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Abstract

We consider a financial model where the prices of risky assets are quoted by a representative market maker who takes into account an exogenous demand. We characterize these prices in terms of a system of BSDEs with quadratic growth. We show that this system admits a unique solution for every bounded demand if and only if the market maker's risk-aversion is sufficiently small. The uniqueness is established in the natural class of solutions, without any additional norm restrictions. To the best of our knowledge, this is the first study that proves such (global) uniqueness result for a system of fully coupled quadratic BSDEs.

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1 Introduction

In the classical problem of optimal investment, an economic agent trades at exogenous stock prices and looks for a strategy maximizing his expected utility. This problem has been extensively studied in the literature with various approaches. For example, Merton [13] relied on PDEs, Kramkov and Schachermayer [10] used the methods of convex duality and martingales and Hu et al. [7] employed BSDEs.

In this paper, we consider an inverse problem: find stock prices for which a given strategy is optimal; that is, instead of the usual task of getting "(optimal stocks') quantities from prices" we want to deduce "prices from quantities". This problem naturally arises in the market microstructure theory; see Grossman and Miller [6], Garleanu et al. [4] and German [5]. Here, the strategy represents the continuous demand on the market for a set of divided-paying stocks. The representative dealer, with exponential utility, provides liquidity for these assets and quotes prices in such a way that the market clears. In [4] and [5], the existence and uniqueness of such prices is established for every *simple* demand process, where trades occur only a finite number of times. It is the purpose of this paper to cover the general case.

As a first step, we obtain in Theorem 3.1 an equivalent characterization of the demand-based prices in terms of solutions to a system of BSDEs with quadratic growth. Similar systems appear naturally in economic equilibrium problems with exponential preferences; see Frei and dos Reis [3]. Contrary to the one-dimensional case, which is well-studied and where general criteria for existence and uniqueness are available (see e.g., Kobylanski [9] and Briand and Hu [1]), the situation with a *system* of quadratic BSDEs is more delicate. A counter-example in [3] shows that, in general, such system may not have solutions even for a *bounded* terminal condition. Moreover, although the existence can be guaranteed when the values at maturity are sufficiently small (see Proposition 1 in Tevzadze [14]), the uniqueness is only obtained in a local manner.

Our main results are stated in Theorem 4.1 and Proposition 4.3. In Theorem 4.1, we prove that the solutions to our system of quadratic BSDEs exist and are (globally) unique, provided that the product of the BMO-norm of the stocks' dividends, the $\mathbf{L}_\infty$-norm of the demand and the dealer's risk-aversion is sufficiently small. To the best of our knowledge, this is the first study that proves a (global) uniqueness result for a system of fully coupled quadratic BSDEs. In Proposition 4.3, we show that, in general, such well-posedness may be violated even if the dividends and the demand

are bounded. A crucial role in our study is played by the "sharp" *a priori* estimate given in Lemma 4.5. This estimate is obtained considering the stochastic control problem, which corresponds to the maximization of the dealer's expected utility with respect to demands bounded by 1.

Notations

For a matrix $A = (A^{ij})$, we denote its transpose by A^* and define its norm as

$$|A| \triangleq \sqrt{\operatorname{trace} AA^*} = \sqrt{\sum_{i,j} (A^{ij})^2}.$$

We will work on a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in[0,T]}, \mathbb{P})$ satisfying the standard conditions of right-continuity and completeness; the initial σ-algebra $\mathscr{F}_0$ is trivial, $\mathscr{F} = \mathscr{F}_T$, and the maturity T is finite. The expectation is denoted as $\mathbb{E}[\cdot]$ and the conditional expectation with respect to $\mathscr{F}_t$ as $\mathbb{E}_t[\cdot]$.

We shall use the following spaces of stochastic processes:

$\mathrm{BMO}(\mathbf{R}^m)$ is the Banach space of continuous m-dimensional martingales M with $M_0 = 0$ and the norm

$$\|M\|_{\mathrm{BMO}} \triangleq \sup_\tau \Big\| \Big\{ \mathbb{E}_\tau [|M_T - M_\tau|^2] \Big\}^{1/2} \Big\|_\infty,$$

where the supremum is taken with respect to all stopping times τ.

$\mathscr{H}_0(\mathbf{R}^{m\times d})$ is the vector space of predictable processes ζ with values in $m \times d$-matrices such that $\int_0^T |\zeta_s|^2 \, ds < \infty$. This is precisely the space of $m\times d$-dimensional integrands ζ for a d-dimensional Brownian motion B. We shall identify α and β in $\mathscr{H}_0(\mathbf{R}^{m\times d})$ if $\int_0^T |\alpha_s - \beta_s|^2 \, ds = 0$ or, equivalently, if the stochastic integrals $\alpha \cdot B$ and $\beta \cdot B$ coincide.

$\mathscr{H}_p(\mathbf{R}^{m\times d})$ for $p \geq 1$ consists of $\zeta \in \mathscr{H}_0(\mathbf{R}^{m\times d})$ such that

$$\|\zeta\|_p \triangleq \left\{ \mathbb{E}\left[\left(\int_0^T |\zeta_s|^2 \, ds \right)^{p/2} \right] \right\}^{1/p} < \infty.$$

It is a complete Banach space under this norm.

$\mathscr{H}_{\mathrm{BMO}}(\mathbf{R}^{m\times d})$ consists of $\zeta \in \mathscr{H}_0(\mathbf{R}^{m\times d})$ such that $\zeta \cdot B \in \mathrm{BMO}(\mathbf{R}^m)$ for a d-dimensional Brownian motion B. It is a Banach space under the

norm:

$$\|\zeta\|_{\mathrm{BMO}} \triangleq \|\zeta \cdot B\|_{\mathrm{BMO}} = \sup_{\tau} \| \left\{ \mathbb{E}_\tau \left[\int_\tau^T |\zeta_s|^2 \, ds \right] \right\}^{1/2} \|_\infty .$$

$\mathscr{H}_\infty(\mathbf{R}^n)$ is the Banach space of bounded n-dimensional predictable processes γ with the norm:

$$\|\gamma\|_\infty \triangleq \inf \left\{ c \geq 0 : \ |\gamma_t(\omega)| \leq c, \quad dt \times \mathbb{P}[d\omega] - a.s. \right\}.$$

For an n-dimensional integrable random variable ξ with $\mathbb{E}[\xi] = 0$ set

$$\|\xi\|_{\mathrm{BMO}} \triangleq \|(\mathbb{E}_t[\xi])_{t\in[0,T]}\|_{\mathrm{BMO}}. \tag{1.1}$$

Denote also

$$\|\xi\|_p \triangleq (\mathbb{E}[|\xi|^p])^{1/p}, \quad p \geq 1,$$
$$\|\xi\|_\infty \triangleq \inf \left\{ c \geq 0 : \ |\xi(\omega)| \leq c, \quad \mathbb{P}[d\omega] - a.s. \right\}.$$

Observe that,

$$\|\xi\|_{\mathrm{BMO}} \leq \inf_{x\in\mathbf{R}^n} \|\xi - x\|_\infty. \tag{1.2}$$

2 Model

There is a single *representative* market maker whose preferences regarding terminal wealth are modeled by the exponential utility with the risk aversion coefficient $a > 0$:

$$U(x) = -\frac{1}{a} e^{-ax}, \quad x \in \mathbf{R}. \tag{2.1}$$

The financial market consists of a bank account and n stocks. The bank account pays an *exogenous* interest rate, which we assume to be zero. The stocks pay dividends $\Psi = (\Psi^i)_{i=1,\dots,n}$ at maturity T; each Ψ^i is a random variable. While the terminal stocks' prices S_T are always given by Ψ, their values S_t on $[0, T)$ are determined *endogenously* by the equilibrium mechanism specified below; in particular, they are affected by demand on stocks. Following Garleanu et al. [4] and German [5], we give the following definition.

Definition 2.1. A predictable process $\gamma = (\gamma_t)$ with values in $\mathbf{R}^n$ is called a *demand*. The demand γ is *viable* if there is an n-dimensional semimartingale of *stock prices* $S = (S_t)$ such that $S_T = \Psi$, the probability measure $\mathbb{Q}$, called the *pricing measure*, is well-defined by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} \triangleq \frac{U'(\int_0^T \gamma dS)}{\mathbb{E}[U'(\int_0^T \gamma dS)]} = \frac{e^{-a\int_0^T \gamma dS}}{\mathbb{E}[e^{-a\int_0^T \gamma dS}]},$$

and S and the stochastic integral $\gamma \cdot S$ are uniformly integrable martingales under $\mathbb{Q}$.

