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Gradient discretization of Hybrid Dimensional Darcy Flows in Fractured Porous Media with discontinuous pressures at the matrix fracture interfaces

K. Brenner*, J. Hennicker*†, R. Masson*, P. Samier†

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Abstract

We investigate the discretization of Darcy flow through fractured porous media on general meshes. We consider a hybrid dimensional model, invoking a complex network of planar fractures. The model accounts for matrix-fracture interactions and fractures acting either as drains or as barriers, i.e. we have to deal with pressure discontinuities at matrix-fracture interfaces. The numerical analysis is performed in the general framework of gradient discretizations which is extended to the model under consideration. Two families of schemes namely the Vertex Approximate Gradient scheme (VAG) and the Hybrid Finite Volume scheme (HFV) are detailed and shown to satisfy the gradient scheme framework, which yields, in particular, convergence.

1 Introduction

This work deals with the discretization of Darcy flows in fractured porous media for which the fractures are modelized as interfaces of codimension one. In this framework, the $d-1$ dimensional flow in the fractures is coupled with the d dimensional flow in the matrix leading to the so called, hybrid dimensional Darcy flow model. We consider the case for which the pressure can be discontinuous at the matrix fracture interfaces in order to account for fractures acting either as drains or as barriers as described in [8], [10] and [2].

The discretization of such hybrid dimensional Darcy flow model has been the object of several works. In [8], [9], [2] a cell-centred Finite Volume scheme using a Two Point Flux Approximation (TPFA) is proposed assuming the orthogonality of the mesh and isotropic permeability fields. Cell-centred Finite Volume schemes have been extended to general meshes and anisotropic permeability fields using MultiPoint Flux Approximations (MPFA) in [11], [13], and [1]. In [10], a Mixed Finite Element (MFE) method is proposed and a MFE discretization adapted to non-matching fracture and matrix grids is studied in [4].

In this work, we extend the Gradient scheme framework introduced in [5] to the case of hybrid dimensional Darcy flows with discontinuous pressures. This framework accounts for a large class of non conforming and conforming discretizations including conforming finite element methods, symmetric finite volume schemes, and mixed and mixed hybrid finite element

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methods. The framework is first described, then applied to extend to our model the Vertex Approximate Gradient scheme (VAG) and the Hybrid Finite Volume Scheme (HFV) introduced in respectively [5] and [6] for the finite volume discretization of anisotropic diffusion problems on general meshes.

In section 2 we introduce the geometry of the matrix and fracture domains and present the strong and weak formulation of the model. Section 3 is devoted to the introduction of the general framework of gradient discretizations and the derivation of the error estimate 3.3. In section 4 we define and investigate the families of VAG and HFV discretizations. Having in mind applications to multi-phase flow, we also present a Finite Volume formulation involving conservative fluxes, which applies for both schemes.

2 Hybrid dimensional Darcy Flow Model in Fractured Porous Media

2.1 Geometry and Function Spaces

Let Ω denote a bounded domain of $\mathbb{R}^d$, $d = 2, 3$ assumed to be polyhedral for $d = 3$ and polygonal for $d = 2$. To fix ideas the dimension will be fixed to $d = 3$ when it needs to be specified, for instance in the naming of the geometrical objects or for the space discretization in the next section. The adaptations to the case $d = 2$ are straightforward.

Let $\overline{\Gamma} = \bigcup_{i\in I} \overline{\Gamma}_i$ and its interior $\Gamma = \overline{\Gamma} \setminus \partial\overline{\Gamma}$ denote the network of fractures $\Gamma_i \subset \Omega$, $i \in I$, such that each Γ_i is a planar polygonal simply connected open domain included in a plane $\mathcal{P}_i$ of $\mathbb{R}^d$. It is assumed that the angles of Γ_i are strictly smaller than 2π, and that $\Gamma_i \cap \overline{\Gamma}_j = \emptyset$ for all $i \neq j$.

For all $i \in I$, let us set $\Sigma_i = \partial\Gamma_i$, with $\mathbf{n}_{\Sigma_i}$ as unit vector in $\mathcal{P}_i$, normal to Σ_i and outward to Γ_i. Further $\Sigma_{i,j} = \Sigma_i \cap \Sigma_j$, $j \in I \setminus \{i\}$, $\Sigma_{i,0} = \Sigma_i \cap \partial\Omega$, $\Sigma_{i,N} = \Sigma_i \setminus (\bigcup_{j\in I\setminus\{i\}} \Sigma_{i,j} \cup \Sigma_{i,0})$, $\Sigma = \bigcup_{(i,j)\in I\times I, i\neq j} \Sigma_{i,j}$ and $\Sigma_0 = \bigcup_{i\in I} \Sigma_{i,0}$. It is assumed that $\Sigma_{i,0} = \overline{\Gamma}_i \cap \partial\Omega$.

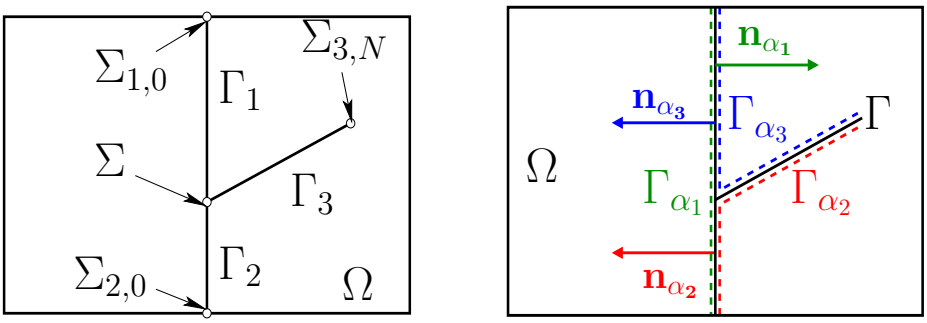


Figure 1: Example of a 2D domain Ω and 3 intersecting fractures $\Gamma_i, i = 1, 2, 3$. We might define the fracture plane orientations by $\alpha^+(1) = \alpha_1, \alpha^-(1) = \alpha_3$ for Γ_1, $\alpha^+(2) = \alpha_1, \alpha^-(2) = \alpha_2$ for Γ_2, and $\alpha^+(3) = \alpha_3, \alpha^-(3) = \alpha_2$ for Γ_3.

We will denote by $d\tau(\mathbf{x})$ the $d-1$ dimensional Lebesgue measure on Γ. On the fracture network Γ, we define the function space $L^2(\Gamma) = \{v = (v_i)_{i\in I}, v_i \in L^2(\Gamma_i), i \in I\}$, endowed with the norm $\|v\|^2_{L^2(\Gamma)} = \sum_{i\in I} \|v_i\|^2_{L^2(\Gamma_i)}$ and its subspace $H^1(\Gamma)$ consisting of functions $v = (v_i)_{i\in I}$ such that $v_i \in H^1(\Gamma_i)$, $i \in I$ with continuous traces at the fracture intersections. The space

$H^1(\Gamma)$ is endowed with the norm $\|v\|^2_{H^1(\Gamma)} = \sum_{i\in I} \|v_i\|^2_{H^1(\Gamma_i)}$. We also define it's subspace with vanishing traces on Σ_0, which we denote by $H^1_{\Sigma_0}(\Gamma)$.

On $\Omega\backslash\overline{\Gamma}$, the gradient operator from $H^1(\Omega\backslash\overline{\Gamma})$ to $L^2(\Omega)^d$ is denoted by ∇. On the fracture network Γ, the tangential gradient, acting from $H^1(\Gamma)$ to $L^2(\Gamma)^{d-1}$, is denoted by ∇_τ, and such that

$$\nabla_\tau v = (\nabla_{\tau_i} v_i)_{i\in I},$$

where, for each $i \in I$, the tangential gradient ∇_{τ_i} is defined from $H^1(\Gamma_i)$ to $L^2(\Gamma_i)^{d-1}$ by fixing a reference Cartesian coordinate system of the plane $\mathcal{P}_i$ containing Γ_i. We also denote by div_{τ_i} the divergence operator from $H_{\mathrm{div}}(\Gamma_i)$ to $L^2(\Gamma_i)$.

We assume that there exists a finite family $(\Gamma_\alpha)_{\alpha\in\chi}$ such that for all $\alpha \in \chi$ holds: $\Gamma_\alpha \subset \Gamma$ and there exists a lipschitz domain $\omega_\alpha \subset \Omega\setminus\overline{\Gamma}$, such that $\Gamma_\alpha = \partial\omega_\alpha \cap \Gamma$. For $\alpha \in \chi$ and an apropriate choice of $I_\alpha \subset I$ we assume that $\overline{\Gamma}_\alpha = \bigcup_{i\in I_\alpha} \overline{\Gamma}_i$. Furthermore should hold $\overline{\Gamma} = \bigcup_{\alpha\in\chi} \overline{\Gamma}_\alpha$. We also assume that each $\Gamma_i \subset \Gamma$ is contained in Γ_α for exactly two $\alpha \in \chi$ and that we can define a unique mapping $i \longmapsto (\alpha^+(i), \alpha^-(i))$, such that $\Gamma_i \subset \Gamma_{\alpha^+(i)} \cap \Gamma_{\alpha^-(i)}$ and $\alpha^+(i) \neq \alpha^-(i)$ (cf. figure 2.1). For all $i \in I$, $\alpha^\pm(i)$ defines the two sides of the fracture Γ_i in $\Omega \setminus \overline{\Gamma}$ and we can introduce the corresponding unit normal vectors $n_{\alpha^\pm(i)}$ at Γ_i outward to $\omega_{\alpha^\pm(i)}$, such that $\mathbf{n}_{\alpha^+(i)} + \mathbf{n}_{\alpha^-(i)} = 0$. We therefore obtain for $\alpha \in \chi$ and a.e. $\mathbf{x} \in \Gamma_\alpha$ a unique unit normal vector $\mathbf{n}_\alpha(\mathbf{x})$ outward to ω_α.

