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# On an asymmetric extension of multivariate Archimedean copulas

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## Abstract

Archimedean copulas are copulas determined by a specific real function, called the generator. Composed with the copula at a given point, this generator can be expressed as a linear form of generators of the considered point components. In this paper, we discuss the case where this function is expressed as a quadratic form (called here *multivariate Archimatrix copulas*). This allows extending Archimedean copulas, in order for example to build asymmetric copulas. Parameters of this new class of copulas are grouped within a matrix, thus facilitating some usual applications as level curve determination or estimation. Some choices as sub-model stability help associating each parameter to one bivariate projection of the copula. We also give some admissibility conditions for the considered Archimatrix copulas. We propose different examples as some natural multivariate extensions of Farlie-Gumbel-Morgenstern, Gumbel-Barnett, or particular Archimax copulas.

**Keywords:** Archimedean copulas; transformations of Archimedean copulas.

## Introduction

Copulas are multivariate distributions on  $[0, 1]^d$  with uniform marginal distributions. Their main interest is that by Sklar's theorem, continuous multivariate distributions can be represented as functions of their marginal distributions through the use of a unique copula. A review on different copula functions is available in [Nelsen \(1999\)](#).

A particular family of copula is the family of Archimedean copulas. Copulas of this family can be expressed in the dimension  $d \in \mathbb{N}^*$  by

$$C_\phi(u_1, \dots, u_d) = \phi(\psi(u_1) + \dots + \psi(u_d)), \quad (1)$$

where  $\phi$  is a real function  $\phi : \mathbb{R}^+ \rightarrow [0, 1]$ , called the generator of the copula, and where  $\psi$  is the generalized inverse function of  $\phi$ ,  $\psi(u) = \inf \{x \in \mathbb{R}^+ : \phi(x) \leq u\}$ ,  $u \in [0, 1]$ . The generator  $\phi$  is continuous, decreasing and convex function, with  $\lim_{x \rightarrow +\infty} \phi(x) = 0$ .

The Archimedean family of copula in (1) is very flexible, since members of this family are indexed by a function  $\phi$  rather than a finite set of parameters. However, a very important limitation is that Archimedean copulas are symmetric: for any permutation of indexes  $p : \{1, \dots, d\} \rightarrow \{1, \dots, d\}$ ,

$$C_\phi(u_1, \dots, u_d) = C_\phi(u_{p(1)}, \dots, u_{p(d)}), \quad u_1, \dots, u_d \in [0, 1]^d. \quad (2)$$

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The symmetry of Archimedean copulas family is often considered to be a rather strong restriction, especially in large dimensional applications. It implies that all multivariate projections of the same dimension are equal, thus, e.g., the dependence among all pairs of components is identical. To circumvent exchangeability, Archimedean copulas can be nested within each other under certain conditions. The resulting copulas are referred to as *nested Archimedean copulas* and allow to model hierarchical dependence structures. In the last decade the nested Archimedean copulas have been studied from different points of view (theoretically, computationally, in view of applications and so on). The interested reader is referred for instance to Hofert (2008, 2010), Hofert and Mächler (2011), Mcneil (2008), Brechmann (2014).

Another strategy developed in the last decade to overcome symmetry of Archimedean copulas is to build multivariate asymmetric dependence structures by generalizing some known families (Farlie-Gumbel-Morgenstern, Extreme-Value copulas and so on). The idea is to generalize the analytical expression of the considered known copula in order to break its symmetric behaviour. In the next paragraph we detail this approach, since the results of the present paper can be seen as a contribution to the corresponding literature.

## Construction of multivariate asymmetric copulas by generalizing some known families

Rodríguez-Lallena and Úbeda Flores (2004) have introduced a class of bivariate copulas  $C^*$  which generalizes some known families such as the Farlie Gumbel Morgenstern distributions of the form:

$$C^*(u, v) = uv + \lambda f(u)g(v), \quad (3)$$

where  $f$  and  $g$  are two non-zero absolutely continuous functions such that  $f(0) = f(1) = g(0) = g(1) = 0$  and the admissible range of the parameter  $\lambda$  can be obtained in terms of the derivatives of  $f$  and  $g$ . Moreover, Dolati and Úbeda Flores (2006) provided procedures to construct parametric families of multivariate distributions which generalize copulas in (3).

Kim et al. (2011) generalized the method of Rodríguez-Lallena and Úbeda Flores (2004) in (3). They define the distorted copula  $C^*$  as

$$C^*(u, v) = C(u, v) + \lambda f(u)g(v),$$

where  $C$  is an arbitrary given copula  $C$ . The method of Kim et al. (2011) gives a sufficient condition for the  $\lambda$  coefficient and it is in general rather difficult to be applied. To overcome this drawback, Mesiar and Najjari (2014) introduced a new method of constructing binary copulas, extending the original study of Rodríguez-Lallena and Úbeda Flores (2004) to new families of symmetric/asymmetric copulas.

Alfonsi and Brigo (2005) describe a new construction method for asymmetric copulas based on periodic functions. Liebscher (2008) introduced two methods to construct asymmetric multivariate copulas. The first is connected with products of copulas, i.e,

$$C^*(u_1, \dots, u_d) = \prod_{j=1}^k C_j(g_{j_1}(u_1), \dots, g_{j_d}(u_d)), \quad \text{for } u_i \in [0, 1],$$

where  $g_{j_i}$  are suitable increasing functions and  $C_j$  are copulas. The second method proposes a generalization of the Archimedean copulas family in (1). Remark that Archimedean copulas can be rewritten in the form:

$$C_\phi(u_1, \dots, u_d) = \bar{\phi}(\bar{\psi}(u_1) \times \dots \times \bar{\psi}(u_d)),$$

using the multiplicative generator  $\bar{\psi}(u) = \exp(\psi(u))$ . Let us replace the product  $\bar{\psi}(u_1) \times \dots \times \bar{\psi}(u_d)$  before by an average of products leading to

$$C_{\phi}^*(u_1, \dots, u_d) = \Psi \left( \frac{1}{m} \sum_{j=1}^m h_{j1}(\varphi(u_1)) \dots h_{jd}(\varphi(u_d)) \right), \quad (4)$$

where  $\varphi = \Psi^{-1}$ . Function in (4) represents a generalisation of Archimedean copulas being asymmetric in general. [Liebscher \(2008\)](#) provides conditions on functions and  $\Psi$  and  $h_{jk}$  such that function in (4) is a proper a copula. Recently, [Wu \(2014\)](#) proposes a new method of constructing asymmetric copulas and a convex-combination of asymmetric copulas that can exhibit different tail dependence along different directions.

An interesting class of copulas presented in the recent literature is the *extreme-value copulas* family. This class not only arise naturally in the domain of extreme events, but it can also be a convenient choice to model data with positive dependence. An advantage with respect to the much more popular class of Archimedean copulas, for instance, is that they are not symmetric. Incidentally, a hybrid class containing both the Archimedean and the extreme-value copulas as a special case are the *Archimax* copulas (see [Capéraà et al. \(2000\)](#) for the bivariate case, [Charpentier et al. \(2014\)](#) for the multivariate case). Following [Capéraà et al. \(2000\)](#), a bivariate copula is said to be Archimax if it can be written, for all  $u_1, u_2 \in (0, 1)$ , in the form:

$$C_{\phi, A}(u_1, u_2) = \phi \circ \left( \psi(u_1) + \psi(u_2) A \left\{ \frac{\psi(u_1)}{\psi(u_1) + \psi(u_2)} \right\} \right),$$

using the Pickand function  $A : [0, 1] \rightarrow [0.5, 1]$  and the generator  $\phi : \mathbb{R}^+ \rightarrow [0, 1]$ . [Mesiar and Jágr \(2013\)](#) suggest that a suitable  $d$ -variate extension of the notion of bivariate Archimax copula  $C_{\phi, A}$ , would be obtained by setting, for all  $u_1, \dots, u_d \in [0, 1]^d$ ,

$$C^*(u_1, \dots, u_d)_{\phi, \mathcal{L}} = \phi \circ \mathcal{L}(\psi(u_1), \dots, \psi(u_d)),$$

where  $\mathcal{L}$  is a the  $d$ -variate stable tail dependence function and  $\phi$  the generator of a  $d$ -variate Archimedean copula. Remark that, for  $d = 2$ ,  $A(t) = \mathcal{L}(t, 1 - t)$ , for all  $t \in [0, 1]$ , then  $C_{\phi, \mathcal{L}}(u_1, u_2) = C_{\phi, A}(u_1, u_2)$ . [Charpentier et al. \(2014\)](#) proved that, if  $\mathcal{L}$  is a  $d$ -variate stable tail dependence function and  $\phi$  the generator of a  $d$ -variate Archimedean copula, then the function  $C_{\phi, \mathcal{L}}$  is a proper  $d$ -dimensional copula (see Corollary 2.3 in [Charpentier et al. \(2014\)](#)).

The aim of the present paper is to construct multivariate families of asymmetric copulas starting from an initial multivariate Archimedean copula  $C_{\phi}(u_1, \dots, u_d)$  as in (1). This copula will be modified using a distortion matrix  $\Sigma$ . Then, the proposed model will be called *Archimatrix copula*, in order to underline the link with the Archimedean copula and the asymmetry role played by the matrix  $\Sigma$ .

**Organization of the paper** The paper is organized as follows. In Section 1 we present our model to extend the Archimedean family of copula offering the possibility of asymmetric distributions. In Section 2 suitable theoretical characteristics for the considered model are presented. Then we consider in Section 3 sufficient admissibility conditions for the proposed Archimatrix model in some particular cases. Using results of Section 3, we give some examples exhibiting multivariate distorted copulas or valid bivariate projections (see Section 4). Finally in Section 5, some supplementary properties and a sampling procedure with associated numerical illustrations are proposed.

# 1 Considered model

We focus on Archimedean copulas family presented in Equation (1). From Theorem 2.2 in McNeil and Nešlehová (2009),  $C_\phi(u_1, \dots, u_d) = \phi(\psi(u_1) + \dots + \psi(u_d))$ , is a  $d$ -dimensional copula if and only if its generator  $\phi$  is  $d$ -monotone on  $[0, \infty)$ , where the  $d$ -monotony definition is recalled hereafter.

**Definition 1.1 ( $d$ -monotone function)** *A real function  $f$  is called  $d$ -monotone in  $(a, b)$ , where  $a, b \in \mathbb{R}$  and  $d \geq 2$ , if it is differentiable there up to the order  $d - 2$  and the derivatives satisfy*

$$(-1)^k f^{(k)}(x) \geq 0, \quad k = 0, 1, \dots, d - 2$$

*for any  $x \in (a, b)$  and further if  $(-1)^{d-2} f^{(d-2)}$  is non-increasing and convex in  $(a, b)$ . For  $d = 1$ ,  $f$  is called 1-monotone in  $(a, b)$  if it is nonnegative and non-increasing there.*

*If  $f$  has derivatives of all orders in  $(a, b)$  and if  $(-1)^k f^{(k)}(x) \geq 0$ , for any  $x \in (a, b)$ , then  $f$  is called completely monotone.*

In the following, we restrict ourselves to strict generators, where  $\forall x \in \mathbb{R}^+, \phi(x) > 0$ . In this case, the function  $\psi$  is the regular inverse of  $\phi$ .

Consider a random vector  $\mathbf{U} = (U_1, \dots, U_d)$  in  $[0, 1]^d$  with Archimedean distribution  $C_\phi$ . Symmetric property of Archimedean copulas in Equation (2) has consequences in particular for bivariate projections: one have a symmetry *within* any couple of random variable  $(U_i, U_j)$ , and a symmetry *among* different couples of random variables  $(U_i, U_j)$  and  $(U_{i'}, U_{j'})$ , i.e., for  $i, j, i', j' \in \{1, \dots, d\}$ ,

$$(U_i, U_j) \stackrel{d}{=} (U_j, U_i) \quad \text{and} \quad (U_i, U_j) \stackrel{d}{=} (U_{i'}, U_{j'}). \quad (5)$$

where  $\stackrel{d}{=}$  denotes the equality in distribution.