In this definition, $-\gamma_t$ stands for the number of stocks that an external counter-party plans to buy/sell from the market up to time t. The stochastic integral $\gamma \cdot S$ represents the evolution of the losses of the external counter-party or, equivalently, of the gains of the market maker. Note that, as $S = S(\gamma)$, the dependence of $\gamma \cdot S$ on γ is *non-linear*; this is in contrast to the standard, "small agent's", model of mathematical finance.

To clarify the economic meaning of Definition 2.1, we recall a well-known result in the theory of optimal investment, which states that under the stock prices $S = S(\gamma)$ the strategy γ is optimal.

Lemma 2.2. *Let the utility function U be given by (2.1) and γ be a viable demand accompanied by the stock prices S and the pricing measure $\mathbb{Q}$ in the sense of Definition 2.1. Then*

$$\mathbb{E}\left[U\left(\int_0^T \gamma dS\right)\right] \geq \mathbb{E}\left[U\left(\int_0^T \zeta dS\right)\right],$$

for every demand ζ such that the stochastic integral $\zeta \cdot S$ is a $\mathbb{Q}$-supermartingale.

Proof. Define the conjugate function to U:

$$V(y) \triangleq \sup_{x \in \mathbf{R}} \{U(x) - xy\} = \frac{1}{a} y(\ln y - 1), \quad y > 0,$$

and observe that, as

$$V(U'(x)) = U(x) - xU'(x), \quad x \in \mathbf{R},$$

the construction of $\mathbb{Q}$ yields that

$$V\left(y\frac{d\mathbb{Q}}{d\mathbb{P}}\right) = U\left(\int_0^T \gamma dS\right) - y\frac{d\mathbb{Q}}{d\mathbb{P}}\int_0^T \gamma dS, \tag{2.2}$$

where

$$y = \mathbb{E}\left[U'\left(\int_0^T \gamma\, dS\right)\right] = \mathbb{E}\left[e^{-a\int_0^T \gamma dS}\right].$$

On the other side, clearly,

$$V\left(y\frac{d\mathbb{Q}}{d\mathbb{P}}\right) \geq U\left(\int_0^T \zeta dS\right) - y\frac{d\mathbb{Q}}{d\mathbb{P}}\int_0^T \zeta dS. \tag{2.3}$$

Taking expectations (under $\mathbb{P}$) in (2.2) and (2.3), we obtain the conclusion. □

We call a demand γ *simple* if

$$\gamma = \sum_{i=0}^{m-1} \theta_i 1_{(\tau_i, \tau_{i+1}]},$$

where $0 = \tau_0 < \tau_1 < \cdots < \tau_m = T$ are stopping times and θ_i is a $\mathscr{F}_{\tau_i}$-measurable random variable with values in $\mathbf{R}^n$, $i = 0, \ldots, m-1$. Theorem 1 in [5] shows that if the dividends $\Psi = (\Psi^i)$ have all exponential moments, then every bounded simple demand γ is viable. Moreover, the price process $S = S(\gamma)$ is unique and is constructed explicitly, by backward induction.

The goal of this paper is to investigate the case of demands γ with general continuous dynamics. Our main results, Theorem 4.1 and Proposition 4.3, rely on the BSDE-characterization of the stock prices $S = S(\gamma)$ from the next section.

Remark 2.3. To simplify notation, we neglected in our setup the existence of the initial random endowment β_0 for the market maker. Due to the choice of exponential utility in (2.1), this condition does not restrict any generality. Indeed, if $\beta_0 \neq 0$, then, in Definition 2.1 and throughout the paper, the measure $\mathbb{P}$ should just be replaced by the measure $\mathbb{Q}(0)$ with the density

$$\frac{d\mathbb{Q}(0)}{d\mathbb{P}} \triangleq \frac{U'(\beta_0)}{\mathbb{E}[U'(\beta_0)]} = \frac{\exp(-a\beta_0)}{\mathbb{E}[\exp(-a\beta_0)]}.$$

3 Characterization in terms of BSDE

Hereafter, we shall assume that

(A1) There exists a d-dimensional Brownian motion B such that every local martingale M is a stochastic integral with respect to B:

$$M = M_0 + \zeta \cdot B.$$

Of course, this assumption holds if the filtration is generated by B.

For a viable demand γ accompanied by stocks' prices S define the process R such that

$$R_t \triangleq U^{-1}\left(\mathbb{E}_t\left[U\left(\int_t^T \gamma dS\right)\right]\right) = -\frac{1}{a}\log\left(\mathbb{E}_t\left[e^{-a\int_t^T \gamma dS}\right]\right), \tag{3.1}$$

is the market maker's *certainty equivalent value* at time t of the *remaining gain* $\int_t^T \gamma dS$. Observe that the density process Z of the pricing measure $\mathbb{Q}$ has the form

$$Z_t \triangleq \mathbb{E}_t\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\right] = e^{-a(R_t - R_0 + \int_0^t \gamma dS)}, \quad t \in [0,T].$$

Jensen's inequality and the martingale property of $\gamma \cdot S$ under $\mathbb{Q}$ imply that $Z^{-1}e^{-aR} = e^{-a(R_0 - \gamma\cdot S)}$ is a $\mathbb{Q}$-submartingale. Hence, e^{-aR} is a submartingale (under $\mathbb{P}$) and, as $R_T = 0$, we obtain that

$$e^{-aR} \leq 1 \quad \text{or, equivalently,} \quad R \geq 0. \tag{3.2}$$

Under (A1), there is $\alpha \in \mathscr{H}_0(\mathbf{R}^d)$, the *market price of risk*, such that

$$Z = \mathscr{E}(-\alpha \cdot B) = e^{-\alpha\cdot B - \frac{1}{2}\int |\alpha|^2 dt}.$$

From Girsanov's theorem we deduce that

$$W \triangleq B + \int \alpha dt$$

is a Brownian motion under $\mathbb{Q}$ and that every local martingale under $\mathbb{Q}$ is a stochastic integral with respect to W. In particular, there is $\sigma \in \mathscr{H}_0(\mathbf{R}^{n\times d})$, the *volatility* of stocks' prices, such that

$$S = S_0 + \sigma \cdot W = S_0 + \int \sigma\alpha dt + \sigma \cdot B.$$

We now characterize S, R, α and σ in terms of solutions to the multi-dimensional quadratic BSDE (3.3)–(3.4).

Theorem 3.1. *Assume (A1). An n-dimensional predictable process γ is a viable demand if and only if there are processes (S, R, η, θ), where S is a n-dimensional semi-martingale, R is a semi-martingale, $\eta \in \mathscr{H}_0(\mathbf{R}^d)$, and $\theta \in \mathscr{H}_0(\mathbf{R}^{n\times d})$, such that, for every $t \in [0,T]$,*

$$aR_t = \frac{1}{2}\int_t^T (|\theta_s^*\gamma_s|^2 - |\eta_s|^2)ds - \int_t^T \eta dB, \tag{3.3}$$

$$aS_t = a\Psi - \int_t^T \theta_s(\eta_s + \theta_s^*\gamma_s)ds - \int_t^T \theta dB, \tag{3.4}$$

and such that the stochastic exponential $Z \triangleq \mathscr{E}(-(\eta + \theta^*\gamma) \cdot B)$ *and the processes* ZS *and* $Z(\gamma \cdot S)$ *are (uniformly integrable) martingales.*

In this case, S *represents stocks' prices which accompany* γ*,* R *is the certainty equivalent value,* Z *is the density process of the pricing measure* $\mathbb{Q}$*, and the market price of risk* α *and the volatility* σ *are given by*

$$\alpha = \eta + \theta^*\gamma, \tag{3.5}$$

$$\sigma = \theta/a. \tag{3.6}$$

Proof. Let γ be a viable demand accompanied by stocks' prices S and the certainty equivalent value R. Define the martingales

$$L_t = \mathbb{E}_t\left[U'\left(\int_0^T \gamma dS\right)\right] = \mathbb{E}_t\left[e^{-a\int_0^T \gamma dS}\right],$$
$$M_t = a\mathbb{E}_t\left[\Psi U'\left(\int_0^T \gamma dS\right)\right] = a\mathbb{E}_t\left[\Psi e^{-a\int_0^T \gamma dS}\right],$$

and observe that the pricing measure $\mathbb{Q}$ has the density L_T/L_0 and

$$aS_t = a\mathbb{E}_t^{\mathbb{Q}}[\Psi] = M_t/L_t,$$
$$aR_t = aR_0 - \log(L_t/L_0) - \int_0^t \gamma d(M/L),$$

or, in a "backward" form, as $S_T = \Psi$ and $R_T = 0$,

$$aS_t = a\Psi - \int_t^T d(M/L),$$
$$aR_t = \int_t^T (d\log L + \gamma d(M/L)).$$

