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Variational formulation of divergence stability for constrained systems

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This paper deals with both divergence and second order work criteria and the kinematical structural stability called ki.s.s. In this context, kinematical structural stability means that the criterion remains valid even if the system is subjected to additional kinematic constraints. First some developments about the effect of additional kinematics constraints are presented on divergence instability. Secondly, divergence and second order work criteria are addressed. Using a variational formulation, previous results from a usual algebraic formulation using Schur's complement formula are highlighted and finally translated through the ki.s.s. concept: unconditional ki.s.s. for the second order work and the divergence of conservative systems and conditional ki.s.s. for the divergence of nonconservative elastic systems.

Keywords:
Variational formulation
Divergence
Nonconservative systems
Second order work criterion
Kinematical structural stability

1. Introduction

This paper deals with the statics or the dynamics of a discrete or discretized mechanical system Σ_{free} whose any coordinate system is usually noted $q = (q_1, \dots, q_n)$ which especially means that Σ_{free} is a n degree of freedom system. In this paper, it is assumed that, after having started with different possible non linear settings, convenient assumptions and approximations lead to a dynamic evolution governed by the following equation of motion of the free system Σ_{free} :

$$M\ddot{X} + K(p)X = 0, \quad (1)$$

where $K(p) = K_s(p) + K_a(p)$ with $K_s(p) = \frac{1}{2}(K(p) + K^T(p))$ and $K_a(p) = \frac{1}{2}(K(p) - K^T(p))$ is generally a non-symmetric matrix ($K_a(p) \neq 0$), due to the possible non conservativeness of Σ_{free} whereas M is symmetric definite positive. M is the mass matrix and $K = K(p)$ is the stiffness matrix.

We may assume that there is an equilibrium configuration q_e and (1) is derived from Lagrange equations by a linearization process about q_e . Then, $X = q - q_e \in \mathbb{R}^n$ and all its derivatives with respect to the time are supposed first order small quantities. The so-called circulatory elastic systems like Ziegler systems used in the last part for illustrating the results are relevant examples of such systems. The load parameter p is a possible continuously increasing real parameter. In Ziegler systems this is the value of the external follower force or an equivalent dimensionless parameter. Obviously, in this case, M and K depend also on q_e .

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Nonconservativity of Σ_{free} means in this framework that Σ_{free} is a system whose at least one of the internal or external actions does not derive from a potential or equivalently such that the total work of these actions between two states depends on the path between these states or again such that there are closed loops with not nil energy balance. But we specifically deal with systems whose nonconservative actions (internal and external) are only positional, namely depend only on the positions of the particles of the system. For us, describing a system obviously involves the external actions acting on the considered mechanical system. The source of the nonconservativity of positional actions lies in the geometry properties of these actions. As example, thinking again to the usual 2 degree of freedom Ziegler system with a complete follower force, it is well-known that this force does not derive from a potential and that there is a mechanical cycle with a not nil energy balance. In a linear framework as the one used in the paper, the mathematical signature of the nonconservativity of Σ_{free} is the non symmetry of the stiffness matrix. From a dynamical point of view, such mechanical systems can exhibit flutter type instabilities. This point is deeply investigating in the following.

The notion of stability that we refer to in this paper is the most common namely the Lyapunov stability: a small perturbation of initial conditions leads to a small (kinematic) response of the system (around the equilibrium configuration). Throughout this paper, this is always understood in a linear meaning, namely for the dynamical system governed by (1). On the other hand, the types of stability we investigate are mainly the two types of linear stability: the divergence type and the flutter one. The third used criterion we will also refer to is the so-called second order work criterion (SOWC).

We now introduce the new concept of kinematical structural stability (ki.s.s.). We say that a (stability) criterion is kinematical structurally stable (ki.s.s.) if this criterion is still satisfied for any constrained system Σ_{cons} built from Σ_{free} by adding kinematic constraints. Here kinematic constraints mean holonomic linear kinematic constraints. For example unilateral constraints that may be only piecewise linear and then non smooth are not taken into account. Moreover, it is supposed that additional kinematic constraints do not perturb the validity of the linear setting of (1). Especially, when an equilibrium configuration q_e is involved, q_e does not change under these additional kinematic constraints. For example, the well-known counter-example of Tarnai [1] for conservative systems does not fall within the areas of our investigations. The associated concept is the kinematical structural stability (ki.s.s.) of the corresponding criterion. We also speak about the ki.s.s. of the corresponding type of stability. Thus we will speak about divergence ki.s.s. and flutter ki.s.s. Mathematically speaking, if $[0, p_{free}^*]$ is the stability domain of Σ_{free} for the investigated criterion and if $[0, p_{cons}^*]$ is the corresponding stability domain of Σ_{cons} , the ki.s.s. issue deals with the link between $[0, p_{free}^*]$ and $[0, p_{cons}^*]$ for any Σ_{cons} . The ki.s.s. holds when $[0, p_{free}^*] \subset [0, p_{cons}^*]$ or when, $p_{free}^* \leq p_{cons}^*$ for any Σ_{cons} . In such a case, we also speak about universal ki.s.s.

Since Rayleigh's works, it is well-known that adding kinematic constraints to a stable conservative system improves its stability and the only mode of instability is divergence instability. Thanks to the Rayleigh quotient and more generally to the variational formulation of the divergence criterion by using quadratic form language, the kinematical structural stability for the divergence criterion of conservative elastic systems is ensured: the divergence stability criterion is then kinematically structurally stable for conservative systems. Another way to sum up the results for conservative systems is to say that divergence ki.s.s. is universal for conservative systems.

For a couple of years, the same ki.s.s. issue of divergence ki.s.s. but for elastic non-conservative systems has been investigated. Several results have been proved and are now summarized (see for example [2,3] for detailed developments).

Let p_{sw} be the smaller positive root of $\det(K_s(p)) = 0$. As long as $0 \leq p < p_{sw}$ then $\det(K(p)) > 0$ and we then say that the second order work criterion (SOWC) holds. For $p = p_{sw}$ the SOWC then fails. This criterion involving only the stiffness matrix $K(p)$ and especially its symmetric parts $K_s(p)$ has first been proposed by Hill [4] for the investigation of uniqueness issues for incremental formulation for non associative plasticity and has been studied extensively since Hill's work (see for example [5-7]) in the framework of continuous media.

This criterion has also been recently reinvestigated independently for discrete systems [8] especially for elastic systems subjected to non-conservative follower forces namely the so-called nonconservative systems in this paper. Although these kinds of systems had been considered as unrealistic for many years [9], they have been rehabilitated in a recent paper and its experimental associated device [10] (see also [11]).

As mentioned above, surprisingly, the SOWC more recently reappeared in a completely different way in the framework of the divergence ki.s.s. for nonconservative systems.

Roughly speaking, the method presented in [2,3] for investigating the stability of a system subjected to m additional constraints is the following:

- First, the size of the dynamic system is extended by introducing Lagrange's multiplier Λ , the $n \times n$ stiffness matrix $K(p)$ in (1) becoming an $(n + m) \times (n + m)$ matrix $\mathcal{K}(p, \Lambda)$.
- Secondly, the stability is characterized by an algebraic condition on $\det(\mathcal{K}(p, \Lambda))$.
- Thirdly, the trick of Schur's complement formula on $\det(\mathcal{K}(p, \Lambda))$ is used.
- Finally, the nature of $K_s(p)$ "naturally" emerges after these transformations and the constrained system remains divergence stable if $\det K_s(p) \neq 0$. Continuity arguments lead more precisely to $\det K_s(p) > 0$: this is the SOWC for the initial free system Σ_{free} .