Then, for $\alpha \in \chi$, we can define the trace operator on Γ_α:

$$\gamma_\alpha : H^1(\Omega \setminus \overline{\Gamma}) \to L^2(\Gamma_\alpha),$$

and the normal trace operator on Γ_α outward to the side α:

$$\gamma_{\mathbf{n},\alpha} : H_{\mathrm{div}}(\Omega \setminus \overline{\Gamma}) \to \mathcal{D}'(\Gamma_\alpha).$$

We now define the hybrid dimensional function spaces that will be used as variational spaces for the Darcy flow model in the next subsection:

$$V = H^1(\Omega \setminus \overline{\Gamma}) \times H^1(\Gamma),$$

and its subspace

$$V^0 = H^1_{\partial\Omega}(\Omega \setminus \overline{\Gamma}) \times H^1_{\Sigma_0}(\Gamma),$$

where

$$H^1_{\partial\Omega}(\Omega \setminus \overline{\Gamma}) = \{v \in H^1(\Omega\backslash\overline{\Gamma}) \mid v = 0 \text{ on } \partial\Omega\},$$

as well as

$$W = W_m \times W_f,$$

where

$$\begin{aligned} W_m &= \left\{\mathbf{q}_m \in H_{\mathrm{div}}(\Omega \setminus \overline{\Gamma}) \mid \gamma_{\mathbf{n},\alpha}\mathbf{q}_m \in L^2(\Gamma_\alpha) \text{ for all } \alpha \in \chi\right\} \text{ and} \\ W_f &= \{\mathbf{q}_f = (\mathbf{q}_{f,i})_{i\in I} \mid \mathbf{q}_{f,i} \in H_{\mathrm{div}}(\Gamma_i)\ \forall i \in I \\ &\quad \text{and } \sum_{i\in\Gamma} \int_{\Gamma_i} \Big(\nabla_\tau v \cdot \mathbf{q}_{f,i} + v \cdot \mathrm{div}_{\tau_i}\mathbf{q}_{f,i}\Big) = 0\ \forall v \in H^1_{\Sigma_0}(\Gamma)\}. \end{aligned}$$

On V, we define the positive semidefinite, symmetric bilinear form

$$((u_m, u_f), (v_m, v_f))_V = \int_\Omega \nabla u_m \cdot \nabla v_m d\mathbf{x} + \int_\Gamma \nabla_\tau u_f \cdot \nabla_\tau v_f d\tau(\mathbf{x}) + \sum_{\alpha\in\chi} \int_{\Gamma_\alpha} (\gamma_\alpha u_m - u_f)(\gamma_\alpha v_m - v_f) d\tau(\mathbf{x})$$

for $(u_m, u_f), (v_m, v_f) \in V$, which induces the seminorm $|(v_m, v_f)|_V$. Note that $(\cdot,\cdot)_V$ is a scalar product and $|\cdot|_V$ is a norm on V^0.
We define for all $(\mathbf{p}_m, \mathbf{p}_f), (\mathbf{q}_m, \mathbf{q}_f) \in W$ the scalar product

$$\begin{aligned}((\mathbf{p}_m, \mathbf{p}_f), (\mathbf{q}_m, \mathbf{q}_f))_W = &\int_\Omega \mathbf{p}_m \mathbf{q}_m d\mathbf{x} + \int_\Omega \mathrm{div}\mathbf{p}_m \cdot \mathrm{div}\mathbf{q}_m d\mathbf{x} \\ &+ \int_\Gamma \mathbf{p}_f \mathbf{q}_f d\tau(\mathbf{x}) + \int_\Gamma \mathrm{div}_\tau \mathbf{p}_f \cdot \mathrm{div}_\tau \mathbf{q}_f d\tau(\mathbf{x}) \\ &+ \sum_{\alpha\in\chi} \int_{\Gamma_\alpha} (\gamma_{\mathbf{n},\alpha}\mathbf{p}_m \cdot \gamma_{\mathbf{n},\alpha}\mathbf{q}_m) d\tau(\mathbf{x}),\end{aligned}$$

which induces the norm $\|(\mathbf{q}_m, \mathbf{q}_f)\|_W$, and where we have used the notation $\mathrm{div}_\tau \mathbf{p}_f = \mathrm{div}_{\tau_i}\mathbf{p}_{f,i}$ on Γ_i for all $i \in I$ and $\mathbf{p}_f = (\mathbf{p}_{f,i})_{i\in I} \in W_f$.
Using similar arguments as in the proof of [12], example II.3.4, one can prove the following Poincaré type inequality

Proposition 2.1 *The seminorm $|.|_V$ satisfies the following inequality*

$$\|v_m\|_{H^1(\Omega\setminus\overline{\Gamma})} + \|v_f\|_{H^1(\Gamma)} \le C_P |(v_m, v_f)|_V, \tag{1}$$

for all $(v_m, v_f) \in V^0$.

The convergence analysis presented in section 4 requires some results on density of smooth subspaces of V and W, which we state below. Let us define the subspace C^∞_Ω of functions in $C^\infty_b(\Omega\setminus\overline{\Gamma})$ vanishing on a neighbourhood of the boundary $\partial\Omega$, where $C^\infty_b(\Omega\setminus\overline{\Gamma}) \subset C^\infty(\Omega\setminus\overline{\Gamma})$ is the set of functions φ, such that for all $\mathbf{x} \in \Omega$ there exists $r > 0$, such that for all connected components ω of $\{\mathbf{x}+\mathbf{y} \in \mathbb{R}^d \mid |\mathbf{y}| < r\} \cap (\Omega\setminus\overline{\Gamma})$ one has $\varphi \in C^\infty(\overline{\omega})$. Let us also define the subspace C^∞_Γ of functions in $\Pi_{i\in I} C^\infty(\overline{\Gamma}_i)$ vanishing on a neigbourhood of Σ_0 and continuous on Γ.

Proposition 2.2 *$C^\infty_\Omega \times C^\infty_\Gamma$ is dense in V^0.*

Let us further set $C^\infty_{W_m} = C^\infty_b(\Omega\setminus\overline{\Gamma})^d$. On Γ we define the function space $C^\infty_{W_f} = \{\mathbf{q}_f = (\mathbf{q}_{f,i})_{i\in I} \mid \mathbf{q}_{f,i} \in C^\infty(\overline{\Gamma}_i)^{d-1},\ \sum_{i\in I} \mathbf{q}_{f,i}\cdot\mathbf{n}_{\Sigma_i} = 0 \text{ on } \Sigma\setminus\Sigma_0,\ \mathbf{q}_{f,i}\cdot\mathbf{n}_{\Sigma_i} = 0 \text{ on } \Sigma_{i,N},\ i \in I\}$.

Proposition 2.3 *$C^\infty_{W_m} \times C^\infty_{W_f}$ is dense in W.*

2.2 Single Phase Darcy Flow Model

2.2.1 Strong formulation

In the matrix domain $\Omega\setminus\overline{\Gamma}$, let us denote by $\Lambda_m \in L^\infty(\Omega)^{d\times d}$ the permeability tensor such that there exist $\overline{\lambda}_m \ge \underline{\lambda}_m > 0$ with

$$\underline{\lambda}_m|\xi|^2 \le (\Lambda_m(\mathbf{x})\xi, \xi) \le \overline{\lambda}_m|\xi|^2 \text{ for all } \xi \in \mathbb{R}^d, \mathbf{x} \in \Omega,$$

Analogously, in the fracture network Γ, we denote by $\Lambda_f \in L^\infty(\Gamma)^{(d-1)\times(d-1)}$ the tangential permeability tensor, and assume that there exist $\overline{\lambda}_f \geq \underline{\lambda}_f > 0$, such that holds

$$\underline{\lambda}_f|\xi|^2 \leq (\Lambda_f(\mathbf{x})\xi,\xi) \leq \overline{\lambda}_f|\xi|^2 \text{ for all } \xi \in \mathbb{R}^{d-1}, \mathbf{x} \in \Gamma.$$

At the fracture network Γ, we introduce the orthonormal system $(\boldsymbol{\tau}_1(\mathbf{x}), \boldsymbol{\tau}_2(\mathbf{x}), \mathbf{n}(\mathbf{x}))$, defined a.e. on Γ. Inside the fractures, the normal direction is assumed to be a permeability principal direction. The normal permeability $\lambda_{f,\mathbf{n}} \in L^\infty(\Gamma)$ is such that $\underline{\lambda}_{f,\mathbf{n}} \leq \lambda_{f,\mathbf{n}}(\mathbf{x}) \leq \overline{\lambda}_{f,\mathbf{n}}$ for a.e. $\mathbf{x} \in \Gamma$ with $0 < \underline{\lambda}_{f,\mathbf{n}} \leq \overline{\lambda}_{f,\mathbf{n}}$. We also denote by $d_f \in L^\infty(\Gamma)$ the width of the fractures assumed to be such that there exist $\overline{d}_f \geq \underline{d}_f > 0$ with

$$\underline{d}_f \leq d_f(\mathbf{x}) \leq \overline{d}_f$$

for a.e. $\mathbf{x} \in \Gamma$. Let us define the weighted Lebesgue $d-1$ dimensional measure on Γ by $d\tau_f(\mathbf{x}) = d_f(\mathbf{x})d\tau(\mathbf{x})$. We consider the source terms $h_m \in L^2(\Omega)$ (resp. $h_f \in L^2(\Gamma)$) in the matrix domain $\Omega \setminus \overline{\Gamma}$ (resp. in the fracture network Γ). The half normal transmissibility in the fracture network is denoted by $T_f = \frac{2\lambda_{f,\mathbf{n}}}{d_f}$.

