ and consider

$$\begin{aligned} (H_1 w, z)_{\mu_1} &= \int_M \langle \gamma^{-1} H(\gamma w), z \rangle \gamma^2 d\mu = \int_M \langle H(\gamma w), \gamma z \rangle d\mu = (H(\gamma w), \gamma z)_\mu \\ &= (\nabla(\gamma w), \nabla(\gamma z))_\mu + (V\gamma w, \gamma z)_\mu = (\gamma^2 \nabla w, \nabla z)_\mu + (d\gamma \otimes w, d\gamma \otimes z)_{L_\mu^2(T^*M \otimes E)} \\ &\quad + (\gamma \nabla w, d\gamma \otimes z)_{L_\mu^2(T^*M \otimes E)} + (d\gamma \otimes w, \gamma \nabla z)_{L_\mu^2(T^*M \otimes E)} + (V\gamma w, \gamma z)_\mu. \end{aligned} \quad (5.4)$$

Setting $\xi := d(\gamma^2/2) \in T^*M$ and using equation (1.34) in Appendix C of [33] we have

$$(\gamma \nabla w, d\gamma \otimes z)_{L_\mu^2(T^*M \otimes E)} = (\nabla w, \xi \otimes z)_{L_\mu^2(T^*M \otimes E)} = (\nabla_X w, z)_\mu, \quad (5.5)$$

where X is the vector field associated with $\xi \in T^*M$ via the metric g^{TM} .

Furthermore, by equation (1.35) in Appendix C of [33] we have

$$\begin{aligned} (d\gamma \otimes w, \gamma \nabla z)_{L_\mu^2(T^*M \otimes E)} &= (\xi \otimes w, \nabla z)_{L_\mu^2(T^*M \otimes E)} = (\nabla^{*,\mu}(\xi \otimes w), z)_\mu \\ &= -(\text{div}_\mu(X)w, z)_\mu - (\nabla_X w, z)_\mu, \end{aligned} \quad (5.6)$$

where, in local coordinates x^1, x^2, \dots, x^n , for $X = X^j \frac{\partial}{\partial x^j}$, with Einstein summation convention,

$$\text{div}_\mu(X) := \frac{1}{\kappa} \left(\frac{\partial}{\partial x^j} (\kappa X^j) \right).$$

(Recall that $d\mu = \kappa(x) dx^1 dx^2 \dots dx^n$, where $\kappa(x)$ is a positive C^∞ -density.) Since $X^j = (g^{TM})^{jl} \left(\gamma \frac{\partial \gamma}{\partial x^l} \right)$, we have

$$\operatorname{div}_\mu(X) = |d\gamma|^2 - \gamma(\Delta_{M,\mu}\gamma), \quad (5.7)$$

where $|d\gamma(x)|$ is the norm of $d\gamma(x) \in T_x^*M$ induced by g^{TM} , and $\Delta_{M,\mu}$ is as in (1.1) with metric g^{TM} . Combining (5.4)–(5.7) and noting that

$$(d\gamma \otimes w, d\gamma \otimes z)_{L_\mu^2(T^*M \otimes E)} = \int_M |d\gamma|^2 \langle w, z \rangle d\mu,$$

we obtain

$$\begin{aligned} (H_1 w, z)_{\mu_1} &= \int_M \langle \nabla w, \nabla z \rangle \gamma^2 d\mu + \int_M \langle Vw, z \rangle \gamma^2 d\mu + \int_M \gamma(\Delta_{M,\mu}\gamma) \langle w, z \rangle d\mu \\ &= (\nabla w, \nabla z)_{L_{\mu_1}^2(T^*M \otimes E)} + (Vw, z)_{\mu_1} + (\gamma^{-1}(\Delta_{M,\mu}\gamma)w, z)_{\mu_1} \\ &= (\nabla^{*,\mu_1} \nabla w, z)_{\mu_1} + (Vw, z)_{\mu_1} + (\gamma^{-1}(\Delta_{M,\mu}\gamma)w, z)_{\mu_1}, \end{aligned} \quad (5.8)$$

which shows (5.3).

