Enumeration of snakes and cycle-alternating permutations

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ENUMERATION OF SNAKES AND CYCLE-ALTERNATING PERMUTATIONS

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To the memory of Vladimir Arnol'd

ABSTRACT. Springer numbers are analogs of Euler numbers for the group of signed permutations. Arnol'd showed that they count some objects called snakes, which generalize alternating permutations. Hoffman established a link between Springer numbers, snakes, and some polynomials related with the successive derivatives of trigonometric functions.

The goal of this article is to give further combinatorial properties of derivative polynomials, in terms of snakes and other objects: cycle-alternating permutations, weighted Dyck or Motzkin paths, increasing trees and forests. We obtain some exponential generating functions in terms of trigonometric functions, and some ordinary generating functions in terms of J-fractions. We also define natural q-analogs, make a link with normal ordering problems and the combinatorial theory of differential equations.

1. INTRODUCTION

It is well-known that the Euler numbers E_n defined by

$$\sum_{n=0}^{\infty} E_n \frac{z^n}{n!} = \tan z + \sec z \tag{1}$$

count alternating permutations in $\mathfrak{S}_n$, *i.e.* σ such that $\sigma_1 > \sigma_2 < \sigma_3 > \dots \sigma_n$. The study of these, as well as other classes of permutations counted by E_n, is a vast topic in enumerative combinatorics, see the survey of Stanley [27]. By a construction due to Springer [26], there is an integer $K(W)$ defined for any finite Coxeter group W such that $K(\mathfrak{S}_n) = E_n$. As for the groups of signed permutations, Arnol'd [1] introduced some particular kind of signed permutations called snakes, counted by the numbers $K(\mathfrak{S}_n^B)$ and $K(\mathfrak{S}_n^D)$. Thus snakes can be considered as a "signed analog" of alternating permutations. Algorithmically, Arnol'd [1] gives a recursive method to compute these integers with triangular arrays similar to the Seidel-Entriger triangle of Euler numbers E_n (see for example [14]).

In this article, we are mostly interested in the numbers $S_n = K(\mathfrak{S}_n^B)$, which were previously considered by Glaisher [12, §§ 109 and 119] in another context. Springer [26] shows that they satisfy

$$\sum_{n=0}^{\infty} S_n \frac{z^n}{n!} = \frac{1}{\cos z - \sin z}, \tag{2}$$

and we call S_n the nth Springer number (although there is in theory a Springer number associated with each finite Coxeter group, this name without further specification

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usually refers to type B). Another link between snakes and trigonometric functions has been established by Hoffman [16], who studies some polynomials $P_n(t)$ and $Q_n(t)$ related with the successive derivatives of tan and sec by:

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n} \tan x = P_n(\tan x), \qquad \frac{\mathrm{d}^n}{\mathrm{d}x^n} \sec x = Q_n(\tan x) \sec x. \tag{3}$$

Among various other results, Hoffman proves that $Q_n(1) = S_n$ and $P_n(1) = 2^n E_n$, moreover this number $2^n E_n$ counts some objects called β-snakes, which are a superset of type B snakes.

The main goal of this article is to give combinatorial models of these derivative polynomials P_n and Q_n (as well as another sequence $Q_n^{(a)}$ related the with the ath power of sec), in terms of several objects:

- Snakes (see Definition 3.2). Besides type B snakes and β-snakes of Arnol'd, we introduce another variant.
- Cycle-alternating permutations (see Definition 3.7). These are essentially the image of snakes via Foata's fundamental transform [22].
- Weighted Dyck prefixes and weighted Motzkin paths (see Section 3). These are two different generalizations of some weighted Dyck paths counted by E_n.
- Increasing trees and forests (see Section 4). The number E_{2n+1} counts increasing complete binary trees, and E_{2n} "almost complete" ones, our trees and forests are generalizations of these.

Various exponential and ordinary generating functions are obtained combinatorially using these objects. They will present a phenomenon similar to the case of Euler numbers E_n: they are given in terms of trigonometric functions for the exponential generating functions, and in terms of continued fractions for the ordinary ones. For example, in Theorem 3.17, we obtain bijectively a continued fraction for the (formal) Laplace transform of $(\cos z - t \sin z)^{-a}$. While this result was known analytically since long ago [28], we show here that it can be fully understood on the combinatorial point of view.

This article is organized as follows. In Section 2, we give first properties of the derivative polynomials, such as recurrence relations, generating functions, we also introduce a natural q-analog and link this with the normal ordering problem. In Section 3, we give combinatorial models of the derivative polynomials in terms of subset of signed permutations, and obtain the generating functions. In Section 4, we give combinatorial models of the derivative polynomials in terms of increasing trees and forests in two different ways: via the combinatorial theory of differential equations [21] and via the normal ordering problem and results from [11].

2. Derivative polynomials

The derivative polynomials for tangent and secant have been introduced by Knuth and Buckholtz [19], and several results such as the generating functions have been proved by Hoffman [16, 17].

2.1. **Definitions.** We generalize the polynomials $Q_n(t)$ defined in (3), by considering $Q_n^{(a)}(t)$ such that

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n} \sec^a x = Q_n^{(a)}(\tan x) \sec^a x. \tag{4}$$

Besides Q_n which is the case $a = 1$ in $Q_n^{(a)}$, the case $a = 2$ will be particularly interesting. We denote $R_n = Q_n^{(2)}$. Since $\sec^2 x$ is the derivative of $\tan x$, it follows that there is the simple relation $P_{n+1}(t) = (1+t^2)R_n(t)$.

By differentiating (3) and (4), we obtain the relations:

$$P_{n+1} = (1+t^2)P_n', \qquad Q_{n+1}^{(a)} = (1+t^2)\frac{\mathrm{d}}{\mathrm{d}t}Q_n^{(a)} + atQ_n^{(a)}, \tag{5}$$

together with $P_0 = t$ and $Q_0^{(a)} = 1$. Let us give other elementary properties which are partly taken from [16].

Proposition 2.1. *The polynomial $Q_n^{(a)}$ has the same parity as n, i.e. $Q_n^{(a)}(-t) = (-1)^n Q_n^{(a)}(t)$, and P_n has different parity. We have:*

$$P_n(1) = 2^n E_n, \qquad Q_n(1) = S_n, \qquad R_n(1) = 2^n E_{n+1}. \tag{6}$$

(Recall that $Q_n = Q_n^{(1)}$ and $R_n = Q_n^{(2)}$.) The generating functions are:

$$\sum_{n=0}^{\infty} P_n(t)\frac{z^n}{n!} = \frac{\sin z + t\cos z}{\cos z - t\sin z}, \qquad \sum_{n=0}^{\infty} Q_n^{(a)}(t)\frac{z^n}{n!} = \frac{1}{(\cos z - t\sin z)^a}. \tag{7}$$

Proof. The generating function of $\{Q_n^{(a)}\}_{n\geq 0}$ is not present in the reference [16], but can be obtained in the same way as in the particular case $a = 1$ (which is in [16]). By a Taylor expansion and a trigonometric addition formula, we have:

$$\sum_{n=0}^{\infty} Q_n^{(a)}(\tan u)\sec^a u \frac{z^n}{n!} = \sec^a(u+z) = \frac{\sec^a u}{(\cos z - \tan u \sin z)^a}, \tag{8}$$

We can divide on both sides by $\sec^a u$, let $t = \tan u$, and the result follows. □

These exponential generating functions will be obtained combinatorially in Section 3. It is interesting to note that $P(z,t) = \sum P_n z^n/n!$ is a Möbius transformation as a function of t, in such a way that $z \mapsto (t \mapsto P(z,t))$ is a group homomorphism from $\mathbb{R}$ to the elliptic Möbius transformations of $\mathbb{C}$ fixing i and $-i$. This has an explanation through the fact that

$$P(z,t) = \tan(\arctan t + z), \tag{9}$$

and consequently, as observed in [16]:

$$P(z, P(z',t)) = P(z+z',t), \tag{10}$$

which is the concrete way to say that $z \mapsto (t \mapsto P(z,t))$ is a group homomorphism.

2.2. **Operators and q-analogs.** Let D and U be linear operators acting on polynomials in the variable t by

$$D(t^n) = [n]_q t^{n-1}, \qquad U(t^n) = t^{n+1}, \tag{11}$$

where $[n]_q = \frac{1-q^n}{1-q}$ is the usual q-integer. The first one is known as the *q-derivative* or *Jackson derivative*, and an important relation is $DU - qUD = I$, where I is the identity.

Definition 2.2. Our q-analogs of the polynomials Q_n and R_n are defined by:

$$Q_n(t,q) = (D + UDU)^n 1, \qquad R_n(t,q) = (D + DUU)^n 1, \tag{12}$$

where 1 should be seen as t^0, the operators acting on polynomials in t.

Remark 2.3. Of course, there are many choices for defining a q-analog, leading to various different properties. We have to mention that another choice can be obtained as a particular case of the multivariable tangent and secant q-derivative polynomials studied by Foata and Han in [10].

Proposition 2.4. *We have*

$$Q_n(-t,q) = (-1)^n Q_n(t,q), \qquad R_n(-t,q) = (-1)^n R_n(t,q), \tag{13}$$

moreover $Q_n(t,q)$ and $R_n(t,q)$ are polynomials in t and q with nonnegative coefficients such that $Q_n(t,1) = Q_n(t)$ and $R_n(t,1) = R_n(t)$.

