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UNRESTRICTED VIRTUAL BRAIDS, FUSED LINKS AND OTHER QUOTIENTS OF VIRTUAL BRAID GROUPS

VALERIY G. BARDAKOV, PAOLO BELLINGERI, AND CELESTE DAMIANI

ABSTRACT. We consider the group of unrestricted virtual braids, describe its structure and explore its relations with fused links. Also, we define the groups of flat virtual braids and virtual Gauss braids and study some of their properties, in particular their linearity.

1. INTRODUCTION

Fused links were defined by L. H. Kauffman and S. Lambropoulou in [21]. Afterwards, the same authors introduced their “braided” counterpart, the *unrestricted virtual braids*, and extended S. Kamada work ([17]) by presenting a version of Alexander and Markov theorems for these objects [22]. In the *group of unrestricted virtual braids*, which shall be denoted by UVB_n , we consider braid-like diagrams in which we allow two kinds of crossing (classical and virtual), and where the equivalence relation is given by ambient isotopy and by the following transformations: classical Reidemeister moves (Figure 1), virtual Reidemeister moves (Figure 2), a mixed Reidemeister move (Figure 3), and two moves of type Reidemeister III with two real crossings and one virtual crossing (Figure 4). These two last moves are called *forbidden moves*.

The group UVB_n appears also in [16], where it is called *symmetric loop braid group*, being it a quotient of the *loop braid group* LB_n studied in [1], usually known as the welded braid group WB_n .

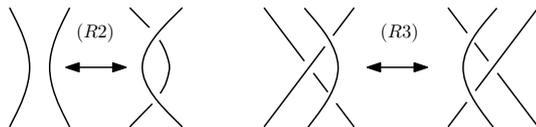


FIGURE 1. Classical Reidemeister moves.

It has been shown that all fused knots are equivalent to the unknot ([18,27]). Moreover, S. Nelson’s proof in [27] of the fact that every virtual knot unknots, when allowing forbidden moves, which is carried on using Gauss diagrams, can be verbatim adapted to links with several components. So, every fused link diagram is fused isotopic to a link diagram where the only crossings (classical or virtual) are the ones involving different components.

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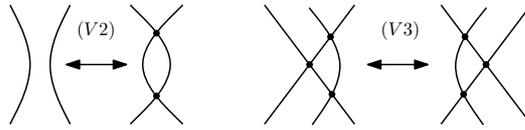


FIGURE 2. Virtual Reidemeister moves.

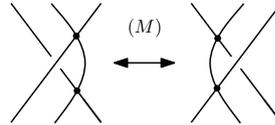


FIGURE 3. Mixed Reidemeister moves.

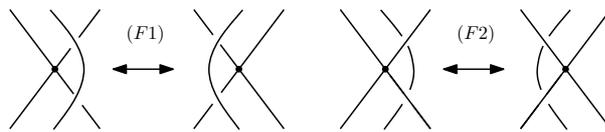


FIGURE 4. Forbidden moves of type F1 (on the left) and type F2 (on the right).

On the other hand, there are non trivial fused links and their classification is not (completely) trivial ([12]): in particular in [11], A. Fish and E. Keyman proved that fused links that have only classical crossings are characterized by their (classical) linking numbers. However, this result does not generalize to links with virtual crossings: as conjectured in [11] it is easy to find non equivalent fused links with the same (classical) linking number (see Section 3).

The first aim of this note is to give a short survey on above knotted objects, describe unrestricted virtual braids and compare more or less known invariants for fused links. In Section 2 we give a description of the structure of the group of unrestricted virtual braids UVB_n (Theorems 2.4 and 2.7), answering a question of Kauffman and Lambropoulou from [22]. In Section 3 we construct a representation for UVB_n in $\text{Aut}(N_n)$, the group of automorphisms of the free 2-step nilpotent group of rank n (Proposition 3.11). Using this representation we define a notion of group of fused links and we compare this invariant to other known invariants (Proposition 3.16). Finally, in Section 4 we describe the structure of other quotients of virtual braid groups: the flat virtual braid group (Proposition 4.1 and Theorem 4.3), the flat welded braid group (Proposition 4.5) and the virtual Gauss braid group (Theorem 4.7). As a corollary we prove that flat virtual braid groups and virtual Gauss braid groups are linear and that have solvable word problem (the fact that unrestricted virtual braid groups are linear and have solvable word problem is a trivial consequence of Theorem 2.7).

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2. UNRESTRICTED VIRTUAL BRAID GROUPS

In this Section, in order to define unrestricted virtual braid groups, we will first introduce virtual and welded braid groups by simply recalling their group presentation; for other definitions, more intrinsic, see for instance [2, 9, 17] for the virtual case and [8, 10, 17] for the welded one.

Definition 2.1. The *virtual braid group* VB_n is the group defined by the group presentation

$$\langle \{\sigma_i, \rho_i \mid i = 1, \dots, n-1\} \mid R \rangle$$

where R is the set of relations:

$$\begin{aligned} \sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1}, & \text{for } i = 1, \dots, n-2; \\ \sigma_i \sigma_j &= \sigma_j \sigma_i, & \text{for } |i-j| \geq 2; \\ \rho_i \rho_{i+1} \rho_i &= \rho_{i+1} \rho_i \rho_{i+1}, & \text{for } i = 1, \dots, n-2; \\ \rho_i \rho_j &= \rho_j \rho_i, & \text{for } |i-j| \geq 2; \\ \rho_i^2 &= 1, & \text{for } i = 1, \dots, n-1; \\ \sigma_i \rho_j &= \rho_j \sigma_i, & \text{for } |i-j| \geq 2; \\ \rho_i \rho_{i+1} \sigma_i &= \sigma_{i+1} \rho_i \rho_{i+1}, & \text{for } i = 1, \dots, n-2. \end{aligned}$$

We define the *virtual pure braid group*, denoted VP_n to be the kernel of the map $VB_n \rightarrow S_n$ sending, for every $i = 1, 2, \dots, n-1$, generators σ_i and ρ_i to s_i , where s_i is the transposition $(i, i+1)$. A presentation for VP_n is given in [3]; it will be recalled in the proof of Theorem 2.7.

The welded braid group WB_n can be defined as a quotient of VB_n via the normal subgroup generated by relations

$$(1) \quad \rho_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \rho_{i+1}, \quad \text{for } i = 1, \dots, n-2.$$

Relations (1) will be referred to as relations of type $F1$.

Remark 2.2. We will see in Section 3 that the symmetrical relations

$$(2) \quad \rho_{i+1} \sigma_i \sigma_{i+1} = \sigma_i \sigma_{i+1} \rho_i, \quad \text{for } i = 1, \dots, n-2$$

called $F2$ relations, do not hold in WB_n . This justifies Definition 2.3.

Definition 2.3. We define the *group of unrestricted virtual braids* UVB_n as the group defined by the group presentation

$$\langle \{\sigma_i, \rho_i \mid i = 1, \dots, n-1\} \mid R' \rangle$$

where R' is the set of relations:

$$(R1) \quad \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \quad \text{for } i = 1, \dots, n-2;$$

$$\begin{aligned}
(R2) \quad & \sigma_i \sigma_j = \sigma_j \sigma_i, & \text{for } |i - j| \geq 2; \\
(R3) \quad & \rho_i \rho_{i+1} \rho_i = \rho_{i+1} \rho_i \rho_{i+1}, & \text{for } i = 1, \dots, n - 2; \\
(R4) \quad & \rho_i \rho_j = \rho_j \rho_i, & \text{for } |i - j| \geq 2; \\
(R5) \quad & \rho_i^2 = 1, & \text{for } i = 1, \dots, n - 1; \\
(R6) \quad & \sigma_i \rho_j = \rho_j \sigma_i, & \text{for } |i - j| \geq 2; \\
(R7) \quad & \rho_i \rho_{i+1} \sigma_i = \sigma_{i+1} \rho_i \rho_{i+1}, & \text{for } i = 1, \dots, n - 2; \\
(F1) \quad & \rho_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \rho_{i+1}, & \text{for } i = 1, \dots, n - 2; \\
(F2) \quad & \rho_{i+1} \sigma_i \sigma_{i+1} = \sigma_i \sigma_{i+1} \rho_i, & \text{for } i = 1, \dots, n - 2.
\end{aligned}$$

The main result of this section is to prove that UVB_n can be described as semi-direct product of a right-angled Artin group and the symmetric group S_n : this way we answer a question posed in [22] about the (non trivial) structure of UVB_n .

