



**HAL**  
open science

# Integrability conditions on coboundary and transfer function for limit theorems

Davide Giraudo

► **To cite this version:**

Davide Giraudo. Integrability conditions on coboundary and transfer function for limit theorems. 2015. hal-01147859v1

**HAL Id: hal-01147859**

**<https://hal.science/hal-01147859v1>**

Preprint submitted on 2 May 2015 (v1), last revised 24 Dec 2015 (v2)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# INTEGRABILITY CONDITIONS ON COBOUNDARY AND TRANSFER FUNCTION FOR LIMIT THEOREMS

DAVIDE GIRAUDDO

ABSTRACT. For a measure preserving automorphism  $T$  of a probability space, we provide conditions on the tail function of  $g: \Omega \rightarrow \mathbb{R}$  and  $g - g \circ T$  which guarantee limit theorems among the weak invariance principle, Marcinkiewicz-Zygmund strong law of large numbers and the law of iterated logarithm to hold for  $f := m + g - g \circ T$ , where  $(m \circ T^i)_{i \geq 0}$  is a martingale difference sequence.

## 1. INTRODUCTION AND NOTATIONS

Let  $(\Omega, \mathcal{F}, \mu)$  be a probability space and  $T: \Omega \rightarrow \Omega$  be a bijective bi-measurable and measure preserving map. We assume that the dynamical system is ergodic (that is, if  $T^{-1}A = A$  for some  $A \in \mathcal{F}$ , then  $\mu(A) \in \{0, 1\}$ ). If  $n \geq 1$  is an integer and  $f: \Omega \rightarrow \mathbb{R}$ , we denote  $S_n(f) := \sum_{j=0}^{n-1} f \circ T^j$  and for a fixed  $t$ , define

$$(1.1) \quad S_n^{\text{pl}}(f, t) := S_{[nt]}(f) + (nt - [nt])f \circ T^{[nt]}, t \in [0, 1],$$

where  $[x]$  denote the integer part of the real number  $x$ . Then for each  $\omega \in \Omega$  and each integer  $n \geq 1$ , the map  $t \mapsto S_n^{\text{pl}}(f, t)$  is an element of the space of continuous functions in  $[0, 1]$ , denoted by  $C[0, 1]$ .

Let us state the limit theorems we are interested in.

**Definition 1.1.** *Let  $f: \Omega \rightarrow \mathbb{R}$  be a measurable function.*

- *We say that the function  $f$  satisfies the invariance principle if the sequence  $(n^{-1/2}S_n^{\text{pl}}(f, \cdot))_{n \geq 1}$  weakly converges in  $C[0, 1]$  to a scalar multiple of a standard Brownian motion.*
- *We say that the function  $f$  satisfies the law of iterated logarithm if for almost every  $\omega \in \Omega$ ,*

$$(1.2) \quad \limsup_{n \rightarrow +\infty} \frac{S_n(f)(\omega)}{\sqrt{n \log \log n}} = 1 \text{ and } \liminf_{n \rightarrow +\infty} \frac{S_n(f)(\omega)}{\sqrt{n \log \log n}} = -1.$$

- *We say that the function  $f$  satisfies the functional law of iterated logarithm if the sequence  $((\sqrt{n \log \log n})^{-1}S_n^{\text{pl}}(f, \cdot))_{n \geq 1}$  is relatively compact and the set of its limit points coincides with the set of all absolutely continuous functions  $x \in C[0, 1]$  such that  $x(0) = 0$  and  $\int_0^1 (x'(t))^2 dt \leq 1$ , where  $x'$  denotes the derivative with respect to the Lebesgue measure.*
- *Let  $1 < p < 2$ . We say that the function  $f$  satisfies the  $p$ -strong law of large numbers if for any  $\alpha \in [1/p, 1]$  if*

$$(1.3) \quad \forall \varepsilon > 0, \quad \sum_{n=1}^{+\infty} n^{\alpha p - 2} \mu \left\{ \max_{1 \leq k \leq n} |S_k(f)| \geq \varepsilon n^\alpha \right\} < +\infty.$$

If it is possible to find a decomposition of the function  $f$

$$(1.4) \quad f = m + g - g \circ T,$$

where  $g: \Omega \rightarrow \mathbb{R}$  is a measurable function and  $m$  satisfies one of the previous limit theorems, then one can wonder if we can deduce the result for  $f$ .