In the following we aim at extending the Archimedean family of copula, while offering the possibility of asymmetric distributions. In order to take into account each interaction  $(U_i, U_j)$ , we consider a model with one parameter  $\sigma_{ij}$  per couple  $(U_i, U_j)$ . It is rather natural to group all these parameters within a matrix  $\Sigma = (\sigma_{ij})_{i,j \in I}$ , where from now on  $I = \{1, \dots, d\}$ . A requirement of our model is to be able to retrieve any member of the Archimedean family of copula for specific values of  $\Sigma$ . It is also very natural to use parameters of  $\Sigma$  with simple matrix products.

**Definition 1.2 (Considered model)** *Let us denote the column vectors of length  $d$ ,  $\mathbf{u} = (u_1, \dots, u_d)$ ,  $\boldsymbol{\psi}(\mathbf{u}) = (\psi(u_1), \dots, \psi(u_d))$  and  $\mathbf{1} = (1, \dots, 1)$ . We define a function  $C_{\phi, \Sigma}$  as*

$$C_{\phi, \Sigma}(\mathbf{u}) = \phi(\mathbf{1}^t \boldsymbol{\psi}(\mathbf{u}) + z(\mathbf{g}(\mathbf{u})^t \Sigma \mathbf{h}(\mathbf{u}))), \quad (6)$$

*where  $\phi$  is a valid strict Archimedean generator with regular inverse function  $\psi$ , where  $\mathbf{g} : [0, 1]^d \rightarrow \mathbb{R}^d$  and  $\mathbf{h} : [0, 1]^d \rightarrow \mathbb{R}^d$  are two vector-valued continuous functions, and  $z : \mathbb{R} \rightarrow [0, 1]$  is a continuous real function. For the sake of simplicity, functions  $z$ ,  $\mathbf{g}$  and  $\mathbf{h}$  do not appear in the notation of the function  $C_{\phi, \Sigma}$ .*

As one can see from Equation (6),  $\psi \circ C_{\phi, \Sigma}(\mathbf{u})$  is the sum of two terms and it corresponds to an Archimedean copula when the second term is zero. This last term uses the distortion matrix  $\Sigma$  with simple matrix multiplications. Then if the second terms is not zero we can obtain different asymmetric non-Archimedean copula structures. The model in Definition 1.2 will be called in the following *Archimatrix copula*, in order to underline the link with the Archimedean copula (that is a particular case of  $C_{\phi, \Sigma}$ ) and the central asymmetry role played by the matrix  $\Sigma$ . Indeed the distortion matrix  $\Sigma$  in Equation (6), with functions  $\mathbf{g}$ ,  $\mathbf{h}$  and  $z$ , permits to leave the symmetric structure typical of any Archimedean copula model. Our model in Definition 1.2 is built in the same spirit as the recent literature about the construction of

multivariate asymmetric copulas by generalizing some known families. Indeed different asymmetric models for copula structures presented in the Introduction section can be related to our model. Some comparisons in this sense will be presented in Section 4.

Furthermore, the second term in the right-hand side of Equation (6) depends on three functions  $\mathbf{g}$ ,  $\mathbf{h}$  and  $z$ , and further requirements are obviously necessary in order to ensure that the function  $C_{\phi, \Sigma}$  is a copula. We deliberately choose first a quite general form for the model, and will try to specify constraints on both parameters and  $\mathbf{g}$ ,  $\mathbf{h}$  and  $z$  functions. These constraints will derive from the validity of  $C_{\phi, \Sigma}$  as a copula function, but also on choices and desired features of the model discussed in the next section.

## 2 Suitable characteristics for the considered model

In this section we discuss requirements that are implied by desired features of the considered model in Definition 1.2. In the following, in order to link requirements implied by chosen specificities of the model, every chosen characteristic is given in a separate proposition. A further section is devoted to admissibility conditions on the copula  $C_{\phi, \Sigma}$  (see Section 3).

### 2.1 Concordance ordering for Archimatrix copulas $C_{\phi, \Sigma}$

For two copulas  $C_1$  and  $C_2$ , recall that one can say that  $C_1$  is smaller than  $C_2$  for the concordance ordering, and we write  $C_1 \prec C_2$ , if for all  $\mathbf{u} \in [0, 1]^d$ ,  $C_1(\mathbf{u}) \leq C_2(\mathbf{u})$  (see, e.g., Definition 2.8.1 in Nelsen (1999) in the bivariate setting; Joe (1990) in the general dimension  $d$ ). Considering copulas indexed by a real parameter  $\theta$ , we recall that a family  $\{C_\theta\}$  of copulas is *positively ordered* for all  $\theta_1 \leq \theta_2$ ,  $C_{\theta_1} \prec C_{\theta_2}$ , and *negatively ordered* if for all  $\theta_1 \leq \theta_2$ ,  $C_{\theta_2} \prec C_{\theta_1}$  (see Nelsen (1999), Section 4.4 in dimension 2 and Dolati and Úbeda Flores (2006) in the general dimension  $d$ ). Most usual Archimedean copulas are (positively or negatively) ordered, even if there exists Archimedean copulas neither negatively or positively ordered (see, e.g. the bivariate copula 4.2.10 in Table 4.1 of Nelsen (1999)).

In the case where the function  $C_{\phi, \Sigma}$  is a copula, a desirable feature is that, for any parameter  $\sigma_{ij}$ ,  $i, j \in I$ , the copula is either positively or negatively ordered. This may ease, for example, the interpretation of the parameters of the copula (i.e., of  $\sigma_{ij}$  for  $i, j \in I$ ). A simple sufficient condition for this is given below.

**Proposition 2.1 (Parameters of  $\Sigma$  matrix and concordance ordering)** *Let  $C_{\phi, \Sigma}$  be defined as in Definition 1.2, and assume that  $C_{\phi, \Sigma}$  is a valid copula. Assume that  $z$  is a monotone function, and that all components of the matrix  $\mathbf{g}(\mathbf{u})\mathbf{h}(\mathbf{u})^t$  have the same sign (i.e. either  $\mathbf{g}(\mathbf{u})_i\mathbf{h}(\mathbf{u})_j \geq 0$ ,  $i, j \in I$ , either  $\mathbf{g}(\mathbf{u})_i\mathbf{h}(\mathbf{u})_j \leq 0$ ,  $i, j \in I$ ). Then the copula is ordered with respect to each parameter  $\sigma_{ij}$ ,  $i, j \in I$ .*

*Proof:* This follows from differentiation of  $C_{\phi, \Sigma}$  with respect to every parameter  $\sigma_{ij}$ ,  $i, j \in I$ .  $\square$

### 2.2 Sub-model stability

Consider a model as in Equation (6), with given functions  $\mathbf{g}$ ,  $\mathbf{h}$ ,  $z$ , and assume that  $C_{\phi, \Sigma}$  is a valid copula. Let a random vector  $\mathbf{U} = (U_1, \dots, U_d)$  be distributed as  $C_{\phi, \Sigma}$ . Among the suitable constraints to impose to our model, we ask that the model remains valid for any subset of random variables among  $\{U_1, \dots, U_d\}$ . In particular, we say that the model is valid for each sub-model if for any non-empty subset  $\Omega = \{\omega_1, \dots, \omega_k\} \subset I$ ,

$$(U_{\omega_1}, \dots, U_{\omega_k}) \sim C_{\phi, \Sigma_\Omega}, \quad (7)$$

where  $\Sigma_\Omega = (\sigma_{ij})_{i, j \in \Omega}$  is a submatrix of  $\Sigma$ . This requirement is a choice that is not compulsory for defining a copula respecting (6), but that seems to us important in order to simplify the model and to interpret

its parameters. Sub-model stability is not ensured by the initial model; it does not hold for example if  $\mathbf{g}(\mathbf{u}) = \mathbf{h}(\mathbf{u}) = \mathbf{u}$ . A sufficient condition for sub-model stability is given in the proposition below.

**Proposition 2.2 (Sub-model stability)** *Consider a model as in Equation (6), assume  $C_{\phi, \Sigma}$  to be a valid copula and let a random vector  $\mathbf{U}$  be distributed as  $C_{\phi, \Sigma}$ . A sufficient condition for the propriety of sub-model stability in (7) is that for all  $\mathbf{u} \in [0, 1]^d$ , for any  $i$ th component,  $i \in I$*

$$[\mathbf{g}(\mathbf{u})]_i = [\mathbf{h}(\mathbf{u})]_i = 0 \text{ as soon as } u_i = 1. \quad (8)$$

*In particular, assume that there exist functions  $g_i$  and  $h_i$ ,  $i \in I$  so that for any  $\mathbf{u} \in [0, 1]^d$ ,  $\mathbf{g}(\mathbf{u}) = (g_1(u_1), \dots, g_d(u_d))$ ,  $\mathbf{h}(\mathbf{u}) = (h_1(u_1), \dots, h_d(u_d))$ . Assume furthermore that  $g_i(1) = h_i(1) = 0$  for all  $i \in I$ . Then the model is valid on any projections on any non-empty subset of indexes  $\Omega \subset I$ , with*

$$\mathbb{P} \left[ \bigcap_{i \in \Omega} U_i \leq u_i \right] = \phi \left( \sum_{i \in \Omega} \psi(u_i) + z \left( \sum_{i, j \in \Omega} g_i(u_i) \sigma_{ij} h_j(u_j) \right) \right). \quad (9)$$

*Proof:* Let  $\Omega \subset I$  and  $\mathcal{U}_\Omega = \{(u_1, \dots, u_d) \in [0, 1]^d : u_i = 1, i \notin \Omega\}$ . Starting from  $\mathbf{U} \sim C_{\phi, \Sigma}$  for any  $\mathbf{u} \in \mathcal{U}_\Omega$ , and using  $\psi(1) = 0$ , a sufficient condition for sub-model stability is that  $\forall \Omega \subset I, \forall \mathbf{u} \in \mathcal{U}_\Omega$ , components  $[\mathbf{g}(\mathbf{u})]_i = [\mathbf{h}(\mathbf{u})]_i = 0$  as soon as  $i \in I \setminus \Omega$ . Other sufficient conditions follows immediately.  $\square$

Despite the reduction of the variety of possible models, the sub-model stability is interesting since it permits to understand any coefficient  $\sigma_{ij}$  in the matrix  $\Sigma$  by considering only the corresponding bivariate distribution  $(U_i, U_j)$ . Under this assumption, the interpretation of these coefficients is thus more straightforward. Notice that the sub-model stability naturally holds for Archimedean copulas when the function  $z$  satisfies:  $z(\cdot) = 0$ .

### 2.3 Boundary conditions

A condition is required when the function  $C_{\phi, \Sigma}$  has to be a copula: the uniform distribution of univariate projections of  $C_{\phi, \Sigma}$  (see Proposition 2.3), and more generally boundary conditions on  $C_{\phi, \Sigma}$  (see Proposition 2.4).

Under the assumption of sub-model stability (see Proposition 2.2 before), it follows directly from Equation (9) that any  $U_i$  has a uniform distribution if and only if for any  $u_i \in [0, 1]$ ,  $z(g_i(u_i) \sigma_{ii} h_i(u_i)) = 0$ ,  $i \in I$ . A sufficient condition to achieve this compulsory requirement is the following.