From (A1) and accounting for the strict positivity of L, we deduce the existence and uniqueness of $\alpha \in \mathscr{H}_0(\mathbf{R}^d)$ and $\beta \in \mathscr{H}_0(\mathbf{R}^{n\times d})$ such that

$$L = L_0 - L\alpha \cdot B,$$
$$M = M_0 + L\beta \cdot B.$$

Direct computations based on the Itô's formula yield

$$d\log L = -\frac{1}{2}|\alpha|^2 dt - \alpha dB,$$
$$\begin{aligned} d(M/L) &= (\beta\alpha + \frac{1}{L}M|\alpha|^2)dt + (\beta + \frac{1}{L}M\alpha^*)dB \\ &= (\beta + aS\alpha^*)\alpha dt + (\beta + aS\alpha^*)dB, \end{aligned}$$

and equations (3.3) and (3.4) hold with

$$\theta = \beta + aS\alpha^*,$$
$$\eta = \alpha - \theta^*\gamma.$$

Observe that

$$Z = \mathscr{E}(-(\eta + \theta^*\gamma) \cdot B) = \mathscr{E}(-\alpha \cdot B) = L/L_0$$

is the density process of $\mathbb{Q}$ and, in particular, is a martingale. The martingale properties of ZS and $Z(\gamma \cdot S)$ under $\mathbb{P}$ then follow from those of S and $\gamma \cdot S$ under $\mathbb{Q}$. Hence, the process (S, R, θ, η) satisfies the conditions of the theorem.

Conversely, let (S, R, θ, η) be as in the statement of the theorem. Define the probability measure $\mathbb{Q}$ with the density process $Z = \mathscr{E}(-(\eta + \theta^*\gamma) \cdot B)$. From (3.3) and (3.4) we deduce that

$$\begin{aligned}\frac{d\mathbb{Q}}{d\mathbb{P}} = Z_T &= e^{-\int_0^T (\eta+\theta^*\gamma)dB - \frac{1}{2}\int_0^T |\eta+\theta^*\gamma|^2 dt} \\ &= e^{-a(R_T - R_0 + \int_0^T \gamma dS)} = \frac{U'\left(\int_0^T \gamma dS\right)}{\mathbb{E}\left[U'\left(\int_0^T \gamma dS\right)\right]}.\end{aligned}$$

Moreover, $S_T = \Psi$ and the martingale properties of S and $\gamma \cdot S$ under $\mathbb{Q}$ follow from those of ZS and $Z(\gamma \cdot S)$ under $\mathbb{P}$. Hence, S satisfies the conditions of Definition 2.1.

Finally, as part of the arguments above, we obtained that, given the stocks' prices S, the linear invertibility relations (3.5) and (3.6) between (η, θ) and (α, σ) hold and equations (3.1) and (3.3) for R are equivalent. □

Remark 3.2. The BSDE characterization in Theorem 3.1 heavily relies on condition (2.1) of exponential preferences. For a general utility function U, one can similarly associate with the stock prices $S = S(\gamma)$ the following system of Forward-Backward Stochastic Differential Equations (FBSDEs):

$$\begin{aligned}S_t &= \Psi - \int_t^T \sigma_u \alpha_u \, du - \int_t^T \sigma dB, \\ Y_t &= \log\left(U'(X_T)\right) + \frac{1}{2}\int_t^T |\alpha_u|^2 \, du + \int_t^T \alpha dB, \\ X_t &= \int_0^t \gamma dS.\end{aligned}$$

Here X is the gain process of the market maker due to the demand γ and $Y = \log Z + \text{const}$ is a normalized log-density process of the pricing measure. If U is of exponential type then, by "decoupling" substitution (3.1), this fully coupled system of FBSDEs can be reduced to the simpler system (3.3)–(3.4) of quadratic BSDEs.

4 Existence and uniqueness

This is our main result.

Theorem 4.1. *Assume (A1). There is a constant $c = c(n) > 0$ (dependent only on the number of stocks n) such that if $\gamma \in \mathscr{H}_\infty(\mathbf{R}^n)$ and*

$$a\|\gamma\|_\infty \|\Psi - \mathbb{E}[\Psi]\|_{\mathrm{BMO}} \le c, \tag{4.1}$$

then γ is a viable demand accompanied by the unique stocks' prices S. Moreover, the BMO*-norms of the volatility σ and of the market price of risk α are bounded by*

$$\begin{aligned} \|\sigma\|_{\mathrm{BMO}} &\le 2\|\Psi - \mathbb{E}[\Psi]\|_{\mathrm{BMO}}, \\ \|\alpha\|_{\mathrm{BMO}} &\le 4a\|\gamma\|_\infty \|\Psi - \mathbb{E}[\Psi]\|_{\mathrm{BMO}}. \end{aligned} \tag{4.2}$$

As the following simple example illustrates, among the dividends Ψ with finite BMO-norm, condition (4.1) is necessary even for the viability of constant demands.

Example 4.2. Suppose that Ψ is a real-valued random variable such that

$$\mathbb{E}[\Psi] = 0, \quad \|\Psi\|_{\mathrm{BMO}} < \infty, \quad \text{but} \quad \mathbb{E}[e^{\Psi}] = \infty;$$

see, for example, Example 3.4 in Kazamaki [8]. It readily follows from Definition 2.1 that the constant demand $\gamma = -1/a$ is not viable. Indeed, in this case, the pricing measure $\mathbb{Q}$ can only be of the form:

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \text{const}\, e^{\Psi},$$

which is not possible, because of the lack of integrability.

It is more delicate to construct a counter-example for *bounded* dividends Ψ. Let $c = c(n) > 0$ be a constant from Theorem 3.1. In view of (1.2), the condition (4.1) holds if

$$a\|\gamma\|_\infty \inf_{x \in \mathbf{R}^n} \|\Psi - x\|_\infty \le c.$$

The following proposition shows that, already in one-dimensional case, the assertions of Theorem 4.1 may fail for bounded Ψ and that $c(1) < 1$. It is stated under a stronger assumption than (A1):

(A2) There exists a one-dimensional Brownian motion B such that the filtration $(\mathscr{F}_t)$ is the completion of the filtration generated by B:

$$\mathscr{F}_t = \mathscr{F}_t^B \vee \mathscr{N}^{\mathbb{P}}, \quad t \in [0, T].$$

Here $\mathscr{F}_t^B \triangleq \sigma\{B_s, s \leq t\}$ and $\mathscr{N}^{\mathbb{P}}$ is the family of all $\mathbb{P}$-null sets in $\mathscr{F}$.

Proposition 4.3. *Assume (A2). There exist a bounded predictable process γ and a bounded random variable Ψ (both γ and Ψ have dimension one) such that*

$$a\|\gamma\|_\infty \|\Psi\|_\infty \leq 1,$$

and such that γ is not supported by a unique semi-martingale S in the sense of Definition 2.1.

Note that, in comparison to the non-existence construction in Example 4.2 for dividends with finite BMO-norm, our result for bounded dividends is weaker. Here we only claim either non-existence or non-uniqueness.

Remark 4.4. In the follow-up paper [11], we show that under (4.1) the prices $S = S(\gamma)$ are stable under small changes in the demand γ; in particular, they can be well approximated by the prices originated from simple demands. We also obtain in [11] a power series expansion of $S = S(\gamma, a)$ with respect to the market's risk-aversion a in a neighborhood of the point $a = 0$ where the price impact effect disappears.

4.1 Outline of the proof of Theorem 4.1

For the reader's convenience, we begin with an outline of the key steps in the proof of Theorem 4.1. To simplify notation, suppose that

$$\mathbb{E}[\Psi] = 0, \quad a = 1, \quad \text{and} \quad |\gamma| \leq 1.$$

By Theorem 4.1, the existence and uniqueness of the price process S, which accompanies the demand γ, is equivalent to the existence and uniqueness of the solution (η, θ) of the multi-dimensional quadratic BSDE:

$$R_t = \frac{1}{2}\int_t^T (|\theta_s^* \gamma_s|^2 - |\eta_s|^2) ds - \int_t^T \eta dB,$$
$$S_t = \Psi - \int_t^T \theta_s(\eta_s + \theta_s^* \gamma_s) ds - \int_t^T \theta dB,$$

such that the stochastic exponential $Z \triangleq \mathscr{E}(-(\eta+\theta^*\gamma)\cdot B)$ and the processes ZS and $Z(\gamma \cdot S)$ are martingales.