Theorem 2.4. *Let X_n be the right-angled Artin group generated by $x_{i,j}$ for $1 \leq i \neq j \leq n$ where all generators commute except the pairs $x_{i,j}$ and $x_{j,i}$ for $1 \leq i \neq j \leq n$. The group UVB_n is isomorphic to $X_n \rtimes S_n$ where S_n acts by permutation on the indices of generators of X_n .*

Let $\nu : UVB_n \rightarrow S_n$ be the map defined as follows:

$$\nu(\sigma_i) = \nu(\rho_i) = s_i, \quad i = 1, 2, \dots, n - 1,$$

where s_i is the transposition $(i, i + 1)$. We will call the kernel of ν *unrestricted virtual pure braid group* and we will denote it by UVP_n . Since ν admits a natural section, we have that $UVB_n = UVP_n \rtimes S_n$.

Let us define some elements of UVB_n . For $i = 1, \dots, n - 1$:

$$\begin{aligned}
(3) \quad & \lambda_{i,i+1} = \rho_i \sigma_i^{-1}, \\
& \lambda_{i+1,i} = \rho_i \lambda_{i,i+1} \rho_i = \sigma_i^{-1} \rho_i.
\end{aligned}$$

For $1 \leq i < j - 1 \leq n - 1$:

$$\begin{aligned}
(4) \quad & \lambda_{ij} = \rho_{j-1} \rho_{j-2} \cdots \rho_{i+1} \lambda_{i,i+1} \rho_{i+1} \cdots \rho_{j-2} \rho_{j-1}, \\
& \lambda_{ji} = \rho_{j-1} \rho_{j-2} \cdots \rho_{i+1} \lambda_{i+1,i} \rho_{i+1} \cdots \rho_{j-2} \rho_{j-1}.
\end{aligned}$$

The next lemma was proved in [3] for the corresponding elements in VB_n and therefore is also true in the quotient UVB_n .

Lemma 2.5. *The following conjugating rules are fulfilled in UVB_n :*

1) for $1 \leq i < j \leq n$ and $k < i - 1$ or $i < k < j - 1$ or $k > j$:

$$\rho_k \lambda_{ij} \rho_k = \lambda_{ij}, \quad \text{and} \quad \rho_k \lambda_{ji} \rho_k = \lambda_{ji};$$

2) for $1 \leq i < j \leq n$:

$$\rho_{i-1} \lambda_{ij} \rho_{i-1} = \lambda_{i-1,j}, \quad \text{and} \quad \rho_{i-1} \lambda_{ji} \rho_{i-1} = \lambda_{j,i-1};$$

3) for $1 \leq i < j - 1 \leq n$:

$$\begin{aligned} \rho_i \lambda_{i,i+1} \rho_i &= \lambda_{i+1,i}, & \rho_i \lambda_{ij} \rho_i &= \lambda_{i+1,j}, \\ \rho_i \lambda_{i+1,i} \rho_i &= \lambda_{i,i+1}, & \rho_i \lambda_{ji} \rho_i &= \lambda_{j,i+1}; \end{aligned}$$

4) for $1 \leq i + 1 < j \leq n$:

$$\rho_{j-1} \lambda_{ij} \rho_{j-1} = \lambda_{i,j-1}, \quad \text{and} \quad \rho_{j-1} \lambda_{ji} \rho_{j-1} = \lambda_{j-1,i};$$

5) for $1 \leq i < j \leq n$:

$$\rho_j \lambda_{ij} \rho_j = \lambda_{i,j+1}, \quad \text{and} \quad \rho_j \lambda_{ji} \rho_j = \lambda_{j+1,i}.$$

Corollary 2.6. *The group S_n acts by conjugation on the set $\{\lambda_{kl} \mid 1 \leq k \neq l \leq n\}$. This action is transitive.*

In particular, the group S_n acts by permutation on the set $\{\lambda_{kl} \mid 1 \leq k \neq l \leq n\}$.

We prove that the group generated by $\{\lambda_{kl} \mid 1 \leq k \neq l \leq n\}$ coincides with UVP_n , and then we will find the defining relations. This will show that UVP_n is a right-angled Artin group.

Let $m_{kl} = \rho_{k-1} \rho_{k-2} \dots \rho_l$ for $l < k$ and $m_{kl} = 1$ in other cases. Then the set

$$\Lambda_n = \left\{ \prod_{k=2}^n m_{k,j_k} \mid 1 \leq j_k \leq k \right\}$$

is a Schreier set of coset representatives of UVP_n in UVB_n .

Theorem 2.7. *The group UVP_n admits a presentation with generators λ_{kl} for $1 \leq k \neq l \leq n$, and defining relations: λ_{ij} commute with λ_{kl} if and only if $k \neq j$ or $l \neq i$.*

Proof. The proof is a straightforward application of Reidemeister–Schreier method (see, for example, [24, Ch. 2.2]); most part of relations were already proven in [3] in the case of the virtual pure braid group VP_n .

Define the map $\bar{\cdot} : UVB_n \rightarrow \Lambda_n$ which takes an element $w \in UVB_n$ to the representative \bar{w} in Λ_n . In this case the element $w\bar{w}^{-1}$ belongs to UVP_n . By Theorem 2.7 from [24] the group UVP_n is generated by

$$s_{\lambda,a} = \lambda a \cdot (\overline{\lambda a})^{-1},$$

where λ runs over the set Λ_n and a runs over the set of generators of UVB_n .

It is easy to establish that $s_{\lambda,\rho_i} = e$ for all representatives λ and generators ρ_i . Consider the generators

$$s_{\lambda,\sigma_i} = \lambda \sigma_i \cdot (\overline{\lambda \sigma_i})^{-1}.$$

For $\lambda = e$ we get $s_{e,\sigma_i} = \sigma_i \rho_i = \lambda_{i,i+1}^{-1}$. Note that $\lambda \rho_i$ is equal to $\overline{\lambda \rho_i}$ in S_n . Therefore,

$$s_{\lambda,\sigma_i} = \lambda (\sigma_i \rho_i) \lambda^{-1}.$$

From Lemma 2.5 it follows that each generator s_{λ,σ_i} is equal to some λ_{kl} , $1 \leq k \neq l \leq n$. By Corollary 2.6, the inverse statement is also true, i. e., each element λ_{kl} is equal to some generator s_{λ,σ_i} . The first part of the theorem is proven.

To find defining relations of UVP_n we define a rewriting process τ . It allows us to rewrite a word which is written in the generators of UVB_n and presents an element in UVP_n as a word in the generators of UVP_n . Let us associate to the reduced word

$$u = a_1^{\varepsilon_1} a_2^{\varepsilon_2} \dots a_\nu^{\varepsilon_\nu}, \quad \varepsilon_l = \pm 1, \quad a_l \in \{\sigma_1, \sigma_2, \dots, \sigma_{n-1}, \rho_1, \rho_2, \dots, \rho_{n-1}\},$$

the word

$$\tau(u) = s_{k_1, a_1}^{\varepsilon_1} s_{k_2, a_2}^{\varepsilon_2} \dots s_{k_\nu, a_\nu}^{\varepsilon_\nu}$$

in the generators of UVP_n , where k_j is a representative of the $(j-1)$ th initial segment of the word u if $\varepsilon_j = 1$ and k_j is a representative of the j -th initial segment of the word u if $\varepsilon_j = -1$.