**Proposition 2.3 (Uniform margins)** *Assume that  $z(0) = 0$  and that  $\sigma_{ii} = 0$  for all  $i \in I$ . Then all margins  $U_i$  follow a uniform distribution on  $[0, 1]$ , i.e.,*

$$U_i \sim U_{[0,1]}, \quad \text{for } i \in I. \quad (10)$$

Other sufficient conditions such as setting  $z(x) = 0$  for all  $x$ , or  $\mathbf{g}(\mathbf{u}) = \mathbf{h}(\mathbf{u}) = 0$  for all  $\mathbf{u}$  could be used, but would imply a simple Archimedean model for  $C_{\phi, \Sigma}$ .

In order to construct a multidimensional copula  $C_{\phi, \Sigma}$ , we provide in Proposition 2.4 two important requirements. Firstly, from assumptions in Proposition 2.2 and 2.3,  $C_{\phi, \Sigma}(1, \dots, 1, u, 1, \dots, 1) = u$ ,  $u \in [0, 1]$ , and in particular  $C_{\phi, \Sigma}(1, \dots, 1) = 1$ . Secondly, for all  $\mathbf{u} \in [0, 1]^d$ ,  $C_{\phi, \Sigma}(u_1, \dots, u_{i-1}, 0, u_{i+1}, \dots, u_d) = 0$ , i.e., the copula must be zero if one of its arguments is zero.

**Proposition 2.4 (Boundary conditions)** Assume that  $C_{\phi, \Sigma}$  is defined as in Definition 1.2. Assume that assumptions in Proposition 2.2 and 2.3 are fulfilled. Assume that  $\phi$  is a valid Archimedean generator and that there exists  $m_z \in \mathbb{R}$  such that for any  $x \in \mathbb{R}^+$ ,  $z(x) > m_z$ . If  $\phi$  is strict, or if  $m_z = 0$ , then

$$C_{\phi, \Sigma}(1, \dots, 1, u, 1, \dots, 1) = u, \quad u \in [0, 1], \quad (11)$$

$$C_{\phi, \Sigma}(u_1, \dots, u_{i-1}, 0, u_{i+1}, \dots, u_d) = 0, \quad \mathbf{u} \in [0, 1]^d \quad (12)$$

Proof: The first part comes down directly from Proposition 2.2 and 2.3. If  $z$  has a lower bound, this ensures that when  $u_i$  tends to zero,  $C_{\phi, \Sigma} \leq \phi(\sum_{j \in I} \psi(u_j) + m_z)$ . If  $\phi$  is strict, then  $\lim_{u_i \rightarrow 0} \psi(u_i) = +\infty$ , and  $\lim_{u_i \rightarrow 0} C_{\phi, \Sigma}(\mathbf{u}) = 0$ . If  $\phi$  is non strict and if  $m_z = 0$ ,  $C_{\phi, \Sigma} \leq \phi(\sum_{j \in I} \psi(u_j))$  which ensures the equality.  $\square$

## 2.4 Resulting chosen assumptions

We summary here the sufficient assumptions that are introduced before to fulfil concordance ordering, sub-model stability and boundary conditions. We only consider in the following of this paper functions  $C_{\phi, \Sigma}$  that satisfy at least Assumption 1.

**Assumption 1 (Basic required assumptions)** Consider a function  $C_{\phi, \Sigma}$  as in Definition 1.2. Basic sufficient assumptions ensuring concordance ordering, sub-model stability and boundary conditions are

- (a)  $\phi$  is a strict  $d$ -monotone Archimedean generator (see Definition 1.1),
- (b)  $\mathbf{g}(\cdot)$  and  $\mathbf{h}(\cdot)$  have monotone components, such that  $[\mathbf{g}(\mathbf{u})]_i(\mathbf{u}) = [\mathbf{h}(\mathbf{u})]_i(\mathbf{u}) = 0$  if  $u_i = 1$ ,  $i \in I$ ,
- (c) all components of the matrix  $\mathbf{g}(\mathbf{u})\mathbf{h}(\mathbf{u})^t$  have the same sign, for any  $\mathbf{u} \in [0, 1]^d$ ,
- (d)  $z$  is a monotone function such that  $z(0) = 0$ ,
- (e)  $\sigma_{ii} = 0$ , for  $i \in I$ .

In the following, we often use a supplementary set of assumptions that simplify the expression of the copula and its derivatives.

**Assumption 2 (More restrictive assumptions)** Consider a function  $C_{\phi, \Sigma}$  as in Definition 1.2. More restrictive assumptions are

- (f)  $C_{\phi, \Sigma}$  satisfies Assumption 1,
- (g)  $\mathbf{g}(\cdot)$  and  $\mathbf{h}(\cdot)$  are identical vector-valued functions with components  $[\mathbf{g}(\mathbf{u})]_i(\mathbf{u}) = [\mathbf{h}(\mathbf{u})]_i(\mathbf{u}) = g(u_i)$ ,  $\mathbf{u} \in [0, 1]^d$ , where  $g$  is a monotone real function on  $[0, 1]$ , such that  $g(1) = 0$ .
- (h)  $\sigma_{ij} = \sigma_{ji}$ , for  $i \in I$ ,  $j \in J$ .

Notice that Conditions (b) and (c) on  $\mathbf{g}(\cdot)$  and  $\mathbf{h}(\cdot)$  in Assumption 1 is automatically ensured by more restrictive Condition (g) in Assumption 2.

## 3 Admissibility conditions

We now discuss admissibility conditions of chosen Archimatrix copulas presented in Sections 1 and 2. We give here a general necessary and sufficient condition ensuring that a function  $C_{\phi, \Sigma}$  in Definition 1.2 is a copula. However, this condition require in practice checking the value of derivatives of possibly large orders on the whole support of the copula, which can be hard to provide. For this raison, the rest of this section will be devoted to the research of simpler sufficient conditions to guarantee admissibility of copula  $C_{\phi, \Sigma}$ .



**Proposition 3.1 (General admissibility condition)** Consider a function  $C_{\phi, \Sigma}$  as in Definition 1.2, satisfying Assumption 1. Assume that  $C_{\phi, \Sigma}$  is  $d$ -times differentiable with respect to successive variables  $u_1, \dots, u_d$ , then  $C_{\phi, \Sigma}$  is a copula if and only if

$$\frac{\partial^d}{\partial u_1 \dots \partial u_d} C_{\phi, \Sigma}(\mathbf{u}) \geq 0, \quad \forall \mathbf{u} \in [0, 1]^d. \quad (13)$$

*Proof:* The density of order  $d$  of the function  $C_{\phi, \Sigma}$  exists by assumption, then its positivity is a necessary condition in order to get a multivariate distribution function. Now if the derivative is positive, then by integration this ensures that the obtained copula is  $d$ -increasing. Furthermore, the function  $C_{\phi, \Sigma}$  satisfies Assumption 1 and thus boundary conditions: the function is zero if one of the arguments is zero, and the function is equal to  $u$  if one argument is  $u$  and all others 1,  $u \in [0, 1]$ . Then the condition is sufficient: under chosen assumption, it ensures that  $C_{\phi, \Sigma}$  is a proper  $d$ -dimensional copula.  $\square$ .

### 3.1 Valid bivariate projections

Bivariate projections are usually easier to represent and to understand than multivariate ones. Furthermore, in the dimension  $d = 2$ , Equation (6) in Definition 1.2 can be seen as a distortion leading to new bivariate families of copulas, which has an interest in this reduced dimension. Bivariate projections are thus treated in this section separately.

The proposed shape of the copula  $C_{\phi, \Sigma}$  aims at modelling different interactions between random variables  $U_i$  and  $U_j$ , when  $i$  and  $j$  vary in  $I$ . However, it could be desirable to have a symmetry within each bivariate projection, so that  $(U_i, U_j)$  and  $(U_j, U_i)$  may have identical distributions. A sufficient condition for symmetric bivariate projections is given in Proposition 3.2 below.

**Proposition 3.2 (Symmetric bivariate projections)** Let  $C_{\phi, \Sigma}$  as in Definition 1.2 be a proper copula, satisfying Assumption 2. Then for all  $i, j \in I$ ,

$$\sigma_{ij} = \sigma_{ji} \Rightarrow (U_i, U_j) \stackrel{d}{=} (U_j, U_i). \quad (14)$$

Remark that, when the matrix  $\Sigma$  is symmetric, any bivariate projection is symmetric, however one can still have different distributions for  $(U_i, U_j)$  and  $(U_{i'}, U_{j'})$ , for  $i, j, i', j' \in I$ .

Now one important point is being able to guarantee the positivity of the density in the dimension 2, so that each bivariate projection is a copula. Let  $\mathbf{u} = (u_1, \dots, u_d) \in [0, 1]^d$ ,  $\Omega = \{i, j\} \subset I$ , for  $i, j \in I$ ,  $i \neq j$  and let  $\mathbf{u}_\Omega$  be the vector with components  $u_i$  if  $i \in \Omega$ , or 1 otherwise. Assume that the model satisfies requirements in Propositions 2.2, 2.3 and 3.2, then

$$C_{\phi, \Sigma}(\mathbf{u}_\Omega) = \phi(\psi(u_i) + \psi(u_j) + z(\mathbf{g}(\mathbf{u})^t \Sigma \mathbf{h}(\mathbf{u}))). \quad (15)$$

As in Proposition 2.2, assume that there exist functions  $g_i$  and  $h_i$ ,  $i \in I$  so that for any  $\mathbf{u} \in [0, 1]^d$ ,  $\mathbf{g}(\mathbf{u}) = (g_1(u_1), \dots, g_d(u_d))$ ,  $\mathbf{h}(\mathbf{u}) = (h_1(u_1), \dots, h_d(u_d))$ . Assume that  $\phi$  and  $z$  are 2-times differentiable and that  $g$  is differentiable, then first order derivative of  $C_{\phi, \Sigma}$  are, for  $i \in I$ ,

$$\frac{\partial}{\partial u_i} C_{\phi, \Sigma}(\mathbf{u}_\Omega) = \phi'(\psi \circ C_{\phi, \Sigma}(\mathbf{u}_\Omega)) [\psi'(u_i) + z'(2\sigma_{ij}g_i(u_i)g_j(u_j))2\sigma_{ij}g_j(u_j)g'_i(u_i)], \quad (16)$$

and second order derivatives  $\frac{\partial^2}{\partial u_i \partial u_j} C_{\phi, \Sigma}(\mathbf{u}_\Omega)$ , for  $i, j \in I$ ,  $i \neq j$ , are

$$\begin{aligned}
& \phi''(\psi \circ C_{\phi, \Sigma}(\mathbf{u}_\Omega)) \left[ \psi'(u_i)\psi'(u_j) + 2\sigma_{ij}z'(\sigma_{ij}\eta_{ij}(\mathbf{u})) (g_j(u_j)g'_i(u_i)\psi'(u_j) + g'_j(u_j)g_i(u_i)\psi'(u_i)) \right] \\
& + \phi''(\psi \circ C_{\phi, \Sigma}(\mathbf{u}_\Omega)) \left[ 4\sigma_{ij}^2(z'(\sigma_{ij}\eta_{ij}(\mathbf{u}))^2 g'_i(u_i)g_i(u_i)g'_j(u_j)g_j(u_j)) \right] \\
& + \phi'(\psi \circ C_{\phi, \Sigma}(\mathbf{u}_\Omega)) \left[ 2\sigma_{ij}g'_i(u_i)g'_j(u_j) [2\sigma_{ij}g_i(u_i)g_j(u_j)z''(\sigma_{ij}\eta_{ij}(\mathbf{u})) + z'(\sigma_{ij}\eta_{ij}(\mathbf{u}))] \right], \tag{17}
\end{aligned}$$

where  $\eta_{ij}(\mathbf{u}) = 2g_i(u_i)g_j(u_j)$ .

