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# ON AN ASYMMETRIC EXTENSION OF MULTIVARIATE ARCHIMEDEAN COPULAS BASED ON QUADRATIC FORM

ELENA DI BERNARDINO AND DIDIER RULLIÈRE

ABSTRACT. An important topic in Quantitative Risk Management concerns the modeling of dependence among risk sources and in this regard Archimedean copulas appear to be very useful. However, they exhibit symmetry, which is not always consistent with patterns observed in real world data. We investigate extensions of the Archimedean copula family that make it possible to deal with asymmetry. Our extension is based on the observation that when applied to the copula the inverse function of the generator of an Archimedean copula can be expressed as a linear form of generator inverses. We propose to add a distortion term to this linear part, which leads to asymmetric copulas. Parameters of this new class of copulas are grouped within a matrix, thus facilitating some usual applications as level curve determination or estimation. Some choices such as sub-model stability help associating each parameter to one bivariate projection of the copula. We also give some admissibility conditions for the considered copulas. We propose different examples as some natural multivariate extensions of Farlie-Gumbel-Morgenstern or Gumbel-Barnett.

Keywords: *Archimedean copulas, transformations of Archimedean copulas, quadratic form.*  
2010 Mathematics Subject Classification: 62H20 .

## 1. INTRODUCTION

Quantitative Risk Management relies strongly on understanding the risk sources, and quantifying risks within a probabilistic framework. This requires the precise determination of dependencies between risks, understood as underlying random variables. The classical copula framework, detailed hereafter, allows to characterize these dependencies. Many parametric copulas allow to summarize these dependencies using a few parameters. Other class of copulas, like the Archimedean class, rely on a whole real-valued function (i.e. an infinite number of parameters), but the class suffers from the symmetric shape of the modelled dependencies. It implies that the dependence among all pairs of the components is identical. However, the dependence between assets from the same industry sector is typically very different from the one that we observe for assets that belong to different sectors. Hence, copulas that accommodate asymmetry are useful. Hereafter, we thus give more details on the copula framework, and investigate a particular extension of the Archimedean copula class, in order to allow asymmetry.

Copulas are multivariate distributions on  $[0, 1]^d$  with uniform marginal distributions. Their main interest is that by Sklar's theorem (see Sklar [31]), every continuous multivariate distribution can be decomposed into its continuous marginal distributions and a unique copula. A review on different copula functions is available in Nelsen [27].

A particular class of copulas is the class of Archimedean copulas. Copulas of this class can be expressed in the dimension  $d \in \mathbb{N}^*$ , where  $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$ , by

$$(1) \quad C_\phi(u_1, \dots, u_d) = \phi(\psi(u_1) + \dots + \psi(u_d)),$$

where  $\phi$  is a real function  $\phi : \mathbb{R}^+ \rightarrow [0, 1]$ , called the generator of the copula, and where  $\psi$  is the generalized inverse function of  $\phi$ , i.e.  $\psi(u) = \inf\{x \in \mathbb{R}^+ : \phi(x) \leq u\}$ , for  $u \in [0, 1]$ . The generator  $\phi$  is continuous, decreasing and convex function, with  $\lim_{x \rightarrow +\infty} \phi(x) = 0$  and  $\phi(0) = 1$  (see, e.g., Definition 2 in McNeil and Nešlehová [22]). Notice that, for strict generators such that  $\phi(x) > 0$  for all  $x \geq 0$ , Equation (1) writes  $\psi \circ C_\phi(u_1, \dots, u_d) = \psi(u_1) + \dots + \psi(u_d)$ : when applied to the copula, the inverse function of the generator can be expressed as a linear form of generator inverses.

The Archimedean class of copula in (1) is very flexible, since members of this class are indexed by a function  $\phi$  rather than a finite set of parameters. However, a very important limitation is that Archimedean copulas are symmetric: for any permutation of indexes  $p : \{1, \dots, d\} \rightarrow \{1, \dots, d\}$ ,

$$(2) \quad C_\phi(u_1, \dots, u_d) = C_\phi(u_{p(1)}, \dots, u_{p(d)}), \quad \text{for } (u_1, \dots, u_d) \in [0, 1]^d.$$

Remark that property in (2) is also called exchangeability in the literature, and a random vector having distribution  $C_\phi$  has necessarily (finite) exchangeable components.

The symmetry of Archimedean copulas class is often considered to be a rather strong restriction, especially in large dimensional applications. It implies that all multivariate projections of the same dimension are equal, thus, e.g., the dependence among all pairs of components is identical. To circumvent exchangeability, Archimedean copulas can be nested within each other under certain conditions. The resulting copulas are referred to as *nested Archimedean copulas*. In the last decade the nested Archimedean copulas have been studied from different points of view (theoretically, computationally, in view of applications and so on). The interested reader is referred to Hofert [14, 12], Hofert and Mächler [15], McNeil [21]. There are other strategies to generalize Archimedean copulas in order to avoid symmetry such as the Hierarchical kendall copulas (see Brechmann [3]), or Liouville copulas (see McNeil and Nešlehová [23]). The interested reader is also referred to Genest and Nešlehová [13] for a survey work on non-exchangeability for bivariate copulas.

A strategy to overcome asymmetry of Archimedean copulas is to modify their analytic expression. In the next paragraph we detail this approach, as the results of the present paper can be seen as a contribution to the corresponding literature.

**Construction of multivariate asymmetric copulas by generalizing some known families.** Rodríguez-Lallena and Úbeda Flores [29] have introduced a class of bivariate copulas  $C^*$  which generalizes some known families such as the Farlie Gumbel Morgenstern distributions of the form:

$$(3) \quad C^*(u, v) = uv + \lambda f(u)g(v),$$

where  $f$  and  $g$  are two non-zero absolutely continuous functions such that  $f(0) = f(1) = g(0) = g(1) = 0$  and the admissible range of the parameter  $\lambda$  can be obtained in terms of the derivatives of  $f$  and  $g$ . Moreover, Dolati and Úbeda Flores [9] have provided procedures to construct parametric families of multivariate distributions which generalize copulas in (3).

Kim et al. [18] generalized the method of Rodríguez-Lallena and Úbeda Flores [29]. They define the distorted copula  $C^*$  as

$$C^*(u, v) = C(u, v) + \lambda f(u)g(v),$$

where  $C$  is an arbitrary given copula. The method of Kim et al. [18] gives a sufficient condition for the  $\lambda$  coefficient and it is in general rather difficult to be applied. To overcome this drawback, Mesiar and Najjari [25] introduced a new method of constructing binary copulas, extending the original study of Rodríguez-Lallena and Úbeda Flores [29] to new families of symmetric/asymmetric copulas.

Alfonsi and Brigo [1] describe a new construction method for asymmetric copulas based on periodic functions. Liebscher [20] introduced two methods to construct asymmetric multivariate copulas. The first is connected with products of copulas, i.e.,

$$C^*(u_1, \dots, u_d) = \prod_{j=1}^k C_j(g_{j1}(u_1), \dots, g_{jd}(u_d)), \quad \text{for } u_i \in [0, 1],$$

where  $g_{j_i}$  are suitable increasing functions and  $C_j$  are copulas. The second method proposes a generalization of the Archimedean copulas class in (1). The aforementioned paper is a generalization of the so-called Khoudraji's device (see Khoudraji [17]).

Remark that Archimedean copulas can be rewritten in the form:

$$C_\phi(u_1, \dots, u_d) = \bar{\phi}(\bar{\psi}(u_1) \times \dots \times \bar{\psi}(u_d)),$$

using the multiplicative generator  $\bar{\psi}(u) = \exp(\psi(u))$ . Let us replace the product  $\bar{\psi}(u_1) \times \dots \times \bar{\psi}(u_d)$  before by an average of products leading to

$$(4) \quad C_\phi^*(u_1, \dots, u_d) = \Psi \left( \frac{1}{m} \sum_{j=1}^m h_{j1}(\varphi(u_1)) \dots h_{jd}(\varphi(u_d)) \right),$$

where  $\varphi = \Psi^{-1}$ . Function in (4) represents a generalisation of Archimedean copulas being asymmetric in general. Liebscher [20] provides conditions on functions and  $\Psi$  and  $h_{jk}$  such that function in (4) is a proper copula. Recently, Wu [33] proposes a new method of constructing asymmetric copulas and a convex-combination of asymmetric copulas that can exhibit different tail dependence along different directions.

A generalization of the Archimedean copula class, containing both the Archimedean and the *extreme-value copulas* as a special case, are the *Archimax* copulas (see Capéraà et al. [4] for the bivariate case, Charpentier et al. [5] for the multivariate case). Following Capéraà et al. [4], a bivariate copula is said to be Archimax if it can be written, for all  $u_1, u_2 \in (0, 1)$ , in the form:

$$(5) \quad C_{\phi, A}(u_1, u_2) = \phi \left( (\psi(u_1) + \psi(u_2)) A \left( \frac{\psi(u_1)}{\psi(u_1) + \psi(u_2)} \right) \right),$$

using the Pickand function  $A : [0, 1] \rightarrow [0.5, 1]$  and the generator  $\phi : \mathbb{R}^+ \rightarrow [0, 1]$  (see Mesiar and Jágr [24] for a suitable  $d$ -variate extension of the notion of bivariate Archimax copula).

**Organization of the paper.** The paper is organized as follows. In Section 2 we present our model in order to extend the Archimedean class of copula offering the possibility of asymmetric distributions. Furthermore, suitable theoretical characteristics for the considered model are presented. Then we consider in Section 3 sufficient admissibility conditions for the proposed Archimatrix model in some particular cases. Using results of Section 3, we give some examples exhibiting multivariate distorted copulas or valid bivariate projections (see Section 4). Finally in Section 5, some supplementary properties and a sampling procedure with associated numerical illustrations are proposed.

## 2. CONSIDERED MODEL

We focus on the class of Archimedean copulas presented in Equation (1). Let  $\phi$  the generator of the Archimedean copula such that  $\lim_{x \rightarrow +\infty} \phi(x) = 0$  and  $\phi(0) = 1$ . In the following, we restrict ourselves to strict generators, where  $\forall x \in \mathbb{R}^+, \phi(x) > 0$ . In this case, the function  $\psi$  is the regular inverse of  $\phi$ . From Theorem 2.2 in McNeil and Nešlehová [22],  $C_\phi(u_1, \dots, u_d) = \phi(\psi(u_1) + \dots + \psi(u_d))$ , is a  $d$ -dimensional copula if and only if its generator  $\phi$  is  $d$ -monotone on  $[0, \infty)$ , where the  $d$ -monotony definition is recalled hereafter, as in McNeil and Nešlehová [22]. A  $d$ -monotone generator will be called in the following *valid* generator.

**Definition 1** (*d-monotone function*). *A real function  $f$  is called  $d$ -monotone in  $(a, b)$ , where  $a, b \in \mathbb{R}$  and  $d \geq 2$ , if it is differentiable there up to the order  $d - 2$  and the derivatives satisfy*

$$(-1)^k f^{(k)}(x) \geq 0, \quad k = 0, 1, \dots, d - 2$$

*for any  $x \in (a, b)$  and further if  $(-1)^{d-2} f^{(d-2)}$  is non-increasing and convex in  $(a, b)$ . For  $d = 1$ ,  $f$  is called 1-monotone in  $(a, b)$  if it is nonnegative and non-increasing there.*

Consider a random vector  $\mathbf{U} = (U_1, \dots, U_d)$  in  $[0, 1]^d$  whose distribution is an Archimedean copula  $C_\phi$ . Symmetric property of Archimedean copulas in Equation (2) has consequences in particular for bivariate projections: one has a symmetry *within* any couple of random variable  $(U_i, U_j)$ , and a symmetry *among* different couples of random variables  $(U_i, U_j)$  and  $(U_{i'}, U_{j'})$ , i.e., for  $i, j, i', j' \in \{1, \dots, d\}$ ,

$$(6) \quad (U_i, U_j) \stackrel{d}{=} (U_j, U_i) \quad \text{and} \quad (U_i, U_j) \stackrel{d}{=} (U_{i'}, U_{j'}),$$

where  $\stackrel{d}{=}$  denotes the equality in distribution.

In the following we aim at extending the Archimedean class of copula, while offering the possibility of asymmetric distributions. In order to take into account each interaction  $(U_i, U_j)$ , we consider a model with one parameter  $\sigma_{ij}$  per couple  $(U_i, U_j)$ . It is rather natural to group all these parameters within a matrix  $\Sigma = (\sigma_{ij})_{i,j \in I}$ , where from now on  $I = \{1, \dots, d\}$ . A chosen requirement of our model is to construct a class that contains at least all Archimedean copulas, for specific values of  $\Sigma$ .