---

*Date:* May 2, 2015.

*2010 Mathematics Subject Classification.* 60F17; 60G10.

*Key words and phrases.* Invariance principle, strictly stationary process.

A known situation is when the sequence  $(m \circ T^i)_{i \geq 0}$  is a square-integrable martingale difference sequence. A necessary and sufficient condition to have (1.4) with such an  $m$  and a square integrable  $g$  is known (see Theorem 2 in [6]). If  $(m \circ T^i)_{i \geq 0}$  is a square-integrable martingale difference sequence, then the functional law of iterated logarithm and the invariance principle take place.

If  $1 < p < 2$  and  $m \in \mathbb{L}^p$ , then Theorem 5 by Dedecker and Merlevède [3] implies that  $m$  satisfies the  $p$ -strong law of large numbers. Actually, their results holds in a more general setting than strictly stationary sequences, as they only require a stochastic domination on the martingale difference sequence  $(X_j)_{j \geq 0}$ . A similar result as (1.3) takes place for  $\alpha = 1$  if we require a conditional stochastic domination (see [2], Theorem 2.2). A necessary and sufficient condition for (1.4) to hold with  $m, g \in \mathbb{L}^p$ ,  $1 < p < 2$ , is given by Volný in Theorem 1 of [7], and in this case, (1.3) is satisfied (see Theorem 6 of [3]).

We call a *coboundary* a function of the form  $g - g \circ T$ , where  $g: \Omega \rightarrow \mathbb{R}$  is a measurable function. The function  $g$  is called a *transfer function*. The following result is Theorem 1 of [8]. It gives a necessary and sufficient condition on the transfer function to preserve the limit theorems mentioned in the previous definition. Sufficiency for the invariance principle and the law of iterated logarithm was established in [5], pages 140-142).

**Theorem 1.2** (The equivalence theorem, [8]). *Let us suppose that for the process  $(m \circ T^i)_{i \in \mathbb{Z}}$  the invariance principle, the law of iterated logarithm (functional law of iterated logarithm) respectively, holds true. Let  $g$  be a measurable function and*

$$(1.5) \quad f = m + g - g \circ T.$$

Then for the process  $(f \circ T^i)_{i \in \mathbb{Z}}$

- the invariance principle holds if and only if

$$(1.6) \quad \frac{1}{\sqrt{n}} \max_{1 \leq k \leq n} |g \circ T^k| \xrightarrow[n \rightarrow \infty]{} 0 \text{ in probability;}$$

- the law of iterated logarithm as well as the functional law of iterated logarithm holds if and only if

$$(1.7) \quad \frac{1}{\sqrt{n \log \log n}} g \circ T^n \xrightarrow[n \rightarrow \infty]{} 0 \text{ a.s.}$$

Both conditions (1.6) and (1.7) take place when the function  $g$  is square-integrable. If  $1 < p < 2$ , Theorem 6 in [3] shows that (1.3) holds if  $g$  belongs to  $\mathbb{L}^p$ .

However, it may happen that we obtain a decomposition (1.4) where  $m \in \mathbb{L}^2$  but the function  $g$  is only integrable (see [8] for explicit counter-examples, and [4] for a condition which guarantees the square integrability of  $m$ ) and in this case, the weak invariance principle does not need to hold. We investigate conditions on the functions  $t \mapsto \mu \{|g| > t\}$  and  $t \mapsto \mu \{|g - g \circ T| > t\}$  which guarantee (1.6) or (1.7). In order to state these conditions in a more concise way, we introduce the so-called weak  $\mathbb{L}^q$ -spaces.

**Definition 1.3.** *Let  $q$  be a real number strictly greater than 1. We denote by  $\mathbb{L}^{q, \infty}$  the space of functions  $h: \Omega \rightarrow \mathbb{R}$  such that*

$$(1.8) \quad \|h\|_{q, \infty}^q := \sup_{t > 0} t^q \mu \{|h| > t\} \text{ is finite.}$$

The subspace of  $\mathbb{L}^{q, \infty}$  which consists of functions  $h$  such that  $\lim_{t \rightarrow +\infty} t^q \mu \{|h| > t\} = 0$  is denoted by  $\mathbb{L}_0^{q, \infty}$ .