One easily checks that the density expressed in Equation 17 is positive when  $\sigma_{ij} = 0$ , and corresponds to the one of the initial Archimedean copula bivariate projections. However, it is not easy to find minimal and maximal values of  $\sigma_{ij}$  in the general case:  $\sigma_{ij}$  appears into  $z'(\cdot)$  and in an hidden way in  $\psi \circ C_{\phi, \Sigma}(\mathbf{u}_\Omega)$ . When  $C_{\phi, \Sigma}$  satisfies Assumption 1, a necessary and sufficient condition for bivariate projection admissibility is simply given by the positivity of the density in Equation (17) (see Proposition 3.1). However, in some cases, a minoration of this density leads to sufficient conditions that are more straightforward to check. Following result gives an example of such a sufficient condition, which is easy to check when  $g'$  is linked with  $\psi'$ .

**Proposition 3.3 (Simplified bivariate admissibility sufficient condition)** *Consider a function  $C_{\phi, \Sigma}$  as in Definition 1.2, satisfying Assumption 2 with  $\phi$  being 2-times differentiable and  $g$  differentiable. Assume that the generator  $\phi$  is such that  $\rho_\phi := \inf \left\{ \left| \frac{\phi''(x)}{\phi'(x)} \right|, x \in \mathbb{R}^+ \right\} > 0$ . Assume that for all  $x \in \mathbb{R}^+$ ,  $(z'(x))^2 \rho_\phi - z''(x) \geq 0$  and that  $z'(x) > 0$ . If for all  $\mathbf{u} \in [0, 1]^d$ ,*

$$0 \leq \sigma_{ij} \leq \frac{1}{2} \frac{1}{z'(2\sigma_{ij}g_i(u_i)g_j(u_j))} \frac{\psi'(u_i)\psi'(u_j)}{g'_i(u_i)g'_j(u_j)} \rho_\phi, \tag{18}$$

then any bivariate projection in (15) of  $C_{\phi, \Sigma}$  is a copula.

*Proof:* Let us denote in a synthetic way  $g'_i = g'_i(u_i)$ ,  $g'_j = g'_j(u_j)$ ,  $\psi'_i = \psi'(u_i)$ ,  $z' = z'(2\sigma_{ij}g_i(u_i)g_j(u_j))$ ,  $z'' = z''(2\sigma_{ij}g_i(u_i)g_j(u_j))$ ,  $\phi' = \phi'(\psi \circ C_{\phi, \Sigma}(\mathbf{u}_\Omega))$ ,  $\phi'' = \phi''(\psi \circ C_{\phi, \Sigma}(\mathbf{u}_\Omega))$ . The density can be written

$$\phi''\psi'_i\psi'_j + 2\sigma_{ij}z' \cdot [(g_j g'_i \psi'_j + g'_j g_i \psi'_i) \phi'' + g'_i g'_j \phi'] + 4\sigma_{ij}^2 g'_i g_i g'_j g_j [(z')^2 \cdot \phi'' + z'' \cdot \phi']$$

which under chosen assumption is greater than

$$\phi''\psi'_i\psi'_j + 2\sigma_{ij}z' [0 + g'_i g'_j \phi'] + 0$$

and we check that this latter quantity is greater than zero under chosen assumptions.  $\square$

### 3.2 Valid multivariate Archimatrix copula for linear function $z$

Proposition 3.3 gives a bivariate admissibility sufficient condition. However, it is more challenging to prove the positivity of the density of  $C_{\phi, \Sigma}$  in any given dimension  $d$ . To this aim, the choice of a linear or affine function  $z$  can help since it leads to vanishing derivatives of  $z(\mathbf{g}(\mathbf{u})^t \Sigma \mathbf{h}(\mathbf{u}))$  when orders are greater than two. Indeed, for distinct  $i, j, k \in I$ , and when  $\mathbf{g}(\mathbf{u}) = (g_1(u_1), \dots, g_d(u_d))$ ,  $\mathbf{h}(\mathbf{u}) = (h_1(u_1), \dots, h_d(u_d))$ ,

$$\frac{\partial^3}{\partial u_i \partial u_j \partial u_k} z(\mathbf{g}(\mathbf{u})^t \Sigma \mathbf{h}(\mathbf{u})) = 0. \tag{19}$$

In the following, we thus give expressions of the density of the copula and admissibility conditions in the simplified model where  $\mathbf{g}(\mathbf{u}) = (g_1(u_1), \dots, g_d(u_d))$ ,  $z(x) = x/2$  and

$$C_{\phi, \Sigma}(\mathbf{u}) = \phi \left( \mathbf{1}^t \boldsymbol{\psi}(\mathbf{u}) + \frac{1}{2} \mathbf{g}(\mathbf{u})^t \Sigma \mathbf{g}(\mathbf{u}) \right). \tag{20}$$

In this case we get the following expression for the density.

**Proposition 3.4 (Density in the simplified model)** Consider a function  $C_{\phi, \Sigma}$  as in Definition 1.2, satisfying Assumption 2. Furthermore consider the simplified multivariate Archimatrix model with linear function  $z$  as in (20), where  $\mathbf{g}(\mathbf{u}) = (g_1(u_1), \dots, g_d(u_d))$  with  $g$  differentiable. Let  $k \in I$  and assume that  $\phi$  is a  $k$ -times differentiable generator. Let  $\lfloor x \rfloor$  the integer part of  $x$ . Then we have

$$\frac{\partial^k}{\partial u_1 \dots \partial u_k} C_{\phi, \Sigma}(\mathbf{u}) = \sum_{\nu=0}^{\lfloor k/2 \rfloor} \phi^{(k-\nu)} \circ \psi \circ C_{\phi, \Sigma}(\mathbf{u}) \cdot R_{\nu, k}(\mathbf{u}), \quad (21)$$

with

$$\begin{cases} R_{0, k}(\mathbf{u}) &= \prod_{l=1}^k G_l(\mathbf{u}) \\ R_{\nu, k}(\mathbf{u}) &= \sum_{\{i_1, j_1, \dots, i_\nu, j_\nu\} \subset \{1, \dots, k\}} G_{i_1 j_1} \dots G_{i_\nu j_\nu} \prod_{l \in \{1, \dots, k\} \setminus \{i_1, j_1, \dots, i_\nu, j_\nu\}} G_l(\mathbf{u}), \quad \text{for } \nu \geq 1, \end{cases} \quad (22)$$

where  $G_i(\mathbf{u}) = \psi'(u_i) + g'(u_i) \sum_{r=1}^d \sigma_{ir} g(u_r)$ ,  $G_{ij}(\mathbf{u}) = g'(u_i) \sigma_{ij} g'(u_j)$ ,  $i, j \in I$ ,  $j \neq i$  and where the sum over  $\{i_1, j_1, \dots, i_\nu, j_\nu\} \subset \{1, \dots, k\}$  refers to all possible distinct choices of  $\nu$  couples in the set  $\{1, \dots, k\}$  (i.e. with  $i_1 < \dots < i_\nu$ , with  $i_r < j_r$  for  $r = 1, \dots, \nu$ , and with all values in  $\{i_1, j_1, \dots, i_\nu, j_\nu\}$  being distinct).

*Proof:* This follows directly from the multivariate version of Faà di Bruno's formula for partial derivatives, using the fact that derivatives of  $\mathbf{g}(\mathbf{u})^t \Sigma \mathbf{g}(\mathbf{u})$  vanish for orders greater than 2 (see Equation (19)).  $\square$

As an example, denoting  $C_{\phi, \Sigma}^{(k)}(\mathbf{u}) := \frac{\partial^k}{\partial u_1 \dots \partial u_k} C_{\phi, \Sigma}(\mathbf{u})$ , one gets for the first four orders

$$\begin{aligned} C_{\phi, \Sigma}^{(1)}(\mathbf{u}) &= \phi^{(1)} \circ \psi \circ C_{\phi, \Sigma}(\mathbf{u}) \cdot G_1(\mathbf{u}), \\ C_{\phi, \Sigma}^{(2)}(\mathbf{u}) &= \phi^{(2)} \circ \psi \circ C_{\phi, \Sigma}(\mathbf{u}) \cdot (G_1 G_2)(\mathbf{u}) + \phi^{(1)} \circ \psi \circ C_{\phi, \Sigma}(\mathbf{u}) \cdot (G_{12})(\mathbf{u}), \\ C_{\phi, \Sigma}^{(3)}(\mathbf{u}) &= \phi^{(3)} \circ \psi \circ C_{\phi, \Sigma}(\mathbf{u}) \cdot (G_1 G_2 G_3)(\mathbf{u}) + \phi^{(2)} \circ \psi \circ C_{\phi, \Sigma}(\mathbf{u}) \cdot (G_{12} G_3 + G_{13} G_2 + G_{23} G_1)(\mathbf{u}), \\ C_{\phi, \Sigma}^{(4)}(\mathbf{u}) &= \phi^{(4)} \circ \psi \circ C_{\phi, \Sigma}(\mathbf{u}) \cdot (G_1 G_2 G_3 G_4)(\mathbf{u}) \\ &\quad + \phi^{(3)} \circ \psi \circ C_{\phi, \Sigma}(\mathbf{u}) \cdot (G_{12} G_3 G_4 + G_{13} G_2 G_4 + G_{14} G_2 G_3 + G_{23} G_1 G_4 + G_{24} G_1 G_3 + G_{34} G_1 G_2)(\mathbf{u}) \\ &\quad + \phi^{(2)} \circ \psi \circ C_{\phi, \Sigma}(\mathbf{u}) \cdot (G_{12} G_{34} + G_{13} G_{24} + G_{14} G_{23})(\mathbf{u}), \end{aligned}$$

In the case where  $\phi(x) = \exp(-x)$ , one can check that  $\phi^{(k)} \circ \psi \circ C_{\phi, \Sigma}(\mathbf{u}) = (-1)^k C_{\phi, \Sigma}(\mathbf{u})$  and the expression can be simplified. Then, the following corollary illustrates the general result provided in Proposition 3.4 in the independence generator case.

**Corollary 3.1 (Density starting from independence)** In the same assumption setting as Proposition 3.4, when  $\phi(x) = \exp(-x)$ ,  $x \in \mathbb{R}^+$  is the independence generator, then for all  $\mathbf{u} \in (0, 1)^d$ ,

$$C_{\phi, \Sigma}^{(k)}(\mathbf{u}) = C_{\phi, \Sigma}(\mathbf{u}) \prod_{l \in I_k} |\psi'(u_l)| \sum_{\nu=0}^{\lfloor k/2 \rfloor} (-1)^\nu \sum_{\{i_1, j_1, \dots, i_\nu, j_\nu\} \subset I_k} \gamma_{i_1 j_1} \dots \gamma_{i_\nu j_\nu}(\mathbf{u}) \prod_{l \in I_k \setminus \{i_1, j_1, \dots, i_\nu, j_\nu\}} \gamma_l(\mathbf{u}) \quad (23)$$

where  $I_k = \{1, \dots, k\}$ ,  $\gamma_i(\mathbf{u}) = G_i(\mathbf{u})/\psi'(u_i)$ ,  $\gamma_{ij}(\mathbf{u}) = G_{ij}(\mathbf{u})/(\psi'(u_i)\psi'(u_j))$  and here  $\psi'(u) = -1/u$ .

*Proof:* Follows directly from Proposition 3.4, using in the independence case  $\phi^{(k)} \circ \psi \circ C_{\phi, \Sigma}(\mathbf{u}) = (-1)^k C_{\phi, \Sigma}(\mathbf{u})$ , and factorizing the product of  $\psi'(u_l) = -|\psi'(u_l)|$ .  $\square$

Also in this case, necessary and sufficient admissibility conditions can be given by requiring the positivity of the density in Equation (21). However, this kind of expression involving partial derivatives of order  $k$  would be of few interest in practice. In following proposition, we give a simplified admissibility condition involving more directly coefficients  $\sigma_{ij}$  of the matrix  $\Sigma$ .