Proof. From $DU - qUD = I$, we can write

$$D + UDU = (I + qU^2)D + U, \qquad D + DUU = (I + q^2U^2)D + (1+q)U, \tag{14}$$

then from (12) and (14) it follows that

$$Q_{n+1}(t,q) = (1+qt^2)D\big(Q_n(t,q)\big) + tQ_n(t,q), \tag{15}$$

$$R_{n+1}(t,q) = (1+q^2t^2)D\big(R_n(t,q)\big) + (1+q)tR_n(t,q), \tag{16}$$

which generalize the recurrences for Q_n and R_n ($a = 1$ and $a = 2$ in (5)). The elementary properties given in the proposition can be proved recursively using (15) and (16). ☐

The fact that $Q_n(t,q)$ and $R_n(t,q)$ have simple expressions in terms of D and U permits to make a link with *normal ordering* [2]. Starting from an expression $f(D,U)$ of D and U, the problem is to find some coefficients $c_{i,j}$ such that we have the *normal form*:

$$f(D,U) = \sum_{i,j\geq 0} c_{i,j} U^i D^j. \tag{17}$$

When $f(D,U)$ is a polynomial, we can always find such coefficients using the commutation relation $DU - qUD = I$, and a finite number of them are non-zero. For example, this is what we have done in (14). In Section 4, we will use some general results on normal ordering to give combinatorial models of P_n and Q_n (when $q = 1$).

What is interesting about normal ordering is that having the coefficients $c_{i,j}$ gives more insight on the quantity $f(D,U)$. For example, consider the case of $Q_n(t,q)$ and let $f(D,U) = (D + UDU)^n$, then we have

$$Q_n(t,q) = (D+UDU)^n 1 = \sum_{i,j\geq 0} c_{i,j} U^i D^j 1 = \sum_{i\geq 0} c_{i,0} t^i, \tag{18}$$

i.e. we can directly obtain the coefficients of $Q_n(t,q)$ through this normal form.

2.3. **Continued fractions.** Let D_1 and U_1 be the matrices of the operators D and U in the basis $\{t^i\}_{i\in\mathbb{N}}$. Moreover, let W_1 be the row vector $(t^i)_{i\in\mathbb{N}}$ and V_1 be the column vector $(\delta_{i,0})_{i\in\mathbb{N}}$ where we use Kronecker's delta. We have $W_1 = (1, t, t^2, \dots)$, and:

$$D_1 = \begin{pmatrix} 0 & [1]_q & & (0) \\ & 0 & [2]_q & \\ & & 0 & \ddots \\ (0) & & & \ddots \end{pmatrix}, \qquad U_1 = \begin{pmatrix} 0 & & & (0) \\ 1 & 0 & & \\ & 1 & 0 & \\ (0) & & \ddots & \ddots \end{pmatrix}, \qquad V_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}. \tag{19}$$

From the previous definitions, it follows that:

$$Q_n(t,q) = W_1(D_1 + U_1D_1U_1)^nV_1, \qquad R_n(t,q) = W_1(D_1 + D_1U_1U_1)^nV_1, \tag{20}$$

$$D_1U_1 - qU_1D_1 = I, \qquad W_1U_1 = tW_1, \qquad D_1V_1 = 0. \tag{21}$$

The point of writing this is that, from (20), we only need the relations in (21) to calculate $Q_n(t,q)$ and $R_n(t,q)$. Even more, if D_2, U_2, W_2, and V_2 are a second set of matrices and vectors satisfying relations in (21), we also have $Q_n(t,q) = W_2(D_2 + U_2D_2U_2)^nV_2$ and $R_n(t,q) = W_2(D_2 + D_2U_2U_2)^nV_2$. Indeed, when we have the relation (17) for a given $f(D,U)$, the coefficients $c_{i,j}$ are obtained only using the commutation relation, so the same identity also holds with either (D_1,U_1) or (D_2,U_2). We can take $W_2 = (1,0,0,\dots)$, $V_2 = V_1$, $D_2 = D_1$, and:

$$U_2 = \begin{pmatrix} t & & & (0) \\ 1 & tq & & \\ & 1 & tq^2 & \\ (0) & & \ddots & \ddots \end{pmatrix}. \tag{22}$$

Then we have indeed $D_2U_2 - qU_2D_2 = I$, $W_2U_2 = tW_2$, and $D_2V_2 = 0$. What is nice about this second set of matrices and vectors is that it enables us to make a link with continued fractions.

Definition 2.5. For any two sequences $\{b_h\}_{h\geq 0}$ and $\{\lambda_h\}_{h\geq 1}$, let $\mathfrak{J}(b_h,\lambda_h)$ denote

$$\mathfrak{J}(b_h,\lambda_h) = \cfrac{1}{1 - b_0z - \cfrac{\lambda_1z^2}{1 - b_1z - \cfrac{\lambda_2z^2}{\ddots}}}. \tag{23}$$

We will always use z as variable and h as index of the two sequences so that there should be no ambiguity in the notation.

These are called J-fractions, or Jacobi continued fractions. They are related with moments of orthogonal polynomials (see [29]), but let us give the results for $Q_n(t,q)$ and $R_n(t,q)$ before we give more details about this.

Proposition 2.6. *We have:*

$$\sum_{n=0}^{\infty} Q_n(t,q)z^n = \mathfrak{J}(b_h^Q,\lambda_h^Q), \qquad \sum_{n=0}^{\infty} R_n(t,q)z^n = \mathfrak{J}(b_h^R,\lambda_h^R), \tag{24}$$

where:

$$b_h^Q = tq^h([h]_q + [h+1]_q), \qquad b_h^R = tq^h(1+q)[h+1]_q, \tag{25}$$

$$\lambda_h^Q = (1 + t^2q^{2h-1})[h]_q^2, \qquad \lambda_h^R = (1+t^2q^{2h})[h]_q[h+1]_q. \tag{26}$$

Proof. First, observe that for any matrix $M = (m_{i,j})_{i,j\in\mathbb{N}}$, the product $W_2M^nV_2$ is the upper-left coefficient $(M^n)_{0,0}$ of M^n. We can obtain this coefficient of $W_2M^nV_2$ in the following way:

$$W_2M^nV_2 = \sum_{i_1,\dots,i_{n-1}\geq 0} m_{0,i_1}m_{i_1,i_2}\dots m_{i_{n-2},i_{n-1}}m_{i_{n-1},0}. \tag{27}$$

When the matrix is tridiagonal, we can restrict the sum to the indices $i_1,\dots,i_{n-1}$ such that $|i_j - i_{j+1}| \leq 1$, so that they are the successive heights in a Motzkin path.

Then (27) shows that $W_2M^nV_2$ is the generating function of Motzkin paths of length n with some weights given by the coefficients of M, and using a classical argument [11] it follows that $\sum_{n=0}^{\infty}(W_2M^nV_2)z^n = \mathfrak{J}(m_{h,h}, m_{h-1,h}m_{h,h-1})$.

In the present case, it suffices to check that $D_2 + U_2D_2U_2$ and $D_2 + D_2U_2U_2$ are tridiagonal and calculate explicitly their coefficients to obtain the result. See also (34) and (36) below for a more general result. □

In particular, it follows from the previous proposition that we have:

$$Q_{2n}(0,q) = E_{2n}(q), \quad R_{2n}(0,q) = E_{2n+1}(q), \tag{28}$$

where $E_{2n}(q)$ and $E_{2n+1}(q)$ are respectively the q-secant and q-tangent numbers defined by Han, Randrianarivony, Zeng [15] using continued fractions.

Remark 2.7. From the fact that $Q_n(1) = S_n$ and $R_n(1) = 2^nE_{n+1}$, the previous proposition implies

$$\sum_{n=0}^{\infty} S_nz^n = \mathfrak{J}(2h+1, 2h^2), \qquad \sum_{n=0}^{\infty} E_{n+1}z^n = \mathfrak{J}(h+1, \tfrac{h(h+1)}{2}). \tag{29}$$

We found only one reference mentioning the latter continued fraction, Sloane's OEIS [25], but there is little doubt it can be proved by classical methods. For example, using a theorem of Stieltjes and Rogers [13, Theorem 5.2.10], the continued fraction can probably be obtained through an addition formula satisfied by the exponential generating function of $\{E_{n+1}\}_{n\geq 0}$ (this function is the derivative of $\tan z + \sec z$, *i.e.* $(1-\sin z)^{-1}$). Combinatorially, the result can be proved using *André trees* [9, Section 5] and the bijection of Françon and Viennot [13, Chapter 5]. More generally, this shows that if we add a parameter x, then $\mathfrak{J}(h+1, x\frac{h(h+1)}{2})$ is the generating function of the *André polynomials* defined by Foata and Schützenberger [9].

An important property of J-fractions is the link with moments of (formal) orthogonal polynomials, and we refer to [11, 29] for the relevant combinatorial facts. We consider here the *continuous dual q-Hahn polynomials* $p_n(x; a, b, c|q)$, or $p_n(x)$ for short. In terms of the Askey-Wilson polynomials which depend on one other parameter d, $p_n(x)$ is just the specialization $d = 0$ (see [20] for the definitions of these classical sequences, but our notations differ, in particular because of a rescaling $x \to x/2$). We have the three-term recurrence relation

$$xp_n(x) = p_{n+1}(x) + \left(a + \tfrac{1}{a} - A_n - C_n\right)p_n(x) + A_{n-1}C_np_{n-1}(x), \tag{30}$$

together with $p_{-1}(x) = 0$ and $p_0(x) = 1$, where

$$A_n = \tfrac{1}{a}(1-abq^n)(1-acq^n) \quad \text{and} \quad C_n = a(1-q^n)(1-bcq^{n-1}). \tag{31}$$

The $p_n(x)$ are orthogonal with respect to the scalar product $(f,g) \mapsto L(fg)$ where L is the linear form such that

$$\sum_{n=0}^{\infty} L(x^n)z^n = \mathfrak{J}\left(a + \tfrac{1}{a} - A_h - C_h, A_{h-1}C_h\right). \tag{32}$$

The quantity $L(x^n)$ is called the nth *moment* of the orthogonal sequence $\{p_n(x)\}_{n\geq 0}$. Let $\mu_n(a,b,c)$ denote this nth moment of the continuous dual q-Hahn polynomials $p_n(x)$. An elementary calculation shows the following:

Proposition 2.8. *We have*

$$Q_n(t,q) = \frac{\mu_n(i\sqrt{q}, -i\sqrt{q}, t)}{(1-q)^n}, \qquad R_n(t,q) = \frac{\mu_n(iq,-iq,t)}{(1-q)^n}. \tag{33}$$