## 2. MAIN RESULTS

In this section, we state the main results of this note. In the first subsection, we give a sufficient condition on the functions  $t \mapsto \mu \{|g| > t\}$  and  $t \mapsto \mu \{|g - g \circ T| > t\}$  in order to preserve the weak invariance principle, the law of iterated logarithm and the  $p$ -strong law of large numbers respectively. In the second subsection, we construct counter-examples

which show that the found conditions are sharp when the considered dynamical system is aperiodic.

Volný and Samek showed in [8] that the conclusion of Theorems 2.1 and 2.2 (see the next subsections) hold when  $p \geq (r+2)/r$  and that of Theorem 2.6 when  $p < (r-1)/(r-3/2)$ . In the case  $r > 2$ , we cannot conclude if  $(r-1)/(r-3/2) \leq p < (r+2)/r$ .

In this section, we assume that  $(m \circ T^i)_{i \geq 0}$  is a square-integrable martingale difference sequence.

### 2.1. Sufficient conditions.

**Theorem 2.1.** *Let  $1 \leq p < 2 \leq r$  be such that  $p \geq r/(r-1)$  and let  $g: \Omega \rightarrow \mathbb{R}$  be a function such that  $g \in \mathbb{L}_0^{p,\infty}$  and  $g - g \circ T \in \mathbb{L}_0^{r,\infty}$ . Then the function  $f = m + g - g \circ T$  satisfies the weak invariance principle in  $C[0, 1]$ .*

A similar result has been obtained for the quenched central limit theorem (see [1], Corollary 7).

**Theorem 2.2.** *Let  $1 \leq p < 2 \leq r$  and let  $g: \Omega \rightarrow \mathbb{R}$  be a function.*

- (i) *If  $p > r/(r-1)$ ,  $g \in \mathbb{L}^{p,\infty}$  and  $g - g \circ T \in \mathbb{L}^{r,\infty}$ , then the function  $f = m + g - g \circ T$  satisfies the law of iterated logarithm;*
- (ii) *if  $p = r/(r-1)$ ,  $g \in \mathbb{L}^p$  and  $g - g \circ T \in \mathbb{L}^r$ , then the function  $f = m + g - g \circ T$  satisfies the law of iterated logarithm.*

**Theorem 2.3.** *Let  $1 \leq q < p < r < 2$  be real numbers and let  $g: \Omega \rightarrow \mathbb{R}$  be a function such that  $g \in \mathbb{L}^q$  and  $g - g \circ T \in \mathbb{L}^r$ . If  $q \geq (p-1)r/(r-1)$ , then the function  $f = m + g - g \circ T$  satisfies (1.3).*

*Remark 2.4.* In [8],  $\mathbb{L}^q$  spaces are involved. It turns out that in the setting of Theorems 2.1 and 2.2, (i), we may work with weak  $\mathbb{L}^q$ -spaces. For the case (ii) in Theorem 2.2, it would be a challenging problem to determine whether strong moments are actually needed.

*Remark 2.5.* Ergodicity of the dynamical system is required for the "only if" direction in the equivalence involving the law of iterated logarithm of Theorem 1.2. Therefore, Theorem 2.1 remains valid in the non-ergodic setting.

### 2.2. Counter-examples.

**Theorem 2.6.** *Assume that the dynamical system  $(\Omega, \mathcal{F}, \mu, T)$  is aperiodic. Let  $1 \leq p < 2 \leq r$  be real numbers such that  $p < r/(r-1)$ . Then there exists a function  $g \in \mathbb{L}^p$  such that  $g - g \circ T \in \mathbb{L}^r$  and the function  $f = m + g - g \circ T$  satisfies neither the invariance principle nor the law of iterated logarithm.*

**Theorem 2.7.** *Assume that the dynamical system  $(\Omega, \mathcal{F}, \mu, T)$  is aperiodic. Let  $1 < p < 2$  and let  $1 \leq q < p < r$  be real numbers such that  $q < (p-1)r/(r-1)$ . Then there exists a function  $g \in \mathbb{L}^q$  such that  $g - g \circ T \in \mathbb{L}^r$  but the sequence  $(n^{-1/p} S_n(g - g \circ T))_{n \geq 1}$  does not converge almost surely to 0.*

## 3. PROOFS

If  $h: \Omega \rightarrow \mathbb{R}$  is a measurable function, we define  $M^*(h) := \sup_{N \geq 1} N^{-1} |S_N(h)|$ .