**Proposition 3.5 (Sufficient admissibility condition in dimension  $k$ )** Consider the multivariate Archimatrix model for linear  $z$  in (20), satisfying Assumption 2. Assume that  $\phi$  is  $k$ -times differentiable,  $g$  differentiable and denote  $\gamma_{ij}(\mathbf{u}) = \frac{g'(u_i)g'(u_j)}{\psi'(u_i)\psi'(u_j)}\sigma_{ij}$ ,  $i, j \in I$ . Let  $I_k = \{1, \dots, k\}$ ,  $\Pi_\nu(I_k)$  be the set of all possible distinct choices of  $\nu$  couples among  $I_k$ . Let  $\rho_{\phi, k} = \inf_{x \in \mathbb{R}^+} \frac{|\phi^{(k)}(x)|}{|\phi^{(k-1)}(x)|}$ . Assume that all  $\sigma_{ij}$  are positive or zero. Then for  $k \in I$  a sufficient condition for the positivity of  $C_{\phi, \Sigma}^{(k)}(\mathbf{u})$  is that for all  $\nu \leq \lfloor k/2 \rfloor - 1$  even, for all  $\pi_\nu \in \Pi_\nu(I_k)$ , for all  $\mathbf{u}$ ,

$$\sum_{\{i, j\} \subset \Pi_1(I_k \setminus \pi_\nu)} \gamma_{ij}(\mathbf{u}) \leq \rho_{\phi, k-\nu}. \quad (24)$$

Setting  $\rho_{\phi, 1..k} = \inf_{\nu \in \{0, \dots, \lfloor k/2 \rfloor - 1\}} \rho_{\phi, k-\nu}$ , a simplified sufficient condition is that  $\sum_{i, j \in I_k, i < j} \gamma_{ij}(\mathbf{u}) \leq \rho_{\phi, 1..k}$ . If  $\phi$  is the independent generator, then  $\rho_{\phi, 1..k} = 1$ . In the case where  $g = \psi$ ,  $\gamma_{ij}(\mathbf{u}) = \sigma_{ij}$  and the admissibility condition does only depend on these parameters.

*Proof:* Using previous notations,  $G_i(\mathbf{u}) = \psi'(u_i)\gamma_i(\mathbf{u})$  and  $G_{ij}(\mathbf{u}) = \psi'(u_i)\psi'(u_j)\gamma_{ij}(\mathbf{u})$ . Let  $\pi_\nu = \{i_1, j_1, \dots, i_\nu, j_\nu\}$  be an (ordered) member of  $\Pi_\nu(I_k)$ , for  $\nu \geq 1$ . Denote

$$S_0^k(\mathbf{u}) = \prod_{l \in I_k} \gamma_l(\mathbf{u}) \quad \text{and} \quad S_\nu^k(\mathbf{u}) = \sum_{\pi_\nu \in \Pi_\nu(I_k)} \gamma_{i_1 j_1}(\mathbf{u}) \dots \gamma_{i_\nu j_\nu}(\mathbf{u}) \prod_{l \in I_k \setminus \pi_\nu} \gamma_l(\mathbf{u}) \quad (25)$$

so that  $R_\nu^k(\mathbf{u}) = \prod_{l \in I_k} \psi'(u_l) S_\nu^k(\mathbf{u})$ . Using  $\phi^{(k-\nu)}(\cdot) = (-1)^{k-\nu} |\phi^{(k-\nu)}(\cdot)|$  from the  $d$ -monotony of  $\phi$ ,

$$\frac{\partial^k}{\partial u_1 \dots \partial u_k} C_{\phi, \Sigma}(\mathbf{u}) = \prod_{l \in I_k} (-\psi'(u_l)) \sum_{\nu=0}^{\lfloor k/2 \rfloor} (-1)^\nu \left| \phi^{(k-\nu)} \circ \psi \circ C_{\phi, \Sigma}(\mathbf{u}) \right| \times S_\nu^k(\mathbf{u}). \quad (26)$$

Since  $\rho_{\phi, k} = \inf_{x \in \mathbb{R}^+} \frac{|\phi^{(k)}(x)|}{|\phi^{(k-1)}(x)|}$ , then  $\frac{\partial^k}{\partial u_1 \dots \partial u_k} C_{\phi, \Sigma}(\mathbf{u})$  is greater than

$$\sum_{\nu=0, \nu \text{ even}}^{\lfloor k/2 \rfloor} \left| \phi^{(k-1-\nu)} \circ \psi \circ C_{\phi, \Sigma}(\mathbf{u}) \right| \left[ \rho_{\phi, k-\nu} S_\nu(\mathbf{u}) - \mathbf{1}_{\{\nu+1 \leq \lfloor k/2 \rfloor\}} S_{\nu+1}(\mathbf{u}) \right]. \quad (27)$$

One can check that for any  $\rho_{\phi, k-\nu} > 0$ , since  $\gamma_{i_{\nu+1}}(\mathbf{u})\gamma_{j_{\nu+1}}(\mathbf{u}) \geq 1$ ,

$$\begin{aligned} \rho_{\phi, k-\nu} S_\nu(\mathbf{u}) - S_{\nu+1}(\mathbf{u}) &\geq \sum_{\pi_\nu \in \Pi_\nu(I_k)} \gamma_{i_1 j_1}(\mathbf{u}) \dots \gamma_{i_\nu j_\nu}(\mathbf{u}) \\ &\cdot \prod_{l \in I_k \setminus \pi_\nu} \gamma_l(\mathbf{u}) \left[ \rho_{\phi, k-\nu} - \mathbf{1}_{\{\nu+1 \leq \lfloor k/2 \rfloor\}} \sum_{\{i_{\nu+1}, j_{\nu+1}\} \subset \Pi_1(I_k \setminus \pi_\nu)} \gamma_{i_{\nu+1} j_{\nu+1}}(\mathbf{u}) \right], \end{aligned}$$

so that a sufficient condition is that for all  $k \in I$ , for all  $\nu$  even such that  $\nu+1 \leq \lfloor k/2 \rfloor$ , for all  $\pi_\nu \in \Pi_\nu(I_k)$ ,

$$\sum_{\{i_{\nu+1}, j_{\nu+1}\} \subset \Pi_1(I_k \setminus \pi_\nu)} \gamma_{i_{\nu+1} j_{\nu+1}}(\mathbf{u}) \leq \rho_{\phi, k-\nu}. \quad (28)$$

Hence the result.  $\square$

As an illustration of Proposition 3.5, we write in the following the sufficient admissibility conditions provided in Equation (24), for different value of the considered dimension  $k$ :

$$\begin{aligned} \text{for } k = 2, & \quad \gamma_{12} \leq \rho_{\phi,2}, \\ \text{for } k = 3, & \quad \gamma_{12} + \gamma_{13} + \gamma_{23} \leq \rho_{\phi,3}, \\ \text{for } k = 4, & \quad \gamma_{12} + \gamma_{13} + \gamma_{14} + \gamma_{23} + \gamma_{24} + \gamma_{34} \leq \rho_{\phi,4}, \\ \text{for } k = 5, & \quad \gamma_{12} + \gamma_{13} + \gamma_{14} + \gamma_{15} + \gamma_{23} + \gamma_{24} + \gamma_{25} + \gamma_{34} + \gamma_{35} + \gamma_{45} \leq \rho_{\phi,5}. \end{aligned}$$

Remark that for  $k = 1, 2, 3, 4, 5$ , Equation (24) provides a single condition since  $\nu = 0$ . Furthermore, remark that in the Independent copula case  $\rho_{\phi,k} = 1$ , for all  $k$ . Conversely, in the Clayton copula case  $\frac{|\phi^{(k)}(x)|}{|\phi^{(k-1)}(x)|} = \frac{|(k-1)\theta+1|}{|\theta x+1|}$ . Then, for  $\theta > 0$ , there is not possible to find a positive lower bound for this quantity. For Joe, Ali-Mikhail-Haq and Frank copula families, at least numerically, it seems possible to find positive constants to inferiorly bound the ratio of the derivatives of the associated generator.

Following result provides the sufficient admissibility condition in dimension  $k$  in Proposition 3.5 in the simplified case when  $g = \psi$  and  $\phi(t) = \exp(-t)$ .

**Corollary 3.2 (Admissibility condition from independence)** *Consider the multivariate Archimatrix model for linear  $z$  in (20), satisfying Assumption 2. Assume conditions of Proposition 3.5 hold true. Furthermore, if  $g = \psi$  and  $\phi(t) = \exp(-t)$ , a sufficient admissibility condition is that*

$$\sum_{\{i,j\} \subset \Pi_1(I_k)} \sigma_{ij} \leq 1. \quad (29)$$

When  $\phi$  is the independence generator, and when  $\mathbf{g} = \mathbf{h} = \psi$ , we have seen that each bivariate projection is a distinct Gumbel-Barnett copula with one parameter per projection. However if one respects this sufficient condition, the sum of parameters is bounded. Thus, the higher the dimension, the more constraint is the copula, with average parameter necessarily closer to zero. In high dimension, this illustrates the fact that the copula  $C_{\phi,\Sigma}$  may in some cases be relatively close to the initial Archimedean copula  $C_{\phi}$ , due to admissibility constraints relying on the parameters. This is obviously one limitation of the resulting multivariate copula.

## 4 Examples and links with existing models

We give hereafter some examples exhibiting valid projections. One shall keep in mind that the related functions  $C_{\phi,\Sigma}$  are not necessarily copulas. Admissibility conditions for bivariate projection or at higher order will be discussed for each example.

When  $C_{\phi,\Sigma}$  is a copula, we now consider in its expression the quantity  $z(\mathbf{g}(\mathbf{u})\Sigma\mathbf{h}(\mathbf{u}))$ .

- If this quantity is always positive, then, as  $\phi$  is decreasing, the copula  $C_{\phi,\Sigma} \prec C_{\phi}$  (see Section 2) and its level curves are nearer to the ones of the lower Fréchet-Hoeffding bound (cf. Figure 2.2 in Nelsen (1999)). The positive dependence is, in this concordance ordering sense, reduced (this is the case for further Examples 1 and 2).
- If it is always negative, then the positive dependence is increased (this will be the case for Example 4).
- This quantity may not be always positive or negative, as shown in Example 3. In this case the dependence is increased for some projections and decreased for others.

**Example 1 (A model with linear  $z$  function)** Let  $z(x) = \frac{1}{2}x$  and  $g_i(x) = h_i(x) = \psi(x)$ ,  $i \in I$ ,  $x \in \mathbb{R}^+$ . The model in Definition 1.2 becomes

$$C_{\phi, \Sigma}(\mathbf{u}) = \phi \left( \mathbf{1}^t \psi(\mathbf{u}) + \frac{1}{2} \psi(\mathbf{u})^t \Sigma \psi(\mathbf{u}) \right), \quad (30)$$

Let  $\rho_\phi = \inf \left\{ \left| \frac{\phi''(x)}{\phi'(x)} \right|, x \in \mathbb{R}^+ \right\}$ . By Proposition 3.3, any bivariate projection is valid if

$$\sigma_{ij} \in [0, \rho_\phi] \quad (31)$$

In the particular case where  $\phi(x) = \exp(-x)$  (i.e., the generator of the independence copula), then  $\rho_\phi = 1$ , the sufficient validity condition becomes  $\sigma_{ij} \in [0, 1]$ . One easily shows that each bivariate projection corresponds here to an Archimedean Gumbel-Barnett copula of parameter  $\sigma_{ij}$ , where

$$P[U_i \leq u_i, U_j \leq u_j] = uv \exp(-\sigma_{ij} \ln(u_i) \ln(u_j)), \quad \sigma_{ij} \in [0, 1]. \quad (32)$$

Each projection has its own parameter  $\sigma_{ij}$ , which allows bivariate projections  $(U_i, U_j)$  to have different distributions, for  $i, j \in I$ . In Section 5, we will generate a 3-dimensional Archimatrix copula as in (30) with  $\phi(x) = \exp(-x)$  (see Figure 1, left). Remark that in this case condition (24) in Proposition 3.5 becomes:  $\sigma_{12} + \sigma_{13} + \sigma_{23} \leq 1$ , since  $\nu = 0$ ,  $\gamma_{ij}(\mathbf{u}) = \sigma_{ij}$ ,  $k = 3$  and  $\rho_{\phi,3} = 1$  (see Corollary 3.2.)