Thus, the natural emergence of the SOWC by considerations about constrained systems leads us here to investigate the issue of the stability behavior of any constrained system Σ_{cons} by referring to the stability behavior of the free system Σ_{free} . As

mentioned above, the unconditional stability of the constrained systems Σ_{cons} is called the universal divergence ki.s.s. Even though the concept of ki.s.s. has never been defined until now – to the best of our knowledge –, the already known results about the ki.s.s. may then be summarized as follows:

- For a conservative system Σ_{free} , no additional kinematic constraint may destabilize a (divergence) stable system for any load parameter. One will then talk about universal ki.s.s. for the divergence of conservative systems.
- For a non-conservative system, as far as the second order work criterion is satisfied ($p < p_{sw}$), no additional kinematic constraint may destabilize a (divergence) stable system by divergence. When ($p = p_{sw}$), the non-conservative system may be destabilized (for the divergence criterion) by adding one or several kinematic constraints and this (or these) constraint(s) may be chosen on the image by $K^{-1}(p_{sw})$ of the isotropic cone of the quadratic form associated with $K_s(p)$. One will then talk about conditional ki.s.s. for the divergence of elastic non conservative systems (see again [2,3]).
- The second order work criterion itself is always kinematically structurally stable and coincides with the divergence criterion for conservative systems. Therefore, the second order work criterion is universally ki.s.s.

In this paper, we systematically tackle the divergence ki.s.s. issue for non conservative systems through a completely different approach from the one used in our previous papers [2,3]: a variational formulation will be used to handle the problem. There are at least three underlying reasons to do so:

1. The first one is historical. Indeed, the first results about the ki.s.s. issue date back to the XIXth century (see Rayleigh's work, see [12] for example). They come from considerations about Rayleigh's quotient $R(X) = \frac{X^T K X}{X^T M X}$ for conservative systems which is nothing else than a variational formulation allowing to evaluate especially the lower eigenfrequency $\omega_1^2 = \min_{X \in \mathbb{R}^n} R(X)$. The strict positivity of ω_1^2 may be, as is well-known, linked to the divergence stability. It can be noticed that, in this case, the (divergence) ki.s.s. seems to be an obvious consequence of this formulation because the minimum of R on a subspace cannot be lower than the one on the whole space. The fact that the elimination of Lagrange's multiplier does not affect Rayleigh's quotient is rarely highlighted although it is absolutely necessary to validate the reasoning. This fact will be highlighted hereafter.
2. The second one is geometrical. Indeed, as mentioned in the previous reasoning, adding constraints is clearly associated with the geometric operation of a projection. In fact, it is apparently more than a simple projection as it consists more precisely of a double projection (namely which is called a compression in mathematics) that appears clearly through a variational formulation of the criteria. This provides a complete coordinate free formulation of the ki.s.s. according to any kind of criterion.
3. The third one is epistemological: a double and complementary (more than opposite) way to tackle an issue may shed light on the matter.

We first start by recalling the usual forms of the criteria involved and by building a variational formulation for the divergence stability especially for non conservative systems with non symmetric stiffness matrices. The second part deals with the issue of additional kinematic constraints, the systematic elimination of Lagrange's multiplier and finally the general rule to study any ki.s.s. We then take the opportunity of this rule to derive a coordinate free geometric formulation. The third section focuses on the application of this rule to divergence ki.s.s. As previously claimed, this original formulation allows one to find again, but in a completely independent way, the main results recently obtained in [2,3] for example. In the last section, general considerations on the flutter stability criterion conclude this paper, this difficult issue being dealt with in a forthcoming paper. Let us specify what means a variational form at once. It simply means that the criterion is formulated with a variable (noted X in the text because it is often a usual vector X of \mathbb{R}^n in the applications) so that this variable is any in a set and so that the formulation always begins by " $\forall X \in \dots$ ". For us, it does not necessary imply calculus of variations (for example the variable does not necessary allow differential calculation and could belong to a discrete set).

2. General considerations about stability criteria of elastic systems

2.1. Usual approaches

Let us remember that we assume that for $p = 0$ the corresponding system is a conservative stable system with a symmetric positive definite stiffness matrix $K(0)$. The load parameter p is continuously increasing. We shall distinguish three criteria in investigating of stability of the dynamics governed by (1). The first one, called instability by divergence, also corresponds to the loss of uniqueness of the solutions of

$$K(p)X = 0, \tag{2}$$

where the inertial terms (here the mass matrix M) are not involved. p_d is then the corresponding critical load of (in) stability and may be defined as the minimal positive value of the p solution of

$$\det(K(p)) = 0. \tag{3}$$

Because of the assumption on the loading path, the usual criterion of divergence stability reads:

$$\det(K(p)) > 0. \quad (4)$$

The second criterion of stability is not, strictly speaking, a stability criterion because there is no direct relationship between this criterion and the Lyapunov stability of (1). As previously mentioned in the introduction, it was first formulated by Hill in the years 1958–1959 in an incrementally non linear framework for associated and/or non associated plasticity [13,4] and independently proposed in 2004 by Absi and Lerbet [8] in the framework of structural stability. It is however directly linked with the divergence stability criterion because as long as it holds, the divergence stability of the system is ensured. The so-called second order work criterion consists in assuming that the quadratic form $X \mapsto X^T K(p) X$ is definite positive. That criterion only involves the symmetric part $K_s(p)$ of $K(p)$ and the critical load p_{sw} for the second order work criterion can be viewed as the minimal positive value such that there is $X \neq 0 \in \mathbb{R}^n$ with $X \mapsto X^T K(p_{sw}) X = 0$. It may be also algebraically defined by the equation $\det(K_s(p)) = 0$ and p_{sw} is then the minimal positive value of the p solution of $\det(K_s(p)) = 0$. It is worth noting that the second order work criterion is first defined by the variational formulation.

$$X^T K(p) X > 0 \quad \forall X \neq 0. \quad (5)$$

The algebraic equation

$$\det(K_s(p)) = 0 \quad (6)$$

characterizing the failing of the second order work criterion is deduced as a consequence and by use of continuity of the increasing loading, (5) is equivalent to

$$\det(K_s(p)) > 0. \quad (7)$$

The characteristic equation or the eigenvalues equation of $\Sigma_{free}(p)$ reads

$$P(p, s) = \det(K(p) + sM) = 0.$$

Taking advantage of definite positivity of M allows us to consider a square root S of M and putting $\tilde{K}(p) = S^{-1}K(p)S^{-1}$ leads to the equivalent characteristic equation:

$$\det(\tilde{K}(p) + sI) = 0. \quad (8)$$

Investigating flutter is often presented as evaluating a critical value p_f of the loading parameter p such that $P(p_f, s)$ has a multiple root s^* . The flutter critical load must preferably be thought as a value p_f of p when $\tilde{K}(p_f)$ fails to be diagonalizable in \mathbb{R} . However, we first follow the usual way involving the vanishing of the discriminant of $P(p, s)$ assuming that the eigenvalues are simple for $p = 0$.