The following lemma about Birkhoff averages will be used in the proof.

**Lemma 3.1.** *Let  $q > 1$  and let  $h: \Omega \rightarrow \mathbb{R}$  be a measurable function.*

- (i) *the inequality*

$$(3.1) \quad \|M^*(h)\|_{q,\infty} \leq \frac{q}{q-1} \|h\|_{q,\infty},$$

*takes place;*

- (ii) *if  $h$  belongs to  $\mathbb{L}_0^{q,\infty}$  then the function  $M^*(h)$  belongs to  $\mathbb{L}_0^{q,\infty}$ ;*
- (iii) *if  $h$  belongs to  $\mathbb{L}^q$ , then so does  $M^*(h)$ .*

We now give the proofs of the main results, which combine Lemma 3.1 with the ideas of [8].

### 3.1. Proof of sufficient conditions.

*Proof of Theorem 2.1.* In view of Theorem 1.2, we have to show the convergence in (1.6).

Let  $\varepsilon$  be a positive fixed number. Let  $k, n$  be positive integers such that  $k < n$ . Denoting  $p_n := \mu \left\{ \max_{1 \leq j \leq n} |g \circ T^j| > 2\varepsilon n^{1/2} \right\}$ , the following estimates take place:

$$\begin{aligned} p_n &\leq \mu \left\{ \max_{1 \leq i \leq \left[\frac{n}{k}\right]+1} \max_{ik \leq j < (i+1)k} |g \circ T^{ik}| + |g \circ T^j - g \circ T^{ik}| > \varepsilon n^{1/2} \right\} \\ &\leq \mu \left\{ \max_{1 \leq i \leq \left[\frac{n}{k}\right]+1} |g \circ T^{ik}| > \varepsilon \sqrt{n} \right\} + \sum_{i=1}^{\left[\frac{n}{k}\right]+1} \mu \left\{ \max_{ik \leq j < (i+1)k} |g \circ T^j - g \circ T^{ik}| > \varepsilon n^{1/2} \right\} \\ &\leq \left( \left[\frac{n}{k}\right] + 1 \right) \mu \left\{ |g| > \varepsilon n^{1/2} \right\} + \left( \left[\frac{n}{k}\right] + 1 \right) \mu \left\{ \max_{0 \leq j < k} |g \circ T^j - g| > \varepsilon n^{1/2} \right\} \\ &\leq \left( \left[\frac{n}{k}\right] + 1 \right) \mu \left\{ |g| > \varepsilon n^{1/2} \right\} + \left( \left[\frac{n}{k}\right] + 1 \right) \mu \left\{ \max_{0 \leq j < k} \frac{1}{j} |S_j(g - g \circ T)| > \varepsilon \frac{n^{1/2}}{k} \right\}. \end{aligned}$$

This yields

$$(3.2) \quad p_n \leq \left( \left[\frac{n}{k}\right] + 1 \right) \mu \left\{ |g| > \varepsilon n^{1/2} \right\} + \left( \left[\frac{n}{k}\right] + 1 \right) \mu \left\{ M^*(g - g \circ T) > \varepsilon \frac{n^{1/2}}{k} \right\}$$

By the assumption on  $p$  and  $r$ , we may choose a number  $\alpha$  such that

$$(3.3) \quad 1 - \frac{p}{2} \leq \alpha \leq \frac{r/2 - 1}{r - 1}.$$

We now use (3.2) with  $k := [n^\alpha]$ . This yields, for some constant  $c$  depending only on  $p$  and  $r$ :

$$(3.4) \quad p_n \leq c \cdot n^{1-\alpha} \mu \left\{ |g| > \varepsilon n^{1/2} \right\} + c \cdot n^{1-\alpha} \mu \left\{ M^*(g - g \circ T) > \varepsilon n^{1/2-\alpha} \right\},$$

and, by (3.3),

$$(3.5) \quad p_n \leq cn^{p/2} \mu \left\{ |g| > \varepsilon n^{1/2} \right\} + cn^{1-\alpha-r(1/2-\alpha)} n^{(1/2-\alpha)r} \mu \left\{ M^*(g - g \circ T) > \varepsilon n^{1/2-\alpha} \right\}$$

$$(3.6) \quad = cn^{p/2} \mu \left\{ |g| > \varepsilon n^{1/2} \right\} + cn^{1-r/2+\alpha(r-1)} n^{(1/2-\alpha)r} \mu \left\{ M^*(g - g \circ T) > \varepsilon n^{1/2-\alpha} \right\}$$

$$(3.7) \quad \leq cn^{p/2} \mu \left\{ |g| > \varepsilon n^{1/2} \right\} + cn^{(1/2-\alpha)r} \mu \left\{ M^*(g - g \circ T) > \varepsilon n^{1/2-\alpha} \right\}.$$