The first step is standard. Using a rather straightforward extension of the results of Tevzadze [14] (see Theorem A.1 in Appendix A), we deduce the existence of a constant $b = b(n)$ such that if

$$\|\Psi\|_{\mathrm{BMO}} \leq b,$$

then the BSDE admits only one solution (η, θ) such that

$$\|(\eta, \theta)\|_{\mathrm{BMO}} \leq 2b.$$

Local existence and *local* uniqueness then readily follow.

The delicate part is to verify the *global* uniqueness. For that, we need to find a constant $0 < c \leq b$ such that

$$\|\Psi\|_{\mathrm{BMO}} \leq c \implies \|(\eta, \theta)\|_{\mathrm{BMO}} \leq 2b,$$

for *every* solution (η, θ) for which $Z = \mathscr{E}(-\int(\eta+\theta^*\gamma)dB)$, ZS, and $Z(\gamma\cdot S)$ are martingales. Using basic BMO-inequalities, we first deduce the existence of an increasing function $f = f(x)$, $x \geq 0$, such that

$$\|(\eta, \theta)\|_{\mathrm{BMO}} \leq f(\|R\|_\infty)\|\Psi\|_{\mathrm{BMO}}.$$

To conclude the argument, we need to find a constant $K > 0$ and an increasing function $g = g(x)$ on $[0, K)$, such that

$$\|R\|_\infty \leq g(\|\Psi\|_{\mathrm{BMO}}), \quad \text{if} \quad \|\Psi\|_{\mathrm{BMO}} < K.$$

A sharp version of the above *a priori* estimate is obtained in Lemma 4.5 and is based on the verification arguments for the stochastic control problem:

$$u_t^* \triangleq \operatorname*{ess\,sup}_{|\gamma|\leq 1}\left(-e^{-R_t(\gamma)}\right) = \operatorname*{ess\,sup}_{|\gamma|\leq 1}\mathbb{E}_t[-e^{-\int_t^T \gamma dS(\gamma)}],$$

where we maximize the market maker's expected utility over all viable demands γ with $|\gamma| \leq 1$. Later, this estimate is also used in Proposition 4.3 to produce a counter-example.

4.2 Proof of Theorem 4.1

From Definition 2.1, we deduce that the dependence of stocks' prices $S = S(\gamma, a, \Psi)$ on the viable demand γ, on the risk-aversion coefficient a, and on the dividend Ψ has the following homogeneity properties: for $b > 0$,

$$S(b\gamma, a, \Psi) = S(\gamma, ba, \Psi) = \frac{1}{b}S(\gamma, a, b\Psi). \tag{4.3}$$

This yields similar properties of the volatilities $\sigma = \sigma(\gamma, a, \Psi)$ and of the market prices of risk $\alpha = \alpha(\gamma, a, \Psi)$ which correspond to $S = S(\gamma, a, \Psi)$:

$$\begin{aligned} \sigma(b\gamma, a, \Psi) &= \sigma(\gamma, ba, \Psi) = \frac{1}{b}\sigma(\gamma, a, b\Psi), \\ \alpha(b\gamma, a, \Psi) &= \alpha(\gamma, ba, \Psi) = \alpha(\gamma, a, b\Psi). \end{aligned} \tag{4.4}$$

In view of these identities, it is sufficient to prove Theorem 4.1 for the case

$$a = 1 \geq \|\gamma\|_\infty. \tag{4.5}$$

Define the function $H = H(u)$ on $[0, \infty)$ as

$$H(u) = e^u(u-1) + 1, \quad u \geq 0.$$

Observe that H is an N-function in the theory of Orlicz spaces, that is, it is convex, strictly increasing, $H(0) = H'(0) = 0$, and $H'(\infty) = \infty$; see Krasnosel′skiĭ and Rutickiĭ [12]. For a later use, we also note that for any $\varepsilon > 0$ there is a constant $C(\varepsilon) > 0$ such that

$$\frac{1}{2}u^2 \leq H(u) \leq C(\varepsilon)e^{(1+\varepsilon)u}, \quad u \geq 0. \tag{4.6}$$

For an n-dimensional martingale M with $M_0 = 0$ set

$$\|M\|_H \triangleq \inf\left\{\lambda > 0 : \sup_\tau \|\mathbb{E}_\tau\left[H\left(\frac{|M_T - M_\tau|}{\lambda}\right)\right]\|_\infty \leq 1\right\},$$

where the upper bound is taken with respect to all stopping times τ. Observe that, by the monotone convergence theorem,

$$\sup_\tau \|\mathbb{E}_\tau\left[H\left(\frac{|M_T - M_\tau|}{\|M\|_H}\right)\right]\|_\infty \leq 1. \tag{4.7}$$

The family of n-dimensional martingales M with $M_0 = 0$ and $\|M\|_H < \infty$ is a Banach space under $\|\cdot\|_H$ and this norm is equivalent to the BMO-norm: there is a constant $C_H = C_H(n) > 0$ such that

$$\frac{1}{\sqrt{2}}\|M\|_{\mathrm{BMO}} \leq \|M\|_H \leq C_H\|M\|_{\mathrm{BMO}}. \tag{4.8}$$

Here, the first inequality follows from the left-hand side of (4.6), while the second one holds by Remark 2.1 on page 28 of Kazamaki [8].

For an n-dimensional integrable random variable ξ with $\mathbb{E}[\xi] = 0$ denote

$$\|\xi\|_H \triangleq \|(\mathbb{E}_t[\xi])_{t\in[0,T]}\|_H.$$

Lemma 4.5. *Let $\gamma \in \mathscr{H}_\infty(\mathbf{R}^n)$ be a viable demand accompanied by stocks' prices S and the certainty equivalent value R. Assume (A1), (4.5), and that*

$$\mathbb{E}[\Psi] = 0, \quad \|\Psi\|_H < 1.$$

Then for every $x \in \mathbf{R}^n$ the process

$$V_t(x) \triangleq (1 - H(|S_t - x|))e^{-R_t}, \quad t \in [0, T],$$

is a supermartingale and the following estimate holds:

$$e^{-R_t} \geq 1 - \|\Psi\|_H, \quad t \in [0, T]. \tag{4.9}$$

Proof. To simplify notation, set

$$F(u) \triangleq 1 - H(u) = e^u(1 - u), \quad u \geq 0.$$

As the density process of the pricing measure $\mathbb{Q}$ has the form:

$$Z_t \triangleq \mathbb{E}_t\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\right] = e^{-(R_t - R_0 + \int_0^t \gamma dS)}, \quad t \in [0, T],$$

the $\mathbb{P}$-supermartingale property of $V(x)$ is equivalent to the $\mathbb{Q}$-supermartingale property of

$$\widetilde{V}(x) \triangleq e^{R_0} Z^{-1} V(x) = F(|S - x|)e^{\gamma \cdot S}.$$

Recall that under $\mathbb{Q}$ the price process S evolves as

$$S = S_0 + \sigma \cdot W,$$

where W is a Brownian motion under $\mathbb{Q}$. Using the fact that $F'(0) = 0$, we deduce from the Itô's formula that

$$\widetilde{V}_t(x) = M_t(x) + \int_0^t e^{(\gamma \cdot S)_r} A_r(x) dr,$$

where $M(x)$ is a local martingale under $\mathbb{Q}$ and

$$A(x) = 1_{\{|S-x|>0\}}\Big(\frac{1}{2}F''(|S - x|)\frac{|\sigma^*(S - x)|^2}{|S - x|^2} + \frac{1}{2}F(|S - x|)\,|\sigma^*\gamma|^2$$
$$+ F'(|S - x|)\Big(\frac{\langle \sigma^*(S - x), \sigma^*\gamma\rangle}{|S - x|} + \frac{1}{2\,|S - x|}\Big(|\sigma|^2 - \frac{|\sigma^*(S - x)|^2}{|S - x|^2}\Big)\Big)\Big).$$

As $\|\gamma\|_\infty \le 1$, $F' \le 0$, and

$$F - 2F' + F'' = 0,$$

we deduce that

$$A(x) \le 1_{\{|S-x|>0\}} \frac{|\sigma|^2}{2} \Big(F'' - 2F' + F \Big)(|S-x|) = 0,$$

thus proving the local supermartingale property of $\widetilde{V}(x)$ under $\mathbb{Q}$.

