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A real p -homogeneous seminorm with square property is submultiplicative

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Abstract. We give a functional representation theorem for a class of real p -Banach algebras. This theorem is used to show that every p -homogeneous seminorm with square property on a real associative algebra is submultiplicative.

Keywords. Functional representation, p -homogeneous seminorm, square property, submultiplicative.

1. Introduction

J. Arhippainen [1] has obtained the following result:

Theorem 1 of [1]. Let q be a p -homogeneous seminorm with square property on a complex associative algebra A . Then

1. $\text{Ker}(q)$ is an ideal of A ;
2. the quotient algebra $A/\text{Ker}(q)$ is commutative ;
3. q is submultiplicative ;
4. $q^{\frac{1}{p}}$ is a submultiplicative seminorm on A .

This result is a positive answer to a problem posed in [2] and considered in [3], [4] and [5]. The proofs of (3) and (4) depend on (2) which is obtained by using a locally bounded version of the Hirschfeld-Zelazko Theorem [1, Lemma 1]. This method can not be used in a real algebra; if q is the usual norm defined on the real algebra H of quaternions, $\text{Ker}(q) = \{0\}$ and $H/\text{Ker}(q) \cong H$ is noncommutative, then the assertion (2) does not hold in the real case.

The purpose of this paper is to provide a real algebra analogue of the above Arhippainen Theorem, this improves the result in [6]. Our method is based on a functional representation theorem which we will establish; it is an extension of the Abel-Jarosz Theorem [7, Theorem 1] to real p -Banach algebras. We also give a functional representation theorem for a class of complex p -Banach algebras. As a consequence, we obtain the main result in [8].

2. Preliminaries

Let A be an associative algebra over the field $K=\mathbb{R}$ or \mathbb{C} . Let $p \in]0, 1]$, a map $\|\cdot\| : A \rightarrow [0, \infty[$ is a p -homogeneous seminorm if for a, b in A and α in K , $\|a+b\| \leq \|a\| + \|b\|$ and $\|\alpha a\| = |\alpha|^p \|a\|$. Moreover, if $\|a\| = 0$ imply that $a = 0$, $\|\cdot\|$ is called a p -homogeneous norm. A 1-homogeneous seminorm (resp. norm) is called a seminorm (resp. norm). $\|\cdot\|$ is submultiplicative if $\|ab\| \leq \|a\| \|b\|$ for all a, b in A . $\|\cdot\|$ has the square property if $\|a^2\| = \|a\|^2$ for all $a \in A$. If $\|\cdot\|$ is a submultiplicative p -homogeneous norm on A , then $(A, \|\cdot\|)$ is called a p -normed algebra, we denote by $M(A)$ the set of all nonzero continuous multiplicative linear functionals on A . A complete p -normed algebra is called a p -Banach algebra. A uniform p -normed algebra is a p -normed algebra $(A, \|\cdot\|)$ such that $\|a^2\| = \|a\|^2$ for all $a \in A$. Let A be a complex algebra with unit e , the spectrum of an element $a \in A$ is defined by $Sp(a) = \{\alpha \in \mathbb{C}, \alpha e - a \notin A^{-1}\}$ where A^{-1} is the set of all invertible elements of A . Let A be a real algebra with unit e , the spectrum of $a \in A$ is defined by $Sp(a) = \{s + it \in \mathbb{C}, (a - se)^2 + t^2 e \notin A^{-1}\}$. Let A be an algebra, the spectral radius of an element $a \in A$ is defined by $r(a) = \sup\{|\alpha|, \alpha \in Sp(a)\}$. Let $(A, \|\cdot\|)$ be a p -normed algebra, the limit $\lim_{n \rightarrow \infty} \|a^n\|^{\frac{1}{pn}}$ exists for each $a \in A$, and if A is complete, we have $r(a) = \lim_{n \rightarrow \infty} \|a^n\|^{\frac{1}{pn}}$ for all $a \in A$. A $*$ -algebra is a complex algebra with a mapping $*$: $A \rightarrow A, a \rightarrow a^*$, such that, for a, b in A and $\alpha \in \mathbb{C}, (a^*)^* = a, (a+b)^* = a^* + b^*, (\alpha a)^* = \bar{\alpha} a^*, (ab)^* = b^* a^*$. The map $*$ is called an involution on A . An element $a \in A$ is said to be hermitian if $a^* = a$. The set of all hermitian elements of A is denoted by $H(A)$.