By (2.5) and (5.2) it follows that

$$\tilde{V}(x) = \frac{\Delta_{M,\mu}\gamma}{\gamma} \operatorname{Id}(x) + V(x) \geq (C-1)\operatorname{Id}(x), \quad \text{for all } x \in M,$$

where C is as in (2.6). Thus, by Theorem 1 the operator $(H_1)^k|_{C^\infty(E)}$ is essentially self-adjoint in $L_{\mu_1}^2(E)$ for all $k \in \mathbb{Z}_+$. \square

6. PROOF OF THEOREM 3

Throughout the section, we assume that the hypotheses of Theorem 3 are satisfied. In subsequent discussion, the notation \widehat{D} is as in (3.1) and the operators H_{\min} and H_{\max} are as in Section 4.1. We begin with the following lemma, whose proof is a direct consequence of the definition of H_{\max} and local elliptic regularity.

Lemma 6.1. *Under the assumption $V \in L_{\text{loc}}^\infty(\operatorname{End} E)$, we have the following inclusion: $\operatorname{Dom}(H_{\max}) \subset W_{\text{loc}}^{2,2}(E)$.*

The proof of the next lemma is given in Lemma 8.10 of [5].

Lemma 6.2. *For any $u \in \operatorname{Dom}(H_{\max})$ and any Lipschitz function with compact support $\psi: M \rightarrow \mathbb{R}$, we have:*

$$(D(\psi u), D(\psi u)) + (V\psi u, \psi u) = \operatorname{Re}(\psi H u, \psi u) + \|\widehat{D}(d\psi)u\|^2. \quad (6.1)$$

Corollary 6.3. *Let H be as in (2.3), let $u \in L^2(E)$ be a weak solution of $Hu = 0$, and let $\psi: M \rightarrow \mathbb{R}$ be a Lipschitz function with compact support. Then*

$$(\psi u, H(\psi u)) = \|\widehat{D}(d\psi)u\|^2, \quad (6.2)$$

where (\cdot, \cdot) on the left-hand side denotes the duality between $W_{\text{loc}}^{1,2}(E)$ and $W_{\text{comp}}^{-1,2}(E)$.

Proof. Since $u \in L^2(E)$ and $Hu = 0$, we have $u \in \text{Dom}(H_{\max}) \subset W_{\text{loc}}^{2,2}(E) \subset W_{\text{loc}}^{1,2}(E)$, where the first inclusion follows by Lemma 6.1. Since ψ is a Lipschitz compactly supported function, we get $\psi u \in W_{\text{comp}}^{1,2}(E)$ and, hence, $H(\psi u) \in W_{\text{comp}}^{-1,2}(E)$. Now the equality (6.2) follows from (6.1), the assumption $Hu = 0$, and

$$(\psi u, H(\psi u)) = (\psi u, D^*D(\psi u)) + (V\psi u, \psi u) = (D(\psi u), D(\psi u)) + (V\psi u, \psi u),$$

where in the second equality we used integration by parts; see Lemma 8.8 in [5]. Here, the two leftmost symbols (\cdot, \cdot) denote the duality between $W_{\text{comp}}^{1,2}(E)$ and $W_{\text{loc}}^{-1,2}(E)$, while the remaining ones stand for L^2 -inner products. \square

The key ingredient in the proof of Theorem 3 is the Agmon-type estimate given in the next lemma, whose proof, inspired by an idea of [25], is based on the technique developed in [10] for magnetic Laplacians on an open set with compact boundary in \mathbb{R}^n .

Lemma 6.4. *Let $\lambda \in \mathbb{R}$ and let $v \in L^2(E)$ be a weak solution of $(H - \lambda)v = 0$. Assume that there exists a constant $c_1 > 0$ such that, for all $u \in W_{\text{comp}}^{1,2}(E)$,*

$$(u, (H - \lambda)u) \geq \lambda_0^2 \int_M \max\left(\frac{1}{r(x)^2}, 1\right) |u(x)|^2 d\mu(x) + c_1 \|u\|^2, \quad (6.3)$$

where $r(x)$ is as in (2.7), λ_0 is as in (2.2), the symbol (\cdot, \cdot) on the left-hand side denotes the duality between $W_{\text{comp}}^{1,2}(E)$ and $W_{\text{loc}}^{-1,2}(E)$, and $|\cdot|$ is the norm in the fiber E_x .