Proof. We have to identify the ordinary generating functions of both sides in each identity. This is possible because we know in each case the explicit form of the J-fraction expansion, from (32) and Proposition 2.6. See also (34) and (35) below for a more general result. □

This very simple link between our polynomials and the moments $\mu_n(a,b,c)$ is one of the properties indicating that $Q_n(t,q)$ and $R_n(t,q)$ are interesting q-analogs of $Q_n(t)$ and $R_n(t)$. Note that in view of (15) and (16), it is tempting to define a q-analog $Q_n^{(a)}(t,q)$ of $Q_n^{(a)}(t)$ by the recurrence:

$$Q_{n+1}^{(a)}(t,q) = (1+q^a t^2) D\Big(Q_n^{(a)}(t,q)\Big) + [a]_q t Q_n^{(a)}(t,q), \tag{34}$$

together with $Q_0^{(a)}(t,q) = 1$. Though we will not study these apart from the particular cases $a=1$ and $a=2$, it is worth mentioning that a natural generalization of the previous proposition holds, more precisely we have:

$$Q_n^{(a)}(t,q) = \frac{\mu_n(iq^{a/2}, -iq^{a/2}, t)}{(1-q)^n}, \tag{35}$$

which can be proved by calculating explicitly the matrix $(I+q^aU_2^2)D_2 + [a]_q U_2$. The entries of this matrix show that the generating function of $\{Q_n^{(a)}(t,q)\}$ has a J-fraction expansion with coefficients:

$$b_h = tq^h\big([h-1+a]_q + q^{a-1}[h+1]_q\big), \qquad \lambda_h = (1+t^2q^{2h-2+a})[h]_q[h-1+a]_q. \tag{36}$$

Indeed we have the same coefficients if we replace (a,b,c) with $(iq^{a/2}, -iq^{a/2}, t)$ in (31) and (32).

3. Combinatorial models via signed permutations

Definition 3.1. We denote $[n] = \{1\dots n\}$ and $[[n]] = \{-n\dots -1\}\cup\{1\dots n\}$. A *signed permutation* is a permutation π of $[[n]]$ such that $\pi(-i) = -\pi(i)$ for any $i\in[[n]]$. It will be denoted $\pi = \pi_1\dots\pi_n$ where $\pi_i = \pi(i)$, and $\mathfrak{S}_n^{\pm}$ is the set of all such π. We will also use the cycle notation, indicated by parenthesis: for example $\pi = 3,-1,2,4$ is $(1,3,2,-1,-3,-2)(4)(-4)$ in cycle notation.

We consider here two classes of signed permutations: snakes and cycle-alternating permutations. These will give combinatorial models of the derivative polynomials. As for the q-analog, going through other objects (weighted Dyck prefixes) we can obtain a statistic on cycle-alternating permutations following the parameter q.

We note that Elizalde and Deutsch defined *cycle up-down permutations* in [7], our cycle-alternating permutations are a variant.

3.1. Snakes.

There are several types of snakes to be distinguished, so let us first give the definition:

Definition 3.2. A signed permutation $\pi = \pi_1\dots\pi_n$ is a *snake* if $\pi_1 > \pi_2 < \pi_3 > \dots\pi_n$. We denote by $\mathcal{S}_n \subset \mathfrak{S}_n^{\pm}$ the set of snakes of size n. Let $\mathcal{S}_n^0 \subset \mathcal{S}_n$ be the subset containing the snakes π satisfying $\pi_1 > 0$, and $\mathcal{S}_n^{00} \subset \mathcal{S}_n^0$ be the subset containing the snakes π satisfying $\pi_1 > 0$ and $(-1)^n\pi_n < 0$.

The sets $\mathcal{S}_n$ and $\mathcal{S}_n^0$ were introduced by Arnol'd [1], under the name "β-snake" and "snakes of type B", respectively. The third set $\mathcal{S}_n^{00}$ is a new variant.

It is practical to make a convention for the values π_0 and π_{n+1}. If $\pi \in \mathcal{S}_n$ (respectively, $\pi \in \mathcal{S}_n^0$, $\pi \in \mathcal{S}_n^{00}$) we define $\pi_0 = -(n+1)$ and $\pi_{n+1} = (-1)^n(n+1)$ (respectively, $\pi_0 = 0$ and $\pi_{n+1} = (-1)^n(n+1)$, $\pi_0 = \pi_{n+1} = 0$). This is not consistent with the inclusions $\mathcal{S}_n^{00} \subset \mathcal{S}_n^0 \subset \mathcal{S}_n$, so we can think of these sets to be disjoint to avoid ambiguity in the sequel. Then, an element of any of these three sets is a $\pi \in \mathfrak{S}_n^{\pm}$ such that $\pi_0 < \pi_1 > \pi_2 < \pi_3 > \dots \pi_{n+1}$. For example in the case of $\mathcal{S}_n^{00}$, the condition $(-1)^n\pi_n < 0$ means that if n is odd, $\pi_n > \pi_{n+1} = 0$, and if n is even, $\pi_n < \pi_{n+1} = 0$. For clarity, we will write $\pi = (\pi_0), \pi_1, \dots, \pi_n, (\pi_{n+1})$.

Definition 3.3. Let π be a snake in one of the sets $\mathcal{S}_n$, $\mathcal{S}_n^0$, or $\mathcal{S}_n^{00}$, and π_0, π_{n+1} as above. We define a statistic $\mathrm{cs}(\pi)$ as the number of changes of sign through the values $\pi_0, \dots, \pi_{n+1}$, *i.e.* $\mathrm{cs}(\pi) = \#\{\, i \mid 0 \leq i \leq n,\ \pi_i\pi_{i+1} < 0 \,\}$.

Using this statistic, we have combinatorial models of the derivative polynomials.

Theorem 3.4. *For any $n \geq 0$, we have:*

$$P_n(t) = \sum_{\pi \in \mathcal{S}_n} t^{\mathrm{cs}(\pi)}, \qquad Q_n(t) = \sum_{\pi \in \mathcal{S}_n^0} t^{\mathrm{cs}(\pi)}, \qquad R_n(t) = \sum_{\pi \in \mathcal{S}_{n+1}^{00}} t^{\mathrm{cs}(\pi)}. \tag{37}$$

Proof. We check that the right-hand sides satisfy the recurrences in (5) (case $a = 1$ or $a = 2$), but we only detail the case of R_n (case $a = 2$), the other ones being similar. We have $R_0 = 1$, and this corresponds to the snake $(0), 1, (0)$. Also $R_1 = 2t$, and this corresponds to $(0), 1, -2, (0)$ and $(0), 2, -1, (0)$. Suppose the result is proved for R_{n-1}. Then, to prove it for R_n, we distinguish three kinds of elements in $\mathcal{S}_{n+1}^{00}$ (the convention is $\pi_0 = \pi_{n+2} = 0$). We will denote by π_i^- the number $\pi_i - 1$ if $\pi_i > 0$ and $\pi_i + 1$ if $\pi_i < 0$.

- First, suppose that $|\pi_{n+1}| = 1$, hence $\pi_{n+1} = (-1)^n$. Let $\pi' = \pi_1^- \dots \pi_n^-$. Depending on the parity of n, we have either $\pi_n < \pi_{n+1} = 1$ or $\pi_n > \pi_{n+1} = -1$, and it follows on one hand that $(-1)^n\pi_n < 0$, on the other hand that $\pi_n\pi_{n+1} < 0$. Thus $\pi' \in \mathcal{S}_n^{00}$, and $\mathrm{cs}(\pi') = \mathrm{cs}(\pi) - 1$. The map $\pi \mapsto \pi'$ is bijective, and with the recurrence hypothesis, it comes that the set of $\pi \in \mathcal{S}_{n+1}^{00}$ with $|\pi_{n+1}| = 1$ has generating function tR_{n-1}.
- Then, suppose that $\pi_1 = 1$, and let $\pi' = -\pi_2^-, \dots, -\pi_{n+1}^-$. From $\pi_1 > \pi_2$, we obtain $\pi_2 < 0$, hence $-\pi_2^- > 0$. It follows that $\pi' \in \mathcal{S}_n^{00}$. Moreover, since $\pi_1\pi_2 < 0$ we obtain $\mathrm{cs}(\pi') = \mathrm{cs}(\pi) - 1$. With the recurrence hypothesis, it comes that the set of $\pi \in \mathcal{S}_{n+1}^{00}$ with $\pi_1 = 1$ has generating function tR_{n-1}.
- Eventually, suppose that there is $j \in \{2 \dots n\}$ such that $|\pi_j| = 1$. We have either $\pi_{j-1} > \pi_j < \pi_{j+1}$ or $\pi_{j-1} < \pi_j > \pi_{j+1}$, and it follows that π_{j-1} and π_{j+1} have the same sign. We will obtain the term R'_{n-1} in the subcase where π_j have also the same sign as π_{j-1} and π_{j+1}, and the term $t^2R'_{n-1}$ otherwise. Let us prove the first subcase, the second will follow since it suffices to consider snakes of the first subcase where π_j is replaced with $-\pi_j$.

  Let $\pi' = \pi_1^-, \dots, \pi_{j-1}^-, -\pi_{j+1}^-, \dots, -\pi_{n+1}^-$. Then we can check that $\pi \mapsto (\pi', j)$ is a bijection between π of the first subcase, and couples (π', j) where $\pi' \in \mathcal{S}_n^0$ and $j \in \{2 \dots n\}$ is such that $\pi'_{j-1}\pi'_j < 0$. At the level of generating function, choosing a sign change in π' is done by differentiation with respect to t, and this explains why π of the first subcase are counted by R'_{n-1}.