By [24, Theorem 2.9], the group UVP_n is defined by relations

$$r_{\mu, \lambda} = \tau(\lambda r_\mu \lambda^{-1}), \quad \lambda \in \Lambda_n,$$

where r_μ is the defining relation of UVB_n .

In [3, Theorem 1] was proven that relations (R1) – (R7) imply the following set of relations:

$$\begin{aligned} (RS1) \quad & \lambda_{ij} \lambda_{kl} = \lambda_{kl} \lambda_{ij} \\ (RS2) \quad & \lambda_{ki} (\lambda_{kj} \lambda_{ij}) = (\lambda_{ij} \lambda_{kj}) \lambda_{ki}, \end{aligned}$$

where distinct letters stand for distinct indices (this a complete set of relations for the virtual pure braid group VP_n). Consider now relation (F1). We get the element

$$(f_1) := \rho_i \sigma_{i+1} \sigma_i \rho_{i+1} \sigma_i^{-1} \sigma_{i+1}^{-1}.$$

Using the rewriting process, we obtain

$$\begin{aligned} r_{f_1, e} &= \tau(f_1) = s_{e, \rho_i} s_{\sigma_i, \sigma_{i+1}} s_{\rho_i \sigma_{i+1}, \sigma_i} s_{\rho_i \sigma_{i+1} \sigma_i, \rho_{i+1}} s_{\rho_i \sigma_{i+1} \sigma_i \rho_{i+1}, \sigma_i^{-1}} s_{\rho_i \sigma_{i+1} \sigma_i \rho_{i+1} \sigma_i^{-1}, \sigma_i} s_{(f_1), \sigma_{i+1}}^{-1} \\ &= e (\rho_i \lambda_{i+1, i+2}^{-1} \rho_i) (\rho_i \rho_{i+1} \lambda_{i, i+1}^{-1} \rho_{i+1} \rho_i) e (\rho_{i+1} \lambda_{i, i+1} \rho_{i+1}) \lambda_{i+1, i+2}. \end{aligned}$$

Using the conjugating rules from Lemma 2.5, we get

$$r_{f_1, e} = \lambda_{i, i+2}^{-1} \lambda_{i+1, i+2}^{-1} \lambda_{i, i+2} \lambda_{i+1, i+2}.$$

Therefore, the following relation

$$\lambda_{i, i+2} \lambda_{i+1, i+2} = \lambda_{i+1, i+2} \lambda_{i, i+2}$$

is fulfilled in UVP_n . The remaining relations $r_{f_1, \lambda}$, $\lambda \in \Lambda_n$, can be obtained from this relation using conjugation by λ^{-1} . By the formulas from Lemma 2.5, we obtain the set of relations $\lambda_{i, j} \lambda_{k, j} = \lambda_{k, j} \lambda_{i, j}$, where i, j, k are distinct letters. On the other hand rewriting relations of type (F2) we get the set of relations $\lambda_{i, j} \lambda_{i, k} = \lambda_{i, k} \lambda_{i, j}$, where i, j, k are distinct letters. Using the proven relations $\lambda_{ki} \lambda_{kj} = \lambda_{kj} \lambda_{ki}$ and $\lambda_{ij} \lambda_{kj} = \lambda_{kj} \lambda_{ij}$ we can rewrite relation (RS2) in the form

$$\lambda_{kj} (\lambda_{ki} \lambda_{ij}) = \lambda_{kj} (\lambda_{ij} \lambda_{ki}).$$

After cancelation we have relations $[\lambda_{ki}, \lambda_{ij}] = [\lambda_{ij}, \lambda_{ki}] = 1$. This complete the proof. \square

Proof of Theorem 2.4. The group X_n is evidently isomorphic to UVP_n (sending any $x_{i,j}$ into the corresponding $\lambda_{i,j}$). Recall that UVP_n is the kernel of the map $\nu : UVB_n \rightarrow S_n$ defined as $\nu(\sigma_i) = \nu(\rho_i) = s_i$ for $i = 1, \dots, n-1$. Recall also that ν has a natural section $s : S_n \rightarrow UVB_n$, defined as $s(s_i) = \rho_i$ for $i = 1, \dots, n-1$. Therefore UVB_n is isomorphic to $UVP_n \rtimes S_n$ where S_n acts by permutation on the indices of generators of UVP_n (see Corollary 2.6). \square

We recall that the pure braid group P_n is the kernel of the homomorphism from B_n to the symmetric group S_n sending every generator σ_i to the permutation $(i, i+1)$. It is generated by the set $\{a_{i,j} \mid 1 \leq i < j \leq n\}$, where

$$a_{i,i+1} = \sigma_i^2,$$

$$a_{i,j} = \sigma_{j-1}\sigma_{j-2}\cdots\sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1}\cdots\sigma_{j-2}^{-1}\sigma_{j-1}^{-1}, \quad \text{for } i+1 < j \leq n.$$

Corollary 2.8. *Let $p: P_n \rightarrow UVP_n$ be the canonical projection of the pure braid group P_n in UVP_n . Then $p(P_n)$ is the abelianization of P_n .*

Proof. As remarked in ([3, page 6]), generators $a_{i,j}$ of P_n can be rewritten as

$$a_{i,i+1} = \lambda_{i,i+1}^{-1}\lambda_{i+1,i}^{-1}, \quad \text{for } i = 1, \dots, n-1;$$

$$a_{i,j} = \lambda_{j-1,j}^{-1}\lambda_{j-2,j}^{-1}\cdots\lambda_{i+1,j}^{-1}(\lambda_{i,j}^{-1}\lambda_{j,i}^{-1})\lambda_{i+1,j}\cdots\lambda_{j-2,j}\lambda_{j-1,j}, \quad \text{for } 2 \leq i+1 < j \leq n.$$

According to Theorem 2.7, UVP_n is the cartesian product of the free groups of rank 2 $F_{i,j} = \langle \lambda_{i,j}, \lambda_{j,i} \rangle$ for $1 \leq i < j \leq n$.

For every generator $a_{i,j}$ for $1 \leq i < j \leq n$ of P_n we have that its image is in $F_{i,j}$ and it is not trivial. In fact, $p(a_{i,j}) = \lambda_{i,j}^{-1}\lambda_{j,i}^{-1}$. So $p(P_n)$ is isomorphic to $\mathbb{Z}^{n(n-1)/2}$. The statement therefore follows readily since the abelianized of P_n is $\mathbb{Z}^{n(n-1)/2}$. \square

3. UNRESTRICTED VIRTUAL BRAIDS AND FUSED LINKS

Definition 3.1. A *virtual link diagram* is a closed oriented 1-manifold D immersed in \mathbb{R}^2 such that all multiple points are transverse double point, and each double point is provided with an information of being *positive*, *negative* or *virtual* as in Figure 5. We assume that virtual link diagrams are the same if they are isotopic in \mathbb{R}^2 . Positive and negative crossings will also be called *classical crossings*.

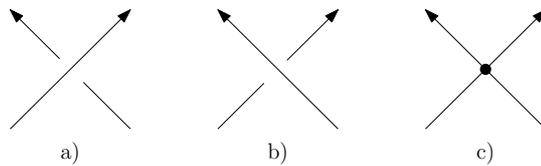


FIGURE 5. a) Positive crossing, b) Negative crossing, c) Virtual crossing.

Definition 3.2. *Fused isotopy* is the equivalence relation on the set of virtual link diagrams given by classical Reidemeister moves, virtual Reidemeister moves, and the forbidden moves $F1$ and $F2$.