Then, the following equality holds: $v = 0$.

Proof. Let ρ and R be numbers satisfying $0 < \rho < 1/2$ and $1 < R < +\infty$. For any $\varepsilon > 0$, we define the function $f_\varepsilon: M \rightarrow \mathbb{R}$ by $f_\varepsilon(x) = F_\varepsilon(r(x))$, where $r(x)$ is as in (2.7) and $F_\varepsilon: [0, \infty) \rightarrow \mathbb{R}$ is the continuous piecewise affine function defined by

$$F_\varepsilon(s) = \begin{cases} 0 & \text{for } s \leq \varepsilon \\ \rho(s - \varepsilon)/(\rho - \varepsilon) & \text{for } \varepsilon \leq s \leq \rho \\ s & \text{for } \rho \leq s \leq 1 \\ 1 & \text{for } 1 \leq s \leq R \\ R + 1 - s & \text{for } R \leq s \leq R + 1 \\ 0 & \text{for } s \geq R + 1. \end{cases}$$

Let us fix $x_0 \in M$. For any $\alpha > 0$, we define the function $p_\alpha: M \rightarrow \mathbb{R}$ by

$$p_\alpha(x) = P_\alpha(d_{g_{TM}}(x_0, x)),$$

where $P_\alpha: [0, \infty) \rightarrow \mathbb{R}$ is the continuous piecewise affine function defined by

$$P_\alpha(s) = \begin{cases} 1 & \text{for } s \leq 1/\alpha \\ -\alpha s + 2 & \text{for } 1/\alpha \leq s \leq 2/\alpha \\ 0 & \text{for } s \geq 2/\alpha. \end{cases}$$

Since $\widehat{d}_{g_{TM}}(x_0, x) \leq d_{g_{TM}}(x_0, x)$, it follows that the support of $f_\varepsilon p_\alpha$ is contained in the set $B_\alpha := \{x \in M: \widehat{d}_{g_{TM}}(x_0, x) \leq 2/\alpha\}$. By assumption (A1) we know that \widehat{M} is a geodesically complete Riemannian manifold. Hence, by Hopf–Rinow Theorem the set B_α is compact. Therefore, the

support of $f_\varepsilon p_\alpha$ is compact. Additionally, note that $f_\varepsilon p_\alpha$ is a β -Lipschitz function (with respect to the distance corresponding to the metric g^{TM}) with $\beta = \frac{\rho}{\rho - \varepsilon} + \alpha$.

Since $v \in L^2(E)$ and $(H - \lambda)v = 0$, we have $v \in \text{Dom}(H_{\max}) \subset W_{\text{loc}}^{2,2}(E) \subset W_{\text{loc}}^{1,2}(E)$, where the first inclusion follows by Lemma 6.1. Since $f_\varepsilon p_\alpha$ is a Lipschitz compactly supported function, we get $f_\varepsilon p_\alpha v \in W_{\text{comp}}^{1,2}(E)$ and, hence, $((H - \lambda)(f_\varepsilon p_\alpha v)) \in W_{\text{comp}}^{-1,2}(E)$.

Using (2.2) we have

$$\|\widehat{D}(d(f_\varepsilon p_\alpha)v)\|^2 \leq \lambda_0^2 \int_M |d(f_\varepsilon p_\alpha)(x)|^2 |v(x)|^2 d\mu(x), \quad (6.4)$$

where $|d(f_\varepsilon p_\alpha)(x)|$ is the norm of $d(f_\varepsilon p_\alpha)(x) \in T_x^*M$ induced by g^{TM} .