It is well-known since Galois' work that, for polynomials of degree ≥ 5 , there is no formula allowing one to calculate the roots of $s \mapsto P(p, s)$ as functions of coefficients $a_k(p)$ of $P(p, \cdot)$ defined by:

$$P(p, X) = \sum_{k=0}^n a_{n-k}(p) X^k = a_0(p) X^n + \dots + a_n(p). \quad (9)$$

Nevertheless, the concept of discriminant remains valid for investigating the existence of multiple roots of a polynomial (see for example [14]). It is formally possible to calculate the discriminant $\Delta(p)$ of $P(p, X)$ but the mathematical expression is cumbersome. It is, roughly speaking, the resultant of $P(p, X)$ and of its derivative $P'(p, X)$; this is a determinant built from coefficients of $P(p, X)$ and $P'(p, X)$. The algebraic criterion for calculating the flutter critical load p_f then reads: p_f is the minimal positive root of

$$\Delta(p) = 0. \quad (10)$$

Thus, as mentioned above, supposing that the eigenfrequencies are all simple for $p = 0$, by continuity of the determinant and of the loading path the flutter stability criterion reads

$$\Delta(p) > 0 \quad (11)$$

without forgetting that (10) is only a necessary condition of flutter instability whereas (11) is only a sufficient condition of flutter stability.

We then have three criteria in hand. For one of these, meaning the second order work criterion, (5) and (7) are respectively the algebraic and variational formulations. For the other two criteria, the divergence and flutter (in) stabilities criteria, (4) and (11) are respectively their algebraic formulation. We will now focus on the divergence criterion.

2.2. Variational and minimization formulations of divergence stability

As a rule, the variational formulation may be split into a pure variational form and a so-called minimization formulation. The variational form always reads:

$$\forall X \in \mathcal{D}, \quad Pr(X), \quad (12)$$

where $\mathcal{D} \subset \mathbb{R}^n$ and $Pr(X)$ a logical property right that is true for the concerned X . This often looks like

$$\forall X \in \mathcal{D}_h, \quad h(X) > 0, \quad (13)$$

where h is a real valued function defined on $\mathcal{D} \supset \mathcal{D}_h$.

Minimization formulation of linear criteria often occurs as follows:

$$\min_{X \in \mathcal{D}_m} j(X) > 0, \quad (14)$$

where j is a real values function defined on $\mathcal{D} \supset \mathcal{D}_m$.

For example, for the SOWC, the variational formulation (13) reads with $\mathcal{D} = \mathbb{R}^n \setminus \{0\}$ and $h(X) = X^T \tilde{K} X = X^T \tilde{K}_s X$ whereas the minimization formulation (14) involves $\mathcal{D}_m = \mathcal{S}^n(\mathbb{R})$ and $j(X) = X^T \tilde{K} X = X^T \tilde{K}_s X = h(X)$ or $\mathcal{D}_m = \mathcal{D} = \mathbb{R}^n \setminus \{0\}$ and $j(X) = \frac{j(X)}{X^T X}$. The same frame is the one of divergence of conservative systems but, to the best of our knowledge, it has never been noted that it can be extended to the divergence of non conservative systems when K is no longer symmetric.

Indeed, (4) means that $K(p)$ remains invertible or that $K(p)X = 0 \Rightarrow X = 0$ or equivalently:

$$\|K(p)X\| = 0 \Rightarrow X = 0.$$

The divergence algebraic criterion can therefore be converted into the variational form (13) with $\mathcal{D} = \mathbb{R}^n \setminus \{0\}$ but now with $h(X) = X^T \tilde{K}^T \tilde{K} X$ and the minimization formulation (14) with $\mathcal{D}_m = \mathcal{S}^n(\mathbb{R})$ and again $j(X) = X^T \tilde{K}^T \tilde{K} X$. The SOWC and the divergence criterion involve three quadratic forms q , a and b defined by:

$$q(X) = X^T X, \quad (15)$$

$$a(X) = X^T \tilde{K} X = X^T \tilde{K}_s X, \quad (16)$$

$$b(X) = X^T \tilde{K}^T \tilde{K} X. \quad (17)$$

Note that for the SOW and divergence criteria, the mass matrix is not involved and \tilde{K} may be replaced by K . Likewise, we can systematically substitute each map j by $\frac{j}{q}$ and each $\mathcal{D}_m = \mathcal{S}^n(\mathbb{R})$ by $\mathcal{D}_m = \mathbb{R}^n \setminus \{0\}$. Finally a variational approach for flutter does not exist so far. Some remarks about flutter ki.s.s are reported at the end of this paper and this problem will definitively be tackled in a forthcoming paper.

3. Additional kinematic constraints and the quadratic forms q , a , b

3.1. Elimination of Lagrange's multipliers

Consider now a family of m independent additional kinematic constraints $C = (C^1, \dots, C^m)$ acting on the free system Σ also denoted Σ_{free} . The resulting constrained system is called $\Sigma_{cons}(C)$.

In a first step, we systematically eliminate Lagrange's multiplier Λ . This has already been done in previous other papers like in [15] but we quickly recall the main calculations.

The linear framework leads to model each kinematic constraint C^j by a linear relationship $\sum_{k=1}^n c_k^j x_k = 0$. Thus C^j is represented by and identified with a vector $C^j = (c_1^j, \dots, c_n^j)$ of \mathbb{R}^n (actually it is a linear form on \mathbb{R}^n but, through the canonical scalar product, we can identify \mathbb{R}^n and its dual space). The family of m constraints $\{C^1, \dots, C^m\}$ can be considered as an element of an nm -dimensional vector space and for instance as an $n \times m$ matrix $C = (C^1 \dots C^m) \in \mathcal{M}_{nm}(\mathbb{R})$, more precisely $C \in \mathcal{G}_{nm}(\mathbb{R})$, the open subset of matrices of $\mathcal{M}_{nm}(\mathbb{R})$ with rank m because of the independency of the constraints. Thus

$$C^T = \begin{pmatrix} C^{1T} \\ \vdots \\ C^{mT} \end{pmatrix} = \begin{pmatrix} c_1^1 & \dots & c_n^1 \\ \vdots & \vdots & \vdots \\ c_1^m & \dots & c_n^m \end{pmatrix}$$

(every vector $C^{iT} = (c_1^i \dots c_n^i)$ could be normalized by $C^{iT} C^i = 1$).

Let $\Lambda \in \mathcal{M}_m(\mathbb{R})$ with $\Lambda^T = (\lambda_1 \dots \lambda_m)$ the Lagrange multiplier attached to the constraints. The equation of motion (1) valid for the non constrained system Σ_{free} changes and reads, for the constrained system $\Sigma_{cons}(C)$:

$$C^T X = 0, \quad (18)$$

$$M\ddot{X} + K(p)X + C\Lambda = 0. \quad (19)$$

Using the change of variables $Y = SX$ where S is a square root of M and multiplying by the left by S^{-1} the Eqs. (18) and (19) leads to:

$$A^T \cdot Y = 0, \quad (20)$$

$$\ddot{Y} + \tilde{K}(p)Y + A\Lambda = 0 \quad (21)$$

with $A = S^{-1}C$ and α_k the k th column of A for $k = 1, \dots, m$.