In Definition 2 we present the considered copula model and specify constraints on both parameters and  $\mathbf{g}$ ,  $\mathbf{h}$  and  $z$  functions. These constraints come from the choices and desired features of the proposed model. A further section is devoted to admissibility conditions on the proposed copula (see Section 3).

**Definition 2** (Considered model and basic required assumptions). *Let us denote the column vectors of length  $d$  by  $\mathbf{u} = (u_1, \dots, u_d)$ ,  $\psi(\mathbf{u}) = (\psi(u_1), \dots, \psi(u_d))$  and  $\mathbf{1} = (1, \dots, 1)$ . We define,*

for all  $\mathbf{u} \in [0, 1]^d$ , a function  $C_{\phi, G, \Sigma}$  as

$$(7) \quad C_{\phi, G, \Sigma}(\mathbf{u}) = \phi(\mathbf{1}^t \psi(\mathbf{u}) + z(\mathbf{g}(\mathbf{u})^t \Sigma \mathbf{h}(\mathbf{u}))),$$

where  $\phi$  is a valid strict Archimedean generator with regular inverse function  $\psi$ , where  $\mathbf{g} : [0, 1]^d \rightarrow \mathbb{R}^d$  and  $\mathbf{h} : [0, 1]^d \rightarrow \mathbb{R}^d$  are two vector-valued continuous functions,  $z : \mathbb{R} \rightarrow \mathbb{R}$  is a continuous real function. The index  $G$  in  $C_{\phi, G, \Sigma}$  is a vector function combining  $\mathbf{g}$ ,  $\mathbf{h}$ ,  $z$ , i.e.,  $G := (\mathbf{g}, \mathbf{h}, z)$ . In the following, we will denote  $\mathcal{F}$  the class of functions  $C_{\phi, G, \Sigma}$  as in (7). Moreover, we require that

- i. (Sub-model stability) for all  $\mathbf{u} \in [0, 1]^d$ ,  $[\mathbf{g}(\mathbf{u})]_i = [\mathbf{h}(\mathbf{u})]_i = 0$  as soon as  $u_i = 1$ ;
- ii. (Boundary conditions)  $z(0) = 0$  and  $\sigma_{ii} = 0$ , for all  $i \in I$ ;
- iii. (Concordance ordering)  $z$  is a monotone and  $d$  times differentiable function and all components of the matrix  $\mathbf{g}(\mathbf{u})\mathbf{h}(\mathbf{u})^t$  have the same sign (i.e. either  $\mathbf{g}(\mathbf{u})_i\mathbf{h}(\mathbf{u})_j \geq 0$ ,  $i, j \in I$ , either  $\mathbf{g}(\mathbf{u})_i\mathbf{h}(\mathbf{u})_j \leq 0$ ,  $i, j \in I$ ), for all  $\mathbf{u} \in [0, 1]^d$ .

An interesting feature of this model is that it is based on a linear expression. This will ease the determination of the level curves of the copula (see Property 2) as well as the estimation of its parameters (see Property 5). However, one limitation is that it will be difficult to determine the range of admissible parameters. Some results will give valid ranges in the dimension  $d = 2$ , or in special cases in any dimension  $d$  (see Section 4).

As one can see from Equation (7),  $\psi \circ C_{\phi, G, \Sigma}(\mathbf{u})$  is the sum of two terms and it corresponds to an Archimedean copula when the second term is zero. This last term uses the distortion matrix  $\Sigma$  with simple matrix multiplications. If the second term is not zero we can obtain several asymmetric non-Archimedean copula structures. The model in Definition 2 will be called in the following *Archimatrix copula*, in order to underline the link with the Archimedean copula (that is a particular case of  $C_{\phi, G, \Sigma}$ ) and the central asymmetry role played by the matrix  $\Sigma$ . Indeed the distortion matrix  $\Sigma$  in Equation (7), with functions  $\mathbf{g}$ ,  $\mathbf{h}$  and  $z$ , permits to leave the symmetric structure typical of any Archimedean copula model. Our model in Definition 2 is built in the same spirit as the recent literature about the construction of multivariate asymmetric copulas by generalizing some known families. Indeed different asymmetric models for copula structures presented in the Introduction section can be related to our model. Some comparisons in this sense will be presented in Section 4.

In the following we discuss Assumptions *i*, *ii* and *iii* in Definition 2 and we illustrate how required conditions are implied by desired features of our multivariate copula model.

**2.1. Sub-model stability.** First we consider the *sub-model stability* assumption (see Assumption *i* in Definition 2). This requirement is a choice that is not compulsory for defining a valid copula respecting Equation (7), but that seems to us important in order to simplify the model and to interpret its parameters. Remark that sub-model stability is not ensured by the initial model; it does not hold for example if  $\mathbf{g}(\mathbf{u}) = \mathbf{h}(\mathbf{u}) = \mathbf{u}$ .

**Proposition 1** (Sub-model stability). *Consider  $C_{\phi, G, \Sigma} \in \mathcal{F}$  as in Equation (7), with given functions  $\mathbf{g}$ ,  $\mathbf{h}$ ,  $z$ ,  $\Sigma$  satisfying Assumption *i* in Definition 2. Assume that  $C_{\phi, G, \Sigma}$  is a valid copula and let a random vector  $\mathbf{U}$  be distributed as  $C_{\phi, G, \Sigma}$ . Then, the model is valid for each sub-model, i.e., for any non-empty subset  $\Omega = \{\omega_1, \dots, \omega_k\} \subset I$ ,*

$$(U_{\omega_1}, \dots, U_{\omega_k}) \sim C_{\phi, G, \Sigma_{\Omega}},$$

where  $\Sigma_\Omega = (\sigma_{ij})_{i,j \in \Omega}$  is a submatrix of  $\Sigma$ .

In particular, assume that there exist functions  $g_i$  and  $h_i$ ,  $i \in I$  so that for any  $\mathbf{u} \in [0, 1]^d$ ,  $\mathbf{g}(\mathbf{u}) = (g_1(u_1), \dots, g_d(u_d))$ ,  $\mathbf{h}(\mathbf{u}) = (h_1(u_1), \dots, h_d(u_d))$ . Assume furthermore that  $g_i(1) = h_i(1) = 0$  for all  $i \in I$ . Then the model is valid on any projections on any non-empty subset of indexes  $\Omega \subset I$ , with

$$(8) \quad \mathbb{P} \left[ \bigcap_{i \in \Omega} U_i \leq u_i \right] = \phi \left( \sum_{i \in \Omega} \psi(u_i) + z \left( \sum_{i,j \in \Omega} g_i(u_i) \sigma_{ij} h_j(u_j) \right) \right).$$

The proof of Proposition 1 is postponed to the Appendix.

Despite the reduction of the variety of possible models, the sub-model stability in Proposition 1 is interesting since it permits to understand any coefficient  $\sigma_{ij}$  in the matrix  $\Sigma$  by considering only the corresponding bivariate distribution  $(U_i, U_j)$ . Under this assumption, the interpretation of these coefficients is thus more straightforward. Notice that the sub-model stability naturally holds for Archimedean copulas when the function  $z$  satisfies  $z(\cdot) = 0$ .

**2.2. Boundary conditions.** Secondly, we focus on the *boundary conditions* (see Assumption *ii* in Definition 2). Indeed, a condition is required when a function  $C_{\phi,G,\Sigma} \in \mathcal{F}$  has to be a copula: the uniform distribution of univariate projections of  $C_{\phi,G,\Sigma}$  and more generally boundary conditions on  $C_{\phi,G,\Sigma}$  (see Proposition 2 below).

**Proposition 2.** Consider  $C_{\phi,G,\Sigma} \in \mathcal{F}$  as in Equation (7), with given functions  $\mathbf{g}$ ,  $\mathbf{h}$ ,  $z$ ,  $\Sigma$  satisfying Assumptions *i* and *ii* in Definition 2, then

- (1)  $C_{\phi,G,\Sigma}(1, \dots, 1, u, 1, \dots, 1) = u$ ,  $u \in [0, 1]$ , and in particular  $C_{\phi,G,\Sigma}(1, \dots, 1) = 1$ ;
- (2)  $C_{\phi,G,\Sigma}(u_1, \dots, u_{i-1}, 0, u_{i+1}, \dots, u_d) = 0$ , for  $\mathbf{u} \in [0, 1]^d$ .

The proof of Proposition 2 is postponed to the Appendix.

**2.3. Concordance ordering.** Finally, we consider the *concordance ordering* assumption (see Assumption *iii* in Definition 2). For two copulas  $C_1$  and  $C_2$ , recall that one can say that  $C_1$  is smaller than  $C_2$  for the concordance ordering, and we write  $C_1 \prec C_2$ , if for all  $\mathbf{u} \in [0, 1]^d$ ,  $C_1(\mathbf{u}) \leq C_2(\mathbf{u})$  (see, e.g., Definition 2.8.1 in Nelsen [27] in the bivariate setting; Joe [16] in the general dimension  $d$ ). Considering copulas indexed by a real parameter  $\theta$ , we recall that a class  $\{C_\theta\}$  of copulas is *positively ordered* for all  $\theta_1 \leq \theta_2$ ,  $C_{\theta_1} \prec C_{\theta_2}$ , and *negatively ordered* if for all  $\theta_1 \leq \theta_2$ ,  $C_{\theta_2} \prec C_{\theta_1}$  (see Nelsen [27], Section 4.4 in dimension 2 and Dolati and Úbeda Flores [9] in the general dimension  $d$ ). Most usual Archimedean copulas are (positively or negatively) ordered, even if there exists Archimedean copulas neither negatively or positively ordered (see, e.g, the bivariate copula 4.2.10 in Table 4.1 of Nelsen [27]).

In the case where the function  $C_{\phi,G,\Sigma}$  is a copula, a desirable feature is that, for any parameter  $\sigma_{ij}$ ,  $i, j \in I$ , the copula is either positively or negatively ordered. This may ease, for example, the interpretation of elements of the distortion matrix  $\Sigma$  of the Archimatrix copula (i.e., of  $\sigma_{ij}$  for  $i, j \in I$ ). As the interpretation of one coefficient  $\sigma_{ij}$  should not depend on indexes  $i$  and  $j$ , one asks that parameters are ordered in the same way. A simple sufficient condition for this is given below.

**Proposition 3** (Parameters of  $\Sigma$  matrix and concordance ordering). Consider  $C_{\phi,G,\Sigma} \in \mathcal{F}$  as in Equation (7), with given functions  $\mathbf{g}$ ,  $\mathbf{h}$ ,  $z$ ,  $\Sigma$  satisfying Assumption *iii* in Definition 2 and

assume that  $C_{\phi,G,\Sigma}$  is a valid copula. Then the considered copulas of the class are ordered in the same way with respect to each parameter  $\sigma_{ij}$ , for  $i, j \in I$  (i.e. all negatively ordered, or all positively ordered).

The concordance order is an important aspect for copulas. In particular, some Archimedean copulas can be ordered via single parameters (see Section 4.4 in Nelsen [27]). Let us consider an Archimatrix copula satisfying Proposition 3. Due to Proposition 1, any bivariate projection associated to  $(U_i, U_j)$  is also a bivariate Archimatrix copula with parameter  $\sigma_{ij}$  and same concordance ordering. For instance, bivariate projections of  $C_{\phi,G,\Sigma}$  in Examples 1-2 (resp. Example 3) are negatively (resp. positively) ordered with respect to the only parameter  $\sigma_{ij}$ .

### 3. ADMISSIBILITY CONDITIONS

We now discuss admissibility conditions of chosen Archimatrix copulas presented in Section 2. Consider the model function  $C_{\phi,G,\Sigma} \in \mathcal{F}$  as in Equation (7) satisfying conditions in Definition 2. Assume that  $C_{\phi,G,\Sigma}$  is  $d$ -times differentiable with respect to successive variables  $u_1, \dots, u_d$  and define  $c_{\phi,G,\Sigma}(\mathbf{u}) = \frac{\partial^d}{\partial u_1 \dots \partial u_d} C_{\phi,G,\Sigma}(\mathbf{u})$ . Throughout the paper, one assumes furthermore that  $\int_{[0,1]^d} c_{\phi,G,\Sigma}(\mathbf{u}) d\mathbf{u} = 1$ , so that  $C_{\phi,G,\Sigma}(\mathbf{u})$  cannot have a singular component. Then as by construction  $C_{\phi,G,\Sigma}$  satisfies bounding conditions, it is a copula if and only if

$$(9) \quad c_{\phi,G,\Sigma}(\mathbf{u}) \geq 0, \quad \forall \mathbf{u} \in [0, 1]^d,$$

(see for instance Billingsley [2], Chapter 4). However in practice, this condition requires checking the value of derivatives of possibly large orders on the whole support of the copula, which can hardly be done. For this reason, the rest of this section will be devoted to the research of simpler sufficient conditions to guarantee admissibility of copula  $C_{\phi,G,\Sigma}$ . To this aim, we use a supplementary set of assumptions that simplify the expression of the copula and its derivatives (see Assumption 1 below). In particular, this Assumption 1 will be a sufficient condition to prove a symmetry property *within* each bivariate projection of  $C_{\phi,G,\Sigma}$  (see Proposition 4), and will be suited when we focus on asymmetry *among* bivariate projections of  $C_{\phi,G,\Sigma}$  (cf. Equation 6 for the distinction between symmetries *within* or *among*).