Adding the different terms, we obtain $(1+t^2)R'_{n-1}+2tR_{n-1} = R_n$. This completes the recurrence. The difference for Q_n (respectively, P_n) is roughly that only the second and third cases (respectively, only the third case) are to be considered. $\square$

The exponential generating function of $\{Q_n(t)\}_{n\geq 0}$ can be directly obtained using snakes in the form

$$\sum_{n=0}^{\infty} Q_n(t)\frac{z^n}{n!} = \frac{\sec z}{1 - t\tan z}. \tag{38}$$

To prove this, let $\pi \in \mathcal{S}_n^0$, and consider its unique factorization $\pi = f_1 \dots f_k$, where each f_k is a maximal non-empty factor such that all its entries have the same sign. Each of these factor has odd length, except possibly the last one. So it is convenient to think that if the last factor f_k has odd size, there is also an empty factor f_{k+1} at the end, and consider the new factorization $\pi = f_1 \dots f_j$ (with $j = k$ or $j = k+1$). Note that $\operatorname{cs}(\pi) = j - 1$. The factorization proves (38), because π is built by assembling alternating permutations of odd size $f_1 \dots f_{j-1}$, and an alternating permutation of even size f_j.

Using snakes, we can also directly obtain that

$$\sum_{n=0}^{\infty} R_n(t)\frac{z^n}{n!} = \left(\sum_{n=0}^{\infty} Q_n(t)\frac{z^n}{n!}\right)^2. \tag{39}$$

Indeed, let $\pi = (0), \pi_1 \dots \pi_{n+1}, (0)$ be among snakes counted by $R_n(t)$, *i.e.* $\pi \in \mathcal{S}_{n+1}^{00}$. There is $j \in \{1 \dots n+1\}$ such that $|\pi_j| = n+1$. We define $\pi' = (0), \pi_1 \dots \pi_{j-1}, (\pi_j)$ and $\pi'' = (0), (-1)^n\pi_{n+1}, \dots, (-1)^n\pi_{j+1}, ((-1)^n\pi_j)$. After some relabelling, these define two snakes in $\mathcal{S}_{j-1}^0$ and $\mathcal{S}_{n-j+1}^0$. We have $\operatorname{cs}(\pi) = \operatorname{cs}(\pi') + \operatorname{cs}(\pi'')$, and π is built by assembling π' and π'', so this proves (39).

Another interesting question concerning snakes is the following. For a given permutation $\sigma = \sigma_1 \dots \sigma_n \in \mathfrak{S}_n$, the problem is to choose signs $\epsilon = \epsilon_1 \dots \epsilon_n \in \{\pm 1\}^n$ such that $\pi = \epsilon_1\sigma_1, \dots, \epsilon_n\sigma_n$ is a snake. Arnol'd [1] described the possible choices such that $\pi \in \mathcal{S}_n^0$ in terms of ascent and descent sequences in the permutation. Let $i \in \{2 \dots n-1\}$, then a case-by-case argument from [1] shows that:

- if $\sigma_{i-1} < \sigma_i < \sigma_{i+1}$, then $\epsilon_i \neq \epsilon_{i+1}$,
- if $\sigma_{i-1} > \sigma_i > \sigma_{i+1}$, then $\epsilon_{i-1} \neq \epsilon_i$, (40)
- if $\sigma_{i-1} > \sigma_i < \sigma_{i+1}$, then $\epsilon_{i-1} = \epsilon_{i+1}$.

Following this, we can answer the problem of building a snake from a permutation in terms of some permutations statistics. We can apply this argument to the three sets $\mathcal{S}_n$, $\mathcal{S}_n^0$, and $\mathcal{S}_n^{00}$, essentially by varying the conventions on $\sigma(0)$ and $\sigma(n+1)$.

Definition 3.5. Let $\sigma = \sigma_1 \dots \sigma_n \in \mathfrak{S}_n$, we make the convention that $\sigma_0 = \sigma_{n+1} = n+1$. We also need to consider other conventions, so let $\mathfrak{S}_n^0$ and $\mathfrak{S}_n^{00}$ be "copies" of the set $\mathfrak{S}_n$, where:

- If $\sigma \in \mathfrak{S}_n^0$, we set $\sigma_0 = 0$ and $\sigma_{n+1} = n+1$.
- If $\sigma \in \mathfrak{S}_n^{00}$, we set $\sigma_0 = \sigma_{n+1} = 0$.

Let σ be in one of the sets $\mathfrak{S}_n$, $\mathfrak{S}_n^0$, or $\mathfrak{S}_n^{00}$. An integer $i \in [n]$ is a *valley* of σ if $\sigma_{i-1} > \sigma_i < \sigma_{i+1}$, a *peak* if $\sigma_{i-1} < \sigma_i > \sigma_{i+1}$, a *double descent* if $\sigma_{i-1} > \sigma_i > \sigma_{i+1}$, and a *double ascent* if $\sigma_{i-1} < \sigma_i < \sigma_{i+1}$. Let $\operatorname{va}(\sigma)$ denote the number of valleys in σ, $\operatorname{pk}(\sigma)$ denote the number of peaks in σ, and $\operatorname{dda}(\sigma)$ denote the number of double descents and double ascents in σ.

For example, we see $1,3,2 \in \mathfrak{S}_3^0$ as $(0),1,3,2,(4)$. Then 3 is a valley. On the other hand, we see $1,3,2 \in \mathfrak{S}_3^{00}$ as $(0),1,3,2,(0)$. Then 3 is a double descent.

We note that the result below has also been obtained by Ma [23] (in the case of P_n and Q_n).

Proposition 3.6. *When $n \geq 1$, we have:*

$$P_n(t) = \sum_{\sigma \in \mathfrak{S}_n} t^{\mathrm{dda}(\sigma)}(1+t^2)^{\mathrm{va}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n^{00}} t^{\mathrm{dda}(\sigma)}(1+t^2)^{\mathrm{pk}(\sigma)}, \tag{41}$$

$$Q_n(t) = \sum_{\sigma \in \mathfrak{S}_n^{0}} t^{\mathrm{dda}(\sigma)}(1+t^2)^{\mathrm{va}(\sigma)}, \qquad R_n(t) = \sum_{\sigma \in \mathfrak{S}_{n+1}^{00}} t^{\mathrm{dda}(\sigma)}(1+t^2)^{\mathrm{va}(\sigma)}. \tag{42}$$

The last two identities are also true when $n = 0$.

Proof. This follows from Theorem 3.4 and (40), which now can be extended to the cases $i = 1$ or $i = n$ using the conventions on σ_0 and σ_{n+1} (note that we also need to consider ϵ_0 and ϵ_{n+1}). Let us first prove the case of P_n, $\sigma \in \mathcal{S}_n$ and $\pi \in \mathfrak{S}_n$. We have $\epsilon_0 = -1$ since we require that $\pi_0 = -(n+1)$. The first two rules in (40) show that to each double ascent or double descent we can associate a sign change in π, and that the only remaining choices to be done concern the valleys. At a valley $\sigma_{i-1} > \sigma_i < \sigma_{i+1}$, we can have either $\epsilon_{i-1} = \epsilon_i = \epsilon_{i+1}$, or $\epsilon_{i-1} = -\epsilon_i = \epsilon_{i+1}$, *i.e.* 0 or 2 sign changes. This proves the first equality in (41), and the second follows by taking the complement permutation. The cases of Q_n and R_n are proved similarly, except that we do not consider ϵ_0 and we have $\epsilon_1 = 1$. □

An important property of the statistics $\mathrm{va}(\sigma)$ and $\mathrm{dda}(\sigma)$ is that they can be followed through the bijection of Françon and Viennot (see for example the book [13]). Omitting details, this gives combinatorial proof of:

$$\sum_{n=0}^{\infty} Q_n(t) z^n = \mathfrak{J}\big(\ (2h+1)t,\ (1+t^2)h^2\ \big), \tag{43}$$

$$\sum_{n=0}^{\infty} R_n(t) z^n = \mathfrak{J}\big(\ (2h+2)t,\ (1+t^2)h(h+1)\ \big), \tag{44}$$

which is the particular case $q = 1$ of Proposition 2.6. Unfortunately, it seems difficult to follow the parameter q through this bijection, and obtain nice statistics on $\mathfrak{S}_n^0$ and $\mathfrak{S}_{n+1}^{00}$ corresponding to $Q_n(t,q)$ and $R_n(t,q)$.

3.2. **Cycle-alternating permutations.** From the combinatorial interpretations of derivative polynomials in terms of snakes, we can derive other ones using simple bijections. We need some general definitions concerning signed permutations.

Definition 3.7. Let $\pi \in \mathfrak{S}_n^{\pm}$. We denote $\mathrm{neg}(\pi)$ the number of $i \in [n]$ such that $\pi_i < 0$. For any orbit O of π, let $-O = \{-x \mid x \in O\}$. A *cycle* of π is an unordered pair of orbits $\{O_1, O_2\}$ such that $-O_1 = O_2$. It is called a *one-orbit cycle* if $O_1 = O_2$ and *two-orbit cycle* otherwise. (This is known to be a natural notion of cycle since there is a decomposition of any signed permutation into a product of disjoint cycles.) For any signed permutation π, its *arch diagram* is defined as follows: draw on the horizontal axis $2n$ nodes labelled by the integers in $[[n]]$ in increasing order, then for any i draw an arch from i to $\pi(i)$ such that the arch is above the horizontal axis if $i \leq \pi(i)$ and below the axis otherwise.

See Figure 1 for an example of an arch diagram. Since the labels from $-n$ to n are in increasing order we do not need to explicitly write them. See also Figure 2 further for another example. We did not specify what happens with the fixed points in the arch diagram. Although this will not be really important in the sequel, we can choose to put a loop at each positive fixed point and a loop at each negative fixed point. This way, the arch diagram of a signed permutation is always centrally symmetric.

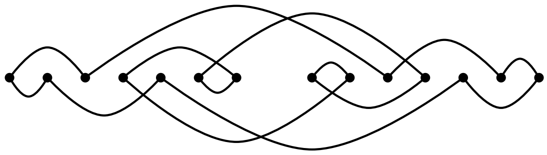


FIGURE 1. The arch diagram of $\pi = 2, -4, 6, 1, -3, 7, 5$.