If B is a subset of a vector space E , we note $\text{Vect}(B)$ the vector subspace spanned by B and B^\perp the subspace of E orthogonal to all the vectors of B . Thus $\text{Vect}(B^\perp) = (\text{Vect}(B))^\perp = B^\perp$ for any subset $B \subset E$. Let $T(A) = \text{Vect}\{\alpha^1, \dots, \alpha^m\}$ and let $H(A) = T(A)^\perp$ be the orthogonal to $T(A)$ in \mathbb{R}^n identified with $\mathcal{M}_{n1}(\mathbb{R})$. Thus $\dim T(A) = m$ and $\dim H(A) = n - m$.

Let $(t_1(A), \dots, t_m(A))$ be an orthonormal basis of $T(A)$ (built by the Gram–Schmidt method from $(\alpha^1, \dots, \alpha^m)$ for example) and another one $(h_{m+1}(A), \dots, h_n(A))$ of $H(A)$ such that $\mathcal{B}(A) = (t_1(A), \dots, t_m(A), h_{m+1}(A), \dots, h_n(A))$ is an orthonormal basis of \mathbb{R}^n and let $P = P(A) \in O_n(\mathbb{R})$ the orthogonal matrix passing from the canonical basis \mathcal{B}_c of \mathbb{R}^n to $\mathcal{B}(A)$:

$$P = P(A) = \text{mat}(t_1(A) \dots t_m(A) h_{m+1}(A) \dots h_n(A), \mathcal{B}_c).$$

Let $Z \in \mathcal{M}_{n1}(\mathbb{R})$ be defined by $Y = P(A)Z$. The previous system (20), (21) reads:

$$(P(A)^T A)^T Z = 0, \quad (22)$$

$$\ddot{Z} + P^T(A)\tilde{K}(p)P(A)Z + P(A)^T A\Lambda = 0 \quad (23)$$

because $P^T(A)P(A) = I_n$. Considering $\tilde{K}_{\text{cons}}(A, p) \in \mathcal{M}_{n-m}(\mathbb{R})$ the square submatrix of $P^T(A)\tilde{K}(p)P(A)$ built by removing the first m rows and the first m columns of $P^T(A)\tilde{K}(p)P(A)$, we get the following equations of the constrained system without Lagrange's multiplier:

$$\ddot{Z}_{\text{cons}} + \tilde{K}_{\text{cons}}(A, p)Z_{\text{cons}} = 0, \quad (24)$$

where $Z_{\text{cons}}^T = (z_{m+1} \dots z_n) \in \mathcal{M}_{1n-m}(\mathbb{R})$.

Parameter p is now omitted to clarify the notations. Decomposing the matrix $P^T(A)\tilde{K}P(A)$ by blocks leads to:

$$P^T(A)\tilde{K}P(A) = \begin{pmatrix} R_{\tilde{K},A} & L_{\tilde{K},A} \\ C_{\tilde{K},A} & \tilde{K}_{\text{cons}}(A) \end{pmatrix}$$

with $R_{\tilde{K},A} \in \mathcal{M}_{mm}(\mathbb{R})$, $L_{\tilde{K},A} \in \mathcal{M}_{mn-m}(\mathbb{R})$, $C_{\tilde{K},A} \in \mathcal{M}_{n-mm}(\mathbb{R})$, $\tilde{K}_{\text{cons}}(A) \in \mathcal{M}_{n-m, n-m}(\mathbb{R})$. More accurately, the block decomposition reads here (p is omitted):

$$\tilde{K}_{\text{cons}}(A) = \begin{pmatrix} h_{m+1}^T(A)\tilde{K}h_{m+1}(A) & \dots & h_{m+1}^T(A)\tilde{K}h_n(A) \\ \vdots & \ddots & \vdots \\ h_n^T(A)\tilde{K}h_{m+1}(A) & \dots & h_n^T(A)\tilde{K}h_n(A) \end{pmatrix}.$$

3.2. Expression of quadratic forms for constrained systems

The main goal of this section is first to give the formulation of Pr_{cons} from the mechanical data of the problem (matrices M , K , \tilde{K} , C , A , $H(A)$, $T(A)$) for both criteria (divergence and SOW) and secondly to find the link between Pr_{free} and Pr_{cons} for the variational formulations of each criterion as well. Obviously Pr_{free} and Pr_{cons} relate to the variational formulations for Σ_{free} and Σ_{cons} respectively.

Because they are functions of the quadratic forms q , a and b , the first and main step is to give the expression of q_{cons} , a_{cons} and b_{cons} in order to find the possible relationships between the quadratic forms q_{free} , a_{free} , b_{free} and q_{cons} , a_{cons} , b_{cons} . The relationship between the restriction of q_{free} , a_{free} and b_{free} to $H(A)$ and q_{cons} , a_{cons} and b_{cons} is now the subject of analysis.

If $Y = (y_1, \dots, y_n) \in \mathbb{R}^n \approx \mathcal{M}_{n1}(\mathbb{R})$, we note $Z = (z_1, \dots, z_n)^T = (Z_1^T Z_2^T)^T \in \mathbb{R}^n \approx \mathcal{M}_{n1}(\mathbb{R})$ such that $Y = P(A)Z$. Thus, $Y \in H(A)$ if and only if $Z_1 = 0$. Because they are quadratic forms, the values of q , a , b are invariant under the action of the orthogonal group $O_n(\mathbb{R})$. This leads to $q(Y) = q(Z)$, $a(Y) = a(Z)$, $b(Y) = b(Z)$. With obvious notations, calculations then give for all $Y \in H(A)$:

$$q_{\text{free}}(Y) = q_{\text{cons}}(Z_2), \quad (25)$$

$$a_{\text{free}}(Y) = a_{\text{cons}}(Z_2), \quad (26)$$

$$b_{\text{free}}(Y) = b_{\text{cons}}(Z_2) + b_{\text{ad}}(Z_2). \quad (27)$$

□ Calculations are carried out only for the quadratic form b . The subscript “ad” refers to “additional”. For $Y \in H(A)$, meaning for $Z = (0 Z_2^T)^T = P(A)^{-1}Y = P(A)^T Y$:

$$b_{free}(Y) = b_{free}(Z) = (0^T Z_2^T \begin{pmatrix} R_{\tilde{K},A}^T & C_{\tilde{K},A}^T \\ L_{\tilde{K},A}^T & \tilde{K}_{cons}^T(A) \end{pmatrix} \begin{pmatrix} R_{\tilde{K},A} & L_{\tilde{K},A} \\ C_{\tilde{K},A} & \tilde{K}_{cons}(A) \end{pmatrix} \begin{pmatrix} 0 \\ Z_2 \end{pmatrix})$$

leading to

$$b_{free}(Z) = (Z_2^T L_{\tilde{K},A}^T Z_2^T \tilde{K}_{cons}^T(A) \begin{pmatrix} L_{\tilde{K},A} Z_2 \\ \tilde{K}_{cons}(A) Z_2 \end{pmatrix}) = Z_2^T L_{\tilde{K},A}^T L_{\tilde{K},A} Z_2 + Z_2^T \tilde{K}_{cons}^T(A) \tilde{K}_{cons}(A) Z_2 = Z_2^T L_{\tilde{K},A}^T L_{\tilde{K},A} Z_2 + b_{cons}(Z_2). \quad (28)$$

It then follows that

$$b_{ad}(Z_2) = Z_2^T L_{\tilde{K},A}^T L_{\tilde{K},A} Z_2.$$

However, even if we may obtain a lot of informations about the SOW and divergence ki.s.s. from these relations, let us go further in investigating the functions n_{cons} , a_{cons} , b_{cons} .