**Example 2 (A model with power-type  $z$  function)** Let  $z(x) = \frac{\alpha}{2}x^\alpha$ , for  $\alpha \in (0, 1]$ , and  $g_i(x) = (\psi(x))^{1/\alpha}$ , for  $i \in I$ ,  $x \in \mathbb{R}^+$ . The model in Definition 1.2 becomes

$$C_{\phi, \Sigma; \alpha}(\mathbf{u}) = \phi \left( \mathbf{1}^t \psi(\mathbf{u}) + \frac{\alpha}{2} \left( \psi^{1/\alpha}(\mathbf{u})^t \Sigma \psi^{1/\alpha}(\mathbf{u}) \right)^\alpha \right). \quad (33)$$

Model in (33) generalizes Example 1, which corresponds to the case  $\alpha = 1$ . One can show that any bivariate projection is valid if

$$\sigma_{ij} \in [0, \rho_\phi^{1/\alpha}], \quad (34)$$

with  $\rho_\phi = \inf \left\{ \left| \frac{\phi''(x)}{\phi'(x)} \right|, x \in \mathbb{R}^+ \right\}$ .

**Example 3 (A model with logarithmic  $z$  function)** Let  $z(x) = -\ln(1 + \frac{x}{2})$  and  $g_i(x) = 1 - x$ , for  $i \in I$ ,  $x \in \mathbb{R}^+$ . The model in Definition 1.2 becomes

$$C_{\phi, \Sigma}(\mathbf{u}) = \phi \left( \mathbf{1}^t \psi(\mathbf{u}) - \ln \left( 1 + \frac{1}{2} (\mathbf{1} - \mathbf{u})^t \Sigma (\mathbf{1} - \mathbf{u}) \right) \right), \quad (35)$$

In the particular case where  $\phi(x) = \exp(-x)$  (i.e., the generator of the independence copula), one easily shows that each bivariate projection corresponds to a Farlie-Gumbel-Morgenstern copula of parameter  $\sigma_{ij}$ , where

$$P[U_i \leq u_i, U_j \leq u_j] = u_i u_j (1 + \sigma_{ij} (1 - u_i)(1 - u_j)), \quad \text{for } \sigma_{ij} \in [-1, 1]. \quad (36)$$

Typically, this is an example where the simplified sufficient condition given in Proposition 3.3 does not suffice to determine the parameter range, which can be obtained from the positivity expression of the density. In Figure 1, right (Section 5) we generate 3-dimensional data from model in (35). We take  $\sigma_{12} = -0.99$ ,  $\sigma_{13} = 0.99$  and  $\sigma_{23} = 0.2$ .

**Example 4 (A particular Archimax case)** Let  $z(x) = -\left(\frac{x}{2}\right)^{\frac{1}{2\alpha}}$  and  $g_i(x) = \psi(x)^\alpha$ , for  $i \in I$ ,  $x \in \mathbb{R}^+$ . The model in Definition 1.2 becomes:

$$C_{\phi, \Sigma; \alpha}(\mathbf{u}) = \phi \left( \mathbf{1}^t \psi(\mathbf{u}) - \left( \frac{1}{2} \psi^\alpha(\mathbf{u})^t \Sigma \psi^\alpha(\mathbf{u}) \right)^{\frac{1}{2\alpha}} \right), \quad (37)$$

with  $\alpha > 0$ . For a bivariate projection on axis  $i$  and  $j$ , a couple of parameter  $(\alpha, \sigma_{ij})$  leads to the same function as a couple  $(1, \sigma_{ij}^{\frac{1}{2\alpha}})$ , thus for varying  $\sigma_{ij}$ , the class of reachable bivariate projections does not depend on  $\alpha$ . The model in (37) can be seen as a particular model of Archimax copulas. Indeed, we have

$$\begin{aligned} C_{\phi, \Sigma}(\mathbf{u}) &= C_{\phi, \mathcal{L}} = \phi \circ \mathcal{L}(\psi(u_1), \dots, \psi(u_d)), \\ \text{with } \mathcal{L}(\mathbf{x}) &= \mathbf{1}^t \mathbf{x} - \left( \frac{1}{2} \mathbf{x}^\alpha \Sigma \mathbf{x}^\alpha \right)^{\frac{1}{2\alpha}}, \quad \mathbf{x} \in \mathbb{R}_+^d, \end{aligned} \quad (38)$$

(see the Introduction section). For instance in a bivariate framework, for  $\alpha = 1/2$  we have the simplified Archimax model:

$$C_{\phi, \Sigma}(u_i, u_j) = \phi \left( \psi(u_i) + \psi(u_j) - \sigma \sqrt{\psi(u_i)} \sqrt{\psi(u_j)} \right). \quad (39)$$

Trivially, the function  $\mathcal{L}$  in (38) is homogeneous of order 1, with  $\mathcal{L}(\alpha \mathbf{x}) = \alpha \mathcal{L}(\mathbf{x})$ , for all  $\alpha \in (0, \infty)$  and  $\mathbf{x} \in \mathbb{R}_+^d$ . Furthermore,  $\mathcal{L}(\mathbf{e}_1) = \dots = \mathcal{L}(\mathbf{e}_d) = 1$ , where  $\mathbf{e}_j$  denotes a  $d$ -dimensional vector whose components are all 0 except the  $j$ th. Finally, for the bivariate Archimax model in (39) one can prove that, if  $\sigma \in [0, 1]$ , then  $\mathcal{L}$  satisfies the fully 2-max decreasing property (see Theorem 2.2 in Charpentier et al. (2014) and Ressel (2013)) and  $\mathcal{L}$  is a 2-variate stable tail dependence function. Then, in this particular case, all conditions of Corollary 2.3 in Charpentier et al. (2014) are satisfied. Then a sufficient condition so that the function  $C_{\phi, \Sigma}$  in Equation (39) is a valid copula is that  $\sigma \in [0, 1]$ . A sampling of the 2-dimensional Archimax model in Equation (39) is provided in Figure 2 (see Section 5). Furthermore, a study of the tail dependent coefficients of this model is given in Property 5.

Example	Model	$g_i(x)$	$z(x)$	$\sigma$ , case $d = 2$	if $\phi(x) = \exp(-x)$
1	Linear-type $z$ function	$\psi(x)$	$\frac{1}{2}x$	$[0, \rho_\phi]$	$[0, 1]$ (Barnett-Gumbel)
2	Power-type $z$ function	$\psi(x)^{1/\alpha}$	$\frac{\alpha}{2}x^\alpha$	$[0, \rho_\phi^{1/\alpha}]$	$[0, 1]$
3	Logarithmic $z$ function	$1 - x$	$-\ln(1 + \frac{x}{2})$	-	$[-1, 1]$ (FGM)
4	Particular Archimax model	$\psi(x)^\alpha$	$-\left(\frac{x}{2}\right)^{\frac{1}{2\alpha}}$	$[0, 1]$	$[0, 1]$

Table 1: Examples 1-4 of Archimatrix copula models for different choices of  $g_i$  and  $z$  functions. Furthermore in the bivariate setting, the range for  $\sigma$  parameter is provided both in the general case and when  $\phi(x) = \exp(-x)$ .

Remark that in Table 1, in the case of logarithmic  $z$  function and general bivariate generator, the explicit condition for the admissibility of the considered transformed copula  $C_{\phi, \Sigma}$  can be difficult to obtain. However in this logarithmic model an explicit condition is given when  $\phi(x) = \exp(-x)$ . The reason of this difficulty is related to the fact that in Example 3 the dependence can increase in some projections and decrease in others.

## 5 Further properties and numerical illustrations

In this section we gathered some properties of the proposed Archimatrix copulas in Equation (6).

**Property 1 (Impact of Archimedean transformations)** Let  $T : [0, 1] \rightarrow [0, 1]$  be an increasing bijection, and denote a transformed copula by

$$\tilde{C}(u_1, \dots, u_d) = T \circ C(T^{-1}(u_1), \dots, T^{-1}(u_d)). \quad (40)$$

For further details on copulas as in (40) the interested reader is referred for example to Durrleman et al. (2000), Valdez and Xiao (2011), Klement et al. (2005), Di Bernardino and Rullière (2013). It is easily seen that if  $C$  is an Archimedean copula with generator  $\phi$ , then when  $\tilde{C}$  is a copula,  $\tilde{C}$  is still Archimedean with transformed generator  $\tilde{\phi} = T \circ \phi$ . Now consider an Archimatrix copula  $C_{\phi, \Sigma}$  as in Equation (6). If there exist functions  $\mathbf{f}_g$  and  $\mathbf{f}_h$  such that  $\mathbf{g}(\mathbf{u}) = \mathbf{f}_g(\psi(\mathbf{u}))$  and  $\mathbf{h}(\mathbf{u}) = \mathbf{f}_h(\psi(\mathbf{u}))$  depend on the chosen generator, then

$$C_{\tilde{\phi}, \Sigma}(\mathbf{u}) = T \circ C_{\phi, \Sigma}(T^{-1}(u_1), \dots, T^{-1}(u_d)) = \tilde{C}_{\phi, \Sigma}(\mathbf{u}). \quad (41)$$

What is noticeable here is that in the Archimedean case,  $T$  is preserving the Archimedean structure, and thus the symmetry. For Archimatrix copulas, the asymmetry depends in this case on the matrix  $\Sigma$ , which is the same in  $C_{\tilde{\phi}, \Sigma}$  or in  $\tilde{C}_{\phi, \Sigma}$ . In a sense,  $\Sigma$  impacts essentially the symmetry of the copula, whereas the transformation  $T$  impacts the position of its level curves (see, e.g., Di Bernardino and Rullière (2013)).

**Property 2 (Level curves)** Consider an Archimatrix copula  $C_{\phi, \Sigma}$  as in Equation (6) and assume that  $\mathbf{g} = \mathbf{h} = \psi$ . Define the level-set  $\partial L_{C_{\phi, \Sigma}}(\alpha)$ , for  $\alpha \in (0, 1)$ , as

$$\partial L_{C_{\phi, \Sigma}}(\alpha) = \{\mathbf{u} \in [0, 1]^d : C_{\phi, \Sigma}(\mathbf{u}) = \alpha\}. \quad (42)$$

One can easily check that

$$\partial L_{C_{\phi, \Sigma}}(\alpha) = \{\mathbf{u} \in [0, 1]^d : \mathbf{u} = \phi(\mathbf{x}), \mathbf{x} \in \mathcal{S}(\psi(\alpha))\}, \quad (43)$$

where the solution set  $\mathcal{S}(\beta) = \{\mathbf{x} \in \mathbb{R}_+^d : \mathbf{1}^t \mathbf{x} + z(\mathbf{x}^t \Sigma \mathbf{x}) = \beta\}$ . In the case where  $z$  is linear, this solution set is easily obtained as a solution of a quadratic form, and explicit parametric forms of the level set can be obtained.