Remark 3.3. These moves are the same moves pictured in Figure 1, 2, 3, and 4, with the addition of Reidemeister moves of type I, both classical and virtual, see Figure 6.

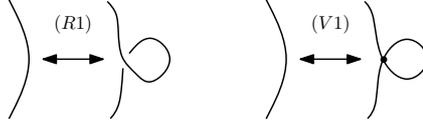


FIGURE 6. Reidemeister moves of type I.

Definition 3.4. A *fused link* is an equivalence class of virtual link diagrams with respect to fused isotopy.

The classical Alexander Theorem generalizes to virtual braids and links, and it directly implies that every oriented welded (resp. fused) link can be represented by a welded (resp. unrestricted virtual) braid, whose Alexander closure is isotopic to the original link. Two braiding algorithms are given in [17] and [21].

Similarly we have the following version of Markov Theorem ([22]):

Theorem 3.5 ([22]). *Two oriented fused links are isotopic if and only if any two corresponding unrestricted virtual braids differ by moves defined by braid relations in UVB_∞ (braid moves) and a finite sequence of the following moves (extended Markov moves):*

- *Virtual and real conjugation:* $\rho_i \beta \rho_i \sim \beta \sim \sigma_i^{-1} \beta \sigma_i \sim \sigma_i \beta \sigma_i^{-1}$;
- *Right virtual and real stabilization:* $\beta \rho_n \sim \beta \sim \beta \sigma_n^{\pm 1}$;

where $UVB_\infty = \bigcup_{n=2}^{\infty} UVB_n$, β is a braid in UVB_n , σ_i, ρ_i generators of UVB_n and $\sigma_n, \rho_n \in UVB_{n+1}$.

A. Fish and E. Keyman proved the following result about fused links.

Theorem 3.6 ([11]). *A fused link with only classical crossings L with n -components is completely determined by the linking numbers of each pair of components under fused isotopy.*

The proof in [11] is quite technical, it involves several computations on generators of the pure braid group and their images in UVP_n .

In order to be self contained let us give an easier proof which uses the structure of UVB_n described in Section 2: the advantage is that no preliminary lemma on the properties of the pure braid group generators is necessary. The main point of the proof is that if L is a fused link with n component that has only classical crossings, then there is a braid $\beta \in UVP_n$ such that L is equivalent to its closure $\hat{\beta}$.

Proof. The case $n = 1$ is trivial because then L is fused isotopic to the unknot ([18, 27]). So let us consider $n > 1$. Let U_j be the subgroup of P_n generated by $\{a_{i,j} \mid 1 \leq i < j\}$, and $B_{i,j} = \sigma_{j-1} \sigma_{j-2} \cdots \sigma_{i+1} \sigma_i$, for $i < j$, and $B_{i,i} = 1$.

We consider a fused link with only classical crossings L with n components, and α in B_m , with $n \leq m$, such that $\hat{\alpha} \sim L$.

Even though B_n and P_n are not subgroups of UVB_n , since L has only classical crossings, we can consider B_n and P_n 's images in UVB_n .

The braid α can be written in the form $\alpha = x_2 B_{k_2,2} \cdots x_m B_{k_m,m}$ where $x_i \in U_i < P_m$, and $1 \leq k_i$ (we refer to [7] for a complete proof).

If $B_{k_i,i} = 1$ for $i = 2, \dots, m$, then α is a pure braid, so $m = n$. So we will assume that $B_{k_s,s} \neq 1$ for some s , and that if $i > s$ then $B_{k_i,i} = 1$. The permutation induced by α is the identity on the strands $s + 1, \dots, m$ so each of these form a component of $\hat{\alpha}$.

Conjugating α by $B_{1,m}^{(m-s)}$ we obtain a braid $\alpha_1 = B_{1,m}^{s-m} \alpha B_{1,m}^{m-s}$ and the s -strand of α is the m -strand of α_1 (whose closure is isotopic to α 's).

We remark that until now we only used classical Alexander theorem and classical isotopy.

Now we can write $\alpha_1 = y_2 B_{t_2,2} \cdots y_m B_{t_m,m}$ with $y_i \in U_i$. Considering the projection $P_n \rightarrow UVP_n$, we rewrite generators $a_{i,j}$ in terms of $\lambda_{i,j}$: then each y_i is in D_{i-1} , the subset of UVP_n generated by $\{\lambda_{i,1}, \lambda_{i,2}, \dots, \lambda_{i,i-1}, \lambda_{1,i}, \lambda_{2,i}, \dots, \lambda_{i-1,i}\}$. From Theorem 2.4 we deduce that: $D_{i-1} = \langle \lambda_{i,1}, \lambda_{1,i} \rangle \times \cdots \times \langle \lambda_{i,i-1}, \lambda_{i-1,i} \rangle$. We can order the words so that y_m is of the form $w_1 \cdots w_{m-1}$ where $w_i \in \langle \lambda_{m,i}, \lambda_{i,m} \rangle$. In addition $B_{t_m,m} = \sigma_{m-1} B_{t_m,m-1}$.

Let γ be $y_2 B_{t_2,2} \cdots y_{m-1} B_{t_{m-1},m-1}$. Then $\alpha_1 = \gamma w_1 \cdots w_{m-1} \sigma_{m-1} B_{t_m,m-1}$, where $w_1 \cdots w_{m-1}$ is a pure braid, and γ does not involve the m -strand.

The m -strand and the other strand involved in the occurrence of σ_{m-1} before $B_{t_m,m-1}$, and hence in all the crossings of w_{m-1} , belong to the same component of $L_1 = \hat{\alpha}_1$ (see Figure 7).

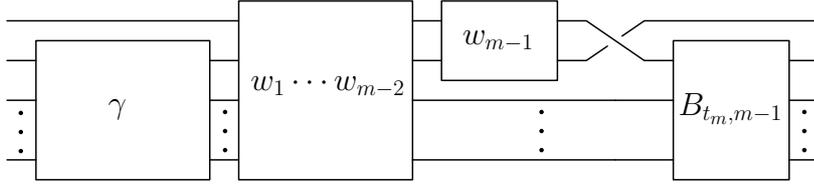


FIGURE 7. Braid α_1 .

We virtualize all classical crossings of $w_{m-1} \sigma_{m-1}$ using Kanenobu's technique ([18, Proof of Theorem 1]): since w_{m-1} has an even total number of generators σ_{m-1} and ρ_{m-1} , after virtualizing $w_{m-1} \sigma_{m-1}$ becomes a word composed by an odd number of ρ_{m-1} . Applying the relation associated with the virtual Reidemeister move of type 2 we obtain a new link L_2 , fused isotopic to L , associated to $\alpha_2 = \gamma w_1 \cdots w_{m-2} \rho_{m-1} B_{t_m,m-1}$.

We remark that for all $\lambda_{i,m}$ and $\lambda_{m,i}$ with $i < m - 1$ we have:

$$\lambda_{i,m} \rho_{m-1} = \rho_{m-1} \lambda_{i,m-1},$$

and

$$\lambda_{m,i} \rho_{m-1} = \rho_{m-1} \lambda_{m-1,i}.$$

Then $\alpha_2 = \gamma \rho_{m-1} w'_1 \cdots w'_{m-2} B_{t_m,m-1}$, where w'_i is a word in $\langle \lambda_{m-1,i}, \lambda_{i,m-1} \rangle$.

In α_2 there is only one (virtual) crossing on the m -strand, so, using Markov moves (conjugation and virtual stabilisation) we obtain a new braid α_3 whose closure is again fused isotopic to L and has $(m - 1)$ strands.

If we continue this process, eventually we will get to a braid β in B_n whose closure is fused isotopic to L . At this point, each strand of β corresponds to a different component of L , so β must be a pure braid.