If $Z_2 = (z_{m+1}, \dots, z_n) \in \mathbb{R}^{n-m} \approx \mathcal{M}_{1 \ n-m}(\mathbb{R})$, put $Y_2 = \sum_{k=m+1}^n z_k h_k(A) \in H(A)$. Remember that, for all k, ℓ , $h_k(A)^T h_\ell(A) = \delta_{k\ell}$ while $h_k(A) h_\ell^T(A) = P_{k\ell}(A)$ with $P_{kk}(A)^2 = P_{kk}(A)$ the projection matrix on $\text{Vect}(h_k(A))$ and $P_{k\ell}(A)^2 = 0$ when $k \neq \ell$. Thus

$$\sum_{k=m+1}^n P_{kk}(A) = P_{H(A)}$$

is the projection matrix on $H(A)$.

For all $1 \leq i, j \leq n-m$, calculations successively give:

$$\begin{aligned} (K_{cons}(A))_{ij} &= h_{m+i}(A) \tilde{K} h_{m+j}(A), \\ (K_{cons}^T(A))_{ij} &= h_{m+j}(A) \tilde{K} h_{m+i}(A) = h_{m+i}(A) \tilde{K}^T h_{m+j}(A), \\ (K_{cons}^T(A) K_{cons}(A))_{ij} &= \sum_{k=1}^{n-m} h_{m+i}(A) \tilde{K}^T h_{m+k}(A) h_{m+k}(A) \tilde{K} h_{m+j}(A) = h_{m+i}(A) \tilde{K}^T \sum_{k=1}^{n-m} h_{m+k}(A) h_{m+k}(A) \tilde{K} h_{m+j}(A) = h_{m+i}(A) \tilde{K}^T P_{H(A)} \tilde{K} h_{m+j}(A). \end{aligned}$$

We then deduce:

$$\begin{aligned} q_{cons}(Z_2) &= Z_2^T Z_2 = Y_2^T Y_2 = q_A(Y_2) = q_{free}(Y_2), \\ a_{cons}(Z_2) &= Z_2^T \tilde{K}_{cons}(A) Z_2 = Y_2^T \tilde{K} Y_2 = a_A(Y_2) = a_{free}(Y_2), \\ b_{cons}(Z_2) &= Z_2^T \tilde{K}_{cons}(A)^T \tilde{K}_{cons}(A) Z_2 = Y_2^T \tilde{K}^T P_{H(A)} \tilde{K} Y_2 = b_A(Y_2) \neq b_{free}(Y_2), \end{aligned}$$

which defines q_A , a_A , b_A .

In fact, the matrix of an orthogonal projection in an orthonormal basis is idempotent and symmetric. The above expressions highlight the relationships between the quadratic forms of the free system and those of the system constrained by the family $C = SA$ of kinematic constraints: q_A , a_A , b_A are exactly q_{free} , a_{free} , b_{free} after substituting \tilde{K} by $P_{H(A)} \tilde{K} P_{H(A)} = P_{H(A)}^T \tilde{K} P_{H(A)} = P_{H(A)} \tilde{K} P_{H(A)}^T = P_{H(A)}^T \tilde{K} P_{H(A)} = \tilde{K}(A)$ when acting on $Y \in H(A)$ satisfying $Y = P_{H(A)} Y$. Indeed, if $Y = P_{H(A)} Y \in H(A)$, then

$$\begin{aligned} a_A(Y) &= Y^T \tilde{K} Y = Y^T P_{H(A)}^T \tilde{K} P_{H(A)} Y = Y^T \tilde{K}(A) Y \\ b_A(Y) &= Y^T \tilde{K}^T P_{H(A)} \tilde{K} Y = Y^T \tilde{K}^T P_{H(A)} P_{H(A)} \tilde{K} Y = Y^T P_{H(A)}^T \tilde{K}^T P_{H(A)}^T P_{H(A)} \tilde{K} P_{H(A)} Y = Y^T P_{H(A)} \tilde{K}^T P_{H(A)} P_{H(A)}^T \tilde{K} P_{H(A)} Y = Y^T \tilde{K}^T(A) \tilde{K}(A) Y. \end{aligned}$$

Finally these last expressions give the same results when adding to $Y = P_{H(A)} Y \in H(A)$ any vector belonging to $T(A) = H(A)^\perp = \text{Ker} P_{H(A)}$ so that these relations remain valid for any vector $Y \in \mathbb{R}^n$.

3.3. Compression of operators

To conclude this section, let us highlight the above formulation from a more geometrical point of view. Let $u \in \mathcal{L}(E)$ be the linear map of $E = \mathbb{R}^n$ whose matrix in the canonical basis of \mathbb{R}^n is \tilde{K} . Forget that $E = \mathbb{R}^n$, suppose only that E is an euclidean space and that $(\cdot | \cdot)$ is the scalar product. If F is a vector subspace of E let us note $u_F = p \circ u|_F \in \mathcal{L}(F)$ where $u|_F$ is the restriction of u to F and p the orthogonal projector on F . Thus, with these notations, $a(x) = (x|u(x))$, $b(x) = (u(x)|u(x))$, $a_A(x) = (x|u_{H(A)}(x))$, $b_A(x) = (u_{H(A)}(x)|u_{H(A)}(x))$ and from now on, we will do the reasoning with these geometric notations.

The element

$$u_F = p \circ u|_F \in \mathcal{L}(F) \quad (29)$$

is usually called the compression of u to F . This concept may be extended to Hilbert spaces (see 4.4) and is the good framework to deal with the ki.s.s. for continuous media. (For more precisions about compressions of operators see for example [16]). From a mechanical viewpoint, the mathematical operation of the compression on F of the operator K or \tilde{K} is

exactly the process the operator must undergo in order that the dynamics takes place on the subspace F , namely for the corresponding degree of freedom reduced dynamics.

4. Consequences and variational formulation

4.1. Second order work criterion

According to the first section, the variational formulation of SOWC involves $j = \frac{a}{q}$. But for every subspace F of E and for every $x \in F$, $a_F(x) = (x|u_F(x)) = (x|p \circ u(x)) = (p(x)|u(x)) = (x|u(x)) = a(x)$ and obviously $q_F(x) = (x|x) = q(x)$. We then deduce that

$$\min_{x \in F \setminus \{0\}} j_F(x) = \min_{x \in F \setminus \{0\}} j(x) \geq \min_{x \in E \setminus \{0\}} j(x),$$

which implies the universal SOWC ki.s.s. Note that the same reasoning is still valid for divergence of conservative elastic systems as is normally used in structural mechanics. The fact that $a_F(x) = a(x)$ or with previous notations that $a_{free}(Y) = a_{cons}(Z_2)$ ((26)) is the key explaining how and why, for conservative elastic systems, the spectrum of the constrained system obtained through Rayleigh's quotient $R_{cons} = \frac{a_{cons}}{q_{cons}}$ may be calculated by the restriction of Rayleigh's quotient $R_{free} = \frac{a_{free}}{q_{free}}$ of the free system to the subspace defined by the constraints. It then also explains -without any calculation whatsoever- why a system constrained from a divergence stable free system may not be divergence unstable, thus signing the universal ki.s.s. of conservative elastic systems.