**Property 3 (Averaging of Archimatrix functions)** Consider a finite set of indexes  $K$ , a sequence of matrices  $\Sigma_k$ , and functions  $C_{\phi, \Sigma_k}$ ,  $k \in K$  as in Equation (6), which are not necessarily copulas. Consider the case where all these functions depend on the same  $z(x) = cx^r$  for some constant  $c \in \mathbb{R}$ ,  $r \in \mathbb{R}^*$ . Let  $\{\alpha_k, k \in K\}$  be a set of real coefficients such that  $\sum_{k \in K} \alpha_k = 1$  and let  $\Sigma = \sum_{k \in K} \alpha_k^{1/r} \Sigma_k$ , then

$$\phi \left( \sum_{k \in K} \alpha_k \cdot \psi \circ C_{\phi, \Sigma_k} \right) = C_{\phi, \Sigma} \quad (44)$$

In particular, for independence generator  $\phi(x) = \exp(-x)$ , when  $z$  is linear, we get  $\Sigma = \sum_{k \in K} \alpha_k \Sigma_k$  and

$$\prod_{k \in K} C_{\phi, \Sigma_k}^{\alpha_k} = C_{\phi, \Sigma}, \quad (45)$$

which in dimension  $d = 2$  is the well known geometric mean property for corresponding Gumbel-Barnett copulas (see Nelsen (1999), Exercice 4.10). This follows directly from the linear properties of the quadratic form in Equation (6).



**Property 4 (Sampling Archimatrix copulas)** A sample of a random vector  $\mathbf{U} = (U_1, \dots, U_d)$  having distribution  $C_{\phi, \Sigma}$  can be obtained by *standard construction*, using Algorithm 2.1. in Embrechts et al. (2003). Let  $C_k(u_k|u_1, \dots, u_{k-1}) = \mathbb{P}[U_k \leq u_k | U_1 = u_1, \dots, U_k = u_k]$ , one have

$$C_k(u_k|u_1, \dots, u_{k-1}) = \frac{\partial}{\partial u_1 \dots \partial u_{k-1}} C_{\phi, \Sigma}(u_1, \dots, u_k, 1, \dots, 1) / \frac{\partial}{\partial u_1 \dots \partial u_{k-1}} C_{\phi, \Sigma}(u_1, \dots, u_{k-1}, 1, \dots, 1)$$

The algorithm is: simulate  $u_1$  from  $U_{[0,1]}$ , simulate  $u_2$  from  $C_2(\cdot|u_1)$ , ..., simulate  $u_d$  from  $C_d(\cdot|u_1, \dots, u_{d-1})$ . As an example, setting  $Q(\mathbf{u}) = \mathbf{g}(\mathbf{u})^t \Sigma \mathbf{h}(\mathbf{u})$ , general trivariate copulas can be sampled from derivatives

$$\begin{aligned} \frac{\partial}{\partial u_1} C_{\phi, \Sigma}(\mathbf{u}) &= \phi'(\psi \circ C_{\phi, \Sigma}(\mathbf{u})) \left( \psi'(u_1) + z'(Q(\mathbf{u})) \frac{\partial}{\partial u_1} Q(\mathbf{u}) \right) \\ \frac{\partial^2}{\partial u_1 \partial u_2} C_{\phi, \Sigma}(\mathbf{u}) &= \phi''(\psi \circ C_{\phi, \Sigma}(\mathbf{u})) \left( \psi'(u_1) + z'(Q(\mathbf{u})) \frac{\partial}{\partial u_1} Q(\mathbf{u}) \right) \left( \psi'(u_2) + z'(Q(\mathbf{u})) \frac{\partial}{\partial u_2} Q(\mathbf{u}) \right) \\ &\quad + \phi'(\psi \circ C_{\phi, \Sigma}(\mathbf{u})) \left( z''(Q(\mathbf{u})) \frac{\partial}{\partial u_1} Q(\mathbf{u}) \frac{\partial}{\partial u_2} Q(\mathbf{u}) + z'(Q(\mathbf{u})) \frac{\partial^2}{\partial u_1 \partial u_2} Q(\mathbf{u}) \right) \end{aligned}$$

For linear expressions of  $z$ , all derivatives of  $C_{\phi, \Sigma}$  are given in Proposition 3.4.

Using Property 4, in Figure 1 (left) we provide a scatterplot of 3-dimensional data from copula presented in Example 1. We take  $\sigma_{12} = 0.001$ ,  $\sigma_{13} = 0.32$  and  $\sigma_{23} = 0.65$ . The parameters  $\sigma_{ij}$  are chosen in such a way that  $\sigma_{12} + \sigma_{13} + \sigma_{23} < 1$ , in order to guarantee the admissibility of the considered Archimatrix copula (see Corollary 3.2). We know that each bivariate projection corresponds here to an Archimedean *Gumbel-Barnett* copula of parameter  $\sigma_{ij}$ . Indeed in Figure 1 we can observe the anti-comonotonic behavior of the sampling data. Furthermore, we give estimates of the Kendall's  $\tau$  for different parameters  $\sigma$  and we compare them with the theoretical ones in the case of bivariate Gumbel-Barnett copula. Results are gathered in Table 2 (first column).

In Figure 1 (right) we provide a scatterplot of 3-dimensional data from copula presented in Example 3. We take  $\sigma_{12} = -0.99$ ,  $\sigma_{13} = 0.99$  and  $\sigma_{23} = 0.2$ . The parameters  $\sigma_{ij}$  are chosen in such a way that  $\sigma_{ij} < 1$  for all  $i, j$ , in order to guarantee that each bivariate projection corresponds to a Farlie–Gumbel–Morgenstern (FGM) copula with parameter  $\sigma_{ij}$ . Also in this case the comparison between theoretical and estimated Kendall's  $\tau$  is provided (see Table 2, second column).

Bivariate Gumbel-Barnett Copula	Bivariate FGM Copula
$\tau_{0.001} = -0.00049$ $\widehat{\tau}_{0.001} = -0.00043$	$\tau_{-0.99} = -0.220$ $\widehat{\tau}_{-0.99} = -0.229$
$\tau_{0.32} = -0.14011$ $\widehat{\tau}_{0.32} = -0.14511$	$\tau_{0.99} = 0.220$ $\widehat{\tau}_{0.99} = 0.215$
$\tau_{0.65} = -0.25671$ $\widehat{\tau}_{0.65} = -0.24789$	$\tau_{0.2} = 0.0444$ $\widehat{\tau}_{0.2} = 0.0441$

Table 2: Theoretical and estimated Kendall's  $\tau$  for bivariate Gumbel-Barnett copula (first column) and bivariate FGM copula (second column) for different choises of parameters  $\sigma_{ij}$ .

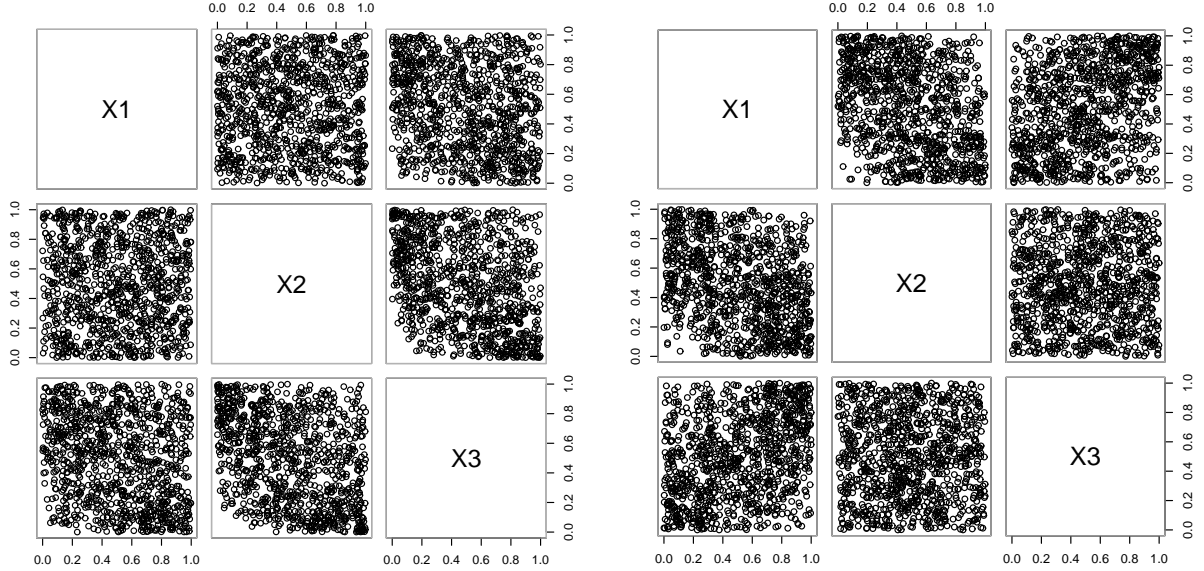


Figure 1: **Left:** Sample of size  $n = 1000$  from the 3-dimensional copula  $C_{\phi, \Sigma}$  as in Equation (30) with  $z(x) = \frac{1}{2}x$ ,  $\phi(x) = \exp(-x)$  and  $g_i(x) = \psi(x)$ ,  $i \in \{1, 2, 3\}$ ,  $x \in \mathbb{R}^+$ . We take  $\sigma_{12} = 0.001$ ,  $\sigma_{13} = 0.32$  and  $\sigma_{23} = 0.65$ . **Right:** Sample of size  $n = 1000$  from the 3-dimensional function  $C_{\phi, \Sigma}$  as in Equation (35) with  $z(x) = -\ln(1 + \frac{x}{2})$ ,  $\phi(x) = \exp(-x)$  and  $g_i(x) = 1 - x$ ,  $i \in \{1, 2, 3\}$  for  $x \in \mathbb{R}^+$ . We take  $\sigma_{12} = -0.99$ ,  $\sigma_{13} = 0.99$  and  $\sigma_{23} = 0.2$ .

Finally, using Property 4, in Figure 2 we provide a plot of 2-dimensional data from the bivariate Archimax model in Equation (39). We take  $\sigma = -0.3$  (Figure 2, left) and  $\sigma = -0.495$  (Figure 2, right). Remark that the chosen parameters guarantee the admissibility of the proposed copula.

**Property 5 (Bivariate tail dependence coefficients)** We consider the general model introduced in Definition 1.2 and the associated bivariate projection

$$C_{\phi, \Sigma}(u_i, u_j) = \phi(\psi(u_i) + \psi(u_j) + z(\sigma_{ij} g_i(u_i) h_j(u_j) + \sigma_{ij} g_j(u_j) h_i(u_i))). \quad (46)$$

The associated bivariate diagonal is

$$\delta(u) := C_{\phi, \Sigma}(u, u) = \phi(2\psi(u) + z(\sigma_{ij} g_i(u) h_j(u) + \sigma_{ij} g_j(u) h_i(u))).$$

Remark that, the bivariate lower and upper tail coefficients  $\lambda_U$  and  $\lambda_L$  (see Sibuya (1960)) associated to the copula  $C_{\phi, \Sigma}$  can be written using the diagonal section  $\delta_{C_{\phi, \Sigma}}(u) = C_{\phi, \Sigma}(u, u)$  (see, e.g., Nelsen et al. (2008), Nelsen (1999)):

$$\begin{aligned} \lambda_L(C_{\phi, \Sigma}) &= \lim_{u \rightarrow 0^+} \frac{d}{du} \delta_{C_{\phi, \Sigma}}(u) = \delta'_{C_{\phi, \Sigma}}(0^+), \\ \lambda_U(C_{\phi, \Sigma}) &= 2 - \lim_{u \rightarrow 1^-} \frac{d}{du} \delta_{C_{\phi, \Sigma}}(u) = 2 - \delta'_{C_{\phi, \Sigma}}(1^-). \end{aligned}$$

**Lemma 1** Let consider the general model introduced in Definition 1.2 and the associated bivariate projection in (46). Under Assumption 1 and if  $|z'(0)| < +\infty$ ,  $|g'_i(1)| < +\infty$ ,  $|g'_j(1)| < +\infty$ ,  $|h'_i(1)| < +\infty$  and

$|h'_j(1)| < +\infty$ , then

$$\lambda_U(C_{\phi, \Sigma}) = \lambda_U(C_{\phi}), \quad (47)$$

where  $\lambda_U(C_{\phi})$  is the bivariate upper tail coefficient associated to the Archimedean copula  $C_{\phi}$ .