At this point, we can rewrite the pure braid in terms of $a_{i,j}$ generators, and conclude as Fish and Keyman do, defining a group homomorphism $\delta_{i,j} : P_n \rightarrow \mathbb{Z}$ by

$$a_{s,t} \mapsto \begin{cases} 1 & \text{if } s = i \text{ and } t = j; \\ 0 & \text{otherwise} \end{cases}$$

which is the classical linking number $lk_{i,j}$ of L 's i -th and j -th components. Any fused link with only classical crossings L with n components can be obtained as a closure of a pure braid $\beta = x_2 \cdots x_n$ where each x_i can be written in the form $x_i = a_{1,i}^{\delta_{1,i}} \cdots a_{i-1,i}^{\delta_{i-1,i}}$ (Corollary 2.8). This shows that β only depends on the linking number of the components. \square

In [14, Section 1] a *virtual* version of the *linking number* is defined in the following way: to a 2-component link we associate a couple of integers $(v lk_{1,2}, v lk_{2,1})$ where $v lk_{1,2}$ is the sum of signs of real crossings where the first component passes over the second one, while $v lk_{2,1}$ is computed by exchanging the components in the definition of $v lk_{1,2}$. Clearly the classical linking number $lk_{1,2}$ is equal to the sum of $v lk_{1,2}$ and $v lk_{2,1}$.

Fish and Keyman in [11] suggest that their theorem cannot be extended to links with virtual crossings between different components. They consider the unlink on two components U_2 and $L = \hat{\alpha}$, where $\alpha = \sigma_1 \rho_1 \sigma_1^{-1} \rho_1$, they remark that their classical linking number is 0 but they conjecture that these two links are not fused isotopic. In fact, considering the virtual linking number we can see that $(v lk_{1,2}, v lk_{2,1})(U_2) = (0, 0)$, while $(v lk_{1,2}, v lk_{2,1})(L) = (-1, 1)$.

Using this definition of virtual linking number, we could be tempted to extend Fish and Keyman results, claiming that a fused link L is completely determined by the virtual linking numbers of each pair of components under fused isotopy.

However for the unrestricted case the previous argument cannot be straightforward applied: the virtual linking number is able to distinguish $\lambda_{i,j}$ from $\lambda_{j,i}$, but it is still an application from UVP_n to $(\mathbb{Z}^2)^{n(n-1)/2} = \mathbb{Z}^{n(n-1)}$ that counts the exponents (i.e., the number of appearances) of each generator. Since UVP_n isn't abelian, this is not sufficient to completely determine the braid.

3.1. A representation for the unrestricted virtual braid group. Let us recall that the braid group B_n may be represented as a subgroup of $\text{Aut}(F_n)$ by associating to any generator σ_i , for $i = 1, 2, \dots, n-1$, of B_n the following automorphism of F_n :

$$(5) \quad \sigma_i : \begin{cases} x_i \mapsto x_i x_{i+1} x_i^{-1}, \\ x_{i+1} \mapsto x_i, \\ x_l \mapsto x_l, \end{cases} \quad l \neq i, i+1.$$

Moreover Artin provided (see for instance [15, Theorem 5.1]) a characterization of braids as automorphisms of free groups: an automorphism $\beta \in \text{Aut}(F_n)$ lies in B_n if and only if β satisfies the following conditions:

- i) $\beta(x_i) = a_i^{-1} x_{\pi(i)} a_i$, $1 \leq i \leq n$
- ii) $\beta(x_1 x_2 \dots x_n) = x_1 x_2 \dots x_n$

where $\pi \in S_n$ and $a_i \in F_n$.

According to [10] we call *group of automorphisms of permutation conjugacy type*, denoted PC_n , the group of automorphisms satisfying the first condition. The group PC_n is isomorphic to WB_n [10]; more precisely to each generator σ_i of WB_n we associate the previous automorphisms of F_n while to each generator ρ_i , for $i = 1, 2, \dots, n-1$, we associate the following automorphism of F_n :

$$(6) \quad \rho_i : \begin{cases} x_i \mapsto x_{i+1} \\ x_{i+1} \mapsto x_i, \\ x_l \mapsto x_l, \quad l \neq i, i+1. \end{cases}$$

We have thus a faithful representation $\psi: WB_n \rightarrow \text{Aut}(F_n)$.

Remark 3.7. The group PC_n admits also other equivalent definitions in terms of mapping classes and configuration spaces: it appears often in the literature with different names and notations, such as group of flying rings [2, 8], McCool group [6], motions group [13] and loop braid group [1].

Remark 3.8. Kamada remarks in [17] that, through the canonical epimorphism $VB_n \rightarrow WB_n$, the classical braid group B_n embeds in VB_n . It can be seen via an argument in [10] that B_n is isomorphic to the subgroup of VB_n generated by $\{\sigma_1, \dots, \sigma_n\}$.

Remark 3.9. As a consequence of the isomorphism between WB_n and PC_n , we can show that relation $F2$ does not hold in WB_n . In fact applying $\rho_{i+1}\sigma_i\sigma_{i+1}$ one gets

$$\rho_{i+1}\sigma_i\sigma_{i+1} : \begin{cases} x_i \mapsto x_i \mapsto x_i x_{i+1} x_i^{-1} \mapsto x_i x_{i+1} x_{i+2} x_{i+1}^{-1} x_i^{-1}, \\ x_{i+1} \mapsto x_{i+2} \mapsto x_{i+2} \mapsto x_{i+1}, \\ x_{i+2} \mapsto x_{i+1} \mapsto x_i \mapsto x_i, \end{cases}$$

while applying $\sigma_i\sigma_{i+1}\rho_i$ one gets

$$\sigma_i\sigma_{i+1}\rho_i : \begin{cases} x_i \mapsto x_i x_{i+1} x_i^{-1} \mapsto x_i x_{i+1} x_{i+2} x_{i+1}^{-1} x_i^{-1} \mapsto x_{i+1} x_i x_{i+2} x_i^{-1} x_{i+1}^{-1}, \\ x_{i+1} \mapsto x_i \mapsto x_i \mapsto x_{i+1}, \\ x_{i+2} \mapsto x_{i+2} \mapsto x_{i+1} \mapsto x_i. \end{cases}$$

Since $x_i x_{i+1} x_{i+2} x_{i+1}^{-1} x_i^{-1} \neq x_{i+1} x_i x_{i+2} x_i^{-1} x_{i+1}^{-1}$ in F_n we deduce that relation $F2$ does not hold in WB_n .

Our aim is to find a representation for unrestricted virtual braids as automorphisms of a group G . Since the map $\psi: WB_n \rightarrow \text{Aut}(F_n)$ does not factor to the quotient UVB_n (Remark 3.9) we need to find a representation in the group of automorphisms of a quotient of F_n in which relation $F2$ is preserved.

Remark 3.10. In [16] the authors look for representations of the braid group B_n that can be extended to the loop braid group WB_n but do not factor over UVB_n , which is its quotient via relations of type $F2$, while we look for a representation that does factor.

Let $F_n = \gamma_1 F_n \supseteq \gamma_2 F_n \supseteq \dots$ be the lower central series of F_n , the free group of rank n , where $\gamma_{i+1} F_n = [F_n, \gamma_i F_n]$. Let us consider its third term, $\gamma_3 F_n = [F_n, [F_n, F_n]]$; the free 2-step nilpotent group N_n of rank n is defined to be the quotient $F_n / \gamma_3 F_n$.

There is an epimorphism from F_n to N_n that induces an epimorphism from $\text{Aut}(F_n)$ to $\text{Aut}(N_n)$. Then, let $\phi: UVB_n \rightarrow \text{Aut}(N_n)$ be the composition of $\varphi: UVB_n \rightarrow \text{Aut}(F_n)$ and $\text{Aut}(F_n) \rightarrow \text{Aut}(N_n)$.