To verify that $\widetilde{V}(x)$ is a (global) $\mathbb{Q}$-supermartingale, it is sufficient to show that $\widetilde{V}(x)$ is bounded below by some $\mathbb{Q}$-martingale. With this goal in mind, take $\varepsilon > 0$ such that

$$\|\Psi\|_H < \frac{1}{1+\varepsilon} < 1$$

and observe that, by the construction of the norm $\|\cdot\|_H$,

$$\mathbb{E}[e^{(1+\varepsilon)|\Psi|}] < \infty.$$

It follows that

$$\mathbb{E}^{\mathbb{Q}}[e^{(1+\varepsilon)|\Psi| + (\gamma\cdot S)_T}] = e^{R_0}\mathbb{E}[e^{(1+\varepsilon)|\Psi|}] < \infty$$

and, hence, the $\mathbb{Q}$-martingale

$$N_t \triangleq \mathbb{E}^{\mathbb{Q}}_t[e^{(1+\varepsilon)|\Psi| + (\gamma\cdot S)_T}], \quad t \in [0,T],$$

is well-defined. Recall that S and $\gamma \cdot S$ are $\mathbb{Q}$-martingales. From the right-hand side of (4.6) and the Jensen's inequality we deduce that

$$\begin{aligned} -\widetilde{V}_t(x) &\le H(|S_t - x|)e^{(\gamma\cdot S)_t} \le C(\varepsilon)e^{(1+\varepsilon)|S_t - x| + (\gamma\cdot S)_t} \\ &\le C(\varepsilon)\mathbb{E}^{\mathbb{Q}}_t[e^{(1+\varepsilon)|\Psi - x| + (\gamma\cdot S)_T}] \le C(\varepsilon)N_t e^{(1+\varepsilon)|x|} \end{aligned}$$

and the global supermartingale property of $\widetilde{V}(x)$ under $\mathbb{Q}$ follows.

We thus have shown that $V(x) = F(|S-x|)e^{-R}$ is a supermartingale. As $F \le 1$ and $R_T = 0$ we then obtain

$$e^{-R_t} \ge F(|S_t - x|)e^{-R_t} \ge \mathbb{E}_t[F(|\Psi - x|)], \quad x \in \mathbf{R}^n.$$

Of course, we can replace x in the inequality above with any $\mathscr{F}_t$-measurable random variable and, in particular, with $\mathbb{E}_t[\Psi]$. As H is convex, $H(0) = 0$,

and $\|\Psi\|_H < 1$ we then deduce that

$$\begin{aligned} e^{-R_t} &\geq \mathbb{E}_t[F(|\Psi - \mathbb{E}_t[\Psi]|)] \\ &= 1 - \mathbb{E}_t[H(|\Psi - \mathbb{E}_t[\Psi]|)] \\ &= 1 - \mathbb{E}_t\left[H\left(\|\Psi\|_H \frac{|\Psi - \mathbb{E}_t[\Psi]|}{\|\Psi\|_H}\right)\right] \\ &\geq 1 - \|\Psi\|_H \mathbb{E}_t\left[H\left(\frac{|\Psi - \mathbb{E}_t[\Psi]|}{\|\Psi\|_H}\right)\right] \end{aligned}$$

and the inequality (4.9) follows from (4.7). □

Recall that if L is a BMO-martingale, then the stochastic exponential $\mathscr{E}(L)$ is a martingale, and hence, is the density process of some probability measure $\mathbb{Q}$. Moreover, if $\|L\|_{\mathrm{BMO}} \leq b$ then there is a constant $K = K(n, b)$ such that if $M \in \mathrm{BMO}(\mathbf{R}^n)$ then its Girsanov's transform $N \triangleq M - \langle M, L\rangle$ belongs to $\mathrm{BMO}(\mathbb{Q})$ and

$$\frac{1}{K}\|N\|_{\mathrm{BMO}(\mathbb{Q})} \leq \|M\|_{\mathrm{BMO}} \leq K\|N\|_{\mathrm{BMO}(\mathbb{Q})};$$

see Theorem 3.3 in Kazamaki [8]. If $M = \beta \cdot B$, then the above inequality can be equivalently written as

$$\frac{1}{K}\|\beta\|_{\mathrm{BMO}(\mathbb{Q})} \leq \|\beta\|_{\mathrm{BMO}} \leq K\|\beta\|_{\mathrm{BMO}(\mathbb{Q})}. \tag{4.10}$$

We need a similar inequality for the BMO-norm (1.1) associated with random variables.

Lemma 4.6. *Let L be a* BMO*-martingale with $\|L\|_{\mathrm{BMO}} \leq b$, $\mathbb{Q}$ be the probability measure with the density process $Z = \mathscr{E}(L)$, and ξ be an integrable n-dimensional random variable such that $\mathbb{E}[\xi] = 0$ and $\|\xi\|_{\mathrm{BMO}} < \infty$. Then ξ is integrable under $\mathbb{Q}$ and there is a constant $K = K(n, b)$ such that*

$$\frac{1}{K}\|\xi - \mathbb{E}_{\mathbb{Q}}[\xi]\|_{\mathrm{BMO}(\mathbb{Q})} \leq \|\xi\|_{\mathrm{BMO}} \leq K\|\xi - \mathbb{E}_{\mathbb{Q}}[\xi]\|_{\mathrm{BMO}(\mathbb{Q})}. \tag{4.11}$$

Proof. It is sufficient to prove only the first inequality in (4.11). Recall that by the reverse Hölder inequality there are constants $p_0 = p_0(b) > 1$ and $C_1 = C_1(p_0, b) > 0$ such that

$$\left(\mathbb{E}_\tau[Z_T^{p_0}]\right)^{1/p_0} \leq C_1 Z_\tau,$$

for every stopping time τ; see Theorem 3.1 in Kazamaki [8]. For a random variable $\eta \geq 0$, this yields

$$\mathbb{E}_\tau^{\mathbb{Q}}[\eta] = \frac{1}{Z_\tau}\mathbb{E}_\tau\left[Z_T\eta\right] \leq \frac{1}{Z_\tau}\left(\mathbb{E}_\tau[Z_T^{p_0}]\right)^{1/p_0}\left(\mathbb{E}_\tau[\eta^{q_0}]\right)^{1/q_0} \leq C_1\left(\mathbb{E}_\tau[\eta^{q_0}]\right)^{1/q_0},$$

where $q_0 = \frac{p_0}{p_0-1} > 1$.

Since $\|\xi\|_{\mathrm{BMO}} < \infty$, the estimate above implies that ξ is integrable under $\mathbb{Q}$ and

$$\mathbb{E}_\tau^{\mathbb{Q}}\left[\left|\xi - \mathbb{E}_\tau^{\mathbb{Q}}[\xi]\right|\right] \leq 2\mathbb{E}_\tau^{\mathbb{Q}}\left[|\xi - \mathbb{E}_\tau[\xi]|\right] \leq 2C_1\left(\mathbb{E}_\tau\left[|\xi - \mathbb{E}_\tau[\xi]|^{q_0}\right]\right)^{1/q_0}.$$

This readily yields the result after we recall that for every $p \geq 1$ there is a constant $C_2 = C_2(p,n)$ such that

$$\frac{1}{C_2}\|\zeta\|_{\mathrm{BMO}} \leq \|\zeta\|_{\mathrm{BMO}_p} \triangleq \sup_\tau \|(\mathbb{E}_\tau[|\zeta - \mathbb{E}_\tau[\zeta]|^p])^{1/p}\|_\infty \leq C_2\|\zeta\|_{\mathrm{BMO}},$$

for every n-dimensional random variable ζ with $\mathbb{E}[\zeta] = 0$. □

Lemma 4.7. *Let $\gamma \in \mathscr{H}_\infty(\mathbf{R}^n)$ and suppose that the conditions (A1) and (4.5) hold and that $\mathbb{E}[\Psi] = 0$ and*

$$\|\Psi\|_H \leq b < 1.$$

Then γ is a viable demand accompanied by stocks' prices S and the certainty equivalent value R if and only if there exist $\theta \in \mathscr{H}_{\mathrm{BMO}}(\mathbf{R}^{n\times d})$ and $\eta \in \mathscr{H}_{\mathrm{BMO}}(\mathbf{R}^d)$ such that (S,R,η,θ) is a solution of the BSDE (3.3)–(3.4). Moreover, there is a constant $K = K(n,b) > 0$ such that

$$\|\eta\|_{\mathrm{BMO}} + \|\theta\|_{\mathrm{BMO}} \leq K\|\Psi\|_{\mathrm{BMO}}. \tag{4.12}$$