**Assumption 1.** Consider  $C_{\phi,G,\Sigma} \in \mathcal{F}$  as in Equation (7) satisfying conditions in Definition 2. More restrictive assumptions are

- (a)  $\mathbf{g}(\cdot)$  and  $\mathbf{h}(\cdot)$  are identical vector-valued functions with components  $[\mathbf{g}(\mathbf{u})]_i = [\mathbf{h}(\mathbf{u})]_i = g_i(u_i)$ ,  $\mathbf{u} \in [0, 1]^d$ , where  $g_i$  is a monotone real function on  $[0, 1]$ , such that  $g_i(1) = 0$ , for all  $i \in I$ .
- (b)  $\sigma_{ij} = \sigma_{ji}$ , for  $i \in I, j \in J$ .

Notice that Assumptions *i* and *iii* on  $\mathbf{g}(\cdot)$  and  $\mathbf{h}(\cdot)$  in Definition 2 is automatically ensured by more restrictive Condition (a) in Assumption 1. Notice also that, under Assumption 1, and due to bounding conditions (Assumptions *ii* in Definition 2) ensuring that  $\sigma_{ii} = 0$ ,  $i \in I$ , one can rewrite  $\mathbf{g}(\mathbf{u})^t \Sigma \mathbf{h}(\mathbf{u}) = 2 \mathbf{g}(\mathbf{u})^t \Sigma^> \mathbf{h}(\mathbf{u})$ , where  $\Sigma^>$  is the upper triangular matrix having components  $\sigma_{ij}$  if  $i < j$ , or 0 otherwise. More restrictive conditions introduced in Assumption 1 will be useful in Sections 3.1 and 3.2 below.



**3.1. Valid bivariate projections.** Bivariate projections are usually easier to represent and to understand than multivariate ones. Furthermore, in the dimension  $d = 2$ , Equation (7) in Definition 2 can be seen as a distortion leading to new bivariate copulas, which has an interest in this reduced dimension. Bivariate projections are thus treated in this section separately.

The proposed shape of the copula  $C_{\phi,G,\Sigma}$  aims at modelling different interactions between random variables  $U_i$  and  $U_j$ , when  $i$  and  $j$  vary in  $I$ . However, it could be desirable to have a symmetry within each bivariate projection, so that  $(U_i, U_j)$  and  $(U_j, U_i)$  may have identical distributions. We give here a simple condition ensuring that all bivariate projections are symmetrical. This condition will be satisfied in further Examples 1-3.

**Proposition 4** (Symmetric bivariate projections). *Consider  $C_{\phi,G,\Sigma} \in \mathcal{F}$  as in Equation (7) satisfying conditions in Definition 2 and condition (a) in Assumption 1. Assume that  $C_{\phi,G,\Sigma}$  is a proper copula. Then if for all  $i \in I$ , for all  $\mathbf{u} \in [0, 1]^d$ ,  $g_i(\mathbf{u}) = g(\mathbf{u})$  does not depend on  $i$ , then bivariate projections are symmetric, i.e., for all  $i, j \in I$ ,*

$$(U_i, U_j) \stackrel{d}{=} (U_j, U_i).$$

The proof is postponed to the Appendix. Notice that even in this symmetric projections case, one can still have different distributions for  $(U_i, U_j)$  and  $(U_{i'}, U_{j'})$ , for  $i, j, i', j' \in I$ .

Now one important point is being able to guarantee the positivity of the density in the dimension 2, so that each bivariate projection is a copula. This can be not trivial in a general framework. However, in some particular cases, one can bound  $c_{\phi,G,\Sigma}(\mathbf{u})$  in (9) and obtain sufficient conditions that are more straightforward to check. The following result gives an example of such a sufficient condition, which is easy to check when  $g'$  is linked with  $\psi'$ .

**Proposition 5** (Simplified bivariate admissibility sufficient condition). *Consider  $C_{\phi,G,\Sigma} \in \mathcal{F}$  as in Equation (7) satisfying conditions in Definition 2 and Assumption 1 with  $\phi$  being 2-times differentiable and  $g_i$  differentiable for all  $i \in I$ . Assume that the generator  $\phi$  is such that  $\rho_\phi := \inf\left\{\left|\frac{\phi''(x)}{\phi'(x)}\right|, x \in \mathbb{R}^+\right\} > 0$ . Assume that for all  $x \in \mathbb{R}^+$ ,  $(z'(x))^2 \rho_\phi - z''(x) \geq 0$  and that  $z'(x) > 0$ . If for all  $\mathbf{u} \in [0, 1]^d$ ,*

$$0 \leq \sigma_{ij} \leq \frac{1}{2} \frac{1}{z'(2\sigma_{ij}g_i(u_i)g_j(u_j))} \frac{\psi'(u_i)\psi'(u_j)}{g'_i(u_i)g'_j(u_j)} \rho_\phi,$$

*then any bivariate projection of  $C_{\phi,G,\Sigma}$  in Equation (22) is a proper bivariate copula.*

The proof of Proposition 5 is postponed to the Appendix.

The sufficient admissibility condition in Proposition 5 has an interest when at least one couple  $(i, j) \in I^2$  is such that  $\inf_{(u,v) \in [0,1]^2} \frac{1}{z'(2\sigma_{ij}g_i(u)g_j(v))} \frac{\psi'(u)\psi'(v)}{g'_i(u)g'_j(v)} \rho_\phi > 0$ . As an example, when for all  $u$ , for all  $k \in I$ ,  $g_k(u) = \psi(u)$ , this infimum is strictly greater than zero if  $\sup_{x \in [0,\infty)} z'(2\sigma_{ij}x) < \infty$ .

This is the case for linear  $z$ , when  $z'$  is a constant and when  $\rho_\phi > 0$  (see Examples 1 and 2 below for an application).

However, we point out that the admissibility sufficient condition provided in Proposition 5 can be not precise enough to control the admissibility of the model function  $C_{\phi,G,\Sigma}$ . Then in these cases a further analysis is required to establish the eventual admissibility of  $C_{\phi,G,\Sigma}$  (for instance remark that  $\rho_\phi = 0$  in the Clayton and Gumbel case, see Table 1 below. In some cases, bivariate

projections are admissible FGM copulas, but the sufficient condition is not precise enough, see Example 3).

In Table 1, we study the quantity  $\rho_\phi$  considered in Proposition 5 for classical generators in the case of most popular Archimedean copulas.

Archimedean Copula class	$\phi(t)$	$\psi(t)$	parameter $\theta$	$\rho_\phi$
Ali-Mikhail-Haq	$\frac{1-\theta}{\exp(t)-\theta}$	$\ln\left(\frac{1-\theta+\theta t}{t}\right)$	$\theta \in (0, 1)$	1
Clayton	$(1+\theta t)^{-1/\theta}$	$\frac{1}{\theta}(t^{-\theta}-1)$	$\theta \in (0, \infty)$	0
Frank	$-\frac{1}{\theta}\ln(1-(1-\exp(-\theta))\exp(-t))$	$-\ln\left(\frac{\exp(-\theta t)-1}{\exp(-\theta)-1}\right)$	$\theta \in (0, \infty)$	1
Gumbel	$\exp(-t^{1/\theta})$	$(-\ln(t))^\theta$	$\theta \in (1, \infty)$	0
Independence	$\exp(-t)$	$(-\ln(t))$	none	1
Joe	$1-(1-\exp(-t))^{1/\theta}$	$-\ln(1-(1-t)^\theta)$	$\theta \in (1, \infty)$	1

TABLE 1. Condition on  $\rho_\phi$  for classical generators in the case of most popular Archimedean copulas.

**3.2. Valid multivariate Archimatrix copula for linear function  $z$ .** Proposition 5 gives a bivariate admissibility sufficient condition. However, it is more challenging to prove the positivity of the density of  $C_{\phi,G,\Sigma}$  in any given dimension  $d$ . To this aim, the choice of a linear or affine function  $z$  can help since it leads to vanishing derivatives of  $z(\mathbf{g}(\mathbf{u})^t \Sigma \mathbf{h}(\mathbf{u}))$  when orders are greater than two. Indeed, for distinct  $i, j, k \in I$ , and when  $\mathbf{g}(\mathbf{u}) = \mathbf{h}(\mathbf{u}) = (g_1(u_1), \dots, g_d(u_d))$ ,

$$(10) \quad \frac{\partial^3}{\partial u_i \partial u_j \partial u_k} z(\mathbf{g}(\mathbf{u})^t \Sigma \mathbf{g}(\mathbf{u})) = 0.$$

In the following proposition, we thus give expressions of the density of the copula and admissibility conditions in the simplified model

$$(11) \quad C_{\phi,G,\Sigma}(\mathbf{u}) = \phi\left(\mathbf{1}^t \psi(\mathbf{u}) + \frac{1}{2} \mathbf{g}(\mathbf{u})^t \Sigma \mathbf{g}(\mathbf{u})\right),$$

where  $\mathbf{g}(\mathbf{u}) = (g_1(u_1), \dots, g_d(u_d))$ ,  $z(x) = x/2$ .

**Proposition 6** (Density in the simplified model). *Consider  $C_{\phi,G,\Sigma} \in \mathcal{F}$  as in Equation (7) satisfying conditions in Definition 2 and Assumption 1. Furthermore consider the simplified multivariate Archimatrix model with linear function  $z$  as in (11), where  $\mathbf{g}(\mathbf{u}) = (g_1(u_1), \dots, g_d(u_d))$  with  $g_i$  differentiable for all  $i \in I$ . Let  $k \in I$  and assume that  $\phi$  is a  $k$ -times differentiable generator. Let  $\lfloor x \rfloor$  the integer part of  $x$ . Then we have*

$$(12) \quad \frac{\partial^k}{\partial u_1 \dots \partial u_k} C_{\phi,G,\Sigma}(\mathbf{u}) = \sum_{\nu=0}^{\lfloor k/2 \rfloor} \phi^{(k-\nu)} \circ \psi \circ C_{\phi,G,\Sigma}(\mathbf{u}) \cdot R_{\nu,k}(\mathbf{u}),$$

with

$$\begin{cases} R_{0,k}(\mathbf{u}) &= \prod_{l=1}^k G_l(\mathbf{u}) \\ R_{\nu,k}(\mathbf{u}) &= \sum_{\{i_1, j_1, \dots, i_\nu, j_\nu\} \subset \{1, \dots, k\}} G_{i_1 j_1} \dots G_{i_\nu j_\nu} \prod_{l \in \{1, \dots, k\} \setminus \{i_1, j_1, \dots, i_\nu, j_\nu\}} G_l(\mathbf{u}), \quad \text{for } \nu \geq 1, \end{cases}$$

where  $G_i(\mathbf{u}) = \psi'(u_i) + g'_i(u_i) \sum_{r=1}^d \sigma_{ir} g_r(u_r)$ ,  $G_{ij}(\mathbf{u}) = g'_i(u_i) \sigma_{ij} g'_j(u_j)$ ,  $i, j \in I$ ,  $j \neq i$  and where the sum over  $\{i_1, j_1, \dots, i_\nu, j_\nu\} \subset \{1, \dots, k\}$  refers to all possible distinct choices of  $\nu$  couples in the set  $\{1, \dots, k\}$  (i.e. with  $i_1 < \dots < i_\nu$ , with  $i_r < j_r$  for  $r = 1, \dots, \nu$ , and with all values in  $\{i_1, j_1, \dots, i_\nu, j_\nu\}$  being distinct).

The proof of Proposition 6 is postponed to the Appendix.