Proposition 3.11. *The map $\phi: UVB_n \rightarrow \text{Aut}(N_n)$ is a representation for UVB_n .*

Proof. In N_n we have that $[[x_i, x_{i+1}], x_{i+2}] = 1$, for $i = 1, \dots, n-2$, meaning that $x_i x_{i+1} x_{i+2} x_{i+1}^{-1} x_i^{-1} = x_{i+1} x_i x_{i+2} x_i^{-1} x_{i+1}^{-1}$, i.e., relation F2 is preserved. \square

Proposition 3.12. *The image of the representation $\phi: UVP_n \rightarrow \text{Aut}(N_n)$ is a free abelian group of rank $n(n-1)$.*

Proof. From Theorem 2.4 we have that the only generators that do not commute in UVP_n are λ_{ij} and λ_{ji} with $1 \leq i \neq j \leq n$.

Recalling the expressions of λ_{ij} and λ_{ji} in terms of generators σ_i and ρ_i , we see that the automorphisms associated to λ_{ij} and λ_{ji} are

$$\begin{aligned} \phi(\lambda_{ij}) &: \begin{cases} x_i \mapsto x_j^{-1} x_i x_j = x_i [x_i, x_j] \\ x_k \mapsto x_k, \text{ for } k \neq i; \end{cases} \\ \phi(\lambda_{ji}) &: \begin{cases} x_j \mapsto x_i^{-1} x_j x_i = x_j [x_j, x_i] = x_j [x_i, x_j]^{-1}; \\ x_k \mapsto x_k \text{ for } k \neq j. \end{cases} \end{aligned}$$

It is then easy to check that the automorphisms associated to $\lambda_{ij}\lambda_{ji}$ and to $\lambda_{ji}\lambda_{ij}$ coincide:

$$\phi(\lambda_{ij}\lambda_{ji}) = \phi(\lambda_{ji}\lambda_{ij}) : \begin{cases} x_i \mapsto x_i [x_i, x_j] \\ x_j \mapsto x_j [x_i, x_j]^{-1}. \end{cases}$$

\square

Remark 3.13. As a consequence of the previous calculation we have that the homomorphism ϕ coincides on UVP_n with the abelianization map.

As a consequence of Proposition 3.12, representation ϕ is not faithful. However, according to previous characterization of WB_n as subgroup of $\text{Aut}(F_n)$ it is natural to ask if we can give a characterization of automorphisms of $\text{Aut}(N_n)$ that belong to $\phi(UVB_n)$.

Proposition 3.14. *Let β be an element of $\text{Aut}(N_n)$, then $\beta \in \phi(UVB_n)$ if and only if β satisfies the condition $\beta(x_i) = a_i^{-1} x_{\pi(i)} a_i$ with $1 \leq i \leq n$, where $\pi \in S_n$ and $a_i \in N_n$.*

Proof. Let us denote with $UVB(N_n)$ the subgroup of $\text{Aut}(N_n)$ such that any element $\beta \in UVB(N_n)$ has the form $\beta(x_i) = g_i^{-1} x_{\pi(i)} g_i$, denoted $x_{\pi(i)}^{g_i}$, with $1 \leq i \leq n$, where $\pi \in S_n$ and $g_i \in N_n$. We need to prove that $\phi: UVB_n \rightarrow UVB(N_n)$ is an epimorphism. Let β be an element of $UVB(N_n)$. Since S_n is both isomorphic to the subgroup of UVB_n generated by the ρ_i generators, and to the subgroup of $UVB(N_n)$ generated by the permutation automorphisms, we can assume that for β the permutation is trivial, i.e., $\beta(x_i) = x_i^{g_i}$. We define ε_{ij} to be $\phi(\lambda_{ij})$ as in Proposition 3.12, and we prove that β is a product of such automorphisms. We recall that every element of N_n , can be written in the form

$$x_1^{\alpha_1} \cdots x_n^{\alpha_n} [x_1, x_2]^{\beta_{1,2}} [x_1, x_3]^{\beta_{1,3}} \cdots [x_{n-1}, x_n]^{\beta_{n-1,n}}$$

with $\alpha_i, \beta_{j,k} \in \mathbb{Z}$, so $\beta(x_i)$ can be expressed as

$$\beta(x_i) = x_i^{x_1^{\alpha_1} \cdots x_n^{\alpha_n} [x_1, x_2]^{\beta_{1,2}} [x_1, x_3]^{\beta_{1,3}} \cdots [x_{n-1}, x_n]^{\beta_{n-1,n}}}.$$

In addition $[x_i, x_j]$ for $1 \leq i < j \leq n$ is a basis for $\gamma_2 F_n / \gamma_3 F_n$, that is central in N_n , so commutators commute among themselves and with generators x_i . Using this fact, and the property of commutators that allows us to express $x_j^\alpha x_i^\beta$ as $x_i^\beta x_j^\alpha [x_i, x_j]^{-\alpha\beta}$, we can rewrite

$$\beta(x_i) = x_i^{x_1^{\alpha_1} \cdots x_n^{\alpha_n}}$$

where $\alpha_i = 0$.

In particular we can assume that

$$\beta(x_1) = x_1^{x_2^{a_2} \cdots x_n^{a_n}}.$$

We define a new automorphism β_1 multiplying β for $\varepsilon_{12}^{-a_2} \cdots \varepsilon_{1n}^{-a_n}$. We have that $\beta_1(x_1) = x_1$, and $\beta_1(x_2) = x_2^{x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n}}$, with $b_2 = 0$. Then again we define a new automorphism $\beta_2 = \beta_1 \varepsilon_{21}^{-b_1} \varepsilon_{23}^{-b_3} \cdots \varepsilon_{2n}^{-b_n} = \beta \varepsilon_{12}^{-a_2} \cdots \varepsilon_{1n}^{-a_n} \varepsilon_{21}^{-b_1} \varepsilon_{23}^{-b_3} \cdots \varepsilon_{2n}^{-b_n}$ that fixes x_1 and x_2 .

Carrying on in this way for n steps we get to an automorphism

$$\beta_n = \beta_{n-1} \varepsilon_{n1}^{-z_1} \cdots \varepsilon_{n,n-1}^{-z_n} = \beta \prod_{j=1}^n \varepsilon_{1j}^{-a_j} \prod_{j=1}^n \varepsilon_{2j}^{-b_j} \cdots \prod_{j=1}^n \varepsilon_{nj}^{-z_j}$$

setting $\varepsilon_{ii} = 1$. The automorphism β_n is the identity automorphism: then β is a product of ε_{ij} automorphisms, hence it has a pre-image in UVB_n . \square

3.2. The knot group. Let L be a fused link. Then there exists a unrestricted virtual braid $\hat{\beta}$ such that its closure $\hat{\beta}$ is equivalent to L .

Definition 3.15. The fused link group $G(L)$ is the group given by the presentation

$$\left\langle x_1, \dots, x_n \left| \begin{array}{ll} \phi(\beta)(x_i) = x_i & \text{for } i \in \{1, \dots, n\}, \\ [x_i, [x_k, x_l]] = 1 & \text{for } i, k, l \text{ not necessarily distinct} \end{array} \right. \right\rangle$$

where $\phi: UVB_n \rightarrow \text{Aut}(N_n)$ is the map from Proposition 3.11.