The PDEs model writes: find $(u_m, u_f) \in V^0$, $(\mathbf{q}_m, \mathbf{q}_f) \in W$ such that:

$$\left\{\begin{array}{rll} \operatorname{div}(\mathbf{q}_m) &= h_m & \text{on } \Omega \setminus \overline{\Gamma}, \\ \mathbf{q}_m &= -\Lambda_m \nabla u_m & \text{on } \Omega \setminus \overline{\Gamma}, \\ \gamma_{\mathbf{n},\alpha}\mathbf{q}_m &= T_f(\gamma_\alpha u_m - u_f) & \text{on } \Gamma_\alpha,\ \alpha \in \chi \\ \operatorname{div}_\tau(\mathbf{q}_f) - \sum_{\alpha\in\chi} \gamma_{\mathbf{n},\alpha}\mathbf{q}_m &= d_f h_f & \text{on } \Gamma \\ \mathbf{q}_f &= -d_f\ \Lambda_f \nabla_\tau u_f & \text{on } \Gamma, \end{array}\right. \tag{2}$$

Above and in the following, for all $\mathbf{q}_m \in W_m$ and for all $\alpha \in \chi$, we denote again by $\gamma_{\mathbf{n},\alpha}\mathbf{q}_m$ the extension of $\gamma_{\mathbf{n},\alpha}\mathbf{q}_m$ by 0 on whole Γ.

2.2.2 Weak formulation

The hybrid dimensional weak formulation amounts to find $(u_m, u_f) \in V^0$ satisfying the following variational equality for all $(v_m, v_f) \in V^0$:

$$\left\{\begin{array}{l} \displaystyle\int_\Omega \Lambda_m \nabla u_m \cdot \nabla v_m d\mathbf{x} + \int_\Gamma \Lambda_f \nabla_\tau u_f \cdot \nabla_\tau v_f d\tau_f(\mathbf{x}) \\ \displaystyle + \sum_{\alpha\in\chi} \int_{\Gamma_\alpha} T_f\Big((\gamma_\alpha u_m - u_f)(\gamma_\alpha v_m - v_f)\Big) d\tau(\mathbf{x}) \\ \displaystyle - \int_\Omega h_m v_m d\mathbf{x} - \int_\Gamma h_f v_f d\tau_f(\mathbf{x}) = 0. \end{array}\right. \tag{3}$$

Proposition 2.4 *The variational problem* (3) *has a unique solution* $(u_m, u_f) \in V^0$ *(from Lax Milgram Theorem) which satisfies the a priori estimate*

$$|(u_m, u_f)|_V \leq C\Big(\|h_m\|_{L^2(\Omega)} + \|h_f\|_{L^2(\Gamma)}\Big),$$

with C *depending only on* C_P, $\underline{\lambda}_m$, $\underline{\lambda}_f$, $\underline{d}_f$, $\overline{d}_f$, *and* $\underline{\lambda}_{f,n}$. *In addition* $(\mathbf{q}_m, \mathbf{q}_f) = -(\Lambda_m \nabla u_m, d_f \Lambda_f \nabla_\tau u_f)$ *belongs to* W.

3 Gradient Discretization of the Hybrid Dimensional Model

3.1 Gradient Scheme Framework

A gradient discretization $\mathcal{D}$ of hybrid dimensional Darcy flow models is defined by a vector space of degrees of freedom $X_{\mathcal{D}} = X_{\mathcal{D}_m} \times X_{\mathcal{D}_f}$, its subspace satisfying ad hoc homogeneous boundary conditions $X^0_{\mathcal{D}} = X^0_{\mathcal{D}_m} \times X^0_{\mathcal{D}_f}$, and the following gradient and reconstruction operators:

- Gradient operator on the matrix domain: $\nabla_{\mathcal{D}_m} : X_{\mathcal{D}_m} \to L^2(\Omega)^d$
- Gradient operator on the fracture network: $\nabla_{\mathcal{D}_f} : X_{\mathcal{D}_f} \to L^2(\Gamma)^{d-1}$
- A function reconstruction operator on the matrix domain:
  $\Pi_{\mathcal{D}_m} : X_{\mathcal{D}_m} \to L^2(\Omega)$
- Two function reconstruction operators on the fracture network:
  $\Pi_{\mathcal{D}_f} : X_{\mathcal{D}_f} \to L^2(\Gamma)$ and $\widetilde{\Pi}_{\mathcal{D}_f} : X_{\mathcal{D}_f} \to L^2(\Gamma)$
- Reconstruction operators of the trace on Γ_α for $\alpha \in \chi$:
  $\Pi^\alpha_{\mathcal{D}_m} : X_{\mathcal{D}_m} \to L^2(\Gamma_\alpha)$.

The space $X_{\mathcal{D}}$ is endowed with the semi-norm

$$\begin{aligned}\|(v_{\mathcal{D}_m}, v_{\mathcal{D}_f})\|^2_{\mathcal{D}} = \|\nabla_{\mathcal{D}_m} v_{\mathcal{D}_m}\|^2_{L^2(\Omega)^d} + \|\nabla_{\mathcal{D}_f} v_{\mathcal{D}_f}\|^2_{L^2(\Gamma)^{d-1}} \\ + \sum_{\alpha\in\chi} \int_{\Gamma_\alpha} (\Pi^\alpha_{\mathcal{D}_m} v_{\mathcal{D}_m} - \widetilde{\Pi}_{\mathcal{D}_f} v_{\mathcal{D}_f})^2 d\tau(\mathbf{x}),\end{aligned}$$

which is assumed to define a norm on $X^0_{\mathcal{D}}$.

The following properties of gradient discretizations are crucial for the convergence analysis of the corresponding numerical schemes:

Coercivity: Let $\mathcal{D}$ be a gradient discretization and

$$C_{\mathcal{D}} = \max_{0\neq(v_{\mathcal{D}_m}, v_{\mathcal{D}_f})\in X^0_{\mathcal{D}}} \frac{\|\Pi_{\mathcal{D}_m} v_{\mathcal{D}_m}\|_{L^2(\Omega)} + \|\Pi_{\mathcal{D}_f} v_{\mathcal{D}_f}\|_{L^2(\Gamma)}}{\|(v_{\mathcal{D}_m}, v_{\mathcal{D}_f})\|_{\mathcal{D}}}.$$

A sequence $(\mathcal{D}^l)_{l\in\mathbb{N}}$ of gradient discretizations is said to be coercive, if there exists $\overline{C}_P > 0$ such that $C_{\mathcal{D}} \leq \overline{C}_P$ for all $l \in \mathbb{N}$.

Consistency: Let $\mathcal{D}$ be a gradient discretization. For $u = (u_m, u_f) \in V^0$ and $v_{\mathcal{D}} = (v_{\mathcal{D}_m}, v_{\mathcal{D}_f}) \in X^0_{\mathcal{D}}$ let us define

$$\begin{cases} s(v_{\mathcal{D}}, u) = \|\nabla_{\mathcal{D}_m} v_{\mathcal{D}_m} - \nabla u_m\|_{L^2(\Omega)^d} + \|\nabla_{\mathcal{D}_f} v_{\mathcal{D}_f} - \nabla_\tau u_f\|_{L^2(\Gamma)^{d-1}} \\ \qquad + \|\Pi_{\mathcal{D}_m} v_{\mathcal{D}_m} - u_m\|_{L^2(\Omega)} + \|\Pi_{\mathcal{D}_f} v_{\mathcal{D}_f} - u_f\|_{L^2(\Gamma)} \\ \qquad + \|\widetilde{\Pi}_{\mathcal{D}_f} v_{\mathcal{D}_f} - u_f\|_{L^2(\Gamma)} + \sum_{\alpha\in\chi} \|\Pi^\alpha_{\mathcal{D}_m} v_{\mathcal{D}_m} - \gamma^\alpha u_m\|_{L^2(\Gamma_\alpha)}. \end{cases}$$

and $\mathcal{S}_{\mathcal{D}}(u) = \inf_{v_{\mathcal{D}}\in X^0_{\mathcal{D}}} s(v_{\mathcal{D}}, u)$. A sequence $(\mathcal{D}^l)_{l\in\mathbb{N}}$ of gradient discretizations is said to be consistent, if for all $u = (u_m, u_f) \in V^0$ holds

$$\lim_{l\to\infty} \mathcal{S}_{\mathcal{D}^l}(u) = 0.$$

Limit Conformity: Let $\mathcal{D}$ be a gradient discretization. For all $\mathbf{q} = (\mathbf{q}_m, \mathbf{q}_f) \in W$, $v_{\mathcal{D}} = (v_{\mathcal{D}_m}, v_{\mathcal{D}_f})$ we define

$$\begin{cases} w(v_{\mathcal{D}}, \mathbf{q}) = \displaystyle\int_\Omega \Big(\nabla_{\mathcal{D}_m} v_{\mathcal{D}_m} \cdot \mathbf{q}_m + (\Pi_{\mathcal{D}_m} v_{\mathcal{D}_m}) \mathrm{div} \mathbf{q}_m\Big) d\mathbf{x} \\ \qquad + \displaystyle\int_\Gamma \Big(\nabla_{\mathcal{D}_f} v_{\mathcal{D}_f} \cdot \mathbf{q}_f + (\Pi_{\mathcal{D}_f} v_{\mathcal{D}_f}) \mathrm{div}_\tau \mathbf{q}_f\Big) d\tau(\mathbf{x}) \\ \qquad + \displaystyle\sum_{\alpha \in \chi} \int_{\Gamma_\alpha} \gamma_{\mathbf{n},\alpha} \mathbf{q}_m \Big(\widetilde{\Pi}_{\mathcal{D}_f} v_{\mathcal{D}_f} - \Pi_{\mathcal{D}_f} v_{\mathcal{D}_f} - \Pi^\alpha_{\mathcal{D}_m} v_{\mathcal{D}_m}\Big) d\tau(\mathbf{x}) \end{cases}$$

and $\mathcal{W}_{\mathcal{D}}(\mathbf{q}) = \sup_{0 \neq v_{\mathcal{D}} \in X^0_{\mathcal{D}}} \frac{1}{\|v_{\mathcal{D}}\|_{\mathcal{D}}} |w(v_{\mathcal{D}}, \mathbf{q})|$.
A sequence $(\mathcal{D}^l)_{l \in \mathbb{N}}$ of gradient discretizations is said to be limit conforming, if for all $\mathbf{q} = (\mathbf{q}_m, \mathbf{q}_f) \in W$ holds