On the contrary, for the non conservative elastic systems, the previous Eq. (27)

$$b_{free}(Y) = b_{cons}(Z_2) + b_{ad}(Z_2)$$

and the divergence stability variational formulation involving the quadratic form b show that we cannot consider the simple restriction of the functions h or j involved in this variational formulation. That suggests that the ki.s.s. is probably no more universal but eventually only conditional. It is now tackled in the following paragraph by using the intrinsic geometric language.

4.2. Divergence stability criterion

According to the first section, the variational formulation of divergence involves $j = \frac{b}{q}$ in the general case where the system is no longer elastic conservative. Clearly, there are subspaces F of E and some $x \in F$, with $b_F(x) = (u_F(x)|u_F(x)) = (p \circ u(x)|p \circ u(x)) \neq (u(x)|u(x))$. It follows that the divergence ki.s.s. can only be conditional. In fact, we already know (see for example [2] or [3]) and we already reported in the introduction, that, as long as the second order work criterion holds, the ki.s.s. of the divergence stability criterion is ensured and that when the second order work criterion fails, the ki.s.s. of the divergence stability criterion fails as well. The approach used in [2] or in [3] was then purely algebraic and is very remote from the one used here. We will now show however that we can find the same results thanks to this new approach.

First, suppose that u is no longer positive definite and let $x^* \neq 0$ such that $(x^*|u(x^*)) = 0$. This means that the SOWC fails and that x^* belongs to the isotropic cone $\mathcal{C}(u_s)$. Because the free system Σ_{free} is supposed divergence stable, meaning that u is invertible, we then have $u(x^*) \neq 0$. Let us choose a constraint C such that $F = H(A) = \langle u(x^*) \rangle^\perp$ meaning $T(A) = \langle u(x^*) \rangle$. Thus $x^* \in H(A) = F$ because $(x^*|u(x^*)) = 0$. Thus, $u_F(x^*) = p \circ u(x^*) = 0$ and then $b_F(x^*) = (u_F(x^*)|u_F(x^*)) = 0$. The constrained system is divergence unstable and the constraint C must be chosen so that $A = u^{-1}(x^*)$.

Reciprocally, suppose that divergence ki.s.s. fails for a family A of constraints. Then there is $x^* \in H(A) \setminus \{0\}$ such that $b_{H(A)}(x^*) = 0$ or $u_{H(A)}(x^*) = 0$ in E . Thus $p \circ u(x^*) = 0$ meaning that $u(x^*) \in H(A)^\perp = T(A)$ and then $(x^*|u(x^*)) = 0$ signing the loss of SOWC. This above double implication then gives exactly the same results as those proved in [2] and in [3] through a radically different method. We can finally remark that, contrary to the following flutter criterion, we got an optimal result without ever having evaluated the optimum, either by calculating critical points or by any numerical approach, although the set of optimization variables is a compact differential manifold -as will be seen hereafter. Indeed, the previous reasoning means that (u being invertible)

$$\min_{F \subset E} \min_{x \in S(E) \cap F} b_F(x) = 0 \iff \exists x \in E \setminus \{0\} (u(x)|x) = 0 \iff C(u_s) \neq \{0\} \quad (30)$$

signing the divergence instability thanks to a convenient choice of constraints but without having evaluated $\min_{F \subset E} \min_{x \in S(E) \cap F} b_F(x)$. An analogous remarkable result holds for the divergence of elastic conservative systems with the so-called Euler criterion for divergence ($\det K(p) = 0$) which does not require the calculation of the minimum of Rayleigh's quotient R involving the mass matrix (or equivalently the minimum of $j = \frac{a}{q}$ but involving the matrix \tilde{K} instead K) even if the divergence instability occurs by definition when this minimum is nil.

4.3. Flutter stability criterion

Flutter instabilities lead us to investigate the following algebro-geometric issue: let E be a n dimensional euclidean space, let $u \in \mathcal{L}(E)$ be a morphism of E that we may suppose with n distinct real eigenvalues and then \mathbb{R} -diagonalizable. Does a subspace F^c of E exist (or finding conditions for the existence of such a subspace) such that $u_{F^c} = p_{F^c} \circ u|_{F^c} \in \mathcal{L}(F^c)$ is no longer \mathbb{R} -diagonalizable where p_{F^c} is the orthogonal projection on F^c ? In this formulation, the linear map is associated with the matrix \tilde{K} : mass distribution and therefore mass matrix are necessarily involved in flutter (in) stability.

Let us remark first that if u is symmetric, then u_F is symmetric too and then \mathbb{R} -diagonalizable: flutter does not occur even for constrained systems and the flutter ki.s.s. issue is objectless.

Suppose now that u is any u and note, according to the first section, $\Delta(F) = \Delta(\det(u_F + sI_n)) = \Delta(\chi_{u_F})$ the discriminant of the characteristic polynomial χ_{u_F} of u_F . The flutter ki.s.s. is then equivalent to the condition

$$\min_{F \subset E} \Delta(F) > 0.$$

If $Gr_k(E)$ denotes the Grassmannian manifold of the k - planes of E and if $Gr(E) = \bigcup_{2 \leq k \leq n-1} Gr_k(E)$ the flutter ki.s.s. condition reads:

$$\min_{F \in Gr(E)} \Delta(F) > 0. \quad (31)$$

$Gr(E)$ is a compact set because each Grassmannian manifold is itself compact and $F \mapsto \Delta(F)$ is continuous, which proves that there is a subspace $F^c \in Gr(E)$ such that $\Delta(F^c) = \min_{F \in Gr(E)} \Delta(F)$. Let us remark that, even with the algebraic criterion of flutter, the ki.s.s. leads to a minimization formulation but with subspaces instead of vectors as optimization variables. The issue to calculate F^c and to conclude about flutter ki.s.s. will be reported in a forthcoming paper [17].

4.4. Rewording with the geometric language

To conclude this section, the geometric language proposed in paragraph 3.3 is used for rewording the ki.s.s. issues and the results.

The problems read:

- For the divergence ki.s.s., the issue is: what condition must satisfy an invertible linear map $u \in \mathcal{L}(E)$ in order that its compressions on any (non nil) subspace F remains invertible?
- For the flutter ki.s.s., the issue is: what condition must satisfy an \mathbb{R} -diagonalizable linear map $u \in \mathcal{L}(E)$ (with simple eigenvalues if necessary) in order that its compressions on any (non nil) subspace F remains \mathbb{R} -diagonalizable?