As an example, denoting  $C_{\phi, \Sigma}^{(k)}(\mathbf{u}) := \frac{\partial^k}{\partial u_1 \dots \partial u_k} C_{\phi, G, \Sigma}(\mathbf{u})$ , one gets for the first four orders

$$\begin{aligned} C_{\phi, \Sigma}^{(1)}(\mathbf{u}) &= \phi^{(1)} \circ \psi \circ C_{\phi, G, \Sigma}(\mathbf{u}) \cdot G_1(\mathbf{u}), \\ C_{\phi, \Sigma}^{(2)}(\mathbf{u}) &= \phi^{(2)} \circ \psi \circ C_{\phi, G, \Sigma}(\mathbf{u}) \cdot (G_1 G_2)(\mathbf{u}) + \phi^{(1)} \circ \psi \circ C_{\phi, G, \Sigma}(\mathbf{u}) \cdot (G_{12})(\mathbf{u}), \\ C_{\phi, \Sigma}^{(3)}(\mathbf{u}) &= \phi^{(3)} \circ \psi \circ C_{\phi, G, \Sigma}(\mathbf{u}) \cdot (G_1 G_2 G_3)(\mathbf{u}) + \phi^{(2)} \circ \psi \circ C_{\phi, G, \Sigma}(\mathbf{u}) \cdot (G_{12} G_3 + G_{13} G_2 + G_{23} G_1)(\mathbf{u}), \\ C_{\phi, \Sigma}^{(4)}(\mathbf{u}) &= \phi^{(4)} \circ \psi \circ C_{\phi, G, \Sigma}(\mathbf{u}) \cdot (G_1 G_2 G_3 G_4)(\mathbf{u}) \\ &\quad + \phi^{(3)} \circ \psi \circ C_{\phi, G, \Sigma}(\mathbf{u}) \cdot (G_{12} G_3 G_4 + G_{13} G_2 G_4 + G_{14} G_2 G_3 + G_{23} G_1 G_4 + G_{24} G_1 G_3 + G_{34} G_1 G_2)(\mathbf{u}) \\ &\quad + \phi^{(2)} \circ \psi \circ C_{\phi, G, \Sigma}(\mathbf{u}) \cdot (G_{12} G_{34} + G_{13} G_{24} + G_{14} G_{23})(\mathbf{u}). \end{aligned}$$

In the case where  $\phi(x) = \exp(-x)$ , one can check that  $\phi^{(k)} \circ \psi \circ C_{\phi, G, \Sigma}(\mathbf{u}) = (-1)^k C_{\phi, G, \Sigma}(\mathbf{u})$  and the expression can be simplified. Then, the following proposition illustrates the general result provided in Proposition 6 in the independence generator case.

**Proposition 7** (Density starting from independence). *In the same assumption setting as Proposition 6, when  $\phi(x) = \exp(-x)$ ,  $x \in \mathbb{R}^+$  is the independence generator, then for all  $\mathbf{u} \in (0, 1)^d$ ,*

$$C_{\phi, \Sigma}^{(k)}(\mathbf{u}) = C_{\phi, G, \Sigma}(\mathbf{u}) \prod_{l \in I_k} \frac{1}{u_l} \sum_{\nu=0}^{\lfloor k/2 \rfloor} (-1)^\nu \sum_{\{i_1, j_1, \dots, i_\nu, j_\nu\} \subset I_k} \gamma_{i_1 j_1} \dots \gamma_{i_\nu j_\nu}(\mathbf{u}) \prod_{l \in I_k \setminus \{i_1, j_1, \dots, i_\nu, j_\nu\}} \gamma_l(\mathbf{u})$$

where  $I_k = \{1, \dots, k\}$ ,  $\gamma_i(\mathbf{u}) = -u_i G_i(\mathbf{u})$ ,  $\gamma_{ij}(\mathbf{u}) = u_i u_j G_{ij}(\mathbf{u})$ .

The proof of Proposition 7 is postponed to the Appendix.

Also in this case, necessary and sufficient admissibility conditions can be given by requiring the positivity of the density in Equation (12). However, this kind of expression involving partial derivatives of order  $k$  would be of few interest in practice. In following proposition, we give a simplified admissibility condition involving more directly coefficients  $\sigma_{ij}$  of the distortion matrix  $\Sigma$ . Remark that the sufficient admissibility condition in dimension  $k$  proposed in Proposition 8 below requires checking the value of derivatives of order  $k$  of the generator, which can hardly be done in practice. A possible application of Proposition 8 when the initial copula is the independent one is provided by Corollary 1 below. Furthermore, a numerical illustration for a 3-dimensional Archimatrix copula is given in Example 1 (see Section 4).

**Proposition 8** (Sufficient admissibility condition in dimension  $k$ ). *Consider the multivariate Archimatrix model for linear  $z$  in Equation (11), satisfying Assumption 1. Assume that  $\phi$  is  $k$ -times differentiable and  $g_i$  differentiable for all  $i \in I$ . Denote  $\gamma_{ij}(\mathbf{u}) = \frac{g'_i(u_i) g'_j(u_j)}{\psi'(u_i) \psi'(u_j)} \sigma_{ij}$ ,  $i, j \in I$ . Let  $I_k = \{1, \dots, k\}$ ,  $\Pi_\nu(I_k)$  be the set of all possible distinct choices of  $\nu$  couples among  $I_k$ . Let  $\rho_{\phi, k} = \inf_{x \in \mathbb{R}^+} \frac{|\phi^{(k)}(x)|}{|\phi^{(k-1)}(x)|}$ . Assume that all  $\sigma_{ij}$  are positive or zero. Then, for  $k \in I$ , a sufficient condition for the positivity of  $C_{\phi, \Sigma}^{(k)}(\mathbf{u})$  is that for all  $\nu \leq \lfloor k/2 \rfloor - 1$  even, for all  $\pi_\nu \in \Pi_\nu(I_k)$ , for*

all  $\mathbf{u}$ ,

$$(13) \quad \sum_{\{i,j\} \subset \Pi_1(I_k \setminus \pi_\nu)} \gamma_{ij}(\mathbf{u}) \leq \rho_{\phi, k-\nu}.$$

Setting  $\rho_{\phi, 1..k} = \inf_{\nu \in \{0, \dots, \lfloor k/2 \rfloor - 1\}} \rho_{\phi, k-\nu}$ , a simplified sufficient condition is that  $\sum_{i,j \in I_k, i < j} \gamma_{ij}(\mathbf{u}) \leq \rho_{\phi, 1..k}$ .

The proof of Proposition 8 is postponed to the Appendix.

**Remark 1.** As an illustration of Proposition 8, we write in the following the sufficient admissibility conditions provided in Equation (13), for different values of the considered dimension  $k$ , for all  $\mathbf{u}$ :

$$\begin{aligned} \text{for } k = 2, & \quad \gamma_{12}(\mathbf{u}) \leq \rho_{\phi, 2}, \\ \text{for } k = 3, & \quad \gamma_{12}(\mathbf{u}) + \gamma_{13}(\mathbf{u}) + \gamma_{23}(\mathbf{u}) \leq \rho_{\phi, 3}, \\ \text{for } k = 4, & \quad \gamma_{12}(\mathbf{u}) + \gamma_{13}(\mathbf{u}) + \gamma_{14}(\mathbf{u}) + \gamma_{23}(\mathbf{u}) + \gamma_{24}(\mathbf{u}) + \gamma_{34}(\mathbf{u}) \leq \rho_{\phi, 4}, \\ \text{for } k = 5, & \quad \gamma_{12}(\mathbf{u}) + \gamma_{13}(\mathbf{u}) + \gamma_{14}(\mathbf{u}) + \gamma_{15}(\mathbf{u}) + \gamma_{23}(\mathbf{u}) + \gamma_{24}(\mathbf{u}) + \gamma_{25}(\mathbf{u}) + \gamma_{34}(\mathbf{u}) + \\ & \quad \gamma_{35}(\mathbf{u}) + \gamma_{45}(\mathbf{u}) \leq \rho_{\phi, 5}. \end{aligned}$$

Remark that for  $k = 1, 2, 3, 4, 5$ , Equation (13) provides a single condition since  $\nu = 0$ . Furthermore, remark that in the Independent copula case  $\rho_{\phi, k} = 1$ , for all  $k$ . Conversely, in the Clayton copula case  $\frac{|\phi^{(k)}(x)|}{|\phi^{(k-1)}(x)|} = \frac{|(k-1)\theta+1|}{|\theta x+1|}$ . Then, for  $\theta > 0$ , it is not possible in this Clayton case to find a positive lower bound for this quantity. For Joe, Ali-Mikhail-Haq and Frank copula families, at least numerically, it seems possible to find positive constants to inferiorly bound the ratio of the derivatives of the associated generator. The interested reader is referred to Table 1 for  $\rho_{\phi, 2}$  for classical Archimedean generators.

Following result provides the sufficient admissibility condition in dimension  $k$  in Proposition 8 in the simplified case when  $g = \psi$  and  $\phi(t) = \exp(-t)$ .

**Corollary 1** (Sufficient admissibility condition in dimension  $k$  from independence). *Consider the multivariate Archimatrix model for linear  $z$  in (11), satisfying Assumption 1. Assume conditions of Proposition 8 hold true. Furthermore, if  $g_i = \psi$  for all  $i \in I$  and  $\phi(t) = \exp(-t)$ , a sufficient admissibility condition is that*

$$\sum_{i,j \in \{1, \dots, k\}, i < j} \sigma_{ij} \leq 1.$$

The proof of Corollary 1 is postponed to the Appendix. Remark that the simplified admissibility condition in Corollary 1 does only depend on  $\sigma_{ij}$  parameters.

As we will provide in Example 1 in next section, when  $\phi$  is the independence generator, and when  $\mathbf{g} = \mathbf{h} = \psi$ , we have seen that each bivariate projection is a distinct Gumbel-Barnett copula with one parameter per projection. However if one respects this sufficient condition, the sum of parameters is bounded. Thus, the higher the dimension, the more constraint is the copula, with average parameter necessarily closer to zero. However, it does not exclude that one  $\sigma_{ij}$ , say  $\sigma_{12}$ , can be close to one and the others being close to zero. In the latter case,  $U_i$  and  $U_j$  with  $i, j \geq 3$ , tend to be independent, whereas  $U_1$  and  $U_2$  still exhibit negative dependence (see Example 4.10 in Nelsen [27]).

In high dimension, this illustrates the fact that the copula  $C_{\phi,G,\Sigma}$  may in some cases be relatively close to the initial Archimedean copula  $C_\phi$ , due to admissibility constraints relying on the parameters. This is obviously one limitation of the resulting multivariate copula. Remark also that sufficient conditions in Proposition 8 and Corollary 1 could restrict the possible range of the dependence structure of  $C_{\phi,G,\Sigma}$ . For instance in Example 1 the obtained bivariate Archimedean *Gumbel-Barnett* copula projections exclude the positive dependency (see the negative Kendall's tau in Table 3).

#### 4. EXAMPLES AND LINKS TO KNOWN MODELS

We give hereafter some examples exhibiting valid projections. Admissibility conditions for bivariate projection or at higher order will be discussed for each example.

When  $C_{\phi,G,\Sigma}$  is a copula, we now consider in its expression the quantity  $z(\mathbf{g}(\mathbf{u})\Sigma\mathbf{h}(\mathbf{u}))$ .

- If this quantity is always positive, then, as  $\phi$  is decreasing, the copula  $C_{\phi,G,\Sigma} \prec C_\phi$  (see Section 2) and its level curves are nearer to the ones of the lower Fréchet-Hoeffding bound (cf. Figure 2.2 in Nelsen [27]). The positive dependence is, in this concordance ordering sense, reduced (this is the case for further Examples 1 and 2).
- This quantity may not be always positive or negative, as shown in Example 3. In this case the dependence with respect to the initial Archimedean copula  $C_\phi$  is increased for some projections and decreased for others.

**Example 1** (A model with linear  $z$  function). Let  $z(x) = \frac{1}{2}x$  and  $g_i(x) = h_i(x) = \psi(x)$ ,  $i \in I$ ,  $x \in \mathbb{R}^+$ . The model in Definition 2 becomes

$$(14) \quad C_{\phi,G,\Sigma}(\mathbf{u}) = \phi \left( \mathbf{1}^t \boldsymbol{\psi}(\mathbf{u}) + \frac{1}{2} \boldsymbol{\psi}(\mathbf{u})^t \Sigma \boldsymbol{\psi}(\mathbf{u}) \right),$$

Let  $\rho_\phi = \inf \left\{ \left| \frac{\phi''(x)}{\phi'(x)} \right|, x \in \mathbb{R}^+ \right\}$ . By Proposition 5, any bivariate projection is valid if  $\sigma_{ij} \in [0, \rho_\phi]$ . In the particular case where  $\phi(x) = \exp(-x)$ , then  $\rho_\phi = 1$ , the sufficient validity condition becomes  $\sigma_{ij} \in [0, 1]$ . One easily shows that each bivariate projection corresponds here to an Archimedean *Gumbel-Barnett* copula of parameter  $\sigma_{ij}$ , where

$$\mathbb{P}[U_i \leq u_i, U_j \leq u_j] = u_i u_j \exp(-\sigma_{ij} \ln(u_i) \ln(u_j)), \quad \sigma_{ij} \in [0, 1].$$

This range provided by Corollary 1 corresponds with the parameter range of the Archimedean *Gumbel-Barnett* (see Copula 4.2.9 in Nelsen [27]). Each projection has its own parameter  $\sigma_{ij}$ , which allows bivariate projections  $(U_i, U_j)$  to have different distributions, for  $i, j \in I$ . In a general dimension  $k > 2$ , Proposition 8 gives conditions on parameters  $\sigma_{ij}$  ensuring that  $C_{\phi,G,\Sigma}$  is a valid multivariate copula in dimension  $k$ . As an example, if  $\phi$  is the independence generator, a simple sufficient condition is given by Corollary 1.