Proposition 3.16. *The fused link group is invariant under fused isotopy.*

Proof. According to [22] two unrestricted virtual braids have fused isotopic closures if and only if they are related by *braid moves* and *extended Markov moves*. We should check that under these moves the fused link group $G(L)$ of a fused link L does not change. This is the case. However a quicker strategy to verify the invariance of this group is to remark that it is a projection of the *welded link group* defined in [5, Section 5]. This last one being an invariant for welded links, we only have to do the verification for the second forbidden braid move, coming from relation $F2$. This invariance is guaranteed by the fact that ϕ preserves relation $F2$ as seen in Proposition 3.11. \square

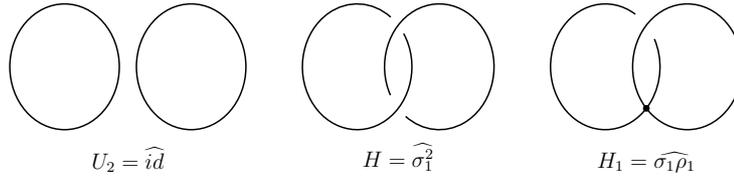


FIGURE 8. The group distinguishes the unlink U_2 from the Hopf link H , but does not distinguish the Hopf link with two classical crossings H from the one with a classical and a virtual crossing H_1 . In fact: $G(U_2) = N_2$, while $G(H) = G(H_1) = \mathbb{Z}^2$. We remark however that H and H_1 are distinguished by the virtual linking number.

Remark 3.17. This invariant does not distinguish the Hopf link with two classical crossings from the one with one virtual crossing (Figure 8). More generally, the knot group does not distinguish the closures of the following braids: let us consider $\lambda_{1,2}^\alpha \lambda_{2,1}^\beta$ and $\lambda_{1,2}^\gamma$, where γ is the greatest common divisor of α and β . The automorphisms associated to them are

$$\phi(\lambda_{1,2}^\alpha \lambda_{2,1}^\beta) : \begin{cases} x_1 \mapsto x_1 [x_1, x_2 [x_1, x_2]^{-\beta}]^\alpha = x_1 [x_1, [x_1, x_2]^{-\beta}]^\alpha [x_1, x_2]^\alpha = x_1 [x_1, x_2]^\alpha \\ x_2 \mapsto x_2 [x_1, x_2]^{-\beta}; \end{cases}$$

$$\phi(\lambda_{1,2}^\gamma) : \begin{cases} x_1 \mapsto x_1 [x_1, x_2]^\gamma \\ x_2 \mapsto x_2. \end{cases}$$

Then

$$G(\lambda_{1,2}^\alpha \lambda_{2,1}^\beta) = G(\lambda_{1,2}^\gamma) = \langle x_1, x_2 \mid [x_1, x_2]^\gamma = 1, [x_i, [x_k, x_l]] = 1 \text{ for } i, k, l \in \{1, 2\} \rangle.$$

However it distinguishes $L = \widehat{\sigma_1 \rho_1 \sigma_1^{-1} \rho_1}$, whose associated group is \mathbb{Z}^2 and U_2 , whose associated group is N_2 . On the other hand, as we saw above, $L = \hat{\beta}$ and U_2 have the same classical linking number.

4. OTHER QUOTIENTS

Several other quotients of virtual braid groups have been studied in the literature: we end this paper with a short survey on them, giving the structure of the corresponding pure subgroups and some results on their linearity.

4.1. Flat virtual braids. The study of flat virtual knots and links was initiated by Kauffman [19] and their braided counterpart was introduced in [20]. The category of flat virtual knots is identical in structure to what are called virtual strings by V. Turaev in [28] (remark that every virtual string is the closure of a flat virtual braid).

The flat virtual braids were introduced in [20] as quotients obtained from VB_n adding relations

$$(7) \quad \sigma_i^2 = 1, \quad \text{for } i = 1, \dots, n-1.$$

It is evident that FVB_n is a quotient of the free product $S_n * S_n$. In addition to relations coming from the two copies of S_n , in FVB_n we have mixed relations

$$(8) \quad s_i \rho_j = \rho_j s_i, \quad \text{for } |i - j| \geq 2,$$

$$(9) \quad \rho_i \rho_{i+1} s_i = s_{i+1} \rho_i \rho_{i+1}, \quad \text{for } i = 1, \dots, n - 1.$$

Let us consider natural map

$$\bullet \pi_V: VB_n \rightarrow FVB_n \text{ defined by } \sigma_i \mapsto s_i \text{ and } \rho_i \mapsto \rho_i,$$

setting s_i (for $i = 1, \dots, n - 1$) to be the image of σ_i in the corresponding quotient.

We call *flat virtual pure braid group* FVP_n the kernel of this map. If we consider the natural projection $f: VB_n \rightarrow FVB_n$, we have that $f(VP_n) = FVP_n$.

$$VP_n^+ = \langle \lambda_{ij} \mid 1 \leq i < j \leq n \rangle.$$

Proposition 4.1. *Let VP_n^+ the subgroup of VP_n defined as*

$$VP_n^+ = \langle \lambda_{ij} \mid 1 \leq i < j \leq n \rangle.$$

There is an isomorphism $FVP_n \cong VP_n^+$.

Proof. Recall that VP_n is generated by elements λ_{ij} as defined in Eq. (3) and (4). and with the following complete set of relations

$$(RS1) \quad \lambda_{ij} \lambda_{kl} = \lambda_{kl} \lambda_{ij},$$

$$(RS2) \quad \lambda_{ki} (\lambda_{kj} \lambda_{ij}) = (\lambda_{ij} \lambda_{kj}) \lambda_{ki}.$$

Hence, FVP_n is generated by the images of these elements under f . Then

$$f(\lambda_{i,i+1}) = \rho_i s_i, \quad f(\lambda_{i+1,i}) = s_i \rho_i$$

and we can check that

$$f(\lambda_{i+1,i}) = (f(\lambda_{i,i+1}))^{-1}, \text{ for } i = 1, \dots, n - 1.$$

In the general case

$$f(\lambda_{j,i}) = (f(\lambda_{i,j}))^{-1}, \text{ for } 1 \leq i < j - 1 \leq n - 1.$$

This means that FVP_n is generated by elements

$$f(\lambda_{i,j}), \text{ for } 1 \leq i < j \leq n.$$

To find the set of defining relations we must replace elements $\lambda_{j,i}$ with elements $\lambda_{i,j}^{-1}$ whenever $i < j$, in all of VP_n 's defining relations. The resulting relations are consequences of relations in VP_n^+ . Hence, the map $FVP_n \rightarrow VP_n^+$, which is defined by the rule

$$f(\lambda_{i,j}) \mapsto \lambda_{i,j}, \text{ for } 1 \leq i < j \leq n$$

is an isomorphism. □

Remark 4.2. For $n = 3$, the group

$$FVP_3 = \langle \lambda_{12}, \lambda_{13}, \lambda_{23} \mid \lambda_{12}^{-1} (\lambda_{23} \lambda_{13}) \lambda_{12} = \lambda_{13} \lambda_{23} \rangle$$

is the HNN-extension of the free group $\langle \lambda_{13}, \lambda_{23} \rangle$ of rank 2 with stable element λ_{12} and with associated subgroups $A = \langle \lambda_{23} \lambda_{13} \rangle$ and $B = \langle \lambda_{13} \lambda_{23} \rangle$, which are isomorphic to the infinite cyclic group. Moreover, the group FVP_3 is isomorphic to the free product $\mathbb{Z}^2 * \mathbb{Z}$.