3. A functional representation theorem for a class of real p -Banach algebras

We will need the following result due to B. Aupetit and J. Zemanek ([9] or [10]), their algebraic approach works for real p -Banach algebras.

Theorem 3.1. Let A be a real p -Banach algebra with unit. If there is a positive constant α such that $r(ab) \leq \alpha r(a)r(b)$ for all a, b in A , then for every irreducible representation π of A on a real linear space E , the algebra $\pi(A)$ is isomorphic (algebraically) to its commutant in the algebra $L(E)$ of all linear transformations on E .

Let A be a real p -Banach algebra with unit such that $\|a\|^{\frac{1}{p}} \leq m r(a)$ for some positive constant m and all $a \in A$. Let $X(A)$ be the set of all nonzero multiplicative linear functionals from A into the noncommutative algebra H of quaternions. For $a \in A$, we consider the map $J(a) : X(A) \rightarrow H, J(a)x = x(a)$ for all $x \in X(A)$. We endow $X(A)$ with the weakest topology such that all the functions $J(a), a \in A$, are continuous. The map $J : A \rightarrow C(X(A), H), a \rightarrow J(a)$, is a homomorphism from A into the real algebra of all continuous functions from $X(A)$ into H .

Theorem 3.2. If π is an irreducible representation of A , then $\pi(A)$ is isomorphic to R , C or H .

Proof. Let $a, b \in A$ and $n \geq 1$, we have $\|(ab)^n\| \leq \|a\|^n \|b\|^n$, then $\|(ab)^n\|^{\frac{1}{pn}} \leq \|a\|^{\frac{1}{p}} \|b\|^{\frac{1}{p}}$. Letting $n \rightarrow \infty$, we obtain $r(ab) \leq m^2 r(a)r(b)$. Let π be an irreducible representation of A on a real linear space E . By Theorem 3.1, $\pi(A)$ is isomorphic to its commutant Q in the algebra $L(E)$ of all linear transformations on E . Let y_0 be a fixed nonzero element in E . For $y \in E$, we consider $\|y\|_E = \inf\{\|a\|, a \in A \text{ and } \pi(a)y_0 = y\}$. By the same proof as in [11, Lemma 6.5], $\|\cdot\|_E$ is a p -norm on E and Q is a real division p -normed algebra of continuous linear operators on E . By [12], Q is isomorphic to R , C or H .

Proposition 3.3. A is semisimple and $X(A)$ is a nonempty set which separates the elements of A .

Proof. By the condition $\|a\|^{\frac{1}{p}} \leq mr(a)$ for all $a \in A$, we deduce that A is semisimple. Let a be a nonzero element in A , since A is semisimple, there is an irreducible representation π of A such that $\pi(a) \neq 0$. By Theorem 3.2, there is $\varphi : \pi(A) \rightarrow H$ an isomorphism (into). We consider the map $T = \varphi \circ \pi, T : A \rightarrow H$ is a multiplicative linear functional. Moreover, $T(a) = \varphi(\pi(a)) \neq 0$ since $\pi(a) \neq 0$ and φ is injective.

Proposition 3.4.

1. $|x(a)| \leq \|a\|^{\frac{1}{p}}$ for all $a \in A$ and $x \in X(A)$;
2. An element a is invertible in A if and only if $J(a)$ is invertible in $C(X(A), H)$;
3. $Sp(a) = Sp(J(a))$ for all $a \in A$.

Proof. (1): Since H is a real uniform Banach algebra under the usual norm $|\cdot|$, $|x(a)| = r_H(x(a)) \leq r_A(a) \leq \|a\|^{\frac{1}{p}}$ for all $a \in A$ and $x \in X(A)$.