In Section 5, we will generate a 3-dimensional Archimatrix copula as in Equation (14) with  $\phi(x) = \exp(-x)$  (see Figure 1, left). Remark that in this case condition in (13) in Proposition 8 becomes:  $\sigma_{12} + \sigma_{13} + \sigma_{23} \leq 1$ , since  $\nu = 0$ ,  $\gamma_{ij}(\mathbf{u}) = \sigma_{ij}$ ,  $k = 3$  and  $\rho_{\phi,3} = 1$  (see Corollary 1). This example is one of the most simple that gives tractable admissibility conditions for both valid bivariate projections and valid multivariate copulas.

**Example 2** (A model with power-type  $z$  function). Let  $z(x) = \frac{\alpha}{2} x^\alpha$ , for  $\alpha \in (0, 1]$ , and  $g_i(x) = h_i(x) = (\psi(x))^{1/\alpha}$ , for  $i \in I$ ,  $x \in \mathbb{R}^+$ . The model in Definition 2 becomes

$$(15) \quad C_{\phi, \Sigma; \alpha}(\mathbf{u}) = \phi \left( \mathbf{1}^t \boldsymbol{\psi}(\mathbf{u}) + \frac{\alpha}{2} \left( \boldsymbol{\psi}^{1/\alpha}(\mathbf{u})^t \boldsymbol{\Sigma} \boldsymbol{\psi}^{1/\alpha}(\mathbf{u}) \right)^\alpha \right).$$

Model in (15) generalizes Example 1, which corresponds to the case  $\alpha = 1$ . One can show that any bivariate projection is valid if  $\sigma_{ij} \in [0, \rho_\phi^{1/\alpha}]$ , with  $\rho_\phi = \inf \left\{ \left| \frac{\phi''(x)}{\phi'(x)} \right|, x \in \mathbb{R}^+ \right\}$ . In the bivariate case with  $\phi(x) = \exp(-x)$ , model in (15) becomes:

$$(16) \quad C_{\phi, \Sigma; \alpha}(u_i, u_j) = u_i u_j \exp \left( -\frac{\alpha (2 \sigma_{ij})^\alpha}{2} \ln(u_i) \ln(u_j) \right)$$

This is an Archimedean Gumbel-Barnett copula of parameter  $\tilde{\sigma}_{ij} := \frac{\alpha (2 \sigma_{ij})^\alpha}{2}$ . Then, the bivariate projections are the same of Example 1 with modified dependence parameter. Conversely, in higher dimension, copulas in Equation (15) are different from copulas in Example 1.

**Example 3** (A model with logarithmic  $z$  function). Let  $z(x) = -\ln(1 + \frac{x}{2})$  and  $g_i(x) = h_i(x) = 1 - x$ , for  $i \in I$ ,  $x \in \mathbb{R}^+$ . The model in Definition 2 becomes

$$(17) \quad C_{\phi, G, \Sigma}(\mathbf{u}) = \phi \left( \mathbf{1}^t \boldsymbol{\psi}(\mathbf{u}) - \ln \left( 1 + \frac{1}{2} (\mathbf{1} - \mathbf{u})^t \boldsymbol{\Sigma} (\mathbf{1} - \mathbf{u}) \right) \right).$$

In the particular case  $\phi(x) = \exp(-x)$ , one easily shows that each bivariate projection corresponds to a Farlie-Gumbel-Morgenstern (FGM) copula of parameter  $\sigma_{ij}$ , where

$$\mathbb{P}[U_i \leq u_i, U_j \leq u_j] = u_i u_j (1 + \sigma_{ij} (1 - u_i) (1 - u_j)), \quad \text{for } \sigma_{ij} \in [-1, 1].$$

Typically, this is an example where the simplified sufficient condition given in Proposition 5 does not suffice to determine the parameter range, which can be obtained from the positivity expression of the density (see Equation (9)). In Figure 1, the bivariate projections of the data are depicted. We take  $\sigma_{12} = -0.99$ ,  $\sigma_{13} = 0.99$  and  $\sigma_{23} = 0.2$ . These parameters are chosen in such a way that  $\sigma_{ij} < 1$  for all  $i, j$ , in order to guarantee that each bivariate projection corresponds to a FGM copula with parameter  $\sigma_{ij}$ . Furthermore, we have checked that the corresponding function  $C_{\phi, G, \Sigma}$  was a valid trivariate copula in this case since the general admissibility condition in Equation (9) is satisfied.

In Table 2 we summarize the obtained results for the models detailed in Examples 1-3.

Example	Model	$g_i(x)$	$z(x)$	$\sigma$ , case $d = 2$	if $\phi(x) = \exp(-x)$
1	Linear-type $z$ function	$\psi(x)$	$\frac{1}{2} x$	$[0, \rho_\phi]$	$[0, 1]$ (Barnett-Gumbel)
2	Power-type $z$ function	$\psi(x)^{1/\alpha}$	$\frac{\alpha}{2} x^\alpha$	$[0, \rho_\phi^{1/\alpha}]$	$[0, 1]$
3	Logarithmic $z$ function	$1 - x$	$-\ln(1 + \frac{x}{2})$	-	$[-1, 1]$ (FGM)

TABLE 2. Examples 1-3 of Archimatrix copula models for different choices of  $g_i$  and  $z$  functions. Furthermore in the bivariate setting, the range for  $\sigma$  parameter is provided both in the general case and when  $\phi(x) = \exp(-x)$ .

## 5. FURTHER PROPERTIES AND NUMERICAL ILLUSTRATIONS

In this section we gathered some properties of the proposed Archimatrix copulas in Equation (7). A general stochastic representation would allow a fast sampling procedure for the proposed copulas  $C_{\phi,G,\Sigma}$ . Unfortunately, we did not find such a representation in the general case. However, as derivatives of a quadratic form are easy to obtain, it is possible to use the following remark for sampling.

**Remark 2** (Sampling Archimatrix copulas). A sample of a random vector  $\mathbf{U} = (U_1, \dots, U_d)$  having distribution  $C_{\phi,G,\Sigma}$  can be obtained by a standard construction, using Algorithm 2.1. in Embrechts et al. [11]. Let  $C_k(u_k|u_1, \dots, u_{k-1}) = \mathbb{P}[U_k \leq u_k | U_1 = u_1, \dots, U_{k-1} = u_{k-1}]$ , one have

$$C_k(u_k|u_1, \dots, u_{k-1}) = \frac{\partial}{\partial u_1 \dots \partial u_{k-1}} C_{\phi,G,\Sigma}(u_1, \dots, u_k, 1, \dots, 1) \Big/ \frac{\partial}{\partial u_1 \dots \partial u_{k-1}} C_{\phi,G,\Sigma}(u_1, \dots, u_{k-1}, 1, \dots, 1).$$

The algorithm is: simulate  $u_1$  from  $U_{[0,1]}$ , simulate  $u_2$  from  $C_2(\cdot|u_1)$ , ..., simulate  $u_d$  from  $C_d(\cdot|u_1, \dots, u_{d-1})$ . As an example, setting  $Q(\mathbf{u}) = \mathbf{g}(\mathbf{u})^t \Sigma \mathbf{h}(\mathbf{u})$ , general trivariate copulas can be sampled from derivatives

$$\begin{aligned} \frac{\partial}{\partial u_1} C_{\phi,G,\Sigma}(\mathbf{u}) &= \phi'(\psi \circ C_{\phi,G,\Sigma}(\mathbf{u})) \left( \psi'(u_1) + z'(Q(\mathbf{u})) \frac{\partial}{\partial u_1} Q(\mathbf{u}) \right); \\ \frac{\partial^2}{\partial u_1 \partial u_2} C_{\phi,G,\Sigma}(\mathbf{u}) &= \phi''(\psi \circ C_{\phi,G,\Sigma}(\mathbf{u})) \left( \psi'(u_1) + z'(Q(\mathbf{u})) \frac{\partial}{\partial u_1} Q(\mathbf{u}) \right) \left( \psi'(u_2) + z'(Q(\mathbf{u})) \frac{\partial}{\partial u_2} Q(\mathbf{u}) \right) \\ &\quad + \phi'(\psi \circ C_{\phi,G,\Sigma}(\mathbf{u})) \left( z''(Q(\mathbf{u})) \frac{\partial}{\partial u_1} Q(\mathbf{u}) \frac{\partial}{\partial u_2} Q(\mathbf{u}) + z'(Q(\mathbf{u})) \frac{\partial^2}{\partial u_1 \partial u_2} Q(\mathbf{u}) \right). \end{aligned}$$

For linear expressions of  $z$ , all derivatives of  $C_{\phi,G,\Sigma}$  are given in Proposition 6.

Using Remark 2, in Figure 1 (left) we provide a scatterplot of data from the distorted 3-dimensional copula  $C_{\phi,G,\Sigma}$  presented in Example 1. We take  $\sigma_{12} = 0.001$ ,  $\sigma_{13} = 0.32$  and  $\sigma_{23} = 0.65$ . We know that each bivariate projection corresponds here to an Archimedean *Gumbel-Barnett* copula of parameter  $\sigma_{ij}$ . Indeed in Figure 1 we can observe the anti-comonotonic behavior of the sampling data. Furthermore, we give estimates of the Kendall's  $\tau$  for different parameters  $\sigma$  and we compare them with the theoretical ones in the case of bivariate Gumbel-Barnett copula. Results are gathered in Table 3 (first column). In Figure 1 (right) we provide a scatterplot of data from the distorted 3-dimensional copula  $C_{\phi,G,\Sigma}$  in Example 3. We take  $\sigma_{12} = -0.99$ ,  $\sigma_{13} = 0.99$  and  $\sigma_{23} = 0.2$ . Also in this case the comparison between theoretical and estimated pair-wise Kendall's  $\tau$  is provided (see Table 3, second column).

Bivariate Gumbel-Barnett Copula	Bivariate FGM Copula
$\tau_{0.001} = -0.00049$ $\widehat{\tau}_{0.001} = -0.00043$	$\tau_{-0.99} = -0.220$ $\widehat{\tau}_{-0.99} = -0.229$
$\tau_{0.32} = -0.14011$ $\widehat{\tau}_{0.32} = -0.14511$	$\tau_{0.99} = 0.220$ $\widehat{\tau}_{0.99} = 0.215$
$\tau_{0.65} = -0.25671$ $\widehat{\tau}_{0.65} = -0.24789$	$\tau_{0.2} = 0.0444$ $\widehat{\tau}_{0.2} = 0.0441$

TABLE 3. Theoretical and estimated pair-wise Kendall's  $\tau$  for bivariate Gumbel-Barnett copula (first column) and bivariate FGM copula (second column) for different choices of parameters  $\sigma_{ij}$ .