The solution for the flutter ki.s.s. is deferred until a forthcoming paper and the solution for the divergence ki.s.s. reads:

Theorem 1 (divergence ki.s.s. in terms of compression). *Let E be an Euclidean space and $u \in \mathcal{GL}(E)$. In order that any compression of u on any subspace $F \neq \{0\}$ of E is invertible (in fact is only an injective map or in other words in finite dimension $u_F \in \mathcal{GL}(F) \forall F \neq \{0\}$), it is necessary and sufficient that its symmetric part $u_s = \frac{1}{2}(u + u^*)$ is definite. Otherwise, one can find (and choose!!) F such that its intersection with the invert image by u of the isotropic cone C of u_s is not reduced to $\{0\}$: $F \cap (u^{-1}(C)) \neq \{0\}$.*

This is a reformulation in terms of compressions of the previous result detailed in 4.2. As already noted, it may be extended to real Hilbert spaces but only with the property of injectivity and assuming that u is bounded:

□ Let H be a real Hilbert space and $u \in \mathcal{L}_c(H)$ an injective linear continuous map. Suppose first that all the compression on (closed) subspaces are injective maps but that u_s is not definite. There is $x \neq 0$ such that $(u_s(x)|x) = (u(x)|x) = 0$ and $u(x) \neq 0$ because u injective. Let $F = \text{Vect}(u(x))^\perp$ and u_F the compression of u on F . Because $(u(x)|x) = 0$, then $x \in F$ and $u_F(x) = p_F(u(x)) = 0$ because $F = \langle u(x) \rangle^\perp$: u_F is then not injective. Conversely suppose that there is a subspace $F \neq \{0\}$ such that u_F is not injective. It means that there is $x \in F$ with $u_F(x) = 0$ namely $p_F(u(x)) = 0$ or equivalently $u(x) \in F^\perp$. But because $x \in F$, then $(u(x)|x) = 0$ and u or u_s is not definite and x is on the isotropic cone C of u_s . In this case $x \in F \cap u^{-1}(C)$ must be not nil which proves the last assertion. Remark finally that we may reasoning only with one dimensional subspaces F of H by choosing $F = \text{Vect}(x)$ in the reasoning.

5. Numerical features of divergence variational formulation

5.1. Systems with follower force

In addition to a new insight about divergence ki.s.s. of non conservative systems, the variational formulation used above to get the result involving the second order work criterion may also be used per se to investigate directly the divergence stability and to get the critical load. In order to compare both methods, we use again the same mechanical systems as

the one used in [3]. It consists in a three degree of freedom Ziegler system Σ as in Fig. 1 made up of three bars OA , AB , BC with $OA = AB = AC = \ell$ linked by three elastic springs of the same stiffness k . The nonconservative external action (the circulatory force) is the follower force \vec{P} .

The elastic energy of the springs is

$$U = \frac{k}{2} \left((\theta_1^2 + (\theta_1 - \theta_2)^2 + (\theta_2 - \theta_3)^2) \right) = \frac{k}{2} (2\theta_1^2 + 2\theta_2^2 + \theta_3^2 - 2\theta_1\theta_2 - 2\theta_2\theta_3)$$

and the virtual power of \vec{P} in any configuration $\theta = (\theta_1, \theta_2, \theta_3)$ reads ($P > 0$ in compression):

$$\mathcal{P}_p^* = P\ell(\sin(\theta_3 - \theta_1)\theta_1^* + \sin(\theta_3 - \theta_2)\theta_2^*).$$

Put $p = \frac{P\ell}{k}$ as dimensionless loading parameter and noting that $(0, 0, 0)$ is the unique equilibrium configuration, the stiffness matrix then reads:

$$K(p) = \begin{pmatrix} 2-p & -1 & p \\ -1 & 2-p & -1+p \\ 0 & -1 & 1 \end{pmatrix}$$

and $\det(K(p)) = 1$ does not depend on p .

We will also use a system with a partial follower force $\vec{P} = \vec{P}(\gamma)$ with $\gamma \in [0, 1]$ ($\gamma = 0$ corresponds to a conservative ‘‘vertical’’ force and $\gamma = 1$ corresponds to a complete follower force). We also consider such a system because the Euler criterion of divergence through the determinant then depends on p . To compare both criteria, following [18], we then use a two degree of freedom Ziegler system with a partial follower force corresponding to $\gamma = \frac{1}{2}$ as in Fig. 2. The stiffness matrix then reads:

$$K_{\gamma=\frac{1}{2}}(p) = \begin{pmatrix} 2-p & -1+\frac{1}{2}p \\ -1 & 1-\frac{1}{2}p \end{pmatrix}$$

and $\det K_{\gamma=\frac{1}{2}}(p) = 1 - \frac{3}{2}p + \frac{1}{2}p^2$.

5.2. Divergence of the free system Σ

5.2.1. 3 Dof system with complete follower force

$\det(K(p)) = 1$ is independent of p proving that the system cannot be unstable by divergence for any value of the loading parameter p . In fact, this result shows how irrelevant this algebraic criterion of divergence stability may be for a practical use. First, it gives no critical load in terms of divergence instability but it does not make any difference between a small value of p and a higher load value. We may say that the discrimination power of this criterion is low as it is often the case for algebraic criteria.

Using the variational approach leads us to consider the quadratic form b on the sphere $S(\mathbb{R}^3)$ of \mathbb{R}^3 . Obviously, this variational approach equivalent to the algebraic one allowing the calculation of the divergence critical load must also not lead to any divergence critical load. That means that the minimum of b on the sphere $S(\mathbb{R}^3)$ must be > 0 for any value $p(\geq 0)$ of the load parameter. However, this approach shows a better discrimination power.

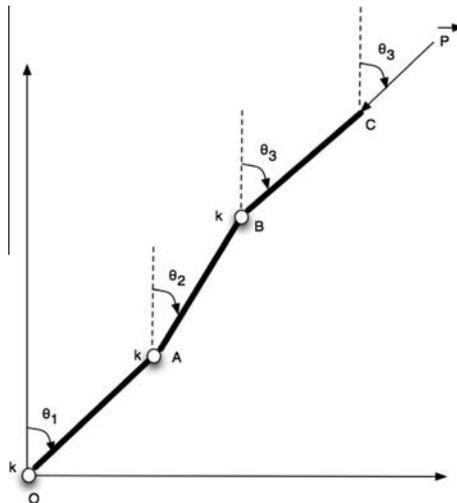


Fig. 1. 3 Dof Ziegler system.

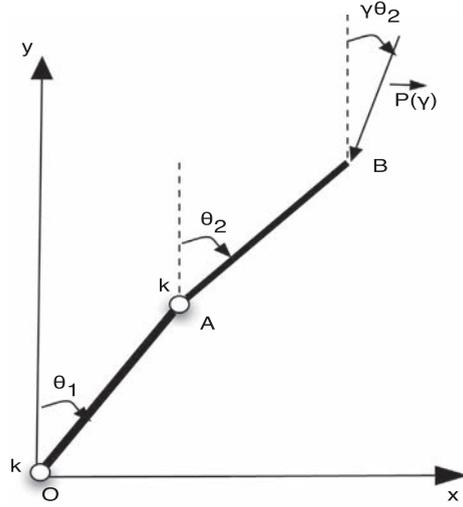


Fig. 2. 2 Dof Ziegler system with partial follower force.

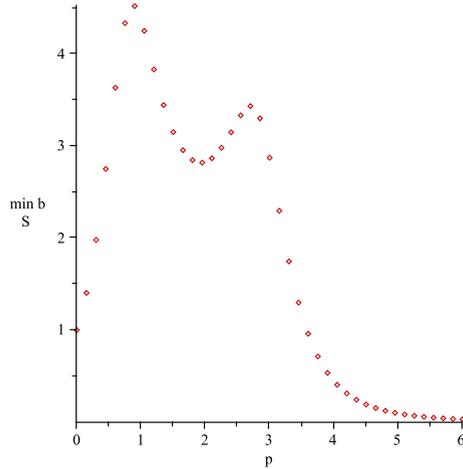


Fig. 3. Minimum of b on the sphere for a 3 dof Ziegler system with a complete follower force.