Definition 3.8. Let $\pi \in \mathfrak{S}_n^{\pm}$. An integer $i \in [[n]]$ is a *cycle peak* of π if $\pi^{-1}(i) < i > \pi(i)$, and it is a *cycle valley* of π if $\pi^{-1}(i) > i < \pi(i)$. The signed permutation π is *cycle-alternating* if every $i \in [[n]]$ is either a cycle peak or a cycle valley. Let $\mathcal{C}_n \subset \mathfrak{S}_n^{\pm}$ denote the subset of cycle-alternating signed permutations, and $\mathcal{C}_n^{\circ} \subset \mathcal{C}_n$ be the subset of $\pi \in \mathcal{C}_n$ with only one cycle.

Lemma 3.9. *Let $\pi \in \mathcal{C}_n^{\circ}$. Then n is even (respectively, odd) if and only if π has a two-orbit cycle (respectively, a one-orbit cycle).*

Proof. In a cycle-alternating permutation, each orbit has even cardinality because there is an alternation between the cycle peaks and cycle valleys. So, if there are two opposite orbits of size n, n is even.

In the cycle-alternating permutation π, $-n$ is a cycle valley and n is a cycle peak. If there is one orbit, it can be written $\pi = (n, i_1, \dots, i_{n-1}, -n, -i_1, \dots, -i_{n-1})$ in cycle notation. Because of the alternation of cycles peaks and cycle valleys, it follows that n is odd. $\square$

Theorem 3.10. *We have :*

$$P_n(t) = \sum_{\pi \in \mathcal{C}_{n+1}^{\circ}} t^{\mathrm{neg}(\pi)}, \quad Q_n(t) = \sum_{\pi \in \mathcal{C}_n} t^{\mathrm{neg}(\pi)}, \quad R_n(t) = \sum_{\substack{\pi \in \mathcal{C}_{n+2}^{\circ} \\ \pi_1 > 0}} t^{\mathrm{neg}(\sigma)}. \tag{45}$$

Proof. There are simple bijections between snakes and cycle-alternating permutations to prove this from Theorem 3.4.

Let us begin with the case of P_n, so let $\pi \in \mathcal{S}_n$. Suppose first that n is odd, hence $\pi_0 = \pi_{n+1} = -(n+1)$. We can read π in cycle notation and consider the two-orbit cycle $\pi' = (\pi_0, \pi_1, \dots, \pi_n)(-\pi_0, -\pi_1, \dots, -\pi_n)$. Then $\pi' \in \mathcal{C}_{n+1}^{\circ}$. When n is even, we have $-\pi_0 = \pi_{n+1} = n+1$, and the bijection is defined by taking $\pi' = (\pi_0, \pi_1, \dots, \pi_n, -\pi_0, -\pi_1, \dots, -\pi_n)$. The map $\pi \mapsto \pi'$ is clearly invertible, and using the previous lemma, we have a bijection between $\mathcal{S}_n$ and $\mathcal{C}_{n+1}^{\circ}$. Let $i \in \{0, \dots, n\}$, then $\pi_i \pi_{i+1} < 0$ is equivalent to $\pi'(|\pi_i|) = -|\pi_{i+1}| < 0$, and it follows that $\mathrm{cs}(\pi) = \mathrm{neg}(\pi')$. This proves the first equality of the theorem.

The case of R_n is slightly different. In this proof, we will denote by π_i^+ the number $\pi_i + 1$ if $\pi_i > 0$, and $\pi_i - 1$ if $\pi_i < 0$. Let $\pi \in \mathcal{S}_{n+1}^{00}$. If $n+1$ is odd, hence $\pi_{n+1} > 0$, we define $\pi' = (1, \pi_1^+, \dots, \pi_{n+1}^+)(-1, -\pi_1^+, \dots, -\pi_{n+1}^+)$. If $n+1$ is even, hence $\pi_{n+1} < 0$, we define $\pi' = (1, \pi_1^+, \dots, \pi_{n+1}^+, -1, -\pi_1^+, \dots, -\pi_{n+1}^+)$. In each case, it is easily checked that this defines a bijection from $\mathcal{S}_{n+1}^{00}$ to the subset of $\pi \in \mathcal{C}_{n+2}^\circ$ satisfying $\pi_1 > 0$, so that a change of sign in π correspond to a $i > 0$ with $\pi'(i) < 0$.

The case of Q_n requires more attention. Let $\pi \in \mathcal{S}_n^0$, we need to split it into blocks that will correspond to the cycles of an element $\pi' \in \mathcal{C}_n$. This can be done by a variant of Foata's fundamental transform [22]. Let $a_1 < \dots < a_k$ be the left-to-right maxima of $|\pi_1| \dots |\pi_n| \in \mathfrak{S}_n$ (recall that u is a left-to-right maximum of $\sigma \in \mathfrak{S}_n$ when $\sigma(i) < \sigma(u)$ for any $i < u$). We consider the factors of π, $f_1 = \pi_{a_1} \dots \pi_{a_2 - 1}$, $f_2 = \pi_{a_2} \dots \pi_{a_3 - 1}$, up to $f_k = \pi_{a_k} \dots \pi_n$ (we can take the convention $a_{k+1} = n+1$). We form π' by putting together some cycles as follows: if f_i has odd length, it gives a one-orbit cycle $(\pi_{a_i} \dots \pi_{a_{i+1}-1}, -\pi_{a_i} \dots -\pi_{a_{i+1}-1})$, and if it has even length, it gives a two-orbit cycle $(\pi_{a_i} \dots \pi_{a_{i+1}-1})(-\pi_{a_i} \dots -\pi_{a_{i+1}-1})$. The definition of a_i shows that π_{a_i} is greater than $\pi_{a_i+1} \dots \pi_{a_{i+1}-1}$, and consequently these cycles are indeed alternating. The inverse bijection is easily deduced: we can write $\pi' \in \mathcal{C}_n$ as a product of cycles $\pi' = c_1 \dots c_k$, where the cycles are sorted so that their maximal entries are increasing. To each c_i we associate a word f_i so that c_i is either $(f_i, -f_i)$ or $(f_i)(-f_i)$, and such that the first letter of f_i is the maximum entry of c_i. Then we can form the snake $\pi = f_1, \epsilon_2 f_2, \dots, \epsilon_k f_k$, where the signs $\epsilon_2, \dots, \epsilon_k \in \{\pm 1\}$ are chosen so that π is alternating (there is a unique choice).

It is in order to give examples of the previous two bijections. If we start from a snake $\pi = (0), 4, -2, -1, -5, 3, -6, (7) \in \mathcal{S}_6^0$, the left-to-right maxima of $|\pi|$ are $1, 4, 6$, and this gives $\pi' = (4, -2, -1, -4, 2, 1)(-5, 3)(5, -3)(6, -6) \in \mathcal{C}_6$. As for the other direction, let $\pi' = (5, -2, 3, -5, 2, -3)(6, -1)(-6, 1)(7, -4)(-7, 4) \in \mathcal{C}_7$. It is a product of three cycles so that the snake π is formed by putting together three words $5, -2, 3$, and $6, -1$, and $7, -4$. We need to change the signs of the second and third words to obtain a snake, which is $\pi = (0), 5, -2, 3, -6, 1, -7, 4, (-8) \in \mathcal{S}_7^0$.

One can check on the examples that in both cases we have $\operatorname{cs}(\pi) = \operatorname{neg}(\pi')$, but this does not immediately follows the construction and need to be proved now. Let $a, b > 0$, such that $\pi'(a) = -b$, we can associate a sign change in π to each such pair (a, b). The two integers are in the same cycle c_i. If the word f_i is of the form $\dots, a, -b, \dots$ or $\dots, -a, b, \dots$, then either the factor $a, -b$ or $-a, b$ appears in the word $\epsilon_i f_i$, and consequently appears in π as a sign change. The other possibility is that $f_i = b, \dots, -a$ which only occur when f_i has even length, and $f_i = b, \dots, a$ which only occur when f_i has odd length. If b (respectively $-b$) appear in π, it is greater (respectively, smaller) than his neighbor entries, and after examining a few cases it comes that the same is true for a. It follows that there is a sign change in π between the last entry of $\epsilon_i f_i$ and its right neighbor. This completes the proof. □

Using cycle-alternating permutations, we can give a bijective proof of the fact that $P_{n+1} = (1 + t^2) R_n$. This can be written

$$\sum_{\pi \in \mathcal{C}_{n+2}^\circ} t^{\operatorname{neg}(\pi)} = (1 + t^2) \sum_{\substack{\sigma \in \mathcal{C}_{n+2}^\circ \\ \sigma_1 > 0}} t^{\operatorname{neg}(\sigma)}. \tag{46}$$

Now, consider the conjugation by the transposition $(1, -1)$, *i.e.* the map that sends π to $(1, -1)\pi(1, -1)$. This is a fixed point free involution on $\mathcal{C}_{n+2}^\circ$ which proves combinatorially (46), as it is easily seen on the arch diagrams. Indeed, in this representation the involution exchanges the dot labelled 1 with the dot labelled -1, see Figure 2. We have $\pi(1) > 0$ if and only if its image $\pi' = (1, -1)\pi(1, -1)$ does not satisfy $\pi'(1) > 0$, and if $\pi(1) > 0$, we have $\operatorname{neg}(\pi') = \operatorname{neg}(\pi) + 2$. This proves (46). Note that it is also possible to prove $P_{n+1} = (1+t^2)R_n$ on the snakes, using Proposition 3.6, but this is more tedious.

FIGURE 2. The arch diagram of the cycle-alternating permutation $\pi = (1,\ 4,\ -5,\ -2,\ -3,\ -1,\ -4,\ 5,\ 2,\ 3) \in \mathcal{C}_5^\circ$, and of its conjugate $(1,-1)\pi(1,-1)$.