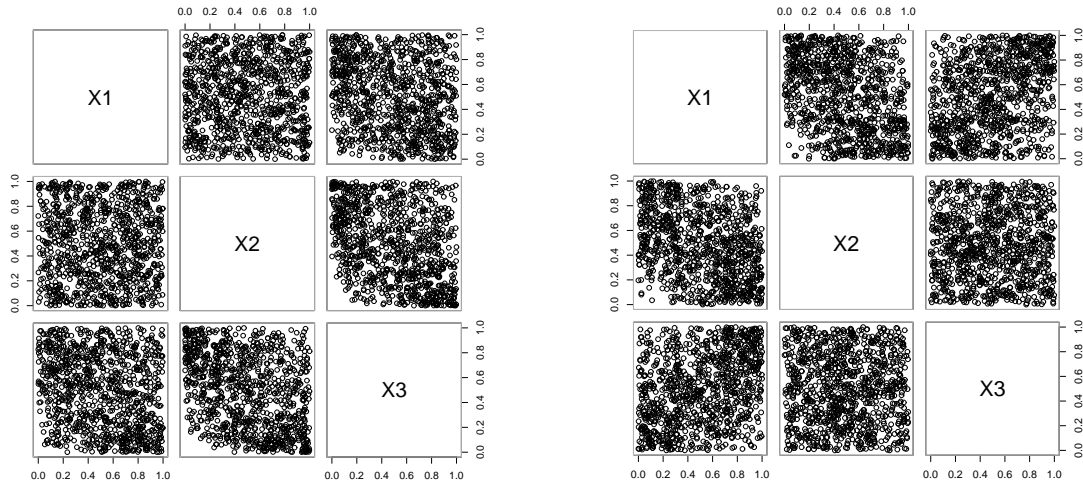


FIGURE 1. **Left:** Sample of size  $n = 1000$  from the 3-dimensional copula  $C_{\phi,G,\Sigma}$  as in Equation (14) with  $z(x) = \frac{1}{2}x$ ,  $\phi(x) = \exp(-x)$  and  $g_i(x) = \psi(x)$ ,  $i \in \{1, 2, 3\}$ ,  $x \in \mathbb{R}^+$ . We take  $\sigma_{12} = 0.001$ ,  $\sigma_{13} = 0.32$  and  $\sigma_{23} = 0.65$  (see Example 1). **Right:** Sample of size  $n = 1000$  from the 3-dimensional copula  $C_{\phi,G,\Sigma}$  as in Equation (17) with  $z(x) = -\ln(1 + \frac{x}{2})$ ,  $\phi(x) = \exp(-x)$  and  $g_i(x) = 1 - x$ ,  $i \in \{1, 2, 3\}$  for  $x \in \mathbb{R}^+$ . We take  $\sigma_{12} = -0.99$ ,  $\sigma_{13} = 0.99$  and  $\sigma_{23} = 0.2$  (see Example 3).

Now, we detail some further properties for the proposed copula  $C_{\phi,G,\Sigma}$ .

**Property 1** (Impact of Archimedean transformations). Let  $T : [0, 1] \rightarrow [0, 1]$  be an increasing bijection, and denote a transformed copula by

$$(18) \quad \widetilde{C}(u_1, \dots, u_d) = T \circ C(T^{-1}(u_1), \dots, T^{-1}(u_d)).$$

For further details on transformed copulas as in (18) the interested reader is referred for example to Durrleman et al. [10], Valdez and Xiao [32], Klement et al. [19], Di Bernardino and



Rullière [8], Morillas [26]. It is easily seen that if  $C$  is an Archimedean copula with generator  $\phi$ , then when  $\tilde{C}$  is a copula,  $\tilde{C}$  is still Archimedean with transformed generator  $\tilde{\phi} = T \circ \phi$ . Now consider an Archimatrix copula  $C_{\phi,G,\Sigma}$  as in Equation (7) and the transformed associated one, i.e.,  $\tilde{C}_{\phi,G,\Sigma}(\mathbf{u}) = T \circ C_{\phi,G,\Sigma}(T^{-1}(\mathbf{u}))$ , where  $T^{-1}(\mathbf{u}) = (T^{-1}(u_1), \dots, T^{-1}(u_d))$ . Consider the multivariate functions  $\tilde{\mathbf{g}}(\mathbf{u}) = \mathbf{g}(T^{-1}(\mathbf{u}))$  and  $\tilde{\mathbf{h}}(\mathbf{u}) = \mathbf{h}(T^{-1}(\mathbf{u}))$ . Then

$$\tilde{C}_{\phi,G,\Sigma}(\mathbf{u}) = C_{\tilde{\phi},\tilde{G},\Sigma}(\mathbf{u}),$$

where  $\tilde{\phi} = T \circ \phi$  and  $\tilde{G} = (\tilde{\mathbf{g}}, \tilde{\mathbf{h}}, z)$ .

The proof of Property 1 is postponed to the Appendix. What is noticeable here is that in the Archimedean case,  $T$  is preserving the Archimedean structure, and thus the symmetry. For Archimatrix copulas, the asymmetry depends in this case on the matrix  $\Sigma$ , which is the same in  $C_{\phi,G,\Sigma}$  or in  $C_{\tilde{\phi},\tilde{G},\Sigma}$ . In a sense,  $\Sigma$  impacts essentially the symmetry of the copula, whereas the transformation  $T$  impacts the position of its level curves (see, e.g., Di Bernardino and Rullière [7]).

**Property 2** (Level curves). Consider an Archimatrix copula  $C_{\phi,G,\Sigma}$  as in Equation (7) and assume that  $\mathbf{g} = \mathbf{h} = \psi$ . Define the level-set  $\partial L_{C_{\phi,G,\Sigma}}(\alpha)$ , for  $\alpha \in (0, 1)$ , as

$$\partial L_{C_{\phi,G,\Sigma}}(\alpha) = \{\mathbf{u} \in [0, 1]^d : C_{\phi,G,\Sigma}(\mathbf{u}) = \alpha\}.$$

One can easily check that

$$\partial L_{C_{\phi,G,\Sigma}}(\alpha) = \{\mathbf{u} \in [0, 1]^d : \mathbf{u} = \phi(\mathbf{x}), \mathbf{x} \in \mathcal{S}(\psi(\alpha))\},$$

where the solution set  $\mathcal{S}(\beta) = \{\mathbf{x} \in \mathbb{R}_+^d : \mathbf{1}^t \mathbf{x} + z(\mathbf{x}^t \Sigma \mathbf{x}) = \beta\}$ . In the case where  $z$  is linear, this solution set is easily obtained as a solution of a quadratic form, and explicit parametric forms of the level set can be obtained. The proof of Property 2 is postponed to the Appendix.

**Property 3** (Averaging of Archimatrix functions). Consider a finite set of indexes  $K$ , a sequence of matrices  $\Sigma_k$ , and functions  $C_{\phi,G,\Sigma_k}$ ,  $k \in K$  as in Equation (7), which are not necessarily copulas. Consider the case where all these functions depend on the same  $z(x) = cx$  for some constant  $c \in \mathbb{R}$ . Let  $\{\alpha_k, k \in K\}$  be a set of real coefficients such that  $\sum_{k \in K} \alpha_k = 1$  and let  $\Sigma = \sum_{k \in K} \alpha_k \Sigma_k$ , then

$$\phi \left( \sum_{k \in K} \alpha_k \cdot \psi \circ C_{\phi,G,\Sigma_k} \right) = C_{\phi,G,\Sigma}.$$

In particular, for independence generator  $\phi(x) = \exp(-x)$ , we get

$$\prod_{k \in K} C_{\phi,G,\Sigma_k}^{\alpha_k} = C_{\phi,G,\Sigma},$$

which in dimension  $d = 2$  is the well known geometric mean property for corresponding *Gumbel-Barnett* copulas (see Nelsen [27], Exercise 4.10). This follows directly from the linear properties of the quadratic form in Equation (7). The proof of Property 3 is postponed to the Appendix.

**Property 4** (Bivariate upper tail dependence coefficient). Consider  $C_{\phi,G,\Sigma} \in \mathcal{F}$  as in Equation (7), with given functions  $\mathbf{g}, \mathbf{h}, z, \Sigma$  satisfying assumptions in Definition 2. Assume that  $[\mathbf{g}(\mathbf{u})]_i = g_i(u_i)$  and  $[\mathbf{h}(\mathbf{u})]_i = h_i(u_i)$ , for all  $i \in I$ . Then, the associated bivariate projection is given by

$$(19) \quad C_{\phi,G,\Sigma_{ij}}(u_i, u_j) = \phi(\psi(u_i) + \psi(u_j) + z(\sigma_{ij} g_i(u_i) h_j(u_j) + \sigma_{ji} g_j(u_j) h_i(u_i))).$$

Recall that the bivariate upper tail coefficient  $\lambda_U$  (see Sibuya [30]) associated to a copula  $C$  can be written, when the limit exists, using the diagonal section  $\delta_C(u) := C(u, u)$  (see, e.g., Nelsen et al. [28], Nelsen [27]):

$$\lambda_U(C) = 2 - \lim_{u \rightarrow 1^-} \frac{d}{du} \delta_C(u).$$

If  $|z'(0)| < +\infty$ ,  $|g'_i(1)| < +\infty$ ,  $|g'_j(1)| < +\infty$ ,  $|h'_i(1)| < +\infty$  and  $|h'_j(1)| < +\infty$ , then

$$\lambda_U(C_{\phi, G, \Sigma_{ij}}) = \lambda_U(C_\phi).$$

The proof of Property 4 is postponed to the Appendix. Using Property 4, we can construct an Archimatrix copula with the same upper tail dependence structure of  $C_\phi$ . This result hold true for Examples 1, 2 and 3 discussed before.

**Property 5** (Linear expression in  $\sigma_{ij}$ ). Due to the sub-model stability, estimation of each parameter  $\sigma_{ij}$  can be done for each bivariate projection  $(U_i, U_j)$ , by classical moment method, regression or by maximum likelihood estimation using given expressions of the copula density. As a consequence of the choice of a quadratic form in the general model, using Equation (8) in the case where  $\mathbf{g} = \mathbf{h}$ , one gets for each couple  $i, j \in I$  a linear expression in  $\sigma_{ij}$ ,

$$(20) \quad \sigma_{ij} \cdot 2g_i(u_i)g_j(u_j) = z^{-1}(\psi \circ \mathbf{P}[U_i \leq u_i, U_j \leq u_j] - (\psi(u_i) + \psi(u_j))).$$

The proof of Property 5 is postponed to the Appendix. This expression can help finding estimators of separate coefficients  $\sigma_{ij}$ . Estimating each coefficient separately can be straightforward since it relies only on one parameter at a time. However, it may result in a global non-admissible copula, due to constraints like those in Corollary 1. The problem of constraint joint estimation of parameters and resulting properties of estimators is not treated here, but constitute an interesting perspective of this work.

For the separate estimation of each  $\sigma_{ij}$ ,  $\mathbf{P}[U_i \leq u_i, U_j \leq u_j] = C_{\phi, G, \Sigma}(1, \dots, 1, u_i, 1, \dots, 1, u_j, 1, \dots, 1)$  is the only unknown quantity in Equation (20). The question is thus how to estimate a quantity  $z^{-1}(\psi \circ C_{\phi, G, \Sigma}(\mathbf{u}) - \psi(\mathbf{u})^t \mathbf{1})$  when  $z(\cdot)$  and  $\psi(\cdot)$  are given. An immediate estimator is the plug-in estimator where  $C_{\phi, G, \Sigma}(\mathbf{u})$  is replaced by the empirical copula  $C_n(\mathbf{u})$  (see, e.g., Deheuvels [6]).

Figure 2 illustrates the possible use of this linearity for estimating each parameter, and for visualizing the dispersion relying on this estimation. We draw two types of set of points:  $\{\alpha_{ij}(\mathbf{u}), \widehat{\beta}_{ij}(\mathbf{u})\}$  (see first and third panels in Figure 2), and  $\{\alpha_{ij}(\mathbf{u}), \beta_{ij}(\mathbf{u})\}$  (see second and fourth panels in Figure 2), where  $\widehat{\beta}_{ij}(\mathbf{u}) = z^{-1}(\psi \circ C_n(u_i, u_j) - (\psi(u_i) + \psi(u_j)))$ ,  $\beta_{ij}(\mathbf{u}) = z^{-1}(\psi \circ \mathbf{P}[U_i \leq u_i, U_j \leq u_j] - (\psi(u_i) + \psi(u_j)))$  and  $\alpha_{ij}(\mathbf{u}) = 2g_i(u_i)g_j(u_j)$  (see Equation (20)). Furthermore, in Figure 2 we present the theoretical regression line (blue line) and the estimated one (red line). In the first and second panels of Figure 2 we choose the Gumbel-Barnett parameters setting, i.e.,  $z(x) = \frac{1}{2}x$ ,  $\phi(x) = \exp(-x)$ ,  $g_i(x) = \psi(x)$  with in particular  $i = 1, j = 3$  (see Example 1). In the third and fourth panels the Farlie-Gumbel-Morgenstern parameters setting is considered, i.e.,  $z(x) = -\ln(1 + \frac{x}{2})$ ,  $g_i(x) = 1 - x$  and  $\phi(x) = \exp(-x)$  with  $i = 1, j = 3$  (see Example 3). The empirical copula  $C_n$  in  $\widehat{\beta}_{ij}(\mathbf{u})$  is estimated on the data-sets of size  $n = 1000$  sampled before (see Property 2 and Figure 1).