(2): The direct implication is obvious. Conversely, let π be an irreducible representation of A . By Theorem 3.2, there is $\varphi : \pi(A) \rightarrow H$ an isomorphism (into). Since $\varphi \circ \pi \in X(A)$ and $J(a)$ is invertible, $0 \neq J(a)(\varphi \circ \pi) = \varphi(\pi(a))$, then $\pi(a) \neq 0$. Consequently, a is invertible.

(3): $s + it \in Sp(a)$ iff $(a - se)^2 + t^2 e \notin A^{-1}$
 Iff $J((a - se)^2 + t^2 e) \notin C(X(A), H)^{-1}$ by (2)
 Iff $(J(a) - sJ(e))^2 + t^2 J(e) \notin C(X(A), H)^{-1}$
 Iff $s + it \in Sp(J(a))$.

Proposition 3.5. $X(A)$ is a Hausdorff compact space.

Proof. Let x_1, x_2 in $X(A), x_1 \neq x_2$, there is an element $a \in A$ such that $x_1(a) \neq x_2(a)$, i.e. $J(a)x_1 \neq J(a)x_2$, so $X(A)$ is Hausdorff. Let $a \in A$ and

$K_a = \{q \in H, |q| \leq \|a\|^{\frac{1}{p}}\}$, K_a is compact in H . Let K be the topological product of K_a for all $a \in A$, K is compact by the Tychonoff Theorem. By Proposition 3.4(1), $X(A)$ is a subset of K . It is easy to see that the topology of $X(A)$ is the relative topology from K and that $X(A)$ is closed in K . Then $X(A)$ is compact.

Theorem 3.6. The map $J : A \rightarrow C(X(A), H), a \rightarrow J(a)$, is an isomorphism (into) such that $m^{-1}\|a\|^{\frac{1}{p}} \leq \|J(a)\|_s \leq \|a\|^{\frac{1}{p}}$ for all $a \in A$, where $\|\cdot\|_s$ is the supnorm on $C(X(A), H)$. If $m = 1$, we have $\|a\|^{\frac{1}{p}} = \|J(a)\|_s$ for all $a \in A$.

Proof. By Proposition 3.3, J is an injective homomorphism. Let $a \in A$, by Proposition 3.4(3), $r(a) = r(J(a)) = \|J(a)\|_s$ since $C(X(A), H)$ is a real uniform Banach algebra under the supnorm $\|\cdot\|_s$. Moreover, $\|J(a)\|_s \leq \|a\|^{\frac{1}{p}}$ by Proposition 3.4(1). Then $m^{-1}\|a\|^{\frac{1}{p}} \leq r(a) = \|J(a)\|_s \leq \|a\|^{\frac{1}{p}}$.

As an application, we obtain an extension of the Kulkarni Theorem [13, Theorem 1] to real p-Banach algebras

Theorem 3.7. Let a be an element in A such that $Sp(a) \subset R$, then a belongs to the center of A .

Proof. By Theorem 3.6, $J : A \rightarrow C(X(A), H)$ is an isomorphism (into). Let $a \in A$ with $Sp(a) \subset R$. Let $x \in X(A)$ and $x(a) = s + t$ where $s \in R$ and $t = t_1i + t_2j + t_3k$. Suppose that $t \neq 0$. We have $(x(a) - s)^2 = t^2 = -(t_1^2 + t_2^2 + t_3^2) = -|t|^2$, then $(x(a) - s)^2 + |t|^2 = 0$. Consequently $s + i|t| \in Sp(x(a)) \subset Sp(a)$ with $|t| \neq 0$, a contradiction. Then $J(a) \in C(X(A), R)$ and $J(a)J(b) = J(b)J(a)$ for all b in A , i.e. $J(ab - ba) = 0$ for all b in A . Since J is injective, $ab - ba = 0$ for all b in A .