Since  $g \in \mathbb{L}_0^{p,\infty}$  and  $g - g \circ T \in \mathbb{L}_0^{r,\infty}$ , we conclude by item (ii) of Lemma 3.1 that the sequence  $(p_n)_{n \geq 1}$  converges to 0.

This concludes the proof of Theorem 2.1.  $\square$

*Proof of Theorem 2.2.* In view of Theorem 1.2, we have to prove that the convergence in (1.7) takes place. Let  $\alpha \in (0, 1)$  be a number which will be specified later. We define

$$(3.8) \quad m_j := \sum_{i=1}^{j-1} [i^\alpha], j \geq 1.$$

Notice that for some constant  $\kappa$  depending only on  $\alpha$ , we have

$$(3.9) \quad \frac{j^{\alpha+1}}{\kappa} \leq m_j \leq \kappa j^{\alpha+1}, j \geq 1.$$

By the Borel-Cantelli lemma, we have to prove the convergence of the series

$$(3.10) \quad \sum_{j=1}^{+\infty} p_j, \text{ with } p_j := \mu \left\{ \max_{0 \leq i \leq [j^\alpha]} \frac{1}{\sqrt{m_j \log \log m_j}} |g \circ T^{m_j+i}| > \varepsilon \right\}$$

for each positive  $\varepsilon$ . To this aim, we start from the inequalities

$$\begin{aligned} p_j &: \leq \mu \left\{ \frac{1}{\sqrt{m_j \log \log m_j}} |g \circ T^{m_j}| > \varepsilon/2 \right\} + \\ &+ \mu \left\{ \max_{0 \leq i \leq [j^\alpha]} \frac{1}{\sqrt{m_j \log \log m_j}} |g \circ T^{m_j+i} - T^{m_j}| > \varepsilon/2 \right\} \\ &= \mu \left\{ \frac{1}{\sqrt{m_j \log \log m_j}} |g| > \varepsilon/2 \right\} + \\ &+ \mu \left\{ \max_{0 \leq i \leq [j^\alpha]} \frac{1}{\sqrt{m_j \log \log m_j}} |g \circ T^i - g| > \varepsilon/2 \right\}, \end{aligned}$$

from which we infer

$$(3.11) \quad p_j \leq \mu \left\{ \frac{1}{\sqrt{m_j \log \log m_j}} |g| > \varepsilon/2 \right\} + \mu \left\{ \frac{1}{\sqrt{m_j \log \log m_j}} M^*(g - g \circ T) > \frac{\varepsilon}{2[j^\alpha]} \right\}.$$

(i) Assume that  $p > r/(r-1)$ . Using the definition of  $\|\cdot\|_{p,\infty}$  and inequality (3.9), we obtain

$$(3.12) \quad \mu \left\{ \frac{1}{\sqrt{m_j \log \log m_j}} |g| > \varepsilon/2 \right\} \leq c(p, \varepsilon, \alpha) \kappa^{p/2} \|g\|_{p,\infty}^p \frac{1}{j^{(\alpha+1)p/2}},$$

where  $c(p, \varepsilon, \alpha)$  is independent of  $j$ . Using (3.9) and (3.1), we derive

$$(3.13) \quad \mu \left\{ \frac{1}{\sqrt{m_j \log \log m_j}} M^*(g - g \circ T) > \frac{\varepsilon}{2[j^\alpha]} \right\} \leq \leq c(r, \varepsilon, \alpha) \kappa^{p/2} \|g - g \circ T\|_{r,\infty}^r j^{\alpha r - (\alpha+1)r/2},$$

where  $c(r, \varepsilon, \alpha)$  is independent of  $j$ .