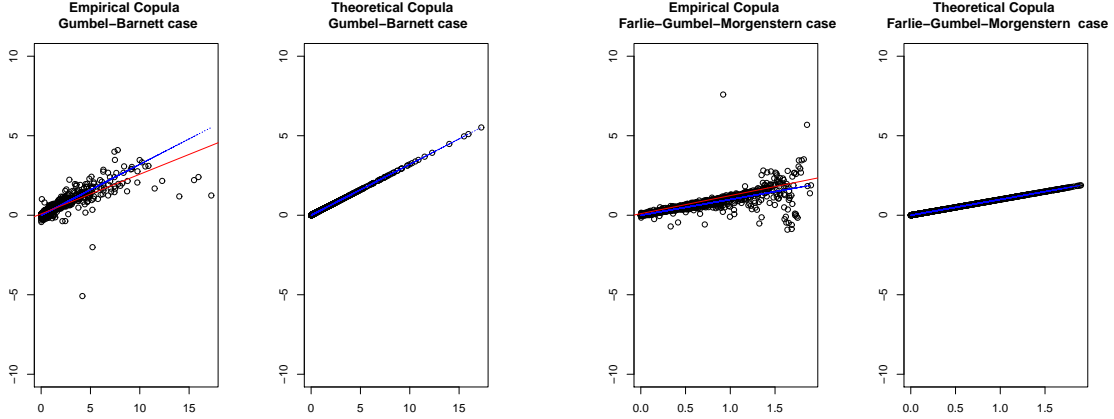


FIGURE 2. Illustration for theoretical and estimated linear expression in  $\sigma_{13}$  in Equation (20). We draw two types of set of points:  $\{\alpha_{ij}(\mathbf{u}), \widehat{\beta}_{ij}(\mathbf{u})\}$  (see first and third panels from the left), and  $\{\alpha_{ij}(\mathbf{u}), \beta_{ij}(\mathbf{u})\}$  (see second and fourth panels). First and second panels: Gumbel-Barnett case with  $\sigma_{13} = 0.32$ . Third and fourth panels: Farlie-Gumbel-Morgenstern case with  $\sigma_{13} = 0.99$ . We present the theoretical regression line (blue line) and the estimated one (red line).

## CONCLUSION

We have proposed a new general class of functions  $C_{\phi, G, \Sigma}$  that permits to build bivariate or multivariate asymmetric copulas. The approach is based on a distortion of Archimedean copulas, involving a linear expression with a parameters matrix  $\Sigma$ . This new class extends the widely used class of Archimedean copulas. The proposed extension is based on a simple linear expression. It thus helps building properties on concordance ordering, on level lines representation or on parameters estimation. One drawback is that, depending on the considered distortion functions, the  $k$ -fold differentiation of the copula is sometimes difficult to simplify, and conditions on the parameters matrix  $\Sigma$  are not always straightforward, especially in the multivariate case. We focused more precisely on multivariate copulas presenting symmetric bivariate projections but asymmetries among bivariate projections. We showed that in a simplified model (with linear function  $z$ ), simple conditions on  $\Sigma$  can be obtained in the multivariate case. Some examples show new constructions of copulas, where the validity condition is easy to check in the bivariate case. In the multivariate case, some examples are given with straightforward validity conditions, some with conditions that are more difficult to check. Even in this last case, we give some illustrations of valid asymmetric 3-variate copulas exhibiting symmetric bivariate (and Archimedean) projections. Natural extensions of this work would be the determination of a stochastic representation of the corresponding random variables, and developments on parameters estimation under validity constraints. Another interesting future study could be the investigation of the link between the proposed Archimatrix copulas and the Archimax ones (see Equation (5)).

**Proof of Proposition 1**

Let  $\Omega = \{\omega_1, \dots, \omega_k\} \subset I$  and  $\mathcal{U}_\Omega = \{(u_1, \dots, u_d) \in [0, 1]^d : u_i = 1, i \in I \setminus \Omega\}$ . Let  $\mathbf{u} = (u_1, \dots, u_d) \in \mathcal{U}_\Omega$ , and recall  $\mathbb{P}[\mathbf{U} \leq \mathbf{u}] = \phi\left(\sum_{i \in I} \psi(u_i) + z\left(\sum_{i,j \in I} [g(\mathbf{u})]_i \sigma_{ij} [h(\mathbf{u})]_j\right)\right)$ . Then,  $\psi(1) = 0$  implies  $\sum_{i \in I} \psi(u_i) = \sum_{i \in \Omega} \psi(u_i)$ , and Assumption *i* in Definition 2 implies  $\sum_{i,j \in I} [g(\mathbf{u})]_i \sigma_{ij} [h(\mathbf{u})]_j = \sum_{i,j \in \Omega} [g(\mathbf{u})]_i \sigma_{ij} [h(\mathbf{u})]_j$ . For  $\mathbf{u} \in \mathcal{U}_\Omega$ ,  $\mathbb{P}[\mathbf{U} \leq \mathbf{u}] = \mathbb{P}[\bigcap_{i \in \Omega} U_i \leq u_i]$ , and finally  $(U_{\omega_1}, \dots, U_{\omega_k}) \sim C_{\phi, G, \Sigma_\Omega}$ . Equation (8) is a direct application when  $[g(\mathbf{u})]_i = g_i(u_i)$  and  $[h(\mathbf{u})]_j = h_j(u_j)$ .  $\square$

**Proof of Proposition 2**

The first item comes down directly from Assumptions *i* and *ii* in Definition 2. Furthermore, since  $\phi$  is a strict generator, then  $\lim_{u_i \rightarrow 0} \psi(u_i) = +\infty$ , and  $\lim_{u_i \rightarrow 0} C_{\phi, G, \Sigma}(\mathbf{u}) = 0$ .  $\square$

**Proof of Proposition 3**

The proof of Proposition 3 comes down from differentiation of  $C_{\phi, G, \Sigma}$  with respect to each parameter  $\sigma_{ij}$ , for  $i, j \in I$ .  $\square$

**Proof of Proposition 4**

Consider  $C_{\phi, G, \Sigma} \in \mathcal{F}$  satisfying conditions in Definition 2 and assume  $C_{\phi, G, \Sigma}$  be a proper copula. Consider  $\mathbf{u} = (u_1, \dots, u_d)$ . From Equation (7), using  $\sigma_{kk} = 0$ , for all  $k \in I$ , any bivariate projection writes

$$(21) \quad \mathbb{P}[U_i \leq u_i, U_j \leq u_j] = \phi(\psi(u_i) + \psi(u_j) + z(Q_{ij}(\mathbf{u}))),$$

where  $Q_{ij}(\mathbf{u}) = [g(\mathbf{u})]_i \sigma_{ij} [h(\mathbf{u})]_j + [g(\mathbf{u})]_j \sigma_{ji} [h(\mathbf{u})]_i$ . If condition (a) in Assumption 1 is satisfied,  $\mathbf{g}(\cdot)$  and  $\mathbf{h}(\cdot)$  are identical vector-valued functions with components  $[g(\mathbf{u})]_i = [h(\mathbf{u})]_i = g_i(u_i)$ ,  $\mathbf{u} \in [0, 1]^d$ , then  $\mathbb{P}[U_i \leq u, U_j \leq v] = \phi(\psi(u) + \psi(v) + z((\sigma_{ij} + \sigma_{ji})g_i(u)g_j(v)))$ . Since  $\mathbb{P}[U_j \leq u, U_i \leq v] = \mathbb{P}[U_i \leq v, U_j \leq u]$ , the condition  $g_i(u) = g(u)$  ensures that  $\mathbb{P}[U_i \leq v, U_j \leq u] = \mathbb{P}[U_i \leq u, U_j \leq v]$ , so that  $(U_i, U_j) \stackrel{d}{=} (U_j, U_i)$ .  $\square$

**Proof of Proposition 5**

Let  $\mathbf{u} = (u_1, \dots, u_d) \in [0, 1]^d$ ,  $\Omega = \{i, j\} \subset I$ , for  $i, j \in I$ ,  $i \neq j$  and let  $\mathbf{u}_\Omega$  be the vector with components  $u_i$  if  $i \in \Omega$ , or 1 otherwise. Then, under Assumption 1, there exist functions  $g_i$  and  $h_i$ ,  $i \in I$  so that for any  $\mathbf{u} \in [0, 1]^d$ ,  $\mathbf{g}(\mathbf{u}) = \mathbf{h}(\mathbf{u}) = (g_1(u_1), \dots, g_d(u_d))$ . In this setting, the model in Definition 2 becomes:

$$(22) \quad C_{\phi, G, \Sigma}(\mathbf{u}_\Omega) = \phi\left(\psi(u_i) + \psi(u_j) + z\left(\sum_{i,j \in I} \sigma_{ij} g_i(u_i) g_j(u_j)\right)\right).$$

Since  $\phi$  and  $z$  are 2-times differentiable and  $g_i$  is differentiable for all  $i \in I$ , then first order derivative of  $C_{\phi, G, \Sigma}$  are, for  $i \in I$ ,

$$\frac{\partial}{\partial u_i} C_{\phi, G, \Sigma}(\mathbf{u}_\Omega) = \phi'(\psi \circ C_{\phi, G, \Sigma}(\mathbf{u}_\Omega)) [\psi'(u_i) + z'(2\sigma_{ij} g_i(u_i) g_j(u_j)) 2\sigma_{ij} g_j(u_j) g'_i(u_i)],$$

and second order derivatives  $\frac{\partial^2}{\partial u_i \partial u_j} C_{\phi, G, \Sigma}(\mathbf{u}_\Omega)$ , for  $i, j \in I$ ,  $i \neq j$ , are

$$\begin{aligned} & \phi''(\psi \circ C_{\phi, G, \Sigma}(\mathbf{u}_\Omega)) [\psi'(u_i)\psi'(u_j) + 2\sigma_{ij}z'(\sigma_{ij}\eta_{ij}(\mathbf{u})) (g_j(u_j)g'_i(u_i)\psi'(u_j) + g'_j(u_j)g_i(u_i)\psi'(u_i))] \\ & + \phi''(\psi \circ C_{\phi, G, \Sigma}(\mathbf{u}_\Omega)) [4\sigma_{ij}^2(z'(\sigma_{ij}\eta_{ij}(\mathbf{u}))^2 g'_i(u_i)g_i(u_i)g'_j(u_j)g_j(u_j))] \\ & + \phi'(\psi \circ C_{\phi, G, \Sigma}(\mathbf{u}_\Omega)) [2\sigma_{ij}g'_i(u_i)g'_j(u_j) [2\sigma_{ij}g_i(u_i)g_j(u_j)z''(\sigma_{ij}\eta_{ij}(\mathbf{u})) + z'(\sigma_{ij}\eta_{ij}(\mathbf{u}))]] , \end{aligned}$$

where  $\eta_{ij}(\mathbf{u}) = 2g_i(u_i)g_j(u_j)$ . One easily checks that the density expressed in previous equation is positive when  $\sigma_{ij} = 0$ , and corresponds to the one of the initial Archimedean copula bivariate projections.

Let us denote in a synthetic way  $g'_i = g'_i(u_i)$ ,  $g'_j = g'_j(u_j)$ ,  $\psi'_i = \psi'(u_i)$ ,  $z' = z'(2\sigma_{ij}g_i(u_i)g_j(u_j))$ ,  $z'' = z''(2\sigma_{ij}g_i(u_i)g_j(u_j))$ ,  $\phi' = \phi'(\psi \circ C_{\phi, G, \Sigma}(\mathbf{u}_\Omega))$ ,  $\phi'' = \phi''(\psi \circ C_{\phi, G, \Sigma}(\mathbf{u}_\Omega))$ . The density can be written

$$(23) \quad \phi''\psi'_i\psi'_j + 2\sigma_{ij}z' \cdot [(g_jg'_i\psi'_j + g'_jg_i\psi'_i)\phi'' + g'_ig'_j\phi'] + 4\sigma_{ij}^2g'_ig_i g'_jg_j [(z')^2 \cdot \phi'' + z'' \cdot \phi']$$

Under Assumption 1,  $g_i$  is monotone with  $g_i(1) = 0$ , so that necessarily  $g_i g'_i \leq 0$  for any  $u_i \in [0, 1]$ . From Assumption *iii* in Definition 2,  $g_i g'_j \leq 0$  for all  $i, j \in I$ . Thus  $(g_j g'_i \psi'_j + g'_j g_i \psi'_i) \phi'' \geq 0$ ,  $4\sigma_{ij}^2 g'_i g_i g'_j g_j \geq 0$  and by assumption  $(z')^2 \cdot \phi'' + z'' \cdot \phi' \geq 0$ . Then the density given by (23) is lower bounded by

$$\phi''\psi'_i\psi'_j + 2\sigma_{ij}z' [0 + g'_ig'_j\phi'] + 0$$

and we check that this latter quantity is greater than zero under chosen assumptions.  $\square$