Indeed, the Figure 3 shows the variations of the minimum of b on the sphere $S = S(\mathbb{R}^3)$ vs the load parameter p and normalized with respect to the pure elastic conservative value corresponding to $p = 0$. Whereas the measure of the divergence instability through the value of the determinant $\det(K(p))$ cannot report on the real structural weakening of the systems as p increases, the decreasing toward 0 of $\min_{x \in S} b(x)$ when $p \rightarrow \infty$ and practically when $p \geq 5$ shows that the variational formulation is more relevant (for a threshold limit at 10%). This property is emphasized by a comparison with the spectrum of the system (here of $K(p)$). The curve $p \mapsto \omega_1(p)$ where $\omega_1(p)$ is the minimum of the spectrum of $K(p)$ (Fig. 4) is in accordance with the previous one about the structural weakening of the system concerning the divergence stability practically for $p \geq 5$ again (for the same threshold limit at 10%) even though, theoretically, there is no divergence of $\Sigma : \min_{x \in S} b(x) > 0$ for all $p > 0$ and $\omega_1(p) > 0$ for all $p > 0$. We emphasize that Figure 3 exhibits a similar behavior as the one of the minimum of the spectrum of K (Fig. 4) only for large values of p : both figures show the structural weakening of the structure when $p \rightarrow \infty$ and the divergence stability for all $p > 0$ as well (See Figs. 3 and 4).

5.2.2. 2 Dof system with partial ($\gamma = \frac{1}{2}$) follower force

Consider now the 2 dof system with a partial follower force measured by $\gamma = 0.5$. The curve giving the determinant of $K_{\gamma=0.5}$ as function of p the divergence of instability of the system for $p = 1$. By comparison, the minimum of b on the sphere (here the circle) vs the load parameter p reads as on Figure 6 which fortunately leads to the same value $p = 1$ of the divergence instability of the system (see Fig. 5, 6).

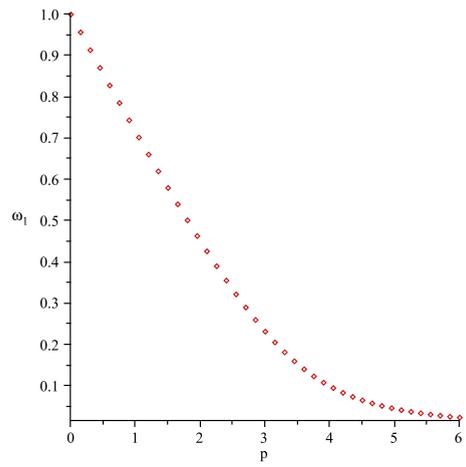


Fig. 4. Minimum of the spectrum of K .

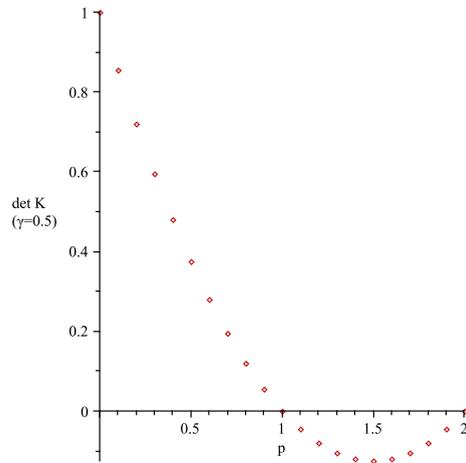


Fig. 5. Determinant of $K_{\gamma=0.5}$.

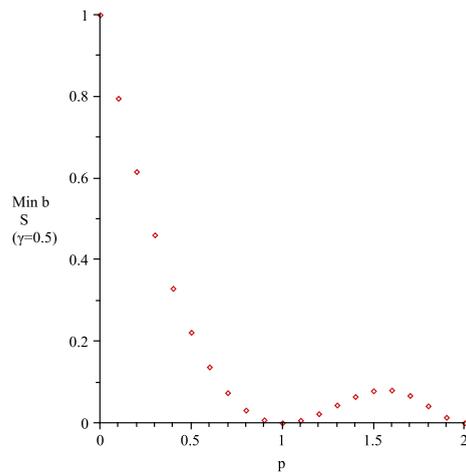


Fig. 6. Minimum of b on the sphere for a 2 dof Ziegler system with a partial follower force.

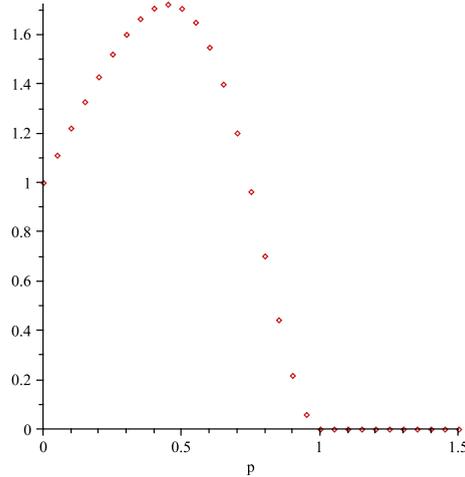


Fig. 7. $\min_{F \subset E} \min_{x \in S(E) \cap F} b_F(x)$.

5.3. Divergence of the constrained system

Going back to the ki.s.s. issue, we can now compare both approaches for getting the solution. We use the 3 dof Ziegler system. In [3], the solution has already been obtained in two different ways both using the concept of m -positive definite matrix. The first one, which is algorithmic, uses only operations of linear algebra and may be deduced from the proof of the main theorem about m -positive matrices. The second one consists in a direct minimization of the map $(B, p) \in S_{32} \times [0, p_b] \mapsto \det(B^T K(p) B)$ where S_{32} is a convenient compact of $\mathcal{M}_{32}(\mathbb{R})$. The common result fortunately gives $p = 1$, which is the smallest positive value of the load parameter that makes the second order work criterion fail.

For the sake of simplicity, we do not take into account the mass matrix meaning $M = I_3$ or $K = \tilde{K}$. The following figure shows the variations of $\min_{F \subset E} \min_{x \in S(E) \cap F} b_F(x)$ vs the load parameter p and normalized with respect to the pure elastic conservative value corresponding to $p = 0$. We consider only the case $\dim(F) = 1$ which corresponds to $3 - 1 = 2$ kinematic constraints. We then find again the singular value $p = 1$ leading to the divergence of a well-constrained system (see Fig. 7).

6. Conclusion

In this paper, the concept of kinematical structural stability for any criterion is defined. Its application to the divergence stability criterion and to the second order work criterion as well, is tackled through two autonomous developments. The first one consists in the building of a variational formulation for the divergence criterion and in the use of the quadratic forms language -as systematically as possible. The second one is the systematic elimination of Lagrange's multipliers that leads to a geometrical coordinate free formulation of the ki.s.s. issue by use of the tool of the compression of a linear map on a subspace. These results then lead to the surprising emergence of the second order work criterion as the optimal solution of the ki.s.s. issue for the divergence stability problem. This new insight improves the geometrical understanding of recent results for example in [2] and in [3]. They also lead us to take a common look at divergence stability for conservative and for non conservative systems as well. Last, it allows us to clearly define the ki.s.s. issue for the flutter criterion with the adequate mathematical tools. A complete forthcoming paper will be devoted to the investigation of this last problem.

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