Combining (3.11), (3.12) and (3.13), we deduce the upper bound

$$(3.14) \quad p_j \leq c(p, r, \alpha, \varepsilon, g) \left( \frac{1}{j^{(\alpha+1)p/2}} + j^{(1-\alpha)r/2} \right).$$

By assumption, we can take  $\alpha$  such that

$$(3.15) \quad (\alpha+1)p/2 > 1 \text{ and } (1-\alpha)r/2 > 1,$$

hence by (3.14), the series defined by (3.10) is convergent for any positive  $\varepsilon$ . This concludes part (i) of Theorem 2.2.

(ii) Assume that  $p = r/(r-1)$ . We pick  $\alpha := 2/p - 1 = 1 - 2/r$ . In this case, for some constant  $c$  depending only on  $p$  and  $r$ , the inequality

$$(3.16) \quad p_j \leq \mu \left\{ |g| > c\varepsilon j^{1/p} \right\} + \mu \left\{ M^*(g - g \circ T) > \varepsilon c j^{-\alpha + (\alpha+1)/2} \right\}$$

$$(3.17) \quad = \mu \left\{ |g| > c\varepsilon j^{1/p} \right\} + \mu \left\{ M^*(g - g \circ T) > \varepsilon c j^{(1-\alpha)/2} \right\}$$

$$(3.18) \quad = \mu \left\{ |g| > c\varepsilon j^{1/p} \right\} + \mu \left\{ M^*(g - g \circ T) > \varepsilon c j^{1/r} \right\}$$

takes place. By item (iii) of Lemma 3.1, we conclude that the series defined by (3.10) is convergent for any positive  $\varepsilon$  and this concludes part (ii) of Theorem 2.2, hence the proof of Theorem 2.2.  $\square$

*Proof of Theorem 2.3.* Let us fix a positive  $\varepsilon$  and  $\alpha \in [1/p, 1]$ . Let  $1 \leq k < n$  be integers. By similar inequalities which led to (3.2) (we replace the exponent  $1/2$  by  $\alpha$ ), we derive

$$(3.19) \quad \mu \left\{ \max_{1 \leq j \leq n} |g - g \circ T^j| > \varepsilon n^\alpha \right\} \leq 2 \left( \left[ \frac{n}{k} \right] + 1 \right) \mu \{ |g| > \varepsilon n^\alpha / 4 \} + \\ + 2 \left( \left[ \frac{n}{k} \right] + 1 \right) \mu \left\{ M^*(g - g \circ T) > \frac{\varepsilon n^\alpha}{2k} \right\}.$$

Let us choose  $k := \lceil n^\beta \rceil$ , where

$$(3.20) \quad (p - q)\alpha \leq \beta \leq \alpha(r - p)/(r - 1)$$

(the existence of such a  $\beta$  is guaranteed by the assumptions on  $p$ ,  $q$  and  $r$ ). Then we have to check that for each positive constant  $c$ , the series

$$(3.21) \quad \Sigma_1 := \sum_{n=1}^{+\infty} n^{p\alpha-1-\beta} \mu \{ |g| \geq cn^\alpha \} \quad \text{and}$$

$$(3.22) \quad \Sigma_2 := \sum_{n=1}^{+\infty} n^{p\alpha-1-\beta} \mu \{ M^*(g - g \circ T) \geq cn^{\alpha-\beta} \}$$

are convergent. The convergence of  $\Sigma_1$  is equivalent to the integrability of the function  $|g|^{p-\beta/\alpha}$ ; this holds since (3.20) implies  $q \geq p - \beta/\alpha$ .

Note that the second series converges if

$$(3.23) \quad \mathbb{E} \left[ (M^*(g - g \circ T))^{\frac{p\alpha-\beta}{\alpha-\beta}} \right] < +\infty.$$

Notice that inequality (3.20) implies that  $(p\alpha - \beta)/(\alpha - \beta) \leq r$ , hence we derive the convergence of  $S_2$  by item (iii) of Lemma 3.1 (with the exponent  $(p\alpha - \beta)/(\alpha - \beta) > 1$  since  $p > 1$ ).