### Proof of Proposition 6

This follows directly from the multivariate version of Faà di Bruno's formula for partial derivatives, using the fact that derivatives of  $\mathbf{g}(\mathbf{u})^t \Sigma \mathbf{g}(\mathbf{u})$  vanish for orders greater than 2 (see Equation (10)).  $\square$

### Proof of Proposition 7

Follows directly from Proposition 6, using in the independence case  $\phi^{(k)} \circ \psi \circ C_{\phi, G, \Sigma}(\mathbf{u}) = (-1)^k C_{\phi, G, \Sigma}(\mathbf{u})$ , factorizing the product of  $\psi'(u_l) = -|\psi'(u_l)|$ , and using here  $\psi'(u) = -1/u$ .  $\square$

### Proof of Proposition 8

Using previous notations,  $G_i(\mathbf{u}) = \psi'(u_i)\gamma_i(\mathbf{u})$  and  $G_{ij}(\mathbf{u}) = \psi'(u_i)\psi'(u_j)\gamma_{ij}(\mathbf{u})$ . Let  $\pi_\nu = \{i_1, j_1, \dots, i_\nu, j_\nu\}$  be an (ordered) member of  $\Pi_\nu(I_k)$ , for  $\nu \geq 1$ . Denote

$$S_0^k(\mathbf{u}) = \prod_{l \in I_k} \gamma_l(\mathbf{u}) \quad \text{and} \quad S_\nu^k(\mathbf{u}) = \sum_{\pi_\nu \in \Pi_\nu(I_k)} \gamma_{i_1 j_1}(\mathbf{u}) \dots \gamma_{i_\nu j_\nu}(\mathbf{u}) \prod_{l \in I_k \setminus \pi_\nu} \gamma_l(\mathbf{u})$$

so that  $R_\nu^k(\mathbf{u}) = \prod_{l \in I_k} \psi'(u_l) S_\nu^k(\mathbf{u})$ . Using  $\phi^{(k-\nu)}(\cdot) = (-1)^{k-\nu} |\phi^{(k-\nu)}(\cdot)|$  from the  $d$ -monotony of  $\phi$ ,

$$\frac{\partial^k}{\partial u_1 \dots \partial u_k} C_{\phi, G, \Sigma}(\mathbf{u}) = \prod_{l \in I_k} (-\psi'(u_l)) \sum_{\nu=0}^{\lfloor k/2 \rfloor} (-1)^\nu \left| \phi^{(k-\nu)} \circ \psi \circ C_{\phi, G, \Sigma}(\mathbf{u}) \right| \times S_\nu^k(\mathbf{u}).$$

Since  $\rho_{\phi,k} = \inf_{x \in \mathbb{R}^+} \frac{|\phi^{(k)}(x)|}{|\phi^{(k-1)}(x)|}$ , then  $\frac{\partial^k}{\partial u_1 \dots \partial u_k} C_{\phi,G,\Sigma}(\mathbf{u})$  is greater than

$$\sum_{\nu=0, \nu \text{ even}}^{\lfloor k/2 \rfloor} \left| \phi^{(k-1-\nu)} \circ \psi \circ C_{\phi,G,\Sigma}(\mathbf{u}) \right| \left[ \rho_{\phi,k-\nu} S_{\nu}(\mathbf{u}) - \mathbf{1}_{\{\nu+1 \leq \lfloor k/2 \rfloor\}} S_{\nu+1}(\mathbf{u}) \right].$$

One can check that for any  $\rho_{\phi,k-\nu} > 0$ , since  $\gamma_{i_{\nu+1}}(\mathbf{u})\gamma_{j_{\nu+1}}(\mathbf{u}) \geq 1$ ,

$$\begin{aligned} \rho_{\phi,k-\nu} S_{\nu}(\mathbf{u}) - S_{\nu+1}(\mathbf{u}) &\geq \sum_{\pi_{\nu} \in \Pi_{\nu}(I_k)} \gamma_{i_1 j_1}(\mathbf{u}) \dots \gamma_{i_{\nu} j_{\nu}}(\mathbf{u}) \\ &\cdot \prod_{l \in I_k \setminus \pi_{\nu}} \gamma_l(\mathbf{u}) \left[ \rho_{\phi,k-\nu} - \mathbf{1}_{\{\nu+1 \leq \lfloor k/2 \rfloor\}} \sum_{\{i_{\nu+1}, j_{\nu+1}\} \subset \Pi_1(I_k \setminus \pi_{\nu})} \gamma_{i_{\nu+1} j_{\nu+1}}(\mathbf{u}) \right], \end{aligned}$$

so that a sufficient condition is that for all  $k \in I$ , for all  $\nu$  even such that  $\nu + 1 \leq \lfloor k/2 \rfloor$ , for all  $\pi_{\nu} \in \Pi_{\nu}(I_k)$ ,

$$\sum_{\{i_{\nu+1}, j_{\nu+1}\} \subset \Pi_1(I_k \setminus \pi_{\nu})} \gamma_{i_{\nu+1} j_{\nu+1}}(\mathbf{u}) \leq \rho_{\phi,k-\nu}.$$

Hence the result.  $\square$

### Proof of Corollary 1

Remark that in the case of independent generator  $\phi$ ,  $\rho_{\phi,1..k} = 1$ . Since, under Assumption of Corollary 1,  $g_i = \psi$ , for all  $i \in I$ , then  $\gamma_{ij}(\mathbf{u}) = \sigma_{ij}$ , for all  $\mathbf{u}$ . Hence the result.  $\square$

### Proof of Property 1

Using transformation in Equation (18), we get

$$\tilde{C}_{\phi,G,\Sigma}(\mathbf{u}) = T \circ C_{\phi,G,\Sigma}(T^{-1}(\mathbf{u})) = T \circ \phi(\psi(T^{-1}(\mathbf{u})) + z(\mathbf{g}(T^{-1}(\mathbf{u}))^t \Sigma \mathbf{h}(T^{-1}(\mathbf{u}))),$$

where  $T^{-1}(\mathbf{u}) = (T^{-1}(u_1), \dots, T^{-1}(u_d))$ . Then  $\tilde{C}_{\phi,G,\Sigma}(\mathbf{u}) = C_{\tilde{\phi}, \tilde{G}, \Sigma}(\mathbf{u})$ , where  $\tilde{\phi} = T \circ \phi$ ,  $\tilde{G} = (\tilde{\mathbf{g}}, \tilde{\mathbf{h}}, z)$  with  $\tilde{\mathbf{g}}(\mathbf{u}) = \mathbf{g}(T^{-1}(\mathbf{u}))$  and  $\tilde{\mathbf{h}}(\mathbf{u}) = \mathbf{h}(T^{-1}(\mathbf{u}))$ .  $\square$

### Proof of Property 2

By using the expression of  $C_{\phi,G,\Sigma}$  in Equation (7), we get the level-set

$$\partial L_{C_{\phi,G,\Sigma}}(\alpha) = \{\mathbf{u} \in [0, 1]^d : (\mathbf{1}^t \boldsymbol{\psi}(\mathbf{u}) + z(\mathbf{g}(\mathbf{u})^t \Sigma \mathbf{h}(\mathbf{u}))) = \psi(\alpha)\}$$

Since, by assumption,  $\mathbf{g} = \mathbf{h} = \boldsymbol{\psi}$ ,

$$\partial L_{C_{\phi,G,\Sigma}}(\alpha) = \{\mathbf{u} \in [0, 1]^d : (\mathbf{1}^t \boldsymbol{\psi}(\mathbf{u}) + z(\boldsymbol{\psi}(\mathbf{u})^t \Sigma \boldsymbol{\psi}(\mathbf{u}))) = \psi(\alpha)\}.$$

Then finally we have

$$\partial L_{C_{\phi,G,\Sigma}}(\alpha) = \{\mathbf{u} \in [0, 1]^d : \mathbf{u} = \phi(\mathbf{x}), \mathbf{x} \in \mathcal{S}(\psi(\alpha))\},$$

where  $\mathcal{S}(\beta) = \{\mathbf{x} \in \mathbb{R}_+^d : \mathbf{1}^t \mathbf{x} + z(\mathbf{x}^t \Sigma \mathbf{x}) = \beta\}$ .  $\square$

### Proof of Property 3

By using the expression of  $C_{\phi,G,\Sigma}$  in Equation (7) and the fact that by assumption  $\sum_{k \in K} \alpha_k = 1$ ,

we get

$$\begin{aligned} \sum_{k \in K} \alpha_k \cdot \psi \circ C_{\phi, G, \Sigma_k} &= \sum_{k \in K} \alpha_k (\mathbf{1}^t \boldsymbol{\psi}(\mathbf{u}) + z (\mathbf{g}(\mathbf{u})^t \Sigma_k \mathbf{h}(\mathbf{u}))) \\ &= \sum_{k \in K} \alpha_k (\mathbf{1}^t \boldsymbol{\psi}(\mathbf{u}) + c \cdot (\mathbf{g}(\mathbf{u})^t \Sigma_k \mathbf{h}(\mathbf{u}))) = \mathbf{1}^t \boldsymbol{\psi}(\mathbf{u}) + c \sum_{k \in K} \alpha_k (\mathbf{g}(\mathbf{u})^t \Sigma_k \mathbf{h}(\mathbf{u})) \end{aligned}$$

Since  $\Sigma = \sum_{k \in K} \alpha_k \Sigma_k$ , then  $\phi(\sum_{k \in K} \alpha_k \cdot \psi \circ C_{\phi, G, \Sigma_k}) = C_{\phi, G, \Sigma}$ . Furthermore, if  $\phi(x) = \exp(-x)$ , then we get

$$\exp\left(-\sum_{k \in K} \alpha_k \cdot \psi \circ C_{\phi, G, \Sigma_k}\right) = \prod_{k \in K} \exp(-\alpha_k \cdot \psi \circ C_{\phi, G, \Sigma_k}) = \prod_{k \in K} C_{\phi, G, \Sigma_k}^{\alpha_k} = C_{\phi, G, \Sigma}.$$

Hence the desired result.  $\square$

#### Proof of Property 4

Let  $\delta_{\phi, G, \Sigma_{ij}}(u) := C_{\phi, G, \Sigma_{ij}}(u, u) = \phi(2\psi(u) + z(\sigma_{ij} g_i(u) h_j(u) + \sigma_{ji} g_j(u) h_i(u)))$ .

Then, we get

$$\begin{aligned} \delta'_{\phi, G, \Sigma_{ij}}(u) &= \phi'(2\psi(u) + z(\sigma_{ij} g_i(u) h_j(u) + \sigma_{ji} g_j(u) h_i(u))) \\ &\quad \cdot [2\psi'(u) + (\sigma_{ij} \xi_{ij}(u) + \sigma_{ji} \xi_{ji}(u)) z'(\sigma_{ij} g_i(u) h_j(u) + \sigma_{ji} g_j(u) h_i(u))], \end{aligned}$$

where  $\xi_{ij}(u) = g'_i(u) h_j(u) + g_i(u) h'_j(u)$ . From conditions in Definition 2 for all  $\mathbf{u} \in [0, 1]^d$ ,  $[\mathbf{g}(\mathbf{u})]_i = [\mathbf{h}(\mathbf{u})]_i = 0$  as soon as  $u_i = 1$  and  $z(0) = 0$ . Furthermore  $\psi(1) = 0$ . Then, if  $|z'(0)| < +\infty$ ,  $|g'_i(1)| < +\infty$ ,  $|g'_j(1)| < +\infty$ ,  $|h'_i(1)| < +\infty$  and  $|h'_j(1)| < +\infty$ , we obtain the desired result.  $\square$

#### Proof of Property 5

Since, by assumption,  $\mathbf{g} = \mathbf{h}$ , one gets for each couple  $i, j \in I$

$$\mathbb{P}[U_i \leq u_i, U_j \leq u_j] = \phi(\psi(u_i) + \psi(u_j) + z(2\sigma_{ij} g(u_i) h(u_j))).$$

Then we get  $\sigma_{ij} \cdot 2g_i(u_i)g_j(u_j) = z^{-1}(\psi \circ \mathbb{P}[U_i \leq u_i, U_j \leq u_j] - (\psi(u_i) + \psi(u_j)))$ .  $\square$

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ELENA DI BERNARDINO, CNAM, PARIS, EA4629, DEPARTEMENT IMATH, 292 RUE SAINT-MARTIN, PARIS CEDEX 03, FRANCE

*E-mail address:* elena.di.bernardino@cnam.fr

DIDIER RULLIERE, UNIVERSITE DE LYON, UNIVERSITE LYON 1, ISFA, LABORATOIRE SAF, 50 AVENUE TONY GARNIER, 69366 LYON, FRANCE

*E-mail address:* didier.rulliere@univ-lyon1.fr