Let us recall that there is another remarkable surjection of the virtual braid group VB_n in the symmetric group S_n , which sends σ_i into 1 and ρ_i into ρ_i : the kernel of this map is denoted by H_n in [4]. In the same way we can define the group FH_n as the kernel of the homomorphism $\mu : FVB_n \rightarrow S_n$, which is defined as follows:

$$\mu(s_i) = 1, \quad \mu(\rho_i) = \rho_i, \quad i = 1, 2, \dots, n-1.$$

Now let us define, for $i = 1, \dots, n-1$:

$$(10) \quad \begin{aligned} y_{i,i+1} &= s_i, \\ y_{i+1,i} &= \rho_i s_i \rho_i. \end{aligned}$$

For $1 \leq i < j-1 \leq n-1$:

$$(11) \quad \begin{aligned} y_{i,j} &= \rho_{j-1} \cdots \rho_{i+1} s_i \rho_{i+1} \cdots \rho_{j-1}, \\ y_{j,i} &= \rho_{j-1} \cdots \rho_{i+1} \rho_i s_i \rho_i \rho_{i+1} \cdots \rho_{j-1}. \end{aligned}$$

It is not difficult to prove that these elements belong FH_n and that:

Theorem 4.3. *The group FH_n admits a presentation with the generators $y_{k,l}$, $1 \leq k \neq l \leq n$, and the defining relations:*

$$(12) \quad y_{k,l}^2 = 1,$$

$$(13) \quad y_{i,j} y_{k,l} = y_{k,l} y_{i,j} \Leftrightarrow (y_{i,j} y_{k,l})^2 = 1,$$

$$(14) \quad y_{i,k} y_{k,j} y_{i,k} = y_{k,j} y_{i,k} y_{k,j} \Leftrightarrow (y_{i,k} y_{k,j})^3 = 1,$$

where distinct letters stand for distinct indices.

Proof. We know that FVB_n is the quotient of VB_n by the set of relations

$$\sigma_i^2 = 1, \quad i = 1, 2, \dots, n-1.$$

Hence, in FH_n relation 12 holds. All other relations follow from the homomorphism $H_n \rightarrow FH_n$ and the group presentation for H_n provided in [4]. \square

Corollary 4.4. *The group FVB_n is linear and it has solvable word problem.*

Proof. From the decomposition $VB_n = VP_n \rtimes S_n$ we have that $FVB_n = FH_n \rtimes S_n$, where FH_n is a finitely generated Coxeter group. The linearity and the solvability of the word problem therefore follow from the fact that all finitely generated Coxeter groups have these properties and that these properties pass to finite extensions. \square

4.2. Flat welded braids. In a similar way we can define the flat welded braid group FWB_n as the quotient obtained from WB_n adding relations

$$(15) \quad \sigma_i^2 = 1, \quad \text{for } i = 1, \dots, n-1,$$

and we can consider the map

- $\pi_W : WB_n \rightarrow FWB_n$ defined by $\sigma_i \mapsto s_i$ and $\rho_i \mapsto \rho_i$;

setting s_i (for $i = 1, \dots, n-1$) to be the image of σ_i in the corresponding quotient.

In FWB_n , in addition to relations (8) and (9), we also have relations coming from relations of type $F1$, *i.e.*,

$$(16) \quad s_{i+1}s_i\rho_{i+1} = \rho_i s_{i+1}s_i, \quad \text{for } i = 1, \dots, n-1.$$

In FWB_n relations (15) and (16) imply that also relations of type $F2$ hold, since from $\rho_i s_{i+1}s_i = s_{i+1}s_i\rho_{i+1}$ one gets $s_i s_{i+1}\rho_i = \rho_{i+1}s_i s_{i+1}$.

Adapting Theorem 2.7 one can easily verify that FWP_n is isomorphic to $\mathbb{Z}^{n(n-1)/2}$. As a straightforward consequence of Theorem 2.4, we can describe the structure of FWB_n .

Proposition 4.5. *Let $\mathbb{Z}^{n(n-1)/2}$ be the free abelian group of rank $n(n-1)/2$. Let us denote by $x_{i,j}$ for $1 \leq i \neq j \leq n$ a possible set of generators. The group FWB_n is isomorphic to $\mathbb{Z}^{n(n-1)/2} \rtimes S_n$, where S_n acts by permutation on the indices of generators of $\mathbb{Z}^{n(n-1)/2}$ (setting $x_{j,i} := x_{i,j}$ for $1 \leq i \neq j \leq n$).*

Proof. Let us recall how elements $\lambda_{i,j}$ in UVB_n were defined.

For $i = 1, \dots, n-1$:

$$\begin{aligned} \lambda_{i,i+1} &= \rho_i \sigma_i^{-1}, \\ \lambda_{i+1,i} &= \rho_i \lambda_{i,i+1} \rho_i = \sigma_i^{-1} \rho_i. \end{aligned}$$

For $1 \leq i < j-1 \leq n-1$:

$$\begin{aligned} \lambda_{ij} &= \rho_{j-1} \rho_{j-2} \cdots \rho_{i+1} \lambda_{i,i+1} \rho_{i+1} \cdots \rho_{j-2} \rho_{j-1}, \\ \lambda_{ji} &= \rho_{j-1} \rho_{j-2} \cdots \rho_{i+1} \lambda_{i+1,i} \rho_{i+1} \cdots \rho_{j-2} \rho_{j-1}. \end{aligned}$$

Relations (15) are therefore equivalent to relations $\lambda_{i,j} \lambda_{j,i} = 1$. Adding these relations and following verbatim the proof of Theorem 2.7 we get the statement. \square

4.3. Virtual Gauss braids. From the notion of flat virtual knot we can get the notion of Gauss virtual knot or simply Gauss knot. Turaev [29] introduced these knots under the name of ‘‘homotopy classes of Gauss words’’, while Manturov [25] used the name ‘‘free knots’’.

The ‘‘braided’’ analogous of Gauss knots, called *free virtual braid group on n strands*, was introduced in [26]. From now on we will be calling it *virtual Gauss braid group* and will denote by GVB_n .

The group of *virtual Gauss braids* GVB_n is the quotient of FVB_n by relations

$$s_i \rho_i = \rho_i s_i, \quad \text{for } i = 1, \dots, n-1.$$

Note also that the virtual Gauss braid group is a natural quotient of the *twisted virtual braid group*, studied for instance in [23].

Once again we can consider the homomorphism from GVB_n to S_n that sends each generator s_i and ρ_i in ρ_i . The *virtual Gauss pure braid group* GVP_n is defined to be the kernel of this map. Since this map admits a natural section GVB_n is isomorphic to $GVP_n \rtimes S_n$.

Adapting the proof of Theorem 2.7, we get the following.

Proposition 4.6. *The group GVP_n admits a presentation with generators λ_{kl} for $1 \leq k < l \leq n$ and the defining relations of FVP_n plus relations*

$$\lambda_{ij}^2 = 1, \quad \text{for } 1 \leq i < j \leq n.$$

Moreover as in the case of FVB_n also in the case of GVB_n we can consider the map $\mu : GVB_n \rightarrow S_n$, defined as follows:

$$\mu(s_i) = 1, \quad \mu(\rho_i) = \rho_i, \quad \text{for } i = 1, 2, \dots, n-1.$$

Let GH_n the kernel of $\mu : GVB_n \rightarrow S_n$ and $y_{k,l}$, defined in subsection 4.1: we can prove the following result.

Theorem 4.7. *The group GH_n admits a presentation with generators $y_{k,l}$, $1 \leq k < l \leq n$, and defining relations:*

$$(17) \quad y_{k,l}^2 = 1,$$

$$(18) \quad (y_{i,j} y_{k,l})^2 = 1,$$

$$(19) \quad (y_{i,k} y_{k,j})^3 = (y_{i,j} y_{k,j})^3 = (y_{i,k} y_{i,j})^3 = 1,$$

where distinct letters stand for distinct indices.

Proof. We know that GVB_n is the quotient of FVB_n by the set of relations

$$s_i \rho_i = \rho_i s_i, \quad i = 1, 2, \dots, n-1.$$

One can easily verify that it implies that $y_{j,i} = y_{i,j}$, $1 \leq i < j \leq n$. Hence, GH_n is generated by elements $y_{k,l}$, $1 \leq k < l \leq n$. If we rewrite the set of relations of FH_n in these generators we get the set of relations given in the statement. \square

As corollary, we have:

Corollary 4.8. *The group GVB_n is linear and it has solvable word problem.*

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