4. A functional representation theorem for a class of complex p-Banach algebras

Let $\|\cdot\|$ be a submultiplicative p-homogeneous seminorm on a complex algebra A . For $a \in A$, $|a|$ is defined as follows: $|a| = \inf \sum_{i=1}^n \|a_i\|^{\frac{1}{p}}$, where the infimum is taken over all decompositions of a satisfying the condition $a = \sum_{i=1}^n a_i$, $a_1, \dots, a_n \in A$. By [14, Theorem 1], $|\cdot|$ is a submultiplicative seminorm on A , it is called the support seminorm of $\|\cdot\|$. Also, it is shown [14] the following result:

Theorem 2 of [14]. Let A be a complex algebra, $\|\cdot\|$ a submultiplicative p-homogeneous seminorm on A , and $|\cdot|$ the support seminorm of $\|\cdot\|$. Then $\lim_{n \rightarrow \infty} \|a^n\|^{\frac{1}{p^n}} = \lim_{n \rightarrow \infty} |a^n|^{\frac{1}{n}}$ for all $a \in A$.

In the proof of this theorem, Xia Dao-Xing uses the following inequality: If $a = a_1 + \dots + a_m$ and $n \geq 1$, then $\|a^n\| \leq \sum_{\alpha_1 + \dots + \alpha_m = n} \left(\frac{n!}{\alpha_1! \dots \alpha_m!}\right)^p \|a_1\|^{\alpha_1} \dots \|a_m\|^{\alpha_m}$.

If the algebra is commutative, $a^n = (a_1 + \cdots + a_m)^n = \sum_{\alpha_1 + \cdots + \alpha_m = n} \frac{n!}{\alpha_1! \cdots \alpha_m!} a_1^{\alpha_1} \cdots a_m^{\alpha_m}$, then $\|a^n\| \leq \sum_{\alpha_1 + \cdots + \alpha_m = n} \left(\frac{n!}{\alpha_1! \cdots \alpha_m!}\right)^p \|a_1\|^{\alpha_1} \cdots \|a_m\|^{\alpha_m}$. This inequality is not justified in the noncommutative case; if the algebra is noncommutative, we only have $\|a^n\| \leq \sum_{\alpha_1 + \cdots + \alpha_m = n} \frac{n!}{\alpha_1! \cdots \alpha_m!} \|a_1\|^{\alpha_1} \cdots \|a_m\|^{\alpha_m}$. For the sequel, we will use Theorem 2 of [14] in the commutative case.

Theorem 4.1. Let $(A, \|\cdot\|)$ be a complex p-normed algebra such that $\|a\|^2 \leq m\|a^2\|$ for some positive constant m and all $a \in A$. Then $|a| \leq \|a\|^{\frac{1}{p}} \leq m^{\frac{1}{p}}|a|$ and $|a|^2 \leq m^{\frac{2}{p}}|a^2|$ for all $a \in A$, where $|\cdot|$ is the support seminorm of $\|\cdot\|$.

Proof. The completion B of $(A, \|\cdot\|)$ is a p-Banach algebra such that $\|b\|^2 \leq m\|b^2\|$ for all $b \in B$, it is commutative by [1, Lemma 1], so A is commutative. By induction, $\|a\| \leq m^{1-2^{-n}} \|a^{2^n}\|^{2^{-n}}$ for all $a \in A$ and $n \geq 1$, then $\|a\| \leq m \lim_{n \rightarrow \infty} \|a^n\|^{\frac{1}{n}}$ for all $a \in A$. By the commutative version of [14, Theorem 2], we have $|a| \leq \|a\|^{\frac{1}{p}} \leq m^{\frac{1}{p}} \lim_{n \rightarrow \infty} \|a^n\|^{\frac{1}{pn}} = m^{\frac{1}{p}} \lim_{n \rightarrow \infty} |a^n|^{\frac{1}{n}} \leq m^{\frac{1}{p}}|a|$ for all $a \in A$. From the above inequalities, $|a|^2 \leq \|a\|^{\frac{2}{p}} \leq (m\|a^2\|)^{\frac{1}{p}} \leq m^{\frac{2}{p}}|a^2|$.

Corollary 4.2. Let $(A, \|\cdot\|)$ be a complex uniform p-normed algebra. Then $|a| = \|a\|^{\frac{1}{p}}$ for all $a \in A$.