Proof. Let γ be a viable demand accompanied by stocks' prices S and the certainty equivalent value R and let η and θ be as in Theorem 3.1. Recall that $a = 1$ and observe that (3.3) can be written as

$$\begin{aligned} R_t &= \frac{1}{2}\int_t^T (|\theta_s^*\gamma_s|^2 - |\eta_s|^2)ds - \int_t^T \eta dB, \\ &= \frac{1}{2}\int_t^T |\alpha_s|^2\, ds - \int_t^T \eta dW, \end{aligned} \tag{4.13}$$

where $\alpha = \eta + \theta^*\gamma$ is the market price of risk and $W = B + \int \alpha dt$ is a Brownian motion under the pricing measure $\mathbb{Q}$. As R is non-negative [see (3.2)], and by Lemma 4.5,

$$R \leq c(b) \triangleq -\log(1-b) > 0,$$

we deduce from the second equality in (4.13) that

$$\|\alpha\|^2_{\mathrm{BMO}(\mathbb{Q})} \leq 2c(b).$$

As the stochastic exponential $\mathscr{E}(\alpha \cdot W)$ is the density of $\mathbb{P}$ with respect to $\mathbb{Q}$ we deduce from Lemma 4.6 that Ψ is $\mathbb{Q}$-integrable and that there is a constant $C_1 = C_1(n, b)$ such that

$$\|\Psi - \mathbb{E}_{\mathbb{Q}}[\Psi]\|_{\mathrm{BMO}(\mathbb{Q})} \leq C_1 \|\Psi\|_{\mathrm{BMO}}.$$

As $S = S_0 + \theta \cdot W$, we have

$$\|\theta\|_{\mathrm{BMO}(\mathbb{Q})} = \|S - S_0\|_{\mathrm{BMO}(\mathbb{Q})} = \|\Psi - \mathbb{E}_{\mathbb{Q}}[\Psi]\|_{\mathrm{BMO}(\mathbb{Q})}.$$

Then, by (4.10), there is a constant $C_2 = C_2(n, b)$ such that

$$\|\theta\|_{\mathrm{BMO}} \leq C_2 \|\theta\|_{\mathrm{BMO}(\mathbb{Q})} \leq C_1 C_2 \|\Psi\|_{\mathrm{BMO}}.$$

Finally, since $\theta \in \mathscr{H}_{\mathrm{BMO}}$ and $R \geq 0$, from the first equality in (4.13) we deduce that $\eta \in \mathscr{H}_{\mathrm{BMO}}$ and, as $\|\gamma\|_\infty \leq 1$, that

$$\|\eta\|_{\mathrm{BMO}} \leq \|\theta^* \gamma\|_{\mathrm{BMO}} \leq \|\theta\|_{\mathrm{BMO}}.$$

This yields (4.12) with $K = 2C_1C_2$.

Conversely, let (S, R, η, θ) be a solution of the BSDE (3.3)–(3.4) with $\theta \in \mathscr{H}_{\mathrm{BMO}}(\mathbf{R}^{n\times d})$ and $\eta \in \mathscr{H}_{\mathrm{BMO}}(\mathbf{R}^d)$. In view of Theorem 3.1, we only have to verify the uniform integrability of the local martingales $Z = \mathscr{E}((\eta + \theta^*\gamma) \cdot B)$, ZS, and $Z(\gamma \cdot S)$. This readily follows from θ and η being in $\mathscr{H}_{\mathrm{BMO}}$. □

Proof of Theorem 4.1. In view of the homogeneity relations (4.3) and (4.4), it is sufficient to prove the result under the extra condition (4.5). Without loss of generality, we can also assume that $\mathbb{E}[\Psi] = 0$.

By Theorem A.1 in Appendix A, there is a constant $b = b(n) > 0$ such that if

$$\|\Psi\|_{\mathrm{BMO}} \leq b,$$

then among $(\eta, \theta) \in \mathrm{BMO}(\mathbf{R}^d \times \mathbf{R}^{n\times d})$ with

$$\|(\eta, \theta)\|_{\mathrm{BMO}} \leq 2b, \tag{4.14}$$

there is only one solution (S, R, η, θ) of (3.3)–(3.4) and this solution satisfies

$$\|(\eta, \theta)\|_{\mathrm{BMO}} \leq 2\|\Psi\|_{\mathrm{BMO}}. \tag{4.15}$$

Lemma 4.7 then implies that γ is a viable demand accompanied by stocks' prices S.

From Lemmas 4.5 and 4.7 and accounting for (4.8), we deduce the existence of a constant $c = c(n, b) \le b$ such that if

$$\|\Psi\|_{\mathrm{BMO}} \le c,$$

then every solution (S, R, η, θ) of (3.3)–(3.4) satisfies (4.14). Hence, there is only one such solution, and thus stocks' prices S are defined uniquely.

Finally, from (4.15) and (3.5)–(3.6) we obtain

$$\begin{aligned}\|\sigma\|_{\mathrm{BMO}} &= \|\theta\|_{\mathrm{BMO}} \le 2\|\Psi\|_{\mathrm{BMO}},\\ \|\alpha\|_{\mathrm{BMO}} &= \|\eta + \theta^*\gamma\|_{\mathrm{BMO}} \le \|\eta\|_{\mathrm{BMO}} + \|\theta\|_{\mathrm{BMO}} \le 4\|\Psi\|_{\mathrm{BMO}},\end{aligned}$$

which, under (4.5), is precisely (4.2). □

4.3 Proof of Proposition 4.3

The proof is divided into lemmas. We begin with a "backward localization" result which does not require either (A1) or (A2).

Lemma 4.8. *Let Ψ be a bounded n-dimensional random variable representing the stocks' dividends and γ be a viable demand for Ψ accompanied by stock's prices S. Let τ be a stopping time taking values in $[0, T]$. Then the predictable process*

$$\gamma'_t \triangleq \gamma_t 1_{\{t>\tau\}}, \quad t \in [0, T],$$

is a viable demand for the stocks' dividends

$$\Psi' = \Psi 1_{\{\tau<T\}}$$

and there are stocks' prices S' for Ψ' and γ' such that

$$S'_t = S_t, \quad t > \tau. \tag{4.16}$$

Proof. To simplify notation, take the risk-aversion $a = 1$. Let $\mathbb{Q}$ be the pricing measure for γ and S, that is,

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \mathrm{const}\, e^{-\int_0^T \gamma dS}.$$

From the martingale property of $\gamma \cdot S$ and Jensen's inequality, we deduce

$$\mathbb{E}^{\mathbb{Q}}[e^{\int_0^\tau \gamma dS}] \le \mathbb{E}^{\mathbb{Q}}[e^{\int_0^T \gamma dS}] < \infty.$$

This allows us to define the probability measure $\mathbb{Q}'$ such that

$$\frac{d\mathbb{Q}'}{d\mathbb{Q}} = \frac{e^{\int_0^\tau \gamma dS}}{\mathbb{E}^{\mathbb{Q}}[e^{\int_0^\tau \gamma dS}]}.$$

Then

$$\frac{d\mathbb{Q}'}{d\mathbb{P}} = \frac{e^{-\int_\tau^T \gamma dS}}{\mathbb{E}[e^{-\int_\tau^T \gamma dS}]} = \frac{e^{-\int_0^T \gamma' dS}}{\mathbb{E}[e^{-\int_0^T \gamma' dS}]}.$$

Define the bounded $\mathbb{Q}'$-martingale

$$S'_t \triangleq \mathbb{E}^{\mathbb{Q}'}[\Psi'|\mathscr{F}_t] = \mathbb{E}^{\mathbb{Q}'}[\Psi 1_{\{\tau<T\}}|\mathscr{F}_t], \quad t \in [0,T].$$

To show that S' is a desired price process for Ψ' and γ', we need to verify (4.16) and the $\mathbb{Q}'$-martingale property of $\gamma' \cdot S'$.