Using cycle-alternating permutations, we can also obtain the exponential generating function of $\{P_n\}_{n\geq 0}$, knowing the one of $\{Q_n\}_{n\geq 0}$. Indeed, a cycle-alternating permutation is an assembly of cycle-alternating cycles, in the sense of combinatorial species, and this gives immediately:

$$\sum_{n=0}^{\infty} P_n(t)\frac{z^{n+1}}{(n+1)!} = \log\left(\sum_{n=0}^{\infty} Q_n(t)\frac{z^n}{n!}\right) = \log\left(\frac{1}{\cos z - t\sin z}\right), \tag{47}$$

and after differentiation, we obtain the formula for $\sum P_n(t)\frac{z^n}{n!}$ as in Equation (7).

3.3. **The q-analogs.** It is possible to give combinatorial interpretations of $Q_n(t,q)$ and $R_n(t,q)$ using cycle-alternating permutations, and the notion of *crossing* for signed permutations which was defined in [6]. But in the case of cycle-alternating permutations, there is a more simple equivalent definition which is the following.

Definition 3.11. A *crossing* of $\pi \in \mathcal{C}_n$ is a pair $(i,j) \in [[n]]^2$ such that $i < j < \pi(i) < \pi(j)$. Equivalently, it is the number of intersection points among the arches above the horizontal axis in the arch diagram of π. We denote by $\operatorname{cr}(\pi)$ the numbers of crossings in π.

Because of the symmetry in the arch diagram, $\operatorname{cr}(\pi)$ is half the total number of intersection points among the arches. For example, the two permutations in Figure 2 have respectively 2 and 3 crossings. This kind of statistic can be followed through bijections between permutations and paths [4], but let us first show that $Q_n(t,q)$ and $R_n(t,q)$ are related with some weighted Dyck prefixes (a Dyck prefix being similar to a Dyck path except that the final height can be non-zero).

Definition 3.12. Let $\mathcal{P}_n$ be the set of weighted Dyck prefixes of length n, where

- each step $\nearrow$ between heights h and $h+1$ has a weight q^i for some $i \in \{0,\dots,h+1\}$,
- each step $\searrow$ between heights h and $h+1$ has a weight q^i for some $i \in \{0,\dots,h\}$.

Let $\mathcal{P}'_n \subset \mathcal{P}_n$ be the subset of paths p such that there is no step $\nearrow$ starting at height h with the maximal weight q^{h+1}. For any $p \in \mathcal{P}_n$, let $\operatorname{fh}(p)$ be its final height, and let $\operatorname{w}(p)$ be its total weight, *i.e.* the product of the weights of each step.

Proposition 3.13. *We have:*

$$Q_n(t,q) = \sum_{p\in\mathcal{P}'_n} t^{\operatorname{fh}(p)}q^{\operatorname{w}(p)}, \qquad R_n(t,q) = \sum_{p\in\mathcal{P}_n} t^{\operatorname{fh}(p)}q^{\operatorname{w}(p)}. \tag{48}$$

Proof. In the proof of Proposition 2.6, we have used the matrices $D_2 + U_2D_2U_2$ and $D_2 + D_2U_2U_2$, thought of as transfer matrices for walks in the non-negative integers. We can do something similar with the matrices

$$D_1 + U_1D_1U_1 = \begin{pmatrix} 0 & [1]_q & & (0) \\ [1]_q & 0 & [2]_q & \\ & [2]_q & 0 & \ddots \\ (0) & & \ddots & \ddots \end{pmatrix}, \; D_1 + D_1U_1U_1 = \begin{pmatrix} 0 & [1]_q & & (0) \\ [2]_q & 0 & [2]_q & \\ & [3]_q & 0 & \ddots \\ (0) & & \ddots & \ddots \end{pmatrix}. \quad (49)$$

Recall that we have (20), where $W_1 = (1, t, t^2, \dots)$. In terms of paths, the coefficients in the matrices and vectors have the following meaning: from $V_1 = (1, 0, \dots)^*$, we only consider paths starting at height 0, and from $W_1 = (1, t, t^2, \dots)$ there is the weight $t^{\mathrm{fh}(p)}$ for the path p. The coefficients $(h+1, h)$ in the matrices give the possible weights on steps $\nearrow$ from height h to $h+1$, the coefficients $(h, h+1)$ in the matrices give the possible weights on steps $\searrow$ from height $h+1$ to h. The result follows. $\square$

Proposition 3.14. *There is a bijection* $\Psi : \mathcal{C}_n \to \mathcal{P}'_n$ *such that* $\mathrm{neg}(\pi) = \mathrm{fh}(p)$ *and* $\mathrm{cr}(\pi) = \mathrm{w}(p)$ *if the image of* π *is* p.

Proof. We can use the bijection Ψ_{FZ} from [4] between permutations and weighted Motzkin paths. To do this, we identify $\mathfrak{S}_n^{\pm}$ with a subset of $\mathfrak{S}_{2n}$ via the order-preserving bijection $[[n]] \to [2n]$. This subset is characterized by the fact that the arrow diagrams are centrally symmetric, and via Ψ_{FZ} from [4], it is in bijection with some weighted Motzkin paths that are vertically symmetric. Cycle-alternating permutations correspond to the case where there is no horizontal step, *i.e.* they are in bijection with some weighted Dyck paths that are vertically symmetric. Keeping the first half of these Dyck paths gives the desired bijection. $\square$

For convenience, we rephrase here explicitly the bijection of the previous proposition. Let $\pi \in \mathcal{C}_n$, then we define a $\Psi(\pi) \in \mathcal{P}'_n$ such that, for any $j \in \{-n \dots -1\}$:

- the $(n+1+j)$th step is $\nearrow$ if $\pi^{-1}(j) > j < \pi(j)$ and $\searrow$ if $\pi^{-1}(j) < j > \pi(j)$,
- if the $(n+1+j)$th step is $\nearrow$, it has a weight q^k where k is the number of i such that (i, j) is a crossing, *i.e.* $i < j < \pi_i < \pi_j$,
- if the $(n+1+j)$th step is $\searrow$, it has a weight q^k where k is the number of i such that $(-i, -j)$ is a crossing, *i.e.* $-i < -j < -\pi_i < -\pi_j$.

See Figure 3 for an example.

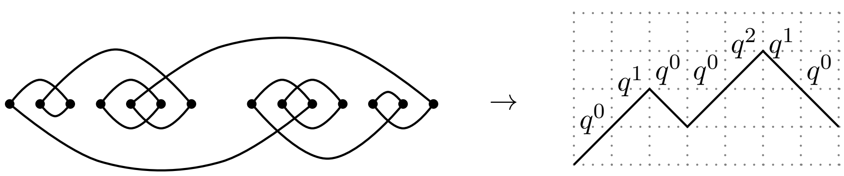


FIGURE 3. The bijection Ψ from $\mathcal{C}_n$ to $\mathcal{P}'_n$ in the case of $\pi = 3, 4, -7, 2, 6, 1, 5$.

Through this bijection we obtain combinatorial models of $Q_n(t, q)$ and $R_n(t, q)$ using the notion of crossing.

Proposition 3.15. *We have*

$$Q_n(t, q) = \sum_{\pi \in \mathcal{C}_n} t^{\mathrm{neg}(\pi)} q^{\mathrm{cr}(\pi)}, \qquad R_n(t, q) = \sum_{\substack{\pi \in \mathcal{C}_{n+1} \\ \pi_{n+1} < 0}} t^{\mathrm{neg}(\pi)-1} q^{\mathrm{cr}(\pi)}. \quad (50)$$

Alternatively, the condition $\pi_{n+1} < 0$ *in the second identity can be replaced with* $\pi^{-1}(n+1) < 0$.

Proof. The first identity follows from Proposition 3.13 and the bijection Ψ between $\mathcal{C}_n$ and $\mathcal{P}'_n$, but the second one is not as immediate.

First, note that the bijection $\pi \mapsto \pi^{-1}$ stabilizes the set $\mathcal{C}_n$ and preserves the number of crossings, since it is just an horizontal symmetry on the arch diagrams. This proves the fact that we can replace $\pi_{n+1} < 0$ with $\pi^{-1}(n+1) < 0$.

Then, let us consider the subset $\mathcal{P}''_{n+1} \subset \mathcal{P}'_{n+1}$ of elements p such that there is no step $\searrow$ from height $h+1$ to h with weight q^h (note that this condition implies there is no return to height 0). There is an obvious bijection from $\mathcal{P}''_{n+1}$ to $\mathcal{P}_n$, because removing the first step in $p \in \mathcal{P}''_{n+1}$ gives a path which is in $\mathcal{P}_n$ with respect to the shifted origin $(1,1)$. This proves:

$$R_n(t,q) = \sum_{p \in \mathcal{P}''_{n+1}} t^{\mathrm{fh}(p)-1} q^{\mathrm{w}(p)}. \tag{51}$$

It remains to show that the image of $\mathcal{P}''_{n+1}$ via Ψ^{-1} is precisely the set of $\pi \in \mathcal{C}_{n+1}$ such that $\pi^{-1}(n+1) < 0$. Essentially, this follows from the fact that there are some steps in the path which characterize the right-to-left minima in π, and we can use Lemma 3.2.2 from [18]. In our case, the subset $\mathcal{P}''_{n+1} \subset \mathcal{P}'_{n+1}$ corresponds to a subset of $\mathfrak{S}_{2n+2}$ (via the identification $\mathfrak{S}^{\pm}_{n+1} \to \mathfrak{S}_{2n+2}$), more precisely to the subset of σ having no right-to-left minima among $1 \dots n+1$, *i.e.* $\sigma^{-1}(1) > n+1$. In terms of $\pi \in \mathfrak{S}^{\pm}_{n+1}$, this precisely means that $\pi^{-1}(-n-1) > 0$ and completes the proof. $\square$

Remark 3.16. Chen, Fan and Jia [3] gave a bijection between snakes of type B and Dyck prefixes such that there are $h+1$ possible choices on each step between height h and $h+1$, and their statistic $\alpha(\pi)$ (see Theorem 4.5) is the same as our statistic $\mathrm{cs}(\pi)$, though defined differently. We can have such a bijection by composing two we have presented (their bijection is not the same as ours and does not seem to be directly related).