By Corollary 6.3 with $H - \lambda$ in place of H and the inequality (6.4), we get

$$(f_\varepsilon p_\alpha v, (H - \lambda)(f_\varepsilon p_\alpha v)) \leq \lambda_0^2 \left(\frac{\rho}{\rho - \varepsilon} + \alpha \right)^2 \|v\|^2. \quad (6.5)$$

On the other hand, using the definitions of f_ε and p_α and the assumption (6.3) we have

$$(f_\varepsilon p_\alpha v, (H - \lambda)(f_\varepsilon p_\alpha v)) \geq \lambda_0^2 \int_{S_{\rho,R,\alpha}} |v(x)|^2 d\mu(x) + c_1 \|f_\varepsilon p_\alpha v\|^2, \quad (6.6)$$

where

$$S_{\rho,R,\alpha} := \{x \in M : \rho \leq r(x) \leq R \text{ and } d_{g^{TM}}(x_0, x) \leq 1/\alpha\}.$$

In (6.6) and (6.5), the symbol (\cdot, \cdot) stands for the duality between $W_{\text{comp}}^{1,2}(E)$ and $W_{\text{loc}}^{-1,2}(E)$. We now combine (6.6) and (6.5) to get

$$\lambda_0^2 \int_{S_{\rho,R,\alpha}} |v(x)|^2 d\mu(x) + c_1 \|f_\varepsilon p_\alpha v\|^2 \leq \lambda_0^2 \left(\frac{\rho}{\rho - \varepsilon} + \alpha \right)^2 \|v\|^2.$$

We fix ρ , R , and ε , and let $\alpha \rightarrow 0+$. After that we let $\varepsilon \rightarrow 0+$. The last step is to do $\rho \rightarrow 0+$ and $R \rightarrow +\infty$. As a result, we get $v = 0$. \square

End of the proof of Theorem 3. Using integration by parts (see Lemma 8.8 in [5]), we have

$$(u, Hu) = (u, D^*Du) + (Vu, u) = (Du, Du) + (Vu, u) \geq (Vu, u), \quad \text{for all } u \in W_{\text{comp}}^{1,2}(E),$$

where the two leftmost symbols (\cdot, \cdot) denote the duality between $W_{\text{comp}}^{1,2}(E)$ and $W_{\text{loc}}^{-1,2}(E)$, while the remaining ones stand for L^2 -inner products. Hence, by assumption (2.8) we get:

$$\begin{aligned} (u, (H - \lambda)u) &\geq \lambda_0^2 \int_M \frac{1}{r(x)^2} |u(x)|^2 d\mu(x) - (\lambda + C) \|u\|^2 \\ &\geq \lambda_0^2 \int_M \max\left(\frac{1}{r(x)^2}, 1\right) |u(x)|^2 d\mu(x) - (\lambda + C + 1) \|u\|^2. \end{aligned} \quad (6.7)$$

Choosing, for instance, $\lambda = -C - 2$ in (6.7) we get the inequality (6.3) with $c_1 = 1$.

Thus, $H_{\min} - \lambda$ with $\lambda = -C - 2$ is a symmetric operator satisfying $(u, (H_{\min} - \lambda)u) \geq \|u\|^2$, for all $u \in C_c^\infty(E)$. In this case, it is known (see Theorem X.26 in [29]) that the essential self-adjointness of $H_{\min} - \lambda$ is equivalent to the following statement: if $v \in L^2(E)$ satisfies

$(H - \lambda)v = 0$, then $v = 0$. Thus, by Lemma 6.4, the operator $(H_{\min} - \lambda)$ is essentially self-adjoint. Hence, H_{\min} is essentially self-adjoint. \square

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DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF NORTH FLORIDA, JACKSONVILLE, FL 32224, USA.

E-mail address: `omilatov@unf.edu`

GRENOBLE UNIVERSITY, INSTITUT FOURIER, UNITÉ MIXTE DE RECHERCHE CNRS-UJF 5582, BP 74, 38402-SAINTE MARTIN D'HÈRES CEDEX, FRANCE.

E-mail address: `francoise.truc@ujf-grenoble.fr`