Theorem 4.3. Let $(A, \|\cdot\|)$ be a complex p-Banach algebra with unit such that $\|a\|^2 \leq m\|a^2\|$ for some positive constant m and all $a \in A$. Then the Gelfand map $G : A \rightarrow C(M(A))$ is an isomorphism (into) such that $m^{-\frac{1}{p}}\|a\|^{\frac{1}{p}} \leq m^{-\frac{1}{p}}|a| \leq \|G(a)\|_s \leq |a| \leq \|a\|^{\frac{1}{p}}$ for all $a \in A$, where $\|\cdot\|_s$ is the supnorm on $C(M(A))$.

Proof. A is commutative by [1, Lemma 1]. By Theorem 4.1, $|a| \leq \|a\|^{\frac{1}{p}} \leq m^{\frac{1}{p}}|a|$ for all $a \in A$, then $(A, |\cdot|)$ is a complex commutative Banach algebra with unit. Clearly $M(A) = M(A, \|\cdot\|) = M(A, |\cdot|)$ is a nonempty compact space. As in the proof of Theorem 4.1, we have $|a| \leq m^{\frac{1}{p}} \lim_{n \rightarrow \infty} |a^n|^{\frac{1}{n}} = m^{\frac{1}{p}} \sup\{|f(a)|, f \in M(A)\} = m^{\frac{1}{p}} \|G(a)\|_s \leq m^{\frac{1}{p}}|a|$. Let $a \in A$, from the above inequalities, $m^{-\frac{2}{p}}\|a\|^{\frac{1}{p}} \leq m^{-\frac{1}{p}}|a| \leq \|G(a)\|_s \leq |a| \leq \|a\|^{\frac{1}{p}}$.

Corollary 4.4. Let $(A, \|\cdot\|)$ be a complex uniform p-Banach algebra with unit. Then the Gelfand map $G : A \rightarrow C(M(A))$ is an isomorphism (into) such that $|a| = \|a\|^{\frac{1}{p}} = \|G(a)\|_s$ for all $a \in A$.

Theorem 4.5. Let $(A, \|\cdot\|)$ be a complex p-normed *-algebra with unit such that

1. $\|a\|^2 \leq m\|a^2\|$ for some positive constant m and all $a \in A$;
2. Every element in $H(A)$ has a real spectrum in the completion B of A .

Then the involution $*$ is continuous on A and the Gelfand map $G : B \rightarrow C(M(B))$ is a $*$ -isomorphism such that $m^{-\frac{2}{p}} \|b\|^{\frac{1}{p}} \leq \|G(b)\|_s \leq \|b\|^{\frac{1}{p}}$ for all b in B .

Proof. By Theorem 4.3, it remains to show that the involution $*$ is continuous on A , $G(b^*) = G(b)^*$ for all $b \in B$, and G is surjective. Let $h \in H(A)$, $Sp_B(h) = \{f(h), f \in M(B)\} \subset R$ by (2). Let $a \in A$, we have $a = h_1 + ih_2$ with $h_1, h_2 \in H(A)$. Let $f \in M(B)$, $f(a^*) = f(h_1 - ih_2) = f(h_1) - if(h_2) = (f(h_1) + if(h_2))^* = f(h_1 + ih_2)^* = f(a)^*$ since $f(h_1)$ and $f(h_2)$ are real. Then $G(a^*) = G(a)^*$ for all $a \in A$. By Theorem 4.3, $m^{-\frac{2}{p}} \|a^*\|^{\frac{1}{p}} \leq \|G(a^*)\|_s = \|G(a)^*\|_s = \|G(a)\|_s \leq \|a\|^{\frac{1}{p}}$ for all $a \in A$, then $\|a^*\| \leq m^2 \|a\|$ for all $a \in A$. Consequently, the involution $*$ is continuous on A and can be extended to a continuous involution on B which we will also denote by $*$. Let $b \in B$, there exists a sequence $(a_n)_n$ in A such that $a_n \rightarrow b$. Since the involution on B and the Gelfand map $G : B \rightarrow C(M(B))$ are continuous, we have $G(a_n^*) \rightarrow G(b^*)$ and $G(a_n)^* \rightarrow G(b)^*$, then $G(b^*) = G(b)^*$. By the Stone-Weierstrass Theorem, we deduce that G is surjective.