3.4. **Another variant of Ψ_{FZ}.** With another adaptation of this bijection, we can link $\mathcal{C}_n$ and some weighted Motzkin path, so that we obtain a combinatorial proof of a J-fraction generalizing (43) and (44) in Theorem 3.17 below. As a consequence of Theorem 3.10, we have:

$$Q_n^{(a)}(t) = \sum_{\pi \in \mathcal{C}_n} a^{\mathrm{cyc}(\pi)} t^{\mathrm{neg}(\pi)} \tag{52}$$

where $\mathrm{cyc}(\pi)$ is the number of cycles, indeed since there is a “cycle structure” in $\mathcal{C}_n$ it suffices to take the ath power of the exponential generating function to have a parameter a counting $\mathrm{cyc}(\pi)$. With the recurrence for $Q_n^{(a)}(t)$ in (5) and with the same method as in the case of Q_n and R_n, we can prove a continued fraction for the ordinary generating function of $\{Q_n^{(a)}(t)\}_{n\geq 0}$. We prove here bijectively this continued fraction, by going from $\mathcal{C}_n$ to weighted Motzkin paths. To this end, we use another representation of a signed permutation, the *signed arch diagram*. Let $\pi \in \mathfrak{S}^{\pm}_n$, this diagram is obtained the following way: draw n dots labelled by $1 \dots n$ from left to right on the horizontal axis, then for each $i \in [n]$, draw an arch from i to $|\pi_i|$, and label this arrow with a $+$ if $\pi_i > 0$ and with a $-$ if $\pi_i < 0$. We understand that as in the previous arch notation, the arch is above the axis if $i \leq |\pi_i|$ and below otherwise. See the left part of Figure 4 for an example. (In the case $|\pi_i| = i$, the arch is a loop attached to the dot i.) A case-by-case study shows that π is cycle-alternating if and only if it avoids certain configurations in the signed arch diagram, which are listed in the right part of Figure 4 ($\pm$ being either $+$ or $-$). Note that these forbidden configurations are reminiscent of (40).

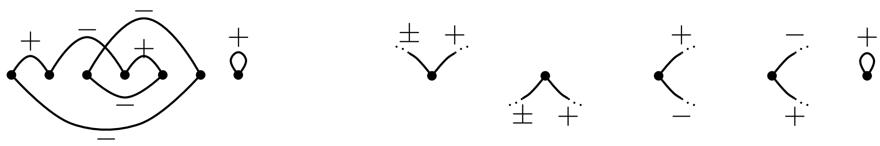


FIGURE 4. The signed arch diagram of $\pi = 2, -4, -6, 5, -3, -1, 7$, and the forbidden configurations in the signed arch diagram of a cycle-alternating permutation.

Once we know the forbidden configurations, it is possible to encode the signed arch diagram of $\pi \in \mathcal{C}_n$ by a weighted Motzkin path of n steps, in a way similar to Ψ_{FZ} from [4]. The diagram of π is scanned from left to right, and the path is built such that:

- If the ith node in π is  then the ith step of p if $\nearrow$, moreover this step has label $+$ (resp. $-$) if the two arches starting from the ith node have label $+$ (resp. $-$).
- If the ith node in π is  then the ith step of p if $\rightarrow$. Moreover this step has label (j) if the left strand in the ith node is connected to the jth strand of the "partial" signed arch diagram (say, from bottom to top). See for example the left part of Figure 5 where there are three possible choices to connect this kind of node. We can see that the possible labels are $(1), \dots, (h)$ where h is the starting height of the ith step in the path.
- Similarly if the ith node in π is  then the ith step of p if $\rightarrow$, and this step has label (j') if the ith node is connected on the left to the jth strand of the "partial" signed arch diagram (say, from top to bottom).
- If the ith node in π is  then the ith step is $\rightarrow$ with label (0).
- If the ith node in π is  then the ith step is $\searrow$ with a label (j,k) where j and k respectively encode where the upper and lower strands are connected, as in the case of  and  above.

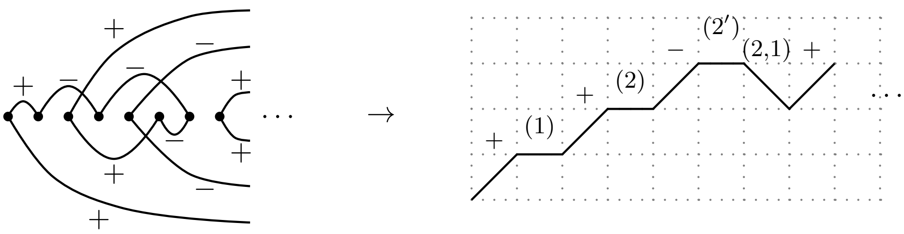


FIGURE 5. Construction of the bijection from the signed arch diagram of $\pi \in \mathcal{C}_n$ to a labelled Motzkin path.

Quite a few bijections of this kind are known so we will not give further details. We have also used this kind bijection in [5], for example. As for the parameters a and t, they are taken into account in the following way:

- For each arch with label $-$ linking i and j where $i < j$, we give a weight t to the ith step in the path. Thus there is a weight t^2 on the step $\nearrow$ with label $-$, and a weight t on each step $\rightarrow$.
- For each i which is the maximal element of a cycle, we give a weight a to the ith step. Thus each step $\rightarrow$ with weight (0) has a weight a, since these correspond to nodes or cycles of the form $(i, -i)$. Besides, there is a weight a on some of the step $\searrow$. More precisely, there are h^2 possible labels on a step $\searrow$ starting at height h, and h of them have this weight a. Indeed, for each choice of where to connect the upper strand of the node there is a unique choice of where to connect the lower strand so that a new cycle is "closed".

Using the classical correspondence between weighted Motzkin paths and J-fractions [11], this bijection proves the following theorem.

Theorem 3.17. *Let $Q_n^{(a)}(t)$ be the polynomials in a and t with exponential generating function $(\cos z - t\sin z)^{-a}$, or defined combinatorially by (52). Then, using the notation in Definition 2.5, we have:*

$$\sum_{n=0}^{\infty} Q_n^{(a)}(t) z^n = \mathfrak{J}\big(\ (2h+a)t,\ h(h-1+a)(1+t^2)\ \big). \tag{53}$$

Apart from the combinatorial side, this result was already known by Stieltjes [28]. Besides his analytical proof, it is also possible to obtain the continued fraction by showing that the exponential generating function satisfies some particular kind of addition formula which permits to make use of a theorem of Stieltjes and Rogers, see for example [13, Theorem 5.2.10]. We thank Philippe Flajolet for communicating these facts and references.

4. Increasing trees and forests

So far, we have only used recurrence relations and bijections to derive our new models of the derivative polynomials. Some more elaborate methods apply as well to give combinatorial models in terms of increasing trees and forests. We first present how to use the combinatorial theory of differential equations, as exposed by Leroux and Viennot in [21], and secondly how to use recent results of Błasiak and Flajolet related with operators and normal ordering [11].

4.1. Increasing trees via differential equations.

An archetype example in the combinatorial theory of differential equation is the one of the tangent and secant functions (see [21]), and it has become a classical method to show that Euler numbers E_n count some increasing trees, *i.e.* labelled trees where labels are increasing from the root to the leaves. Here we have a system of two differential equations similar to the one of tangent and secant, except that an initial condition is given by the parameter t instead of 0. This will give rise to more general trees where we allow some leaves with no label having a weight t.

Lemma 4.1. *Let $f = \sum P_n z^n/n!$ and $g = \sum Q_n z^n/n!$, then we have:*

$$\begin{cases} f' = 1 + f^2 & f(0) = t, \\ g' = fg & g(0) = 1. \end{cases} \tag{54}$$

Proof. Of course, this can be checked on the closed form given in (7). Also, $f' = 1 + f^2$ can be directly checked on the snakes as in the case of alternating permutations [27], and $g' = fg$ follows from $f = (\log g)'$ previously seen. ☐

It is adequate to rewrite the equations in the following way:

$$\begin{cases} f = t + z + \int f^2, \\ g = 1 + \int fg. \end{cases} \tag{55}$$

Let us begin with the case of $f(z)$. From $f = t + z + \int f^2$, and proceeding as in [21], $f(z)$ counts increasing trees that are recursively produced by the following rules, starting from an isolated node marked by f:

- a node marked by f can become a leaf with no label (this corresponds to the term t in $f = t + z + \int f^2$, $f(z)$, so these leaves will have a weight t),
- a node marked by f can become a leaf with an integer label, this label being the smallest integer that does not already appear in the tree (this corresponds to the term z in $f = t + z + \int f^2$),
- a node marked by f can become an internal node having an integer label and two (ordered) sons marked by f (this corresponds to the last term $\int f^2$ in $f = t + z + \int f^2$). As before, the integer label is the smallest integer that does not already appear in the tree.

More precisely, the coefficient of z^n in $f(z)$ counts these trees having n integer labels. See Figure 6 for an example of tree produced by these rules.

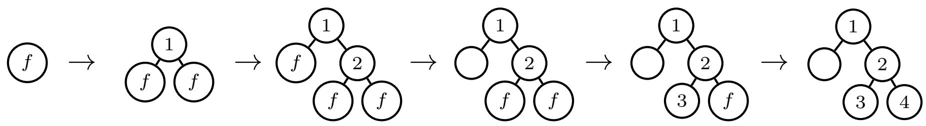


FIGURE 6. Tree produced via the equation $f = t + z + \int f^2$.

The case of $g(z)$ is quite similar. Starting from an isolated node marked by g, each node marked by g can become either a leaf with no label, or an internal node having two sons respectively marked by f and g. Note that we need the production rules for f to build a tree counted by g, and note also that g can produce an empty leaf which has no weight t, contrary to the empty leaves produced by f.

The two kinds of tree can also be given a non-recursive definition.

Definition 4.2. Let $\mathcal{T}_n$ be the set of complete binary trees, such that:

- except some leaves that are empty, the nodes are labelled by integers so that each $i \in [n]$ appears exactly once,
- the labels are increasing from the root to the leaves.