This concludes the proof of Theorem 2.3.  $\square$

### 3.2. Counter-examples.

*Proof of Theorem 2.6.* We recall the construction given in the proof of Theorem 3 of [8]. We choose a real number  $\alpha$  such that

$$(3.24) \quad \frac{r-2}{2(r-1)} < \alpha < 1 - \frac{p}{2}.$$

For each  $i \geq 1$ , we define  $n_i := 2^i$  and  $k_i := \lceil 2^{i\alpha} \rceil$ . By the Rokhlin lemma, one can find a set  $A_i \in \mathcal{F}$  such that

$$(3.25) \quad \text{sets } A_i, TA_i, \dots, T^{n_i-1}A_i \text{ are pairwise disjoint and}$$

$$(3.26) \quad \mu \left( \bigcup_{j=0}^{n_i-1} T^j A_i \right) > 1/2.$$

In particular, the quantity  $\mu(A_i)$  can be bounded as follows:

$$(3.27) \quad \frac{1}{2n_i} \leq \mu(A_i) \leq \frac{1}{n_i}.$$

We then define for  $i \geq 1$ ,

$$(3.28) \quad g_i := \frac{\sqrt{n_i \log \log n_i}}{k_i} \left( \sum_{j=1}^{k_i} j \mathbf{1}(T^{n_i-j} A_i) + \sum_{j=k_i+1}^{2k_i-1} (2k_i - j) \mathbf{1}(T^{n_i-j} A_i) \right),$$

and  $g := \sum_{i=1}^{+\infty} g_i$ .

Since it has been shown in [8] that the function  $f$  satisfies neither the invariance principle nor the law of iterated logarithm, it remains to prove that the constructed function  $g$  belongs to  $\mathbb{L}^p$  and that the coboundary  $g - g \circ T$  belongs to  $\mathbb{L}^r$ .

By (3.25) and (3.28), the equality

$$(3.29) \quad |g_i|^p = \left( \frac{\sqrt{n_i \log \log n_i}}{k_i} \right)^p \left( \sum_{j=1}^{k_i} j^p \mathbf{1}(T^{n_i-j} A_i) + \sum_{j=k_i+1}^{2k_i-1} (2k_i-j)^p \mathbf{1}(T^{n_i-j} A_i) \right)$$

takes place, hence integrating and accounting (3.27), we derive the estimates

$$(3.30) \quad \mathbb{E} |g_i|^p \leq \left( \frac{\sqrt{n_i \log \log n_i}}{k_i} \right)^p \left( \sum_{j=1}^{k_i} j^p + \sum_{j=k_i+1}^{2k_i-1} (2k_i-j)^p \right) \frac{1}{n_i}$$

$$(3.31) \quad \leq 2 \frac{n_i^{p/2-1} (\log \log n_i)^{p/2}}{k_i^p} k_i^{p+1}$$

$$(3.32) \quad = 2n_i^{p/2-1} (\log \log n_i)^{p/2} k_i,$$

hence

$$(3.33) \quad \|g_i\|_p \leq 2^{1/p} n_i^{1/2-1/p} (\log \log n_i)^{1/2} k_i^{1/p}.$$

By definition of  $n_i$  and  $k_i$ , one can find a constant  $c$  depending only on  $\alpha$  (hence on  $p$  and  $r$ ) such that for  $i$  large enough,

$$(3.34) \quad \|g_i\|_p \leq c \cdot 2^{i(1/2-1/p)} (\log i)^{1/2} \cdot 2^{i\alpha/p} = c \cdot (\log i)^{1/2} \cdot 2^{i(\alpha-1+p/2)/p}.$$

By (3.24), the series  $\sum_{i=1}^{+\infty} (\log i)^{1/2} \cdot 2^{i(\alpha-1+p/2)/p}$  is convergent, and we conclude by (3.34) that  $g$  belongs to  $\mathbb{L}^p$ .

It is proved in [8] that by construction, the equality

$$(3.35) \quad |g_i - g_i \circ T| = \frac{\sqrt{n_i \log \log n_i}}{k_i} \cdot \mathbf{1} \left( \bigcup_{j=1}^{2k_i} T^{n_i-j} A_i \right)$$

holds. By (3.25) and (3.28), we have

$$(3.36) \quad \|g_i - g_i \circ T\|_r \leq \frac{\sqrt{n_i \log \log n_i}}{k_i} \left( \frac{2k_i}{n_i} \right)^{1/r},$$

hence by the definition of  $n_i$  and  $k_i$ , we have for  $i$  large enough and a constant  $c$  depending only on  $\alpha$ ,

$$(3.37) \quad \|g_i - g_i \circ T\|_r \leq c \cdot 2^{i(1/2-1/r)} 2^{i\alpha(1/r-1)} (\log i)^{1/2},$$

from which we infer (by (3.24)) the convergence of the series  $\sum_{i=1}^{+\infty} \|g_i - g_i \circ T\|_r$  hence the fact that the function  $g - g \circ T$  belongs to  $\mathbb{L}^r$ .