As a consequence, we obtain the main result in [8].

Corollary 4.6. Let A be a complex uniform p -normed $*$ -algebra with unit such that every element in $H(A)$ has a real spectrum in the completion B of A . then B is a commutative C^* -algebra.

5. The main result

Theorem 5.1. Let A be a real associative algebra. Every p -homogeneous seminorm q with square property on A is submultiplicative and $q^{\frac{1}{p}}$ is a submultiplicative seminorm on A .

Proof. By [1], there exists a positive constant m such that $q(ab) \leq mq(a)q(b)$ for all $a, b \in A$. $Ker(q)$ is an ideal of A , the norm $|\cdot|$ on the quotient algebra $A/Ker(q)$ defined by $|a + Ker(q)| = q(a)$ is a p -norm with square property. Define $\|a + Ker(q)\| = m|a + Ker(q)|$ for all $a \in A$. Let $a, b \in A$, $\|ab + Ker(q)\| = m|ab + Ker(q)| \leq m^2|a + Ker(q)||b + Ker(q)| = \|a + Ker(q)\| \|b + Ker(q)\|$, then $(A/Ker(q), \|\cdot\|)$ is a real p -normed algebra. Let $a \in A$, $\|a^2 + Ker(q)\| = m|a^2 + Ker(q)| = m|a + Ker(q)|^2 = m^{-1}(m|a + Ker(q)|)^2 = m^{-1}\|a + Ker(q)\|^2$ i.e. $\|a + Ker(q)\|^2 = m\|a^2 + Ker(q)\|$. The completion B of $(A/Ker(q), \|\cdot\|)$ satisfies also the property $\|b\|^2 = m\|b^2\|$ for all $b \in B$, and by induction $\|b\| = m^{1-2^{-n}} \|b^{2^n}\|^{2^{-n}}$ for all $b \in B$ and $n \geq 1$, then $\|b\| = mr(b)^p$ for all $b \in B$. We consider two cases:

B is unital: By section 3, $X(B)$ is a nonempty compact space and the map $J : B \rightarrow C(X(B), H)$ is an isomorphism (into). By Proposition 3.4(3), $r(b) = r(J(b))$ for all $b \in B$. Let $b \in B$, $\|b\| = mr(b)^p = mr(J(b))^p = m\|J(b)\|_s^p$ since $C(X(B), H)$ is a real uniform Banach algebra under the supnorm $\|\cdot\|_s$. Then

$|b| = m^{-1}\|b\| = \|J(b)\|_s^p$ for all $b \in A/Ker(q)$, so $|\cdot|$ is submultiplicative and $|\cdot|^{\frac{1}{p}}$ is a submultiplicative norm. Consequently, q is submultiplicative and $q^{\frac{1}{p}}$ is a submultiplicative seminorm.

B is not unital: Let B_1 be the algebra obtained from B by adjoining the unit. By the same proof of [15, Lemma 2] which works for real p-Banach algebras, there exists a p-norm N on B_1 such that

1. (B_1, N) is a real p-Banach algebra with unit;
2. $N(b)^{\frac{1}{p}} \leq m^3 r_{B_1}(b)$ for all $b \in B_1$;
3. N and $\|\cdot\|$ are equivalent on B .

By section 3, $X(B_1)$ is a nonempty compact space and the map $J : B_1 \rightarrow C(X(B_1), H)$ is an isomorphism (into). Let $b \in B$,

$$\|b\| = mr_B(b)^p = mr_{B_1}(b)^p \text{ by (3)}$$

$$= mr(J(b))^p \text{ by Proposition 3.4(3)}$$

$$= m\|J(b)\|_s^p \text{ by the square property of the supnorm.}$$

Then $|b| = m^{-1}\|b\| = \|J(b)\|_s^p$ for all $b \in A/Ker(q)$, so $|\cdot|$ is submultiplicative and $|\cdot|^{\frac{1}{p}}$ is a submultiplicative norm. Consequently, q is submultiplicative and $q^{\frac{1}{p}}$ is a submultiplicative seminorm.

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