Let $\mathrm{em}(T)$ be the number of empty leaves of $T \in \mathcal{T}_n$, and let $\mathcal{T}_n^* \subset \mathcal{T}_n$ be the subset of trees such that the rightmost leaf is empty.

The production rules for f (respectively, g) can be checked on the set $\mathcal{T}_n$ (respectively $\mathcal{T}_n^*$), so that the result of the above discussion is the following.

Theorem 4.3. *We have:*

$$P_n(t) = \sum_{T \in \mathcal{T}_n} t^{\mathrm{em}(T)}, \qquad Q_n(t) = \sum_{T \in \mathcal{T}_n^*} t^{\mathrm{em}(T)-1}. \tag{56}$$

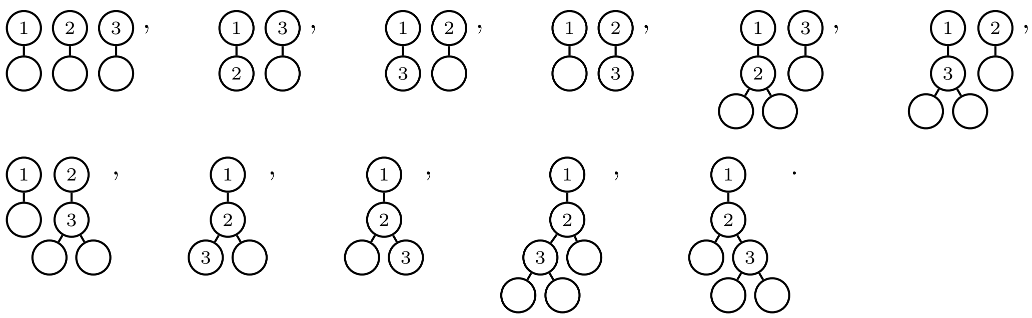
4.2. **Increasing trees via normal ordering.** Some recent results of Błasiak and Flajolet [2] directly apply in the present context and also give models of $P_n(t)$ and $Q_n^{(a)}(t)$ in terms of increasing trees and forests. Let D be the derivation with respect to t and U the multiplication by t (*i.e.* we are in the particular $q=1$ of operators defined in (11)). We have $DU-UD=I$. A general idea in [2] is that the coefficients $c_{i,j}$ in the normal form of $f(D,U)$ as in (17), at least for certain particular forms of $f(D,U)$, naturally counts some labelled directed graphs which are produced by connecting some "gates". In the present case, when $f(D,U)$ is $(D+UUD)^n$ for P_n and $(D+aU+UUD)^n$ for $Q_n^{(a)}$, and we obtain some increasing trees and forests. The theorem of this subsection is a direct application of a main result of Flajolet and Błasiak [2, Theorem 1], so as a proof we will roughly explain some ideas leading to the definition below and refer to [2] for more details.

Definition 4.4. Let $\mathcal{F}_n$ be the set of plane rooted forests satisfying the following conditions:

- each root has exactly one child, and each of the other internal nodes has exactly two (ordered) children,
- there are n nodes labelled by integers from 1 to n, but some leaves can be non-labelled (these are called *empty* leaves), and labels are increasing from each root down to the leaves.

Note that the trees forming a forest are unordered. Let $\mathcal{U}_n \subset \mathcal{F}_n$ be the subset of trees, *i.e.* forests with one connected component. For any tree or forest T, let $\mathrm{em}(T)$ be the number of empty leaves, and let $\mathrm{cc}(T)$ be its number of connected components.

For example, there are 11 elements in $\mathcal{F}_3$ and they are:



Theorem 4.5. *We have:*

$$P_n(t) = \sum_{T\in\mathcal{U}_{n+1}} t^{\mathrm{em}(T)}, \qquad Q_n^{(a)}(t) = \sum_{T\in\mathcal{F}_n} a^{\mathrm{cc}(T)} t^{\mathrm{em}(T)}. \tag{57}$$

For example, the forests in $\mathcal{F}_3$ given above illustrate $Q_3 = 6t^3+5t$. The last four elements of the list are the trees, and illustrate $P_2 = 2t^3+2t$. By counting with a weight 2 on each connected component, we obtain from this list that $R_3 = 24t^3+16t$.

Proof. From the definition of derivative polynomials in terms of D and U, and the relation $DU-UD=I$, we have:

$$P_n(t) = (D+UUD)^n t, \qquad Q_n^{(a)}(t) = (D+aU+UUD)^n 1. \tag{58}$$

Let us consider the case of $Q_n^{(a)}$. Let $f_n(D,U) = (D+aU+UUD)^n$ and $c_{n,i,j}$ the coefficient of U^iD^j in its normal form, as in (17). Then from [2, Theorem 1], $c_{n,i,j}$ counts *labelled diagrams* obtained by connecting three kinds of "gates", one for each term in $D + aU + UUD$. More precisely, to each term U^kD^ℓ we associate a gate consisting of one node with k outgoing strands and ℓ ingoing strands. See the left part of Figure 7. Then, $c_{n,i,j}$ count labelled diagrams obtained by connecting n of these gates such that:

- i outgoing strands and j ingoing strands are not connected,
- all other strands are connected, so that each ingoing strand is connected with an outgoing strand and these form a directed edge,
- the gates are labelled by the integers in $\{1, \dots, n\}$, and labels are increasing when we follow a directed edge,
- at each node, the ingoing strands on one side and the outgoing strands on another side are ordered,
- there is a weight a at each gate corresponding to the term U.

We have

$$Q_n^{(a)}(t) = (D + aU + UUD)^n 1 = \Big(\sum_{i,j \geq 0} c_{i,j} U^i D^j \Big) 1 = \sum_{i \geq 0} c_{i,0} t^i \tag{59}$$

so that we can only consider diagrams with no unconnected ingoing strand, and t counts the unconnected outgoing strands.

The labelled diagrams described by the above rules are essentially the same as elements in $\mathcal{F}_n$: it suffices to add an empty leaf at each unconnected outgoing strand to see the equivalence. It is clear that the node corresponding to the term U will appear exactly once in each connected component of the labelled diagrams, so the parameter a counts indeed the connected components.

In the case of P_n, we can also consider $f_n(D,U) = (D + UUD)^n U$ and $c_{n,i,j}$ the coefficient of U^iD^j in its normal form, as in (17). This case is somewhat different since $f_n(D,U)$ is not the nth power of some expression, but similar arguments apply as well: the labelled diagrams that appear have $n+1$ gates, the gate labelled 1 is of type U, all other gates are of type D or UUD. These labelled diagrams are the same as elements in $\mathcal{U}_{n+1}$, as in the previous case we just have to add an empty leaf to each unconnected outgoing strand to see the equivalence. As previously said, we refer to [2] for precisions about this proof. □

FIGURE 7. The "gates" corresponding to terms in $D + aU + DUU$, and a labelled diagram obtained by connecting them.

There is a simple bijection between $\mathcal{T}_n$ and $\mathcal{U}_{n+1}$: given $T \in \mathcal{T}_n$, relabel the nodes by $i \mapsto i+1$, then add a new node with label 1 on top of the root. There is also a simple bijection between $\mathcal{T}_n^*$ and $\mathcal{F}_n$: let $T \in \mathcal{T}_n^*$, remove the rightmost leaf, as well as all edges in the path from the root to the rightmost leaf, then the remaining components form the desired forest. See Figure 8 for an example.

Thus, Theorems 4.3 and 4.5 are essentially equivalent although obtained by different methods. It is also in order to give a bijection between $\mathcal{U}_n$ and $\mathcal{C}_n^\circ$, and by applying this bijection componentwise it will give a bijection between $\mathcal{F}_n$ and $\mathcal{C}_n$. It is more practical

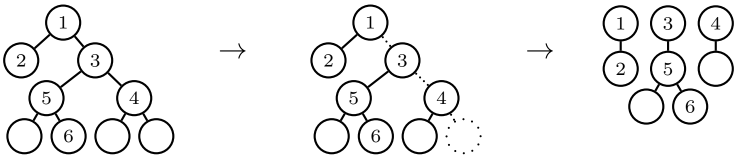
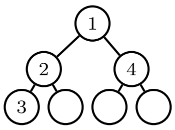


FIGURE 8. The bijection from $\mathcal{T}_n^*$ to $\mathcal{F}_n$.

to give the bijection from $\mathcal{T}_n$ to $\mathcal{S}_n$ (recall that we already have simple bijections $\mathcal{T}_n \to \mathcal{U}_{n+1}$ and $\mathcal{S}_n \to \mathcal{C}^{\circ}_{n+1}$), so let $T \in \mathcal{T}_n$. Consider the "reading word" of this tree (it can be defined by $w(T) = w(T_1)\ i\ w(T_2)$ if the tree T has a root labelled i, left son T_1 and right son T_2). This word contains integers from 1 to n, and some letters (say $\circ$) to indicate the empty leaves. The first step is to replace each i with $n+1-i$ in this word. To obtain the snake, replace each integer i by $(-1)^{j+1}i$ where j is the number of $\circ$ before i in the word, then remove all $\circ$. See Figure 9 for an example.



$$\to\ 3,2,\circ,1,\circ,4,\circ\ \to\ 2,3,\circ,4,\circ,1,\circ\ \to\ (-5),-2,-3,4,-1,(5)$$

FIGURE 9. The bijection from $\mathcal{T}_n$ to $\mathcal{S}_n$.

Actually, this bijection can be highlighted by Proposition 3.6 and the discussion leading to it. Indeed, it is just a variant of a classical bijection between permutations and unary-binary increasing trees [29], where double ascents and double descents correspond to nodes having only one child, and valleys correspond to leaves. Via these bijections, removing the empty leaves of $T \in \mathcal{T}_n$ is the same as taking the absolute value of a snake $S \in \mathcal{S}_n$. In the other direction, T seen as a unary-binary tree together with the data of the empty leaves, is the same as a permutation together with a choice of signs making it into a snake.

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