This concludes the proof of Theorem 2.6.  $\square$

*Proof of Theorem 2.7.* The construction is similar to that of the proof of Theorem 2.6.

For each  $i \geq 1$ , we define  $n_i := 2^i$  and  $k_i := \lceil 2^{i\beta} \rceil$ , where  $\beta$  satisfies

$$(3.38) \quad \frac{r-p}{p(r-1)} < \beta < 1 - \frac{q}{p}.$$

We take a set  $A_i \in \mathcal{F}$  such that (3.25) and (3.26) hold. We then define for  $i \geq 1$ ,

$$(3.39) \quad g_i := \frac{n_i^{1/p}}{k_i} \left( \sum_{j=1}^{k_i} j \mathbf{1}(T^{n_i-j} A_i) + \sum_{j=k_i+1}^{2k_i-1} (2k_i-j) \mathbf{1}(T^{n_i-j} A_i) \right),$$

and  $g := \sum_{i=1}^{+\infty} g_i$ .

The proof will be complete if we show the following three assertions:

- (1) the function  $g$  belongs to  $\mathbb{L}^q$ ;
- (2) the function  $g - g \circ T$  belongs to  $\mathbb{L}^r$ ;
- (3) the sequence  $(n^{-1/p} S_n(g - g \circ T))_{n \geq 1}$  does not converge almost surely to 0.



The first two items follow by completely similar computations as in the proof of Theorem 2.6. To show the last item, we notice that the sequence  $(2^{-i/p} \max_{2^i \leq l \leq 2^{i+1}} g_i \circ T^l)_{i \geq 1}$  does not converge to 0 in probability. To see this, one can note that

$$(3.40) \quad \mu \left\{ 2^{-i/p} \max_{2^i \leq l \leq 2^{i+1}} g_i \circ T^l \geq 1 \right\} \geq \mu \left( \bigcup_{j=1}^{n_i - k_i} T^j(A_i) \right) \geq \frac{1}{2}.$$

This finishes the proof of Theorem 2.7. □

#### REFERENCES

- [1] David Barrera, Costel Peligrad, and Magda Peligrad, *On the functional CLT for stationary Markov Chains started at a point*, (2015).
- [2] Y. Benoist and J-F. Quint, *Central limit theorem for linear groups*, to appear in *Annals of Probability* (2015).
- [3] J. Dedecker and F. Merlevède, *Convergence rates in the law of large numbers for Banach-valued dependent variables*, *Teor. Veroyatn. Primen.* **52** (2007), no. 3, 562–587. MR 2743029 (2011k:60096)
- [4] Carl-Gustav Esseen and Svante Janson, *On moment conditions for normed sums of independent variables and martingale differences*, *Stochastic Process. Appl.* **19** (1985), no. 1, 173–182. MR 780729 (86m:60120)
- [5] P. Hall and C. C. Heyde, *Martingale limit theory and its application*, Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1980, Probability and Mathematical Statistics. MR 624435 (83a:60001)
- [6] Dalibor Volný, *Approximating martingales and the central limit theorem for strictly stationary processes*, *Stochastic Process. Appl.* **44** (1993), no. 1, 41–74. MR 1198662 (93m:28021)
- [7] ———, *Martingale approximation of non-stationary stochastic processes*, *Stoch. Dyn.* **6** (2006), no. 2, 173–183. MR 2239087 (2007g:60047)
- [8] Dalibor Volný and Pavel Samek, *On the invariance principle and the law of iterated logarithm for stationary processes*, *Mathematical physics and stochastic analysis (Lisbon, 1998)*, World Sci. Publ., River Edge, NJ, 2000, pp. 424–438. MR 1893125 (2003b:60049)

NORMANDIE UNIVERSITÉ, UNIVERSITÉ DE ROUEN, LABORATOIRE DE MATHÉMATIQUES RAPHAËL SALEM,  
CNRS, UMR 6085, AVENUE DE L'UNIVERSITÉ, BP 12, 76801 SAINT-ETIENNE DU ROUVRAY CEDEX,  
FRANCE.

*E-mail address:* `davide.giraudoi@univ-rouen.fr`