Since the density of $d\mathbb{Q}'/d\mathbb{Q}$ is $\mathscr{F}_\tau$-measurable, the conditional expectations of $\mathbb{Q}$ and $\mathbb{Q}'$ with respect to the σ-algebras $\mathscr{F}_{\tau\vee t}$, $t \in [0,T]$, coincide. This readily implies (4.16). We also deduce that if N is a $\mathbb{Q}$-martingale then

$$N'_t \triangleq N_t - N_{t\wedge\tau} = \int_0^t 1_{\{s>\tau\}} dN_s, \quad t \in [0,T],$$

is a $\mathbb{Q}'$-martingale. In particular, as

$$\int_0^t \gamma' dS' = \int_0^t 1_{\{r>\tau\}} \gamma_r dS_r, \quad t \in [0,T],$$

we obtain that $\gamma' \cdot S'$ is a $\mathbb{Q}'$-martingale. □

The following lemma contains the main idea behind the proof of Proposition 4.3. In its formulation, all processes and random variables are one-dimensional.

Lemma 4.9. *Let B be a Brownian motion, Ψ be a random variable different from a constant, and γ be a predictable process such that*

$$|\Psi(\omega)| = |\gamma_t(\omega)| = 1, \quad \mathbb{P}[d\omega] \times dt - a.s..$$

Then there is no a solution (S, R, η, θ) of the BSDE

$$R_t = \frac{1}{2}\int_t^T (\theta_s^2 - \eta_s^2) ds - \int_t^T \eta dB, \tag{4.17}$$

$$S_t = \Psi - \int_t^T \theta_s(\eta_s + \theta_s\gamma_s) ds - \int_t^T \theta dB, \tag{4.18}$$

with bounded S, non-negative R, and such that

$$\operatorname{sign}(S_t(\omega)) = -\gamma_t(\omega), \quad \mathbb{P}[d\omega] \times dt - a.s.. \tag{4.19}$$

Proof. Suppose, on the contrary, that (S, R, η, θ) solves (4.17)–(4.18) and that S is bounded, R is non-negative, and (4.19) holds. As in the proof of Lemma 4.5, define the function

$$F(x) \triangleq e^{|x|}(1 - |x|), \quad x \in \mathbf{R},$$

and observe that it is twice continuously differentiable and solves

$$F(x) - 2F'(x)\,\mathrm{sign}(x) + F''(x) = 0. \tag{4.20}$$

From the Itô's formula and the equations (4.17)–(4.18) for R and S, we deduce that

$$\begin{aligned} de^{-R_t} &= e^{-R_t}\big(-\eta_t dB + \frac{1}{2}\theta_t^2 dt\big), \\ dF(S_t) &= F'(S_t)\theta_t dB + \big(F'(S_t)\theta_t(\eta_t + \theta_t\gamma_t) + \frac{1}{2}F''(S_t)\theta_t^2\big)dt. \end{aligned}$$

Applying Itô's formula to

$$V_t = F(S_t)e^{-R_t}, \quad t \in [0, T],$$

we then obtain that

$$V_t = M_t + \int_0^t e^{-R_s} A_s ds,$$

where M is a local martingale and

$$A_t = \frac{1}{2}\theta_t^2 \left(F(S_t) + 2F'(S_t)\gamma_t + F''(S_t)\right) = 0,$$

because of (4.19) and (4.20).

Thus, V is a local martingale. As S is bounded and R is non-negative, V is bounded, and hence, is a martingale. Since

$$V_T = F(S_T)e^{-R_T} = F(\Psi) = 0,$$

we deduce that $V = 0$, and hence, that $|S| = 1$. However, as S is a continuous one-dimensional process, S equals to a constant, which contradicts the assumption that $\Psi = S_T$ is not a constant. □

Proof of Proposition 4.3. In view of the self-similarity relations (4.3), it is sufficient to consider the case $a = 1$. Take

$$\Psi \triangleq \mathrm{sign}(B_T), \quad \gamma \triangleq -\mathrm{sign}(B) \tag{4.21}$$

and assume that γ is accompanied by a price process S. Lemma 4.9 yields the contradiction if

$$\operatorname{sign}(S_r) = \operatorname{sign}(B_r), \quad r \in (0,T). \tag{4.22}$$

Fix $r \in (0,T)$, define the stopping time

$$\tau = \tau(r) \triangleq \inf\{t \geq r : \ B_t = 0\} \wedge T,$$

and observe that (4.22) holds if

$$S_\tau = 0 \quad \text{on the set} \quad \{\tau < T\}. \tag{4.23}$$

Indeed, in this case,

$$S_\tau = \Psi 1_{\{\tau=T\}} = \operatorname{sign}(B_T) 1_{\{\tau=T\}} = \operatorname{sign}(B_r) 1_{\{\tau=T\}}$$

and, as S is a martingale under the pricing measure $\mathbb{Q}$, we obtain

$$S_r = \mathbb{E}^{\mathbb{Q}}[S_\tau|\mathscr{F}_r] = \operatorname{sign}(B_r)\mathbb{Q}[\tau = T|\mathscr{F}_r].$$

This readily implies (4.22) after we observe that, because $r < T$ and $\mathbb{Q}$ is equivalent to $\mathbb{P}$, the conditional probability

$$\mathbb{Q}[\tau = T|\mathscr{F}_r] = \mathbb{Q}[\inf_{t\in[r,T]} |B_t| > 0|\mathscr{F}_r]$$

is strictly positive.

In view of (A2), the stock price S admits the representation

$$S_t(\omega) = X_t(B(\omega)) = X_t((B_s(\omega))_{0\leq s\leq t}),$$

in terms of a continuous adapted process X defined on the canonical Wiener space of continuous functions on $[0,T]$. Define a Brownian motion

$$\widetilde{B}_t \triangleq \int_0^t \operatorname{sign}(\tau - r) dB_r = B_t 1_{\{t\leq\tau\}} - B_t 1_{\{t>\tau\}}, \quad t \in [0,T],$$

and observe that, as S corresponds to Ψ and γ from (4.21), the continuous semi-martingale

$$\widetilde{S}_t \triangleq -X_t(\widetilde{B}), \quad t \in [0,T],$$

accompanies $\widetilde{\Psi}$ and $\widetilde{\gamma}$ given by

$$\widetilde{\Psi} \triangleq -\operatorname{sign}(\widetilde{B}_T), \quad \widetilde{\gamma} \triangleq \operatorname{sign}(\widetilde{B}).$$

By construction,

$$S_t = X_t((B_s)_{s\le t}) = -\widetilde{S}_t, \quad t \le \tau, \tag{4.24}$$

and

$$\begin{aligned}\Psi' &\triangleq \operatorname{sign}(B_T)1_{\{\tau<T\}} = \Psi 1_{\{\tau<T\}} = \widetilde{\Psi}1_{\{\tau<T\}}\\ \gamma'_t &\triangleq -\operatorname{sign}(B_t)1_{\{t>\tau\}} = \gamma_t 1_{\{t>\tau\}} = \widetilde{\gamma}_t 1_{\{t>\tau\}}, \quad t \in [0,T].\end{aligned}$$

If Ψ' and γ' are accompanied by the *unique* price process S' then, by Lemma 4.9,

$$S'_t = S_t = \widetilde{S}_t, \quad t > \tau,$$

and, in particular,

$$S_\tau = \widetilde{S}_\tau, \quad \tau < T,$$

which jointly with (4.24) implies (4.23). Thus, we have a contradiction. □

A BSDE with quadratic growth in BMO

As before, we work on a complete filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in[0,T]}, \mathbb{P})$ where T is a finite time horizon and assume that (A1) holds.

Consider the n-dimensional BSDE:

$$Y_t = \Xi + \int_t^T f(s,\zeta_s)\,ds - \int_t^T \zeta_s\,dB_s, \quad t \in [0,T]. \tag{A.1}$$

Here, Y is an n-dimensional semi-martingale, ζ is a predictable process with values in the space of $n \times d$ matrices, and the terminal condition Ξ and the driver $f = f(t,z)$ satisfy the following assumptions:

(A3) Ξ is an integrable random variable with values in $\mathbf{R}^n$ such that the martingale

$$L_t \triangleq \mathbb{E}_t[\Xi] - \mathbb{E}[\Xi], \quad t \in [0,T],$$

belongs to BMO.

(A4) $t \mapsto f(t,z)$ is a predictable process with values in $\mathbf{R}^n$,

$$f(t,0) = 0,$$

and there is a constant $\Theta > 0$ such that

$$|f(t,u) - f(t,v)| \le \Theta(|u-v|)(|u| + |v|),$$

for all $t \in [0,T]$ and $u, v \in \mathbf{R}^{n\times m}$.

Note that $f = f(t,z)$ has a quadratic growth in z.