Remark that result in (47) holds true for Examples 1, 2 and 3 discussed before. Conversely, for the Archimax model in Example 4 we have  $|g'_i(1)| = |g'_j(1)| = +\infty$ . In this case we get  $\lambda_U(C_{\phi, \Sigma}) = \sigma^{\frac{1}{2\alpha}}$ , with  $\sigma \in [0, 1]$  and  $\alpha > 0$  (see Figure 2).

In the lower case, we get

$$\begin{aligned} \delta'_{C_{\phi, \Sigma}}(0^+) &= \phi'(2\psi(0) + z(\sigma_{ij}g_i(0)h_j(0) + \sigma_{ij}g_j(0)h_i(0))) \\ &\cdot [2\psi'(0) + \sigma_{ij}\xi_{ij}(0)z'(\sigma_{ij}g_i(0)h_j(0) + \sigma_{ij}g_j(0)h_i(0))]. \end{aligned}$$

where  $\xi_{ij}(0) = g'_i(0)h_j(0) + g_i(0)h'_j(0) + g'_j(0)h_i(0) + g_j(0)h'_i(0)$ . For a strict generator (i.e.,  $\psi(0) = +\infty$ ), if  $z(\sigma_{ij}g_i(0)h_j(0) + \sigma_{ij}g_j(0)h_i(0))$ , if furthermore the second line of equation before is bounded in  $(-\infty, +\infty)$ , and  $\phi'(+\infty) = 0$ , then  $\lambda_L(C_{\phi, \Sigma}) = 0$ . Remark that it is exactly the case in Examples 1, 2 and 3 the discussed before. Conversely, for the Archimax model in Example 4, we get  $\lambda_L(C_{\phi, \Sigma}) = 0$  for  $\sigma \in [0, 1)$  (see Figure 2, left) and  $\lambda_L(C_{\phi, \Sigma}) = 1$  for  $\sigma = 1$  (see Figure 2, right).

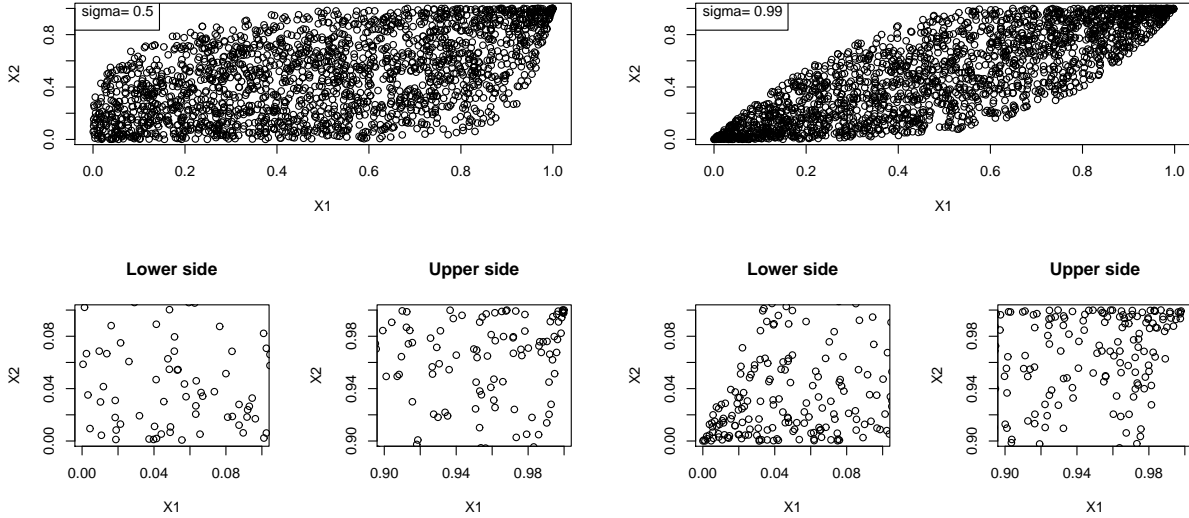


Figure 2: Sample of size  $n = 2000$  from the 2-dimensional Archimax model  $C_{\phi, \Sigma}$  as in Equation (39) with  $z(x) = -(\frac{x}{2})^{\frac{1}{2\alpha}}$ ,  $\phi(x) = \exp(-x)$  and  $g_i(x) = \psi(x)^\alpha$  with  $\alpha = 0.5$ ,  $i \in \{1, 2\}$ ,  $x \in \mathbb{R}^+$ . We take  $\sigma = 0.5$  (left) and  $\sigma = 0.99$  (right). A zoom of the data-set in the lower side and in the upper one is also displayed.

**Property 6 (Linear expression in  $\sigma_{ij}$ )** Due to the sub-model stability, estimation of each parameter  $\sigma_{ij}$  can be done for each bivariate projection  $(U_i, U_j)$ , by classical moment method, regression or by maximum likelihood estimation using given expressions of the copula density. As a consequence of the choice of a quadratic form in the general model, using Equation (9) in the case where  $\mathbf{g} = \mathbf{h}$ , one get for each couple  $i, j \in I$  a linear expression in  $\sigma_{ij}$ ,

$$\sigma_{ij} \cdot 2g_i(u_i)g_j(u_j) = z^{-1}(\psi \circ P[U_i \leq u_i, U_j \leq u_j] - (\psi(u_i) + \psi(u_j))). \quad (48)$$

This expression can help finding estimators of separate coefficients  $\sigma_{ij}$ . Estimating each coefficient separately can be straightforward since it relies only on one parameter at time. However, it may result in a global non-admissible copula, due to constraints like those in Corollary 3.2. The problem of constraint joint estimation of parameters and resulting properties of estimators is not treated here, but constitute an interesting perspective of this work.

For the separate estimation of each  $\sigma_{ij}$ ,  $P[U_i \leq u_i, U_j \leq u_j] = C_{\phi, \Sigma}(1, \dots, 1, u_i, 1, \dots, 1, u_j, 1, \dots, 1)$  is the only unknown quantity in Equation (48). The question is thus how to estimate a quantity  $z^{-1}(\psi \circ C_{\phi, \Sigma}(\mathbf{u}) - \psi(\mathbf{u})^t \mathbf{1})$  when  $z(\cdot)$ ,  $\psi(\cdot)$  and  $\psi(\cdot)$  are given. An immediate estimator is the plug-in estimator where  $C_{\phi, \Sigma}(\mathbf{u})$  is replaced by the empirical copula  $C_n(\mathbf{u})$ .

Figure 3 illustrates the possible use of this linearity for estimating each parameter, and for visualizing the dispersion relying on this estimation. Then, in Figure 3 we draw two types of set of points:  $\{\alpha_{i,j}(\mathbf{u}), \hat{\beta}_{i,j}(\mathbf{u})\}$  (see first and third panels), and  $\{\alpha_{i,j}(\mathbf{u}), \beta_{i,j}(\mathbf{u})\}$  (see second and fourth panels), where  $\hat{\beta}_{i,j}(\mathbf{u}) = z^{-1}(\psi \circ C_n(u_i, u_j) - (\psi(u_i) + \psi(u_j)))$ ,  $\beta_{i,j}(\mathbf{u}) = z^{-1}(\psi \circ P[U_i \leq u_i, U_j \leq u_j] - (\psi(u_i) + \psi(u_j)))$  and  $\alpha_{i,j}(\mathbf{u}) = 2g_i(u_i)g_j(u_j)$  (see Equation (48)). Furthermore, in Figure 3 we present the theoretical regression line (blue line) and the estimated one (red line). In the first and second panels of Figure 3 we choose the Gumbel-Barnett parameters setting, i.e.,  $z(x) = \frac{1}{2}x$ ,  $\phi(x) = \exp(-x)$ ,  $g_i(x) = \psi(x)$  with in particular  $i = 1, j = 3$  (see Example 1). In the third and fourth panels the Farlie-Gumbel-Morgenstern parameters setting is considered, i.e.,  $z(x) = -\ln(1 + \frac{x}{2})$ ,  $g_i(x) = 1 - x$  and  $\phi(x) = \exp(-x)$  with  $i = 1, j = 3$  (see Example 3). The empirical copula  $C_n$  in  $\hat{\beta}_{i,j}(\mathbf{u})$  is estimated on the data-sets of size  $n = 1000$  sampled before (see Property 4, Figure 1).

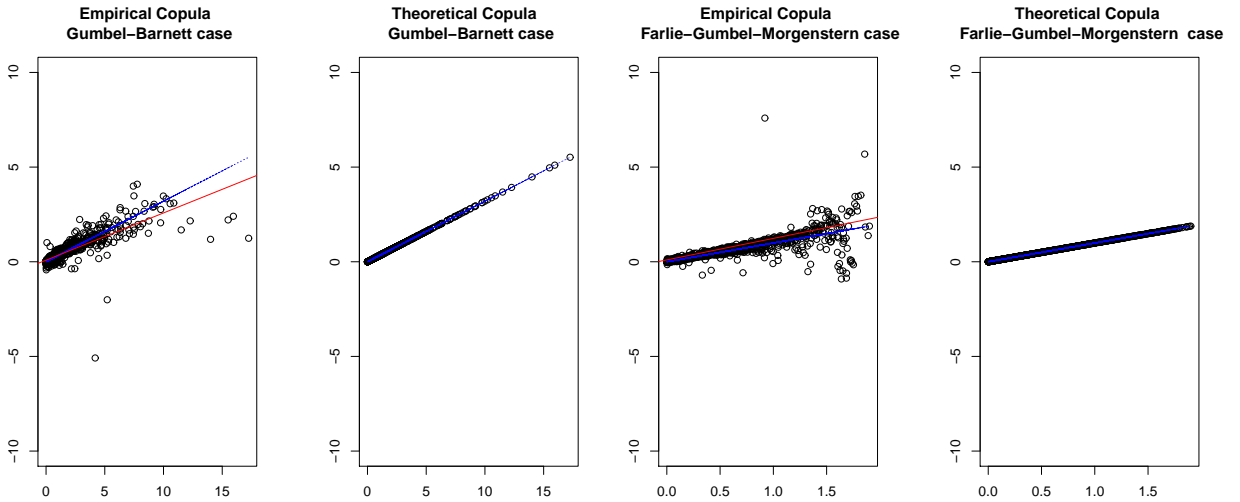


Figure 3: Illustration for theoretical and estimated linear expression in  $\sigma_{13}$  in Equation (48). We draw two types of set of points:  $\{\alpha_{i,j}(\mathbf{u}), \hat{\beta}_{i,j}(\mathbf{u})\}$  (see first and third panels from the left), and  $\{\alpha_{i,j}(\mathbf{u}), \beta_{i,j}(\mathbf{u})\}$  (see second and fourth panels). First and second panels: Gumbel-Barnett case with  $\sigma_{13} = 0.32$ . Third and fourth panels: Farlie-Gumbel-Morgenstern case with  $\sigma_{13} = 0.99$ . We present the theoretical regression line (blue line) and the estimated one (red line).

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