$$\lim_{l \to \infty} \mathcal{W}_{\mathcal{D}^l}(\mathbf{q}) = 0.$$

Proposition 3.1 (Regularity at the Limit) *Let $(\mathcal{D}^l)_{l \in \mathbb{N}}$ be a coercive and limit conforming sequence of gradient discretizations and let $(v_{\mathcal{D}^l_m}, v_{\mathcal{D}^l_f})_{l \in \mathbb{N}}$ be a uniformly bounded sequence in $X^0_{\mathcal{D}^l}$. Then, there exists $(v_m, v_f) \in V^0$ and a subsequence still denoted by $(v_{\mathcal{D}^l_m}, v_{\mathcal{D}^l_f})_{l \in \mathbb{N}}$ such that*

$$\begin{cases} \Pi_{\mathcal{D}^l_m} v_{\mathcal{D}^l_m} \rightharpoonup v_m \quad \text{in } L^2(\Omega), \\ \nabla_{\mathcal{D}_m} v_{\mathcal{D}^l_m} \rightharpoonup \nabla v_m \quad \text{in } L^2(\Omega)^d, \\ \Pi_{\mathcal{D}^l_f} v_{\mathcal{D}^l_f} \rightharpoonup v_f \quad \text{in } L^2(\Gamma), \\ \nabla_{\mathcal{D}_f} v_{\mathcal{D}^l_f} \rightharpoonup \nabla_\tau v_f \quad \text{in } L^2(\Gamma)^{d-1}, \\ \widetilde{\Pi}_{\mathcal{D}_f} v_{\mathcal{D}^l_f} - \Pi^\alpha_{\mathcal{D}_m} v_{\mathcal{D}^l_m} \rightharpoonup v_f - \gamma^\alpha v_m \quad \text{in } L^2(\Gamma_\alpha), \text{ for all } \alpha \in \chi. \end{cases}$$

3.2 Application to (3)

The non conforming discrete variational formulation of the model problem is defined by: find $(u_{\mathcal{D}_m}, u_{\mathcal{D}_f}) \in X^0_{\mathcal{D}}$ such that

$$\begin{cases} \displaystyle\int_\Omega \Lambda_m \nabla_{\mathcal{D}_m} u_{\mathcal{D}_m} \cdot \nabla_{\mathcal{D}_m} v_{\mathcal{D}_m} d\mathbf{x} + \int_\Gamma \Lambda_f \nabla_{\mathcal{D}_f} u_{\mathcal{D}_f} \cdot \nabla_{\mathcal{D}_f} v_{\mathcal{D}_f} d\tau_f(\mathbf{x}) \\ \displaystyle + \sum_{\alpha \in \chi} \int_{\Gamma_\alpha} T_f (\Pi^\alpha_{\mathcal{D}_m} u_{\mathcal{D}_m} - \widetilde{\Pi}_{\mathcal{D}_f} u_{\mathcal{D}_f})(\Pi^\alpha_{\mathcal{D}_m} v_{\mathcal{D}_m} - \widetilde{\Pi}_{\mathcal{D}_f} v_{\mathcal{D}_f}) d\tau(\mathbf{x}) \\ \displaystyle - \int_\Omega h_m \Pi_{\mathcal{D}_m} v_{\mathcal{D}_m} d\mathbf{x} - \int_\Gamma h_f \Pi_{\mathcal{D}_f} v_{\mathcal{D}_f} d\tau_f(\mathbf{x}) = 0, \end{cases} \tag{4}$$

for all $(v_{\mathcal{D}_m}, v_{\mathcal{D}_f}) \in X^0_{\mathcal{D}}$.

Proposition 3.2 *Let $\mathcal{D}$ be a gradient discretization, then (4) has a unique solution $(u_{\mathcal{D}_m}, u_{\mathcal{D}_f}) \in X^0_{\mathcal{D}}$ satisfying the a priori estimate*

$$\|(u_{\mathcal{D}_m}, u_{\mathcal{D}_f})\|_{\mathcal{D}} \leq C\Big(\|h_m\|_{L^2(\Omega)} + \|h_f\|_{L^2(\Gamma)}\Big)$$

with C depending only on $C_{\mathcal{D}}$, $\underline{\lambda}_m$, $\underline{\lambda}_f$, $\underline{d}_f$, $\overline{d}_f$, and $\underline{\lambda}_{f,n}$.

The main theoretical result for gradient schemes is stated by the following Proposition:

Proposition 3.3 (Error Estimate) *Let $(u_m, u_f) \in V^0$, $(\mathbf{q}_m, \mathbf{q}_f) \in W$ the solution of (2). Let $\mathcal{D}$ be a gradient discretization and $(u_{\mathcal{D}_m}, u_{\mathcal{D}_f}) \in X^0_{\mathcal{D}}$ be the solution of (4). Then, there exists $C > 0$ depending only on $C_{\mathcal{D}}$, $\underline{\lambda}_m$, $\underline{\lambda}_f$,$\overline{\lambda}_m$, $\overline{\lambda}_f$, $\underline{d}_f$, $\overline{d}_f$, $\underline{\lambda}_{f,n}$, and $\overline{\lambda}_{f,n}$ such that one has the following error estimate:*

$$\left\{\begin{array}{l} \|\Pi_{\mathcal{D}_m} u_{\mathcal{D}_m} - u_m\|_{L^2(\Omega)} + \|\Pi_{\mathcal{D}_f} u_{\mathcal{D}_f} - u_f\|_{L^2(\Gamma)} \\ +\|\widetilde{\Pi}_{\mathcal{D}_f} u_{\mathcal{D}_f} - u_f\|_{L^2(\Gamma)} + \sum_{\alpha\in\chi} \|\Pi^{\alpha}_{\mathcal{D}_m} u_{\mathcal{D}_m} - \gamma_\alpha u_m\|_{L^2(\Gamma_\alpha)} \\ +\|\nabla u_m - \nabla_{\mathcal{D}_m} u_{\mathcal{D}_m}\|_{L^2(\Omega)^d} + \|\nabla_\tau u_f - \nabla_{\mathcal{D}_f} u_{\mathcal{D}_f}\|_{L^2(\Gamma)^{d-1}} \\ \leq C(\mathcal{S}_{\mathcal{D}}(u_m, u_f) + \mathcal{W}_{\mathcal{D}}(\mathbf{q}_m, \mathbf{q}_f)). \end{array}\right.$$

4 Two Examples of Gradient Schemes

Following [5], we consider generalised polyhedral meshes of Ω. Let $\mathcal{M}$ be the set of cells that are disjoint open subsets of Ω such that $\bigcup_{K\in\mathcal{M}} \overline{K} = \overline{\Omega}$. For all $K \in \mathcal{M}$, $\mathbf{x}_K$ denotes the so-called "center" of the cell K under the assumption that K is star-shaped with respect to $\mathbf{x}_K$. Let $\mathcal{F}$ denote the set of faces of the mesh which are not assumed to be planar, hence the term "generalised polyhedral cells". We denote by $\mathcal{V}$ the set of vertices of the mesh. Let $\mathcal{V}_K$, $\mathcal{F}_K$, $\mathcal{V}_\sigma$ respectively denote the set of the vertices of $K \in \mathcal{M}$, faces of K, and vertices of $\sigma \in \mathcal{F}$. For any face $\sigma \in \mathcal{F}_K$, we have $\mathcal{V}_\sigma \subset \mathcal{V}_K$. Let $\mathcal{M}_s$ (resp. $\mathcal{F}_s$) denote the set of the cells (resp. faces) sharing the vertex $s \in \mathcal{V}$. The set of edges of the mesh is denoted by $\mathcal{E}$ and $\mathcal{E}_\sigma$ denotes the set of edges of the face $\sigma \in \mathcal{F}$. Let $\mathcal{F}_e$ denote the set of faces sharing the edge $e \in \mathcal{E}$, and $\mathcal{M}_\sigma$ denote the set of cells sharing the face $\sigma \in \mathcal{F}$. We denote by $\mathcal{F}_{ext}$ the subset of faces $\sigma \in \mathcal{F}$ such that $\mathcal{M}_\sigma$ has only one element, and we set $\mathcal{E}_{ext} = \bigcup_{\sigma\in\mathcal{F}_{ext}} \mathcal{E}_\sigma$, and $\mathcal{V}_{ext} = \bigcup_{\sigma\in\mathcal{F}_{ext}} \mathcal{V}_\sigma$. It is assumed that for each face $\sigma \in \mathcal{F}$, there exists a so-called "center" of the face $\mathbf{x}_\sigma$ such that

$$\mathbf{x}_\sigma = \sum_{s\in\mathcal{V}_\sigma} \beta_{\sigma,s}\, \mathbf{x}_s, \text{ with } \sum_{s\in\mathcal{V}_\sigma} \beta_{\sigma,s} = 1,$$

where $\beta_{\sigma,s} \geq 0$ for all $s \in \mathcal{V}_\sigma$. The face σ is assumed to match with the union of the triangles $T_{\sigma,e}$ defined by the face center $\mathbf{x}_\sigma$ and each of its edge $e \in \mathcal{E}_\sigma$.

The mesh is assumed to be conforming w.r.t. the fracture network Γ in the sense that there exist subsets $\mathcal{F}_{\Gamma_i}$, $i \in I$ of $\mathcal{F}$ such that

$$\overline{\Gamma}_i = \bigcup_{\sigma\in\mathcal{F}_{\Gamma_i}} \bar{\sigma}. \tag{5}$$

We will denote by $\mathcal{F}_\Gamma$ the set of fracture faces $\bigcup_{i\in I} \mathcal{F}_{\Gamma_i}$.
Similarly, we will denote by $\mathcal{E}_\Gamma$ the set of fracture edges $\bigcup_{\sigma\in\mathcal{F}_\Gamma} \mathcal{E}_\sigma$ and by $\mathcal{V}_\Gamma$ the set of fracture vertices $\bigcup_{\sigma\in\mathcal{F}_\Gamma} \mathcal{V}_\sigma$.

We also define a submesh $\mathcal{T}$ of tetrahedra, where each tetrahedron $D_{K,\sigma,e}$ is the convex hull of the cell center $\mathbf{x}_K$ of K, the face center $\mathbf{x}_\sigma$ of $\sigma \in \mathcal{F}_K$ and the edge $e \in \mathcal{E}_\sigma$. Similarly we define a triangulation Δ of Γ, such that we have:

$$\mathcal{T} = \bigcup_{K\in\mathcal{F},\sigma\in\mathcal{F}_K,e\in\mathcal{E}_\sigma} D_{K,\sigma,e} \quad \text{and} \quad \Delta = \bigcup_{\sigma\in\mathcal{F}_\Gamma,e\in\mathcal{E}_\sigma} T_{\sigma,e}.$$

We introduce for $D \in \mathcal{T}$ the diameter h_D of D and set $h_{\mathcal{T}} = \max_{D\in\mathcal{T}} h_D$. The regularity of our polyhedral mesh will be measured by the shape regularity of the tetrahedral submesh defined

by $\theta_{\mathcal{T}} = \max_{D\in\mathcal{T}} \frac{h_D}{\rho_D}$ where ρ_D is the insphere diameter of $D \in \mathcal{T}$.

The set of matrix $\times$ fracture degrees of freedom is denoted by $dof_{\mathcal{D}_m} \times dof_{\mathcal{D}_f}$. The real vector spaces $X_{\mathcal{D}_m}$ and $X_{\mathcal{D}_f}$ of discrete unknowns in the matrix and in the fracture network respectively are then defined by

$$\begin{aligned} X_{\mathcal{D}_m} &= \mathrm{span}\{\mathfrak{e}_\nu \mid \nu \in dof_{\mathcal{D}_m}\} \\ X_{\mathcal{D}_f} &= \mathrm{span}\{\mathfrak{e}_\nu \mid \nu \in dof_{\mathcal{D}_f}\}, \end{aligned}$$

where

$$\mathfrak{e}_\nu = \begin{cases} (\delta_{\nu\mu})_{\mu\in dof_{\mathcal{D}_m}} & \text{for } \nu \in dof_{\mathcal{D}_m} \\ (\delta_{\nu\mu})_{\mu\in dof_{\mathcal{D}_f}} & \text{for } \nu \in dof_{\mathcal{D}_f}. \end{cases}$$

For $u_{\mathcal{D}_m} \in X_{\mathcal{D}_m}$ and $\nu \in dof_{\mathcal{D}_m}$ we denote by u_ν the νth compoment of $u_{\mathcal{D}_m}$ and likewise for $u_{\mathcal{D}_f} \in X_{\mathcal{D}_f}$ and $\nu \in dof_{\mathcal{D}_f}$.
We also introduce the direct product of these vector spaces

$$X_{\mathcal{D}} = X_{\mathcal{D}_m} \times X_{\mathcal{D}_f},$$

for which we have, by construction, $\dim X_{\mathcal{D}} = \#dof_{\mathcal{D}_m} + \#dof_{\mathcal{D}_f}$.

To account for our homogeneous boundary conditions on $\partial\Omega$ and Σ_0 we introduce the subsets $dof_{Dir_m} \subset dof_{\mathcal{D}_m}$, and $dof_{Dir_f} \subset dof_{\mathcal{D}_f}$, and we set $dof_{Dir} = dof_{Dir_m} \times dof_{Dir_f}$, and

$$X^0_{\mathcal{D}} = \{u \in X_D \,|\, u_\nu = 0 \text{ for all } \nu \in dof_{Dir}\}.$$

4.1 Vertex Approximate Gradient Discretization

We first establish an equivalence relation on each $\mathcal{M}_s,\ s \in \mathcal{V}$, by

$$\begin{aligned} K \equiv_{\mathcal{M}_s} L \quad \Longleftrightarrow \quad & \text{there exists } n \in \mathbb{N} \text{ and a sequence } (\sigma_i)_{i=1,\dots,n} \text{ in } \mathcal{F}_s\backslash\mathcal{F}_\Gamma, \\ & \text{such that } K \in \mathcal{M}_{\sigma_1}, L \in \mathcal{M}_{\sigma_n} \text{ and } \mathcal{M}_{\sigma_{i+1}} \cap \mathcal{M}_{\sigma_i} \neq \emptyset \\ & \text{for } i = 1, \dots, n-1. \end{aligned}$$

Let us then denote by $\overline{\mathcal{M}}_s$ the set of all classes of equivalence of $\mathcal{M}_s$ and by $\overline{K}_s$ the element of $\overline{\mathcal{M}}_s$ containing $K \in \mathcal{M}$. Obviously $\overline{\mathcal{M}}_s$ might have more than one element only if $s \in \mathcal{V}_\Gamma$. Then we define

$$\begin{aligned} dof_{\mathcal{D}_m} &= \mathcal{M} \cup \Big\{K_\sigma \mid \sigma \in \mathcal{F}_\Gamma, K \in \mathcal{M}_\sigma\Big\} \cup \Big\{\overline{K}_s \mid s \in \mathcal{V}, \overline{K}_s \in \overline{\mathcal{M}}_s\Big\}, \\ dof_{\mathcal{D}_f} &= \mathcal{F}_\Gamma \cup \mathcal{V}_\Gamma, \\ dof_{Dir_m} &:= \Big\{\overline{K}_s \mid s \in \mathcal{V}_{ext}, \overline{K}_s \in \overline{\mathcal{M}}_s\Big\}, \\ dof_{Dir_f} &= \mathcal{V}_\Gamma \cap \mathcal{V}_{ext}. \end{aligned}$$

We thus have

$$\begin{aligned} X_{\mathcal{D}_m} &= \Big\{u_K \mid K \in \mathcal{M}\Big\} \cup \Big\{u_{K_\sigma} \mid \sigma \in \mathcal{F}_\Gamma, K \in \mathcal{M}_\sigma\Big\} \\ &\quad \cup \Big\{u_{\overline{K}_s} \mid s \in \mathcal{V}, \overline{K}_s \in \overline{\mathcal{M}}_s\Big\}, \\ X_{\mathcal{D}_f} &= \Big\{u_\sigma \mid \sigma \in \mathcal{F}_\Gamma\Big\} \cup \Big\{u_s \mid s \in \mathcal{V}_\Gamma\Big\}. \end{aligned} \tag{6}$$

Now we can introduce the piecewise affine interpolators (or reconstruction operators)

$$\Pi_{\mathcal{T}} \colon X_{\mathcal{D}_m} \longrightarrow H^1(\Omega \backslash \overline{\Gamma}) \qquad \text{and} \qquad \Pi_{\Delta} \colon X_{\mathcal{D}_f} \longrightarrow H^1(\Gamma),$$

which act linearly on $X_{\mathcal{D}_m}$ and $X_{\mathcal{D}_f}$, such that $\Pi_{\Delta} u_{\mathcal{D}_f}$ is affine on each $T_{\sigma,e} \in \Delta$ and satisfies on each cell $K \in \mathcal{M}$

$$\begin{array}{ll} \Pi_{\mathcal{T}} u_{\mathcal{D}_m}(\mathbf{x}_K) = u_K, & \\ \Pi_{\mathcal{T}} u_{\mathcal{D}_m}(\mathbf{x}_s) = u_{\overline{K}_s} & \forall s \in \mathcal{V}_K, \\ \Pi_{\mathcal{T}} u_{\mathcal{D}_m}(\mathbf{x}_\sigma) = u_{K_\sigma} & \forall \sigma \in \mathcal{F}_K \cap \mathcal{F}_\Gamma, \\ \Pi_{\mathcal{T}} u_{\mathcal{D}_m}(\mathbf{x}_\sigma) = \sum\limits_{s \in \mathcal{V}_\sigma} \beta_{\sigma,s} u_{\overline{K}_s} & \forall \sigma \in \mathcal{F}_K \backslash \mathcal{F}_\Gamma, \end{array}$$

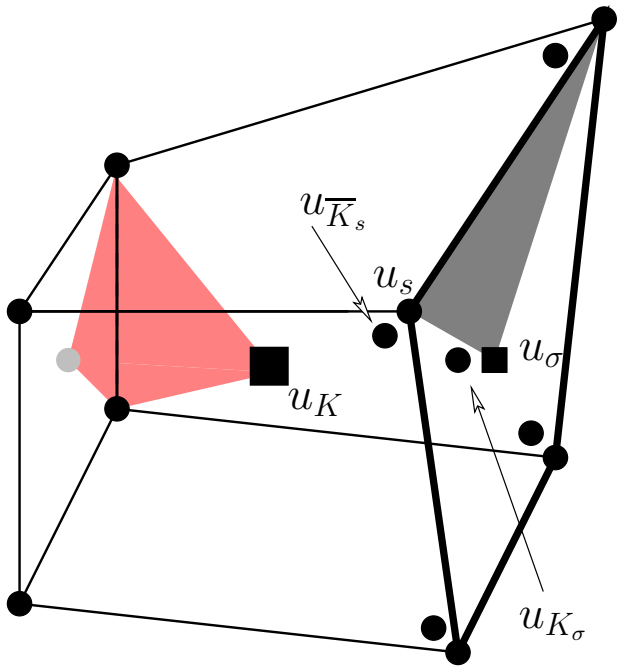
while $\Pi_{\mathcal{T}} u_{\mathcal{D}_m}$ is affine on each $D_{K,\sigma,e} \in \mathcal{T}$ and satisfies for all $\nu \in dof_{\mathcal{D}_f}$

$$\Pi_{\Delta} u_{\mathcal{D}_f}(\mathbf{x}_\nu) = u_\nu,$$

where $\mathbf{x}_\nu \in \overline{\Omega}$ is the grid point associated with the degree of freedom $\nu \in dof_{\mathcal{D}_m} \cup dof_{\mathcal{D}_f}$. The discrete gradients on $X_{\mathcal{D}_m}$ and $X_{\mathcal{D}_f}$ are subsequently defined by

$$\nabla_{\mathcal{D}_m} = \nabla \Pi_{\mathcal{T}} \qquad \text{and} \qquad \nabla_{\mathcal{D}_f} = \nabla_\tau \Pi_{\Delta}. \tag{7}$$

Cell touching a fracture face. Illustration of the simplices on which:
Red: $\nabla_{\mathcal{D}_m}$ is constant.
Grey: $\nabla_{\mathcal{D}_f}$ is constant.



We define the *VAG-FE* scheme's reconstruction operators by

$$\begin{array}{l} \bullet\ \Pi_{\mathcal{D}_m} = \Pi_{\mathcal{T}}, \\ \bullet\ \Pi_{\mathcal{D}_f} = \widetilde{\Pi}_{\mathcal{D}_f} = \Pi_{\Delta}, \\ \bullet\ \Pi^{\alpha}_{\mathcal{D}_m} = \gamma_\alpha \Pi_{\mathcal{T}} \quad \text{for all } \alpha \in \chi. \end{array} \tag{8}$$

For the family of *VAG-CV* schemes, reconstruction operators are constant by volumes. We introduce, for any given $K \in \mathcal{M}$, a partition

$$\overline{K} = \overline{\omega}_K \cup \Big(\bigcup_{s \in \mathcal{V}_K \backslash \mathcal{V}_{ext}} \overline{\omega}_{K,\overline{K}_s} \Big) \cup \Big(\bigcup_{\sigma \in \mathcal{F}_K \cap \mathcal{F}_\Gamma} \overline{\omega}_{K_\sigma} \Big).$$

Similarly, we define for any given $\sigma \in \mathcal{F}_\Gamma$ a partition

$$\overline{\sigma} = \overline{\omega}_\sigma \cup \Big(\bigcup_{s \in \mathcal{V}_\sigma \setminus \mathcal{V}_{ext}} \overline{\omega}_{\sigma,s} \Big).$$

With each $s \in \mathcal{V} \setminus \mathcal{V}_{ext}$ and $\overline{K}_s \in \overline{\mathcal{M}}_s$ we associate an open set $\omega_{\overline{K}_s}$, satisfying

$$\overline{\omega}_{\overline{K}_s} = \bigcup_{K \in \overline{K}_s} \overline{\omega}_{K,\overline{K}_s}.$$

Similarly, for all $s \in \mathcal{V}_\Gamma \setminus \mathcal{V}_{ext}$ we define ω_s by

$$\overline{\omega}_s = \bigcup_{\sigma \in \mathcal{F}_s \cap \mathcal{F}_\Gamma} \overline{\omega}_{\sigma,s}.$$

We obtain the partition

$$\overline{\Omega} = \Big(\bigcup_{\nu \in dof_{\mathcal{D}_m} \setminus dof_{Dir_m}} \overline{\omega}_\nu \Big) \cup \Big(\bigcup_{\nu \in dof_{\mathcal{D}_f} \setminus dof_{Dir_f}} \overline{\omega}_\nu \Big).$$

We also introduce for each $T = T_{\sigma,s,s'} \in \Delta$ a partition $\overline{T} = \bigcup_{i=1}^3 \overline{T}_i$, which we need for the definition of the *VAG-CV* matrix-fracture interaction operators. We assume that holds $|T_1| = |T_2| = |T_3| = \frac{1}{3}|T|$ in order to preserve the first order convergence of the scheme.

Finally, we need a correlation between the degrees of freedom of the matrix domain, which are situated on one side of the fracture network, and the set of indices χ. For $K_\sigma \in dof_{\mathcal{D}_m}$ we have the one-element set $\chi(K_\sigma) = \{\alpha \in \chi \mid \mathbf{n}_{K,\sigma} = \mathbf{n}_\alpha \text{ on } \sigma\}$ and therefore the notation $\alpha(K_\sigma) = \alpha \in \chi(K_\sigma)$.

The *VAG-CV* scheme's reconstruction operators are

$$\begin{aligned}
&\bullet\ \Pi_{\mathcal{D}_m} u_{\mathcal{D}_m} = \sum_{\nu \in dof_{\mathcal{D}_m} \setminus dof_{Dir_m}} u_\nu \mathbb{1}_{\omega_\nu}, \\
&\bullet\ \Pi_{\mathcal{D}_f} u_{\mathcal{D}_f} = \sum_{\nu \in dof_{\mathcal{D}_f} \setminus dof_{Dir_f}} u_\nu \mathbb{1}_{\omega_\nu}, \\
&\bullet\ \widetilde{\Pi}_{\mathcal{D}_f} u_{\mathcal{D}_f} = \sum_{T_{\sigma,s,s'} \in \Delta} (u_\sigma \mathbb{1}_{T_1} + u_s \mathbb{1}_{T_2} + u_{s'} \mathbb{1}_{T_3}), \\
&\bullet\ \Pi^\alpha_{\mathcal{D}_m} u_{\mathcal{D}_m} = \sum_{T_{\sigma,s,s'} \in \Delta} \sum_{K \in \mathcal{M}_\sigma} (u_{K_\sigma} \mathbb{1}_{T_1} + u_{\overline{K}_s} \mathbb{1}_{T_2} + u_{\overline{K}_{s'}} \mathbb{1}_{T_3}) \delta_{\alpha(K_\sigma)\alpha} \mathbb{1}_{\Gamma_\alpha}.
\end{aligned} \tag{9}$$

Remark 4.1 *The* VAG-CV *scheme leads us to recover two-point fluxes for the matrix-fracture interactions.*

Proposition 4.1 *Let us consider a sequence of meshes $(\mathcal{M}^l)_{l \in \mathbb{N}}$ and let us assume that the sequence $(\mathcal{T}^l)_{l \in \mathbb{N}}$ of tetrahedral submeshes is shape regular, i.e. $\theta_{\mathcal{T}^l}$ is uniformly bounded. We also assume that $\lim_{l \to \infty} h_{\mathcal{T}^l} = 0$. Then, the corresponding sequence of gradient discretizations $(\mathcal{D}^l)_{l \in \mathbb{N}}$, defined by* (6), (7), (8), *is coercive, consistent and limit conforming.*

Proposition 4.2 *Let us consider a sequence of meshes $(\mathcal{M}^l)_{l \in \mathbb{N}}$ and let us assume that the sequence $(\mathcal{T}^l)_{l \in \mathbb{N}}$ of tetrahedral submeshes is shape regular, i.e. $\theta_{\mathcal{T}^l}$ is uniformly bounded. We also assume that $\lim_{l \to \infty} h_{\mathcal{T}^l} = 0$. Then, any corresponding sequence of gradient discretizations $(\mathcal{D}^l)_{l \in \mathbb{N}}$, defined by* (6), (7), (9), *is coercive, consistent and limit conforming.*

Remark 4.2 *It can be shown that for solutions* $(u_m, u_f) \in V^0$ *and* $(\mathbf{q}_m, \mathbf{q}_f) \in W$ *of* (2) *such that* $u_m \in C^2(\overline{K})$, $u_f \in C^2(\overline{\sigma})$, $\mathbf{q}_m \in (C^1(\overline{K}))^d$, $\mathbf{q}_f \in (C^1(\overline{\sigma}))^{d-1}$ *for all* $K \in \mathcal{M}$ *and all* $\sigma \in \Gamma_f$, *the* VAG *schemes are consistent and limit conforming of order 1, and therefore convergent of order 1.*

4.2 Hybrid Finite Volume Discretization

We assume here that the faces are planar and that $\mathbf{x}_\sigma$ is the barycenter of σ for all $\sigma \in \mathcal{F}$.

The set of indices $dof_{\mathcal{D}_m} \times dof_{\mathcal{D}_f}$ for the unknowns is defined by

$$\begin{aligned} dof_{\mathcal{D}_m} &= \mathcal{M} \cup \Big(\bigcup_{\sigma \in \mathcal{F}} \overline{\mathcal{M}}_\sigma \Big) \\ dof_{\mathcal{D}_f} &= \mathcal{F}_\Gamma \cup \mathcal{E}_\Gamma, \\ dof_{Dir_m} &= \mathcal{F}_{ext}, \\ dof_{Dir_f} &= \mathcal{E}_\Gamma \cap \mathcal{E}_{ext}, \end{aligned}$$

where for $\sigma \in \mathcal{F}$ and $K \in \mathcal{M}_\sigma$

$$\overline{K}_\sigma = \begin{cases} \mathcal{M}_\sigma & \text{if } \sigma \in \mathcal{F} \setminus \mathcal{F}_\Gamma \\ \{K\} & \text{if } \sigma \in \mathcal{F}_\Gamma. \end{cases}$$

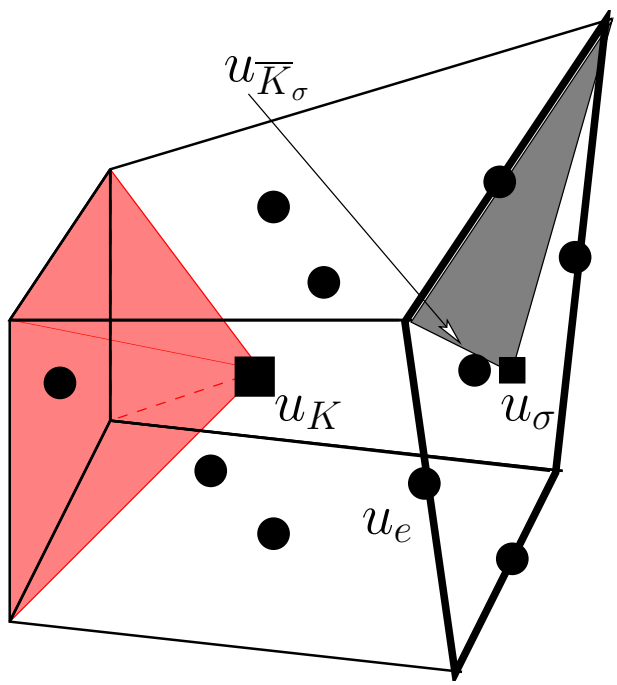
and $\overline{\mathcal{M}}_\sigma = \{\overline{K}_\sigma \mid K \in \mathcal{M}_\sigma\}$. We thus have

$$\begin{aligned} X_{\mathcal{D}_m} &= \Big\{ u_K \mid K \in \mathcal{M} \Big\} \cup \Big\{ u_{\overline{K}_\sigma} \mid \sigma \in \mathcal{F}_\Gamma, \overline{K}_\sigma \in \overline{\mathcal{M}}_\sigma \Big\}, \\ X_{\mathcal{D}_f} &= \Big\{ u_\sigma \mid \sigma \in \mathcal{F}_\Gamma \Big\} \cup \Big\{ u_e \mid e \in \mathcal{E}_\Gamma \Big\}. \end{aligned} \tag{10}$$

The discrete gradients in the matrix (respectively in the fracture domain) are defined in each cell (respectively in each face) by the 3D (respectively 2D) discrete gradients

$$\nabla_{\mathcal{D}_m} \text{ (resp. } \nabla_{\mathcal{D}_f}) \text{ as proposed in [6], pp. 8-9.} \tag{11}$$

The function reconstruction operators are piecewise constant on a partition of the cells and of the fracture faces.



Cell touching a fracture face. Illustration of the simplices on which:
Red: $\nabla_{\mathcal{D}_m}$ is constant.
Grey: $\nabla_{\mathcal{D}_f}$ is constant.

These partitions are respectively denoted, for all $K \in \mathcal{M}$, by

$$\overline{K} \;=\; \overline{\omega}_K \;\cup\; \Big(\bigcup_{\sigma \in \mathcal{F}_K \setminus \mathcal{F}_{ext}} \overline{\omega}_{K,\overline{K}_\sigma} \Big),$$

and, for all $\sigma \in \mathcal{F}_\Gamma$, by

$$\overline{\sigma} \;=\; \overline{\omega}_\sigma \;\cup\; \Big(\bigcup_{e \in \mathcal{E}_\sigma \setminus \mathcal{E}_{ext}} \overline{\omega}_{\sigma,e} \Big).$$

With each $\sigma \in \mathcal{F} \setminus \mathcal{F}_{ext}$ and $\overline{K}_\sigma \in \overline{\mathcal{M}}_\sigma$ we associate an open set $\omega_{\overline{K}_\sigma}$, s.t.

$$\overline{\omega}_{\overline{K}_\sigma} = \bigcup_{K \in \overline{K}_\sigma} \overline{\omega}_{K,\overline{K}_\sigma}.$$

Similarly, for all $e \in \mathcal{E}_\Gamma \setminus \mathcal{E}_{ext}$ we define ω_e by

$$\overline{\omega}_e = \bigcup_{\sigma \in \mathcal{F}_e \cap \mathcal{F}_\Gamma} \overline{\omega}_{\sigma,e}.$$

We obtain the partition $\overline{\Omega} = \Big(\bigcup_{\nu \in dof_{\mathcal{D}_m} \setminus dof_{Dir_m}} \overline{\omega}_\nu \Big) \cup \Big(\bigcup_{\nu \in dof_{\mathcal{D}_f} \setminus dof_{Dir_f}} \overline{\omega}_\nu \Big)$.

We also need a correlation between the degrees of freedom of the matrix domain, which are situated on one side of the fracture network, and the set of indices χ. For $\sigma \in \mathcal{F}_\Gamma$ and $\overline{K}_\sigma \in \overline{\mathcal{M}}_\sigma$ holds by definition $\overline{K}_\sigma = \{K\}$ for a $K \in \mathcal{M}_\sigma$ and hence $\mathbf{n}_{\overline{K}_\sigma} = \mathbf{n}_{K,\sigma}$ is well defined. We obtain the one-element set $\chi(\overline{K}_\sigma) = \{\alpha \in \chi \mid \mathbf{n}_{\overline{K}_\sigma} = \mathbf{n}_\alpha \text{ on } \sigma\}$ and therefore the notation $\alpha(\overline{K}_\sigma) = \alpha \in \chi(\overline{K}_\sigma)$.

We define the *HFV* scheme's reconstruction operators by

$$\begin{aligned}
&\bullet\ \Pi_{\mathcal{D}_m} u_{\mathcal{D}_m} = \sum_{\nu \in dof_{\mathcal{D}_m} \setminus dof_{Dir_m}} u_\nu \mathbb{1}_{\omega_\nu}, \\
&\bullet\ \Pi_{\mathcal{D}_f} u_{\mathcal{D}_f} = \sum_{\nu \in dof_{\mathcal{D}_f} \setminus dof_{Dir_f}} u_\nu \mathbb{1}_{\omega_\nu}, \\
&\bullet\ \widetilde{\Pi}_{\mathcal{D}_f} u_{\mathcal{D}_f} = \sum_{\sigma \in \mathcal{F}_\Gamma} u_\sigma \mathbb{1}_\sigma, \\
&\bullet\ \Pi^\alpha_{\mathcal{D}_m} u_{\mathcal{D}_m} = \sum_{\sigma \in \mathcal{F}_\Gamma} \sum_{\overline{K}_\sigma \in \overline{\mathcal{M}}_\sigma} \delta_{\alpha(\overline{K}_\sigma)\alpha} u_{\overline{K}_\sigma} \mathbb{1}_\sigma \quad \text{for all } \alpha \in \chi.
\end{aligned} \tag{12}$$

Proposition 4.3 *Let us consider a sequence of meshes $(\mathcal{M}^l)_{l\in\mathbb{N}}$ and let us assume that the sequence $(\mathcal{T}^l)_{l\in\mathbb{N}}$ of tetrahedral submeshes is shape regular, i.e. $\theta_{\mathcal{T}^l}$ is uniformly bounded. We also assume that $\lim_{l\to\infty} h_{\mathcal{T}^l} = 0$. Then, any corresponding sequence of gradient discretizations $(\mathcal{D}^l)_{l\in\mathbb{N}}$, defined by* (10), (11) *and definition* (12)*, is coercive, consistent and limit conforming.*

Remark 4.3 *It can be shown that for solutions $(u_m, u_f) \in V^0$ and $(\mathbf{q}_m, \mathbf{q}_f) \in W$ of* (2) *such that $u_m \in C^2(\overline{K})$, $u_f \in C^2(\overline{\sigma})$, $\mathbf{q}_m \in (C^1(\overline{K}))^d$, $\mathbf{q}_f \in (C^1(\overline{\sigma}))^{d-1}$ for all $K \in \mathcal{M}$ and all $\sigma \in \Gamma_f$, the* HFV *schemes are consistent and limit conforming of order 1, and therefore convergent of order 1.*

4.3 Finite Volume Formulation for *VAG* and *HFV* Schemes

For $K \in \mathcal{M}$ let

$$dof_K = \begin{cases} \{\overline{K}_s, s \in \mathcal{V}_K\} \cup \{K_\sigma, \sigma \in \mathcal{F}_K \cap \mathcal{F}_\Gamma\} \text{ for } VAG, \\ \{\overline{K}_\sigma, \sigma \in \mathcal{F}_K\} \text{ for } HFV. \end{cases}$$

Analogously, in the fracture domain, for $\sigma \in \mathcal{F}_\Gamma$ let

$$dof_\sigma = \begin{cases} \mathcal{V}_\sigma \text{ for } VAG, \\ \mathcal{E}_\sigma \text{ for } HFV. \end{cases}$$

The transmissivities $(T_K^{\nu\nu'})_{\nu,\nu' \in dof_K}$ in the matrix domain are then for each $K \in \mathcal{M}$ real $\#dof_K \times \#dof_K$ SPD-tensors, defined by

$$T_K^{\nu\nu'} = \int_K \Lambda_m \nabla_{\mathcal{D}_m} \mathfrak{e}_\nu \nabla_{\mathcal{D}_m} \mathfrak{e}_{\nu'} d\mathbf{x}.$$

Then, for any $\nu \in dof_K$ the discrete *matrix-matrix*-fluxes are defined as

$$F_{K\nu}(u_{\mathcal{D}_m}) = \sum_{\nu' \in dof_K} T_K^{\nu\nu'}(u_K - u_{\nu'}).$$

In the fracture network, the transmissivities $(T_\sigma^{\nu\nu'})_{\nu,\nu' \in dof_\sigma}$ are for each $\sigma \in \mathcal{F}_\Gamma$ real $\#dof_\sigma \times \#dof_\sigma$ SPD-tensors, defined by

$$T_\sigma^{\nu\nu'} = \int_\sigma \Lambda_f \nabla_{\mathcal{D}_f} \mathfrak{e}_\nu \nabla_{\mathcal{D}_f} \mathfrak{e}_{\nu'} d\tau_f(\mathbf{x})$$

and for any $\nu \in dof_\sigma$ the discrete *fracture-fracture*-fluxes are defined as

$$F_{\sigma\nu}(u_{\mathcal{D}_f}) = \sum_{\nu' \in dof_\sigma} T_\sigma^{\nu\nu'}(u_\sigma - u_{\nu'}).$$

To take interactions of the matrix and the fracture domain into account we introduce the set of *matrix-fracture* (mf) connectivities

$$\mathcal{C} = \{(\nu_m, \nu_f) \,|\, \nu_m \in dof^\Gamma_{\mathcal{D}_m}, \nu_f \text{ s.t. } \mathbf{x}_{\nu_m} = \mathbf{x}_{\nu_f}\}$$

with $dof^\Gamma_{\mathcal{D}_m} = \{\nu \in dof_{\mathcal{D}_m} \,|\, \mathbf{x}_\nu \in \overline{\Gamma}\}$. For $(\nu_m, \nu_f) \in \mathcal{C}$ we define $\mathcal{C}_{(\nu_m,\nu_f)} \subset \mathcal{C}$ as the stencil of the corresponding flux defined by the subset C_{ν_m} of $dof^\Gamma_{\mathcal{D}_m}$.

The mf-transmissivities $(T_{\nu_m\nu_f}^{\nu'_m\nu'_f})_{(\nu'_m,\nu'_f) \in \mathcal{C}_{(\nu_m,\nu_f)}}$, $(\nu_m, \nu_f) \in \mathcal{C}$, are defined by

$$T_{\nu_m\nu_f}^{\nu'_m\nu'_f} = \int_{\Upsilon(\nu_m)} T_f \widetilde{\Pi}_{\mathcal{D}_f} \mathfrak{e}_{\nu_f} \widetilde{\Pi}_{\mathcal{D}_f} \mathfrak{e}_{\nu'_f} d\tau(\mathbf{x}),$$

where

$$\Upsilon(\nu_m) = \begin{cases} \Gamma \cap \bigcup_{K \in \overline{K}_s} \overline{K} & \text{for } \nu_m = \overline{K}_s \ (VAG) \\ \Gamma \cap \sigma & \text{for } \nu_m = K_\sigma \ (VAG) \text{ or } \nu_m = \overline{K}_\sigma \ (HFV) \end{cases}$$

The mf-fluxes are subsequently defined as

$$F_{\nu_m\nu_f}(u_{\mathcal{D}_m}, u_{\mathcal{D}_f}) = \sum_{(\nu'_m,\nu'_f) \in \mathcal{C}_{(\nu_m,\nu_f)}} T_{\nu_m\nu_f}^{\nu'_m\nu'_f}(u_{\nu'_m} - u_{\nu'_f}) \quad \text{for all } (\nu_m, \nu_f) \in \mathcal{C}.$$

We observe that for *VAG-CV* and *HFV* schemes, these fluxes are two-point fluxes in the sense that $\mathcal{C}_{(\nu_m,\nu_f)} = \{(\nu_m,\nu_f)\}$. For the *VAG-FE* scheme we obtain multi-point fluxes with the stencils

$$\mathcal{C}_{\nu_m} = \{\overline{K}_s\} \cup \{\overline{L}_{s'}, L_\sigma \,|\, ss' \in \mathcal{E}_L \cap \mathcal{E}_\Gamma, \sigma \in \mathcal{F}_s \cap \mathcal{F}_L \cap \mathcal{F}_\Gamma, L \in \overline{K}_s\},$$

for $\nu_m = \overline{K}_s \in \overline{\mathcal{M}}_s$ and $s \in \mathcal{V}_\Gamma$, $\mathcal{E}_L$ denoting the set of edges of cell L, and

$$\mathcal{C}_{\nu_m} = \{K_\sigma\} \cup \{\overline{K}_s, s \in \mathcal{V}_\sigma\},$$

for $\nu_m = K_\sigma \in \mathcal{M}_\sigma$, $\sigma \in \mathcal{F}_\Gamma$. The discrete source terms are defined by

$$H_\nu = \begin{cases} \displaystyle\int_\Omega h_m \Pi_{\mathcal{D}_m} \mathfrak{e}_\nu d\mathbf{x} & \text{for } \nu \in dof_{\mathcal{D}_m}, \\ \displaystyle\int_\Gamma h_f \Pi_{\mathcal{D}_f} \mathfrak{e}_\nu d\tau_f(\mathbf{x}) & \text{for } \nu \in dof_{\mathcal{D}_f}. \end{cases}$$

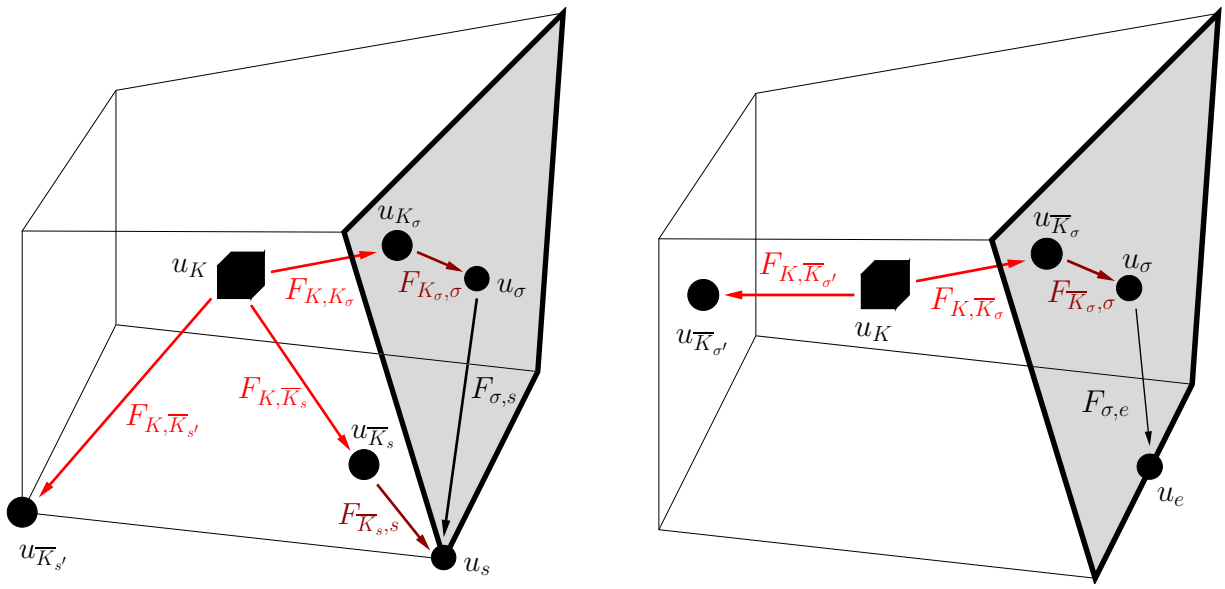


Figure 2: *mm*-fluxes (red), *mf*-fluxes (dark red) and *ff*-fluxes (black) for *VAG* (left) and *HFV* (right) on a 3D cell touching a fracture

The following Finite Volume formulation of (3) is equivalent to the discrete variational formulation (4): find $(u_{\mathcal{D}_m}, u_{\mathcal{D}_f}) \in X^0_{\mathcal{D}}$ such that

$$
\begin{cases}
\text{for all } K \in \mathcal{M} : \quad \displaystyle\sum_{\nu \in dof_K} F_{K\nu}(u_{\mathcal{D}_m}) = H_K \\[2ex]
\text{for all } \sigma \in \mathcal{F}_\Gamma : \quad \displaystyle\sum_{\nu \in dof_\sigma} F_{\sigma\nu}(u_{\mathcal{D}_f}) - \sum_{\substack{\nu_m \in dof_{\mathcal{D}_m} \\ \text{s.t. } (\nu_m,\sigma)\in\mathcal{C}}} F_{\nu_m\sigma}(u_{\mathcal{D}_m}, u_{\mathcal{D}_f}) = H_\sigma \\[2ex]
\text{for all } \nu_m \in dof_{\mathcal{D}_m} \setminus (\mathcal{M} \cup dof_{Dir_m}) : \\[2ex]
\qquad\qquad - \displaystyle\sum_{K \in \mathcal{M}_{\nu_m}} F_{K\nu_m}(u_{\mathcal{D}_m}) - \sum_{\substack{\nu_f \in dof_{\mathcal{D}_f} \\ \text{s.t. } (\nu_m,\nu_f)\in\mathcal{C}}} F_{\nu_m\nu_f}(u_{\mathcal{D}_m}, u_{\mathcal{D}_f}) = H_{\nu_m} \\[2ex]
\text{for all } \nu_f \in dof_{\mathcal{D}_f} \setminus (\mathcal{F}_\Gamma \cup dof_{Dir_f}) : \\[2ex]
\qquad\qquad - \displaystyle\sum_{\sigma \in \mathcal{F}_{\Gamma,\nu_f}} F_{\sigma\nu_f}(u_{\mathcal{D}_f}) - \sum_{\substack{\nu_m \in dof_{\mathcal{D}_m} \\ \text{s.t. } (\nu_m,\nu_f)\in\mathcal{C}}} F_{\nu_m\nu_f}(u_{\mathcal{D}_m}, u_{\mathcal{D}_f}) = H_{\nu_f}.
\end{cases}
$$

Here, $\mathcal{M}_{\nu_m}$ stands for the set of indices $\{K \in \mathcal{M} \mid \nu \in dof_K\}$ and $\mathcal{F}_{\Gamma,\nu_f}$ stands for the set $\{\sigma \in \mathcal{F}_\Gamma \mid \nu \in dof_\sigma\}$.

5 Conclusion

In this work, we extended the framework of gradient schemes (see [5]) to the model problem (2) of stationary Darcy flow through fractured porous media and gave numerical analysis results for this general framework.

The model problem (an extension to a network of fractures of a PDE model presented in [8], [10] and [2]) takes heterogeneities and anisotropy of the porous medium into account and involves a complex network of planar fractures, which might act either as barriers or as drains.

We also extended the *VAG* and *HFV* schemes to our model, where fractures acting as barriers force us to allow for pressure jumps across the fracture network. We developed two versions of *VAG* schemes, the conforming finite element version and the non-conforming control volume version, the latter particularly adapted for the treatment of material interfaces (cf. [7]). We showed, furthermore, that both versions of *VAG* schemes, as well as the proposed non-conforming *HFV* schemes, are incorporated by the gradient scheme's framework. Then, we applied the results for gradient schemes on *VAG* and *HFV* to obtain convergence, and, in particular, convergence of order 1 for "piecewise regular" solutions.

For implementation purposes and in view of the application to multi-phase flow, we also proposed a uniform Finite Volume formulation for *VAG* and *HFV* schemes.

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