Recall that there is a constant $\kappa = \kappa(n)$ such that, for every martingale $M \in \mathrm{BMO}(\mathbf{R}^n)$,

$$\frac{1}{\kappa}\|M\|_{\mathrm{BMO}} \le \|M\|_{\mathrm{BMO}_1} \triangleq \sup_\tau \|\mathbb{E}_\tau[|M_T - M_\tau|]\|_\infty \le \|M\|_{\mathrm{BMO}}; \tag{A.2}$$

see [8], Corollary 2.1, page 28. Hereafter, we fix the constants κ and Θ from (A.2) and (A4) and use the BMO-martingale L from (A3).

Theorem A.1. *Assume (A1), (A3) and (A4). If*

$$\|L\|_{\mathrm{BMO}} < \frac{1}{8\kappa\Theta}, \tag{A.3}$$

then there is $\zeta \in \mathscr{H}_{\mathrm{BMO}}$ solving (A.1) and such that

$$\|\zeta\|_{\mathrm{BMO}} \le 2\|L\|_{\mathrm{BMO}}. \tag{A.4}$$

Moreover, if (A.3) holds and $\zeta' \in \mathscr{H}_{\mathrm{BMO}}$ is another solution to (A.1) such that

$$\|\zeta'\|_{\mathrm{BMO}} \le \frac{1}{4\kappa\Theta}, \tag{A.5}$$

then $\zeta = \zeta'$.

Remark A.2. Theorem A.1 extends Proposition 1 in Tevzadze [14], where the terminal condition Ψ is supposed to have sufficiently small $\mathbf{L}_\infty$-norm. A similar extension to the case $\Psi \in \mathrm{BMO}$ has been obtained independently in Proposition 2.1 of Frei [2], however, with slightly different constants.

We are unaware of any general result on the *global* uniqueness of a *local* solution ζ from Theorem A.1; that is, the uniqueness of ζ in the whole space $\mathscr{H}_{\mathrm{BMO}}$, without the constraint (A.5). This highlights the relevance of Theorem 4.1 which, to the best of our knowledge, is the first example of a coupled system of quadratic BSDEs where such uniqueness is established.

We divide the proof of Theorem A.1 into lemmas.

Lemma A.3. *Assume (A1), (A3), and (A4). For $\zeta \in \mathscr{H}_{\mathrm{BMO}}$, there is unique $\zeta' \in \mathscr{H}_{\mathrm{BMO}}$ such that*

$$(\zeta' \cdot B)_t = \mathbb{E}_t\left[\Xi + \int_0^T f(s,\zeta_s)ds\right] - \mathbb{E}\left[\Xi + \int_0^T f(s,\zeta_s)ds\right]. \tag{A.6}$$

Moreover,

$$\|\zeta'\|_{\mathrm{BMO}} \le \|L\|_{\mathrm{BMO}} + 2\kappa\Theta\|\zeta\|^2_{\mathrm{BMO}}. \tag{A.7}$$

Proof. Define the martingale

$$M_t \triangleq \mathbb{E}_t\left[\int_0^T f(s,\zeta_s)ds\right] - \mathbb{E}\left[\int_0^T f(s,\zeta_s)ds\right].$$

For a stopping time τ, we deduce from (A4) and the Itô's isometry that

$$\begin{aligned}\mathbb{E}_\tau[|M_T - M_\tau|] &= \mathbb{E}_\tau\left[\left|\int_\tau^T f(s,\zeta_s)ds - \mathbb{E}_\tau[\int_\tau^T f(s,\zeta_s)ds]\right|\right] \\ &\leq 2\mathbb{E}_\tau\left[\int_\tau^T |f(s,\zeta_s)|\,ds\right] \leq 2\Theta\mathbb{E}_\tau\left[\int_\tau^T |\zeta_s|^2\,ds\right] \\ &= 2\Theta\mathbb{E}_\tau\left[\left|\int_\tau^T \zeta dB\right|^2\right].\end{aligned}$$

Accounting for (A.2) we obtain

$$\|M\|_{\mathrm{BMO}} \leq 2\kappa\Theta(\|\zeta\cdot B\|_{\mathrm{BMO}})^2 = 2\kappa\Theta\|\zeta\|^2_{\mathrm{BMO}}.$$

This shows that the martingale on the right-hand side of (A.6) belongs to BMO. In view of (A1) it then admits an integral representation as $\zeta'\cdot B$ for some $\zeta' \in \mathscr{H}_{\mathrm{BMO}}$. We clearly have that ζ' is unique in $\mathscr{H}_{\mathrm{BMO}}$ and

$$\|\zeta'\|_{\mathrm{BMO}} = \|\zeta'\cdot B\|_{\mathrm{BMO}} \leq \|L\|_{\mathrm{BMO}} + \|M\|_{\mathrm{BMO}}.$$

□

Lemma A.3 allows us to define the map

$$F:\ \mathscr{H}_{\mathrm{BMO}} \to \mathscr{H}_{\mathrm{BMO}}$$

such that $\zeta' = F(\zeta)$ is given by (A.6).

Lemma A.4. *Assume (A1), (A3) and (A4). Let ζ and ζ' be in $\mathscr{H}_{\mathrm{BMO}}$. Then*

$$\|F(\zeta) - F(\zeta')\|_{\mathrm{BMO}} \leq 2\kappa\Theta\|\zeta-\zeta'\|_{\mathrm{BMO}}(\|\zeta\|_{\mathrm{BMO}} + \|\zeta'\|_{\mathrm{BMO}}).$$

Proof. We have

$$\|F(\zeta) - F(\zeta')\|_{\mathrm{BMO}} = \|M\|_{\mathrm{BMO}},$$

where

$$M_t \triangleq \mathbb{E}_t\left[\int_0^T (f(s,\zeta_s) - f(s,\zeta'_s))ds\right] - \mathbb{E}\left[\int_0^T (f(s,\zeta_s) - f(s,\zeta'_s))ds\right].$$

For a stopping time τ, we deduce from (A4) that

$$\mathbb{E}_\tau[|M_T - M_\tau|] \leq 2\Theta\mathbb{E}_\tau\left[\int_\tau^T \left|\zeta_s - \zeta'_s\right| \left(|\zeta_s| + \left|\zeta'_s\right|\right) ds\right].$$

Cauchy's inequality and Itô's isometry then yield

$$\mathbb{E}_\tau[|M_T - M_\tau|] \leq 2\Theta\|\zeta - \zeta'\|_{\mathrm{BMO}}(\|\zeta\|_{\mathrm{BMO}} + \|\zeta'\|_{\mathrm{BMO}}).$$

The result now follows from (A.2). □

Proof of Theorem A.1. From Lemma A.3 we deduce that F maps the ball of the radius $R \triangleq \frac{1}{4\kappa\Theta}$ into the ball of the radius

$$R' = \|L\|_{\mathrm{BMO}} + 2\kappa\Theta R^2 < R.$$

From Lemma A.4 we obtain that F is a contraction on the ball of the radius R': if $\zeta, \zeta' \in \mathscr{H}^{\mathrm{BMO}}$ and $\max(\|\zeta\|_{\mathrm{BMO}}, \|\zeta'\|_{\mathrm{BMO}}) \leq R'$, then

$$\begin{aligned}\|F(\zeta) - F(\zeta')\|_{\mathrm{BMO}} &\leq 2\kappa\Theta\|\zeta - \zeta'\|_{\mathrm{BMO}}(\|\zeta\|_{\mathrm{BMO}} + \|\zeta'\|_{\mathrm{BMO}}) \\ &\leq \frac{R'}{R}\|\zeta - \zeta'\|_{\mathrm{BMO}}.\end{aligned}$$

Banach's fixed-point theorem now implies the existence and uniqueness of $\zeta \in \mathscr{H}^{\mathrm{BMO}}$ such that $\|\zeta\|_{\mathrm{BMO}} \leq R$ and $F(\zeta) = \zeta$. The estimate (A.4) for ζ follows from (A.7):

$$\|\zeta\|_{\mathrm{BMO}} \leq \|L\|_{\mathrm{BMO}} + 2\kappa\Theta\|\zeta\|^2_{\mathrm{BMO}} \leq \|L\|_{\mathrm{BMO}} + \frac{1}{2}\|\zeta\|_{\mathrm{BMO}}.$$

It only remains to observe that the fixed points of F are in one-to-one correspondence with the solutions ζ to (A.1) such that $\zeta \cdot B \in \mathrm{